DISSIPATIVE DYNAMICAL SYSTEMS

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THEME

A dissipative system absorbs 'supply' (e.g., energy).

How do we formalize this?

Involves the storage function.

How is it constructed? Is it unique?

$\rightsquigarrow$ KYP, LMI's, ARE's.

Where is this notion applied in systems and control?

OUTLINE

- **Lyapunov theory**
- **Dissipative systems**
- **Physical examples**
- **Construction of the storage function**
- **LQ theory** $\rightsquigarrow$ **LMI's, etc.**
- **Applications in systems and control**
- **Extensions**
- **Polynomial spectral factorization**
- **Recapitulation**

LYAPUNOV THEORY

LYAPUNOV FUNCTIONS

Consider the classical 'dynamical system', the *flow*

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^n$, the *state space*, $f : \mathbb{X} \to \mathbb{X}$. Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathfrak{B}$, the 'behavior'. The function

$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a Lyapunov function for Σ if along $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalent to $\qquad \dot{V}^{\Sigma} := \nabla V \cdot f \leq 0$

Typical Lyapunov 'theorem':

$$V(x) > 0 \text{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \text{ for } 0 \neq x \in \mathbb{X}$$

$$\Longrightarrow$$

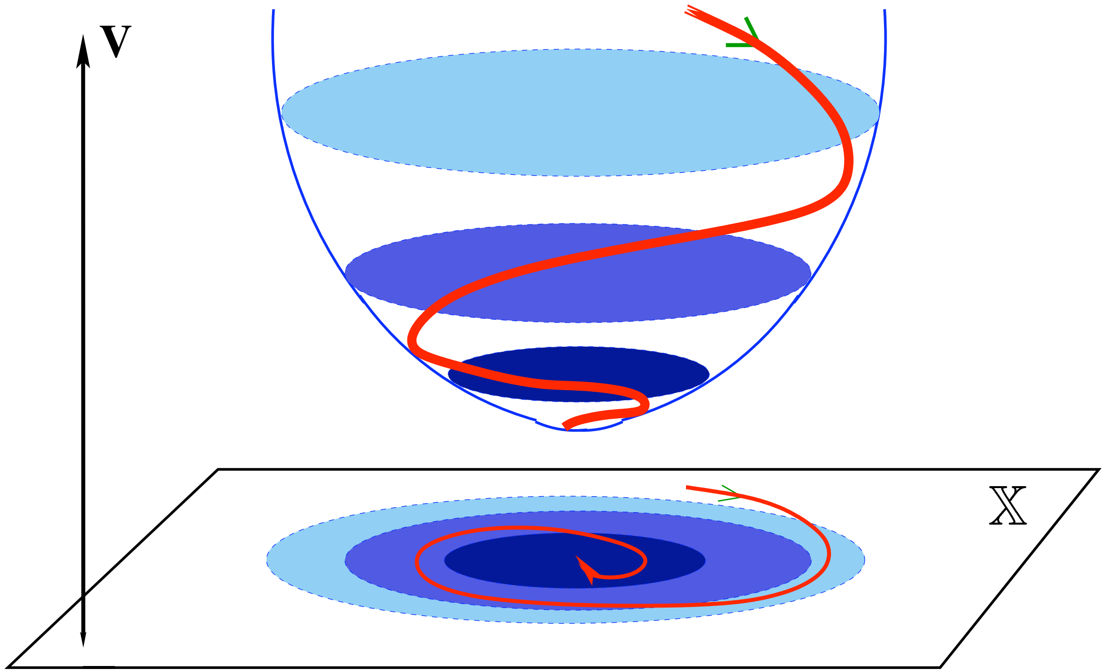
$$\forall\ x \in \mathfrak{B},\ \text{there holds } x(t) \to 0 \text{ for } t \to \infty$$ 'global stability'

Refinements: LaSalle's invariance principle.

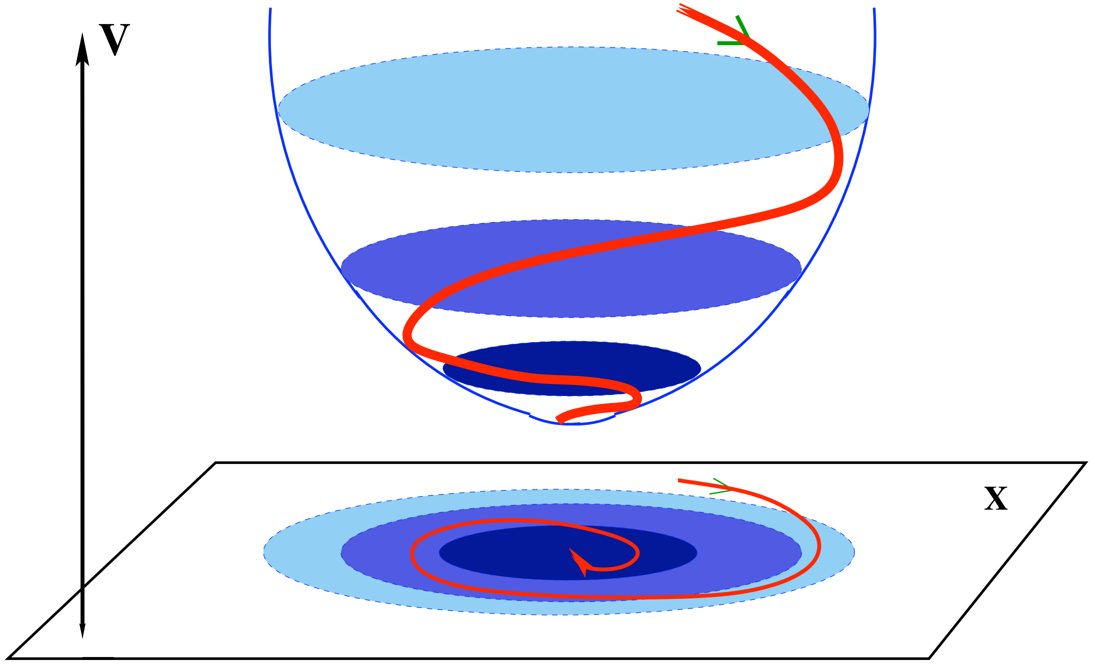
Converse: Kurzweil's thm.

LQ theory *leadsto* $A^\top X + XA = Y$ 'Lyapunov (matrix) equation'. A linear system is stable iff it has a quadratic positive definite Lyapunov function.

Basis for most stability results in control, physics, adaptation, even numerical analysis, system identification.



V
X



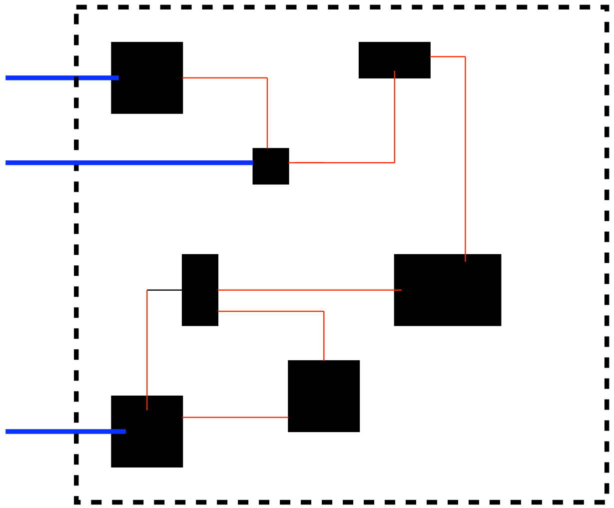
Aleksandr Mikhailovich Lyapunov (1857-1918)

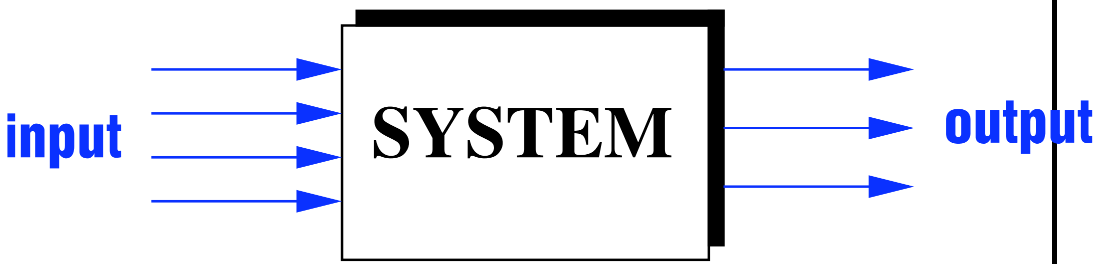
Studied mechanics, differential equations.

Introduced Lyapunov's 'second method' in his Ph.D. thesis (1899).

DISSIPATIVE SYSTEMS

A much more appropriate starting point for the study of dynamics are 'open' systems. ⤳





INPUT/STATE/OUTPUT SYSTEMS

Consider the 'dynamical system'

$$\Sigma : \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **the input, output, state.**

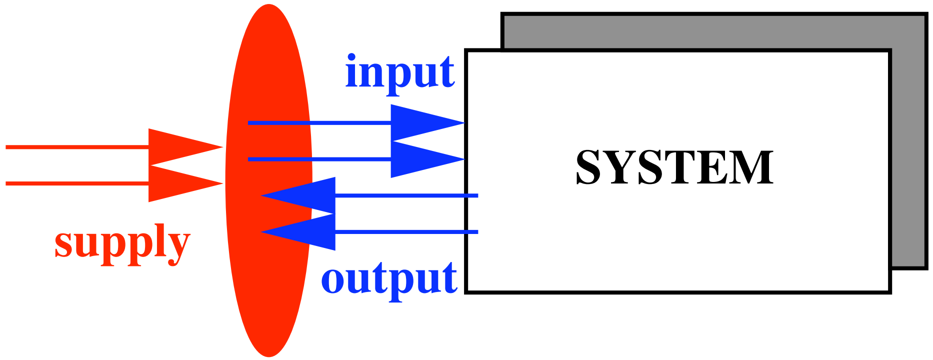
Behavior $\mathfrak{B} =$ **all sol'ns** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Let

$$s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$$

be a function, called the *supply rate*.

$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.



DISSIPATIVITY

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function,* such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

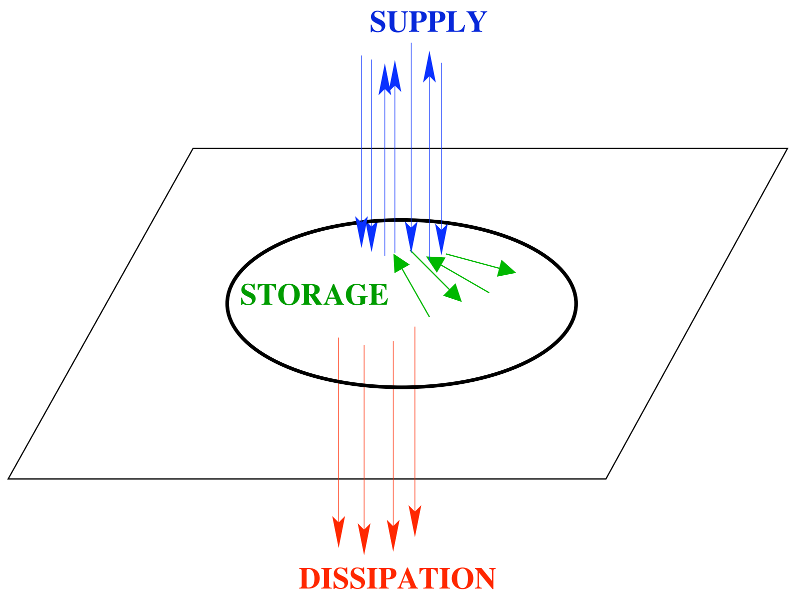
along input/output/state trajectories ($\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

This inequality is called the *dissipation inequality.*

Equivalent to $\quad \overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$
for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: 'conservative' system.

Dissipativeness :$\Leftrightarrow$ Increase in storage $\leq$ Supply.



Special case: 'closed system': $\boxed{s = 0}$

then dissipativeness $\leftrightarrow$ V **is a Lyapunov function.**

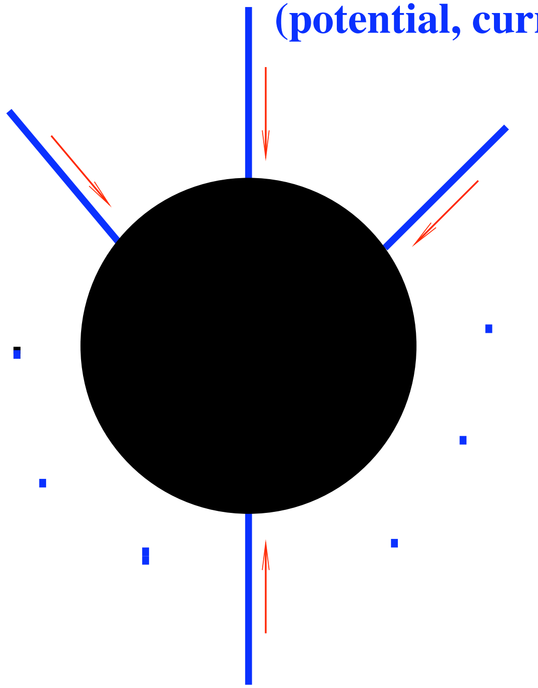
Dissipativity is a natural generalization of LF to open systems.

Stability for closed systems $\simeq$ **Dissipativity for open systems.**

PHYSICAL EXAMPLES

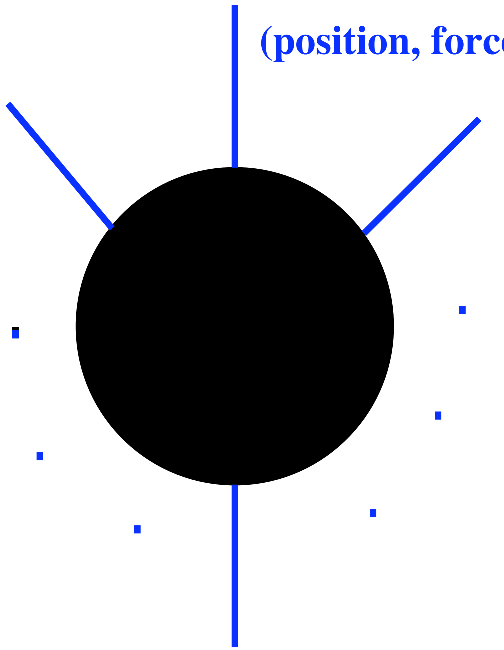
System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy
Thermodynamic system	$Q + W$ Q : heat, W : work	internal energy
Thermodynamic system	$-Q/T$ Q : heat, T : temp.	entropy
etc.	etc.	etc.

Electrical circuit:



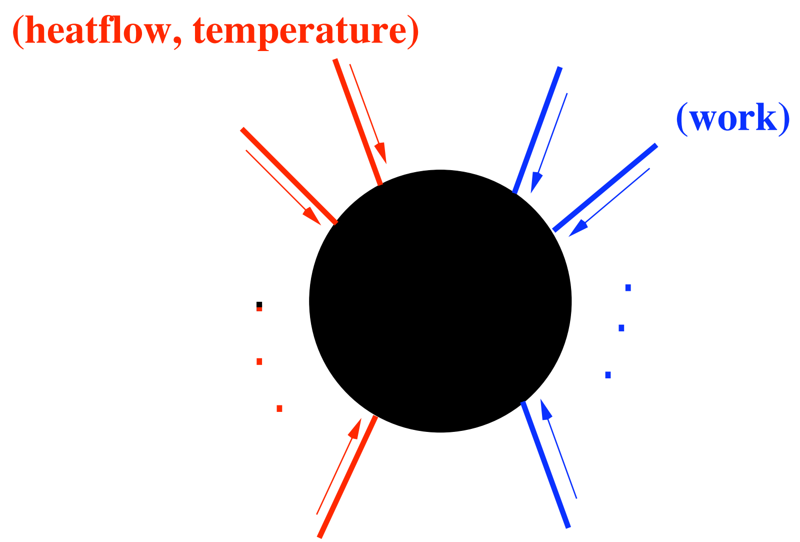
Dissipative w.r.t. $\Sigma_{\ell=1}^{N} V_\ell I_\ell$ **(electrical power).**

Mechanical device:



Dissipative w.r.t. $\Sigma_{\ell=1}^{N}((\frac{d}{dt}q_\ell)^\top F_\ell + (\frac{d}{dt}\theta_\ell)^\top T_\ell)$ **(mech. power).**

Thermodynamic system:



Conservative w.r.t. $\Sigma_{\ell=1}^{N} Q_\ell + \Sigma_{\ell=1}^{N'} W_\ell$,

Dissipative w.r.t. $-\Sigma_{\ell=1}^{N} \frac{Q_\ell}{T\ell}$.

THE CONSTRUCTION OF STORAGE FUNCTIONS

Central question:

Given (a representation of) Σ, the dynamics, and given s, the supply rate, is the system dissipative w.r.t. s, i.e., does there exist a storage function V such that the dissipation inequality holds?

Assume henceforth that a number of (reasonable) conditions hold:

$f(0,0) = 0, h(0,0) = 0, s(0,0) = 0;$

Maps and functions (including V) smooth;

State space $\mathbb{X}$ of Σ 'connected':

every state reachable from every other state;

Observability.

'Thm': Let Σ and s be given.

Then Σ is dissipative w.r.t. w iff

$$\boxed{\oint s(u(\cdot), y(\cdot)) \ dt \geq 0}$$

for all periodic $(u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$.

Two universal storage functions:

1. The available storage

$$V_{\text{available}}(x_0) :=$$

$$\sup_{(u(\cdot),y(\cdot),x(\cdot))\in\mathfrak{B},x(0)=x_0,x(\infty)=0} \quad \{-\int_0^{+\infty} s(u(\cdot),y(\cdot)) \ dt\}$$

2. The required supply

$$V_{\text{required}}(x_0) :=$$

$$\inf_{(u(\cdot),y(\cdot),x(\cdot))\in\mathfrak{B},x(-\infty)=0,x(0)=x_0} \quad \{\int_{-\infty}^{0} s(u(\cdot),y(\cdot)) \ dt\}$$

Storage f'ns form convex set, every storage function satisfies

$$\boxed{V_{\text{available}} \leq V \leq V_{\text{required}}.}$$

LINEAR SYSTEMS with QUADRATIC SUPPLY RATES

Assume Σ linear, time-invariant, finite-dimensional:

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx,$$

and s quadratic: e.g.,

$$s : (u, y) \mapsto ||u||^2 - ||y||^2.$$

E.g., for circuits $u = \frac{V+I}{2}$, $y = \frac{V-I}{2}$, etc.

Assume (A, B) controllable, (A, C) observable.
$G(s) := D + C(Is - A)^{-1}B$, the transfer function of Σ.

Theorem: **The following are equivalent:**

1. **Σ is dissipative w.r.t. s (i.e., there exists a storage function V),**

2. **$\forall (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B} \cap \mathcal{L}_2$,**

$$\boxed{||u(\cdot)||_{\mathcal{L}_2} \leq ||y(\cdot)||_{\mathcal{L}_2},}$$

3. $\boxed{||G(i\omega)|| \leq 1}$ **for all $\omega \in \mathbb{R}$,**

4. **$\exists$ a quadratic storage f'n, $V(x) = x^\top K x, K = K^\top$,**

5. there exists a solution $K = K^\top$ to the
Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^\top K + KA + C^\top C & KB \\ B^\top K & -I \end{bmatrix} \leq 0,$$

6. there exists a solution $K = K^\top$ to the
Algebraic Riccati Inequality (ARIneq)

$$A^\top K + KA + KBB^\top K + C^\top C \leq 0,$$

7. there exists a solution $K = K^\top$ to the
Algebraic Riccati Equation (ARE)

$$A^\top K + KA + KBB^\top K + C^\top C = 0.$$

Solution set (of LMI, ARineq) is convex, compact, and attains its infimum and its supremum:

$$K^{-} \leq K \leq K^{+}$$

These extreme sol'ns K^{-} and K^{+} themselves satisfy the ARE.

Extensive theory, relation with other system representations, many applications, well-understood (also algorithmically).

Connection with optimal LQ control, semi-definite programming.

Important refinement:

Existence of a $V \geq 0$ (i.e., bounded from below) (energy?)

$$\rightsquigarrow \int_{-\infty}^{0} s(u(\cdot), y(\cdot)) \; dt \geq 0.$$

In LQ case $\Leftrightarrow$

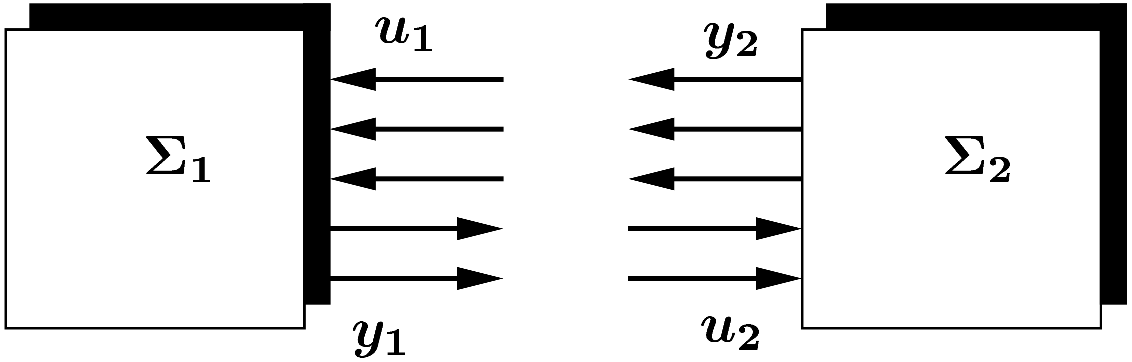
- $\int_{-\infty}^{0} ||u(\cdot)||^2 \, dt \geq \int_{-\infty}^{0} ||y(\cdot)||^2 \, dt,$
- $\sup_{\{s \in \mathbb{C} | \mathrm{Re}(s) > 0\}} ||G(s)|| =: ||G||_{\mathcal{H}_\infty} \leq 1,$

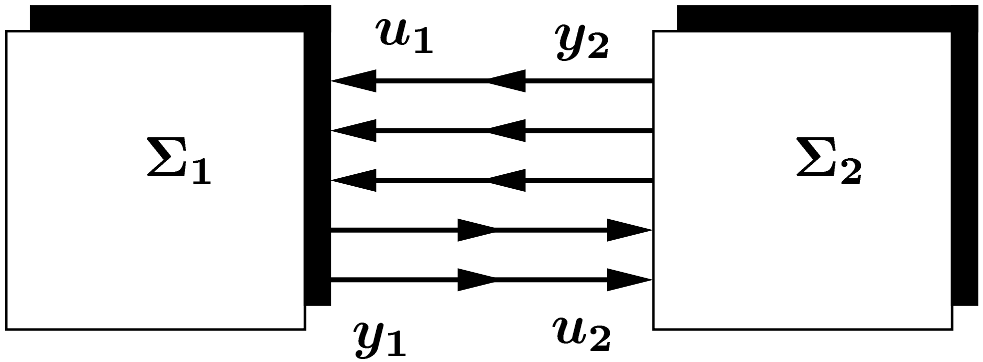
  Note def. of $\mathcal{H}_\infty$-norm !
- $\exists$ sol'n $K = K^\top \geq 0$ to LMI, ARineq, ARE.

$\rightsquigarrow$ KYP-lemma.

APPLICATIONS

- **Synthesis of RLC-circuits**
- $\Rightarrow$ **Robust stability and stabilization**
- $\Rightarrow$ **Norm estimation**
- ...





Interconnection laws: $u_1 = y_2, u_2 = y_1$.

Interconnected system: $\Sigma_1 \wedge \Sigma_2$.

Assume

- (Σ_1, s_1) dissipative, storage f'n V_1,
- (Σ_2, s_2) dissipative, storage f'n V_2,
- $s_1(u_1, y_1) + s_2(y_2, u_2) = 0.$

  For example,

$$s_1 : u_1, y_1 \mapsto ||u_1||^2 - \rho^2||y_1||^2,$$
$$s_2 : u_2, y_2 \mapsto \rho^2||u_2||^2 - ||y_2||^2;$$

or $s_1 : u_1, y_1 \mapsto u_1^\top y_1, s_2 : u_2, y_2 \mapsto -u_2^\top y_2.$

Then $V_1 + V_2$ is a Lyapunov function

for the interconnected system $\Sigma_1 \wedge \Sigma_2.$

Proof:

$$\frac{d}{dt} V_1(x_1(\cdot)) \leq s_1(u_1(\cdot), y_1(\cdot))$$

$$\frac{d}{dt} V_2(x_2(\cdot)) \leq s_2(u_2(\cdot), y_2(\cdot))$$

$$\Rightarrow \frac{d}{dt}(V_1(x_1(\cdot)) + V_2(x_2(\cdot)) \leq 0.$$

$\rightsquigarrow$ **Small gain theorem, Positive operator theorem, Robust stability.**

ROBUST STABILITY

Σ_1: linear, time-invariant, transfer f'n G,

Σ_2: uncertain system,

e.g. memoryless: $u_2 \mapsto y_2 = f(u_2, t)$ with

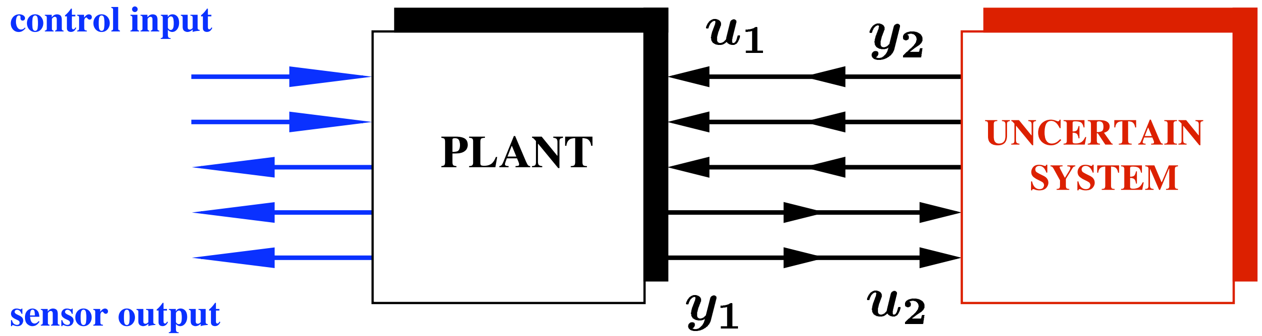
$$\frac{||f(u_2, t)||}{||u_2||} \leq \rho \quad \forall u_2, t.$$

Then

$$||G||_{\mathcal{H}_\infty} < \frac{1}{\rho} \Rightarrow \Sigma_1 \wedge \Sigma_2 \text{ stable.}$$

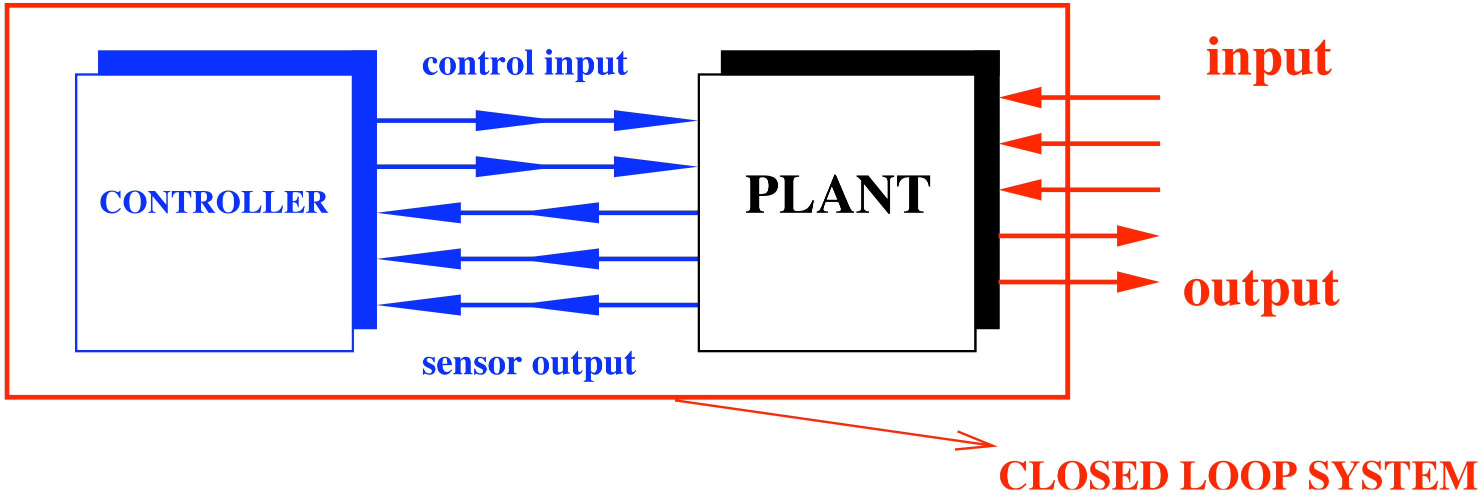
Quadratic LF: $V(x) = x^\top K x$, K from LMI, ARIneq, or ARE.

Leads to:



!! Stabilize robustly

Find a controller that stabilizes for a whole class of systems at once.



!! Given plant Σ_{plant}, find controller $\Sigma_{\mathrm{controller}}$ such that

$$\boxed{||G_{\mathrm{controlled}}||_{\mathcal{H}_\infty} < \frac{1}{\rho}}$$

$\rightsquigarrow$ $\mathcal{H}_\infty$-control theory, synthesis of dissipative systems.

Application:

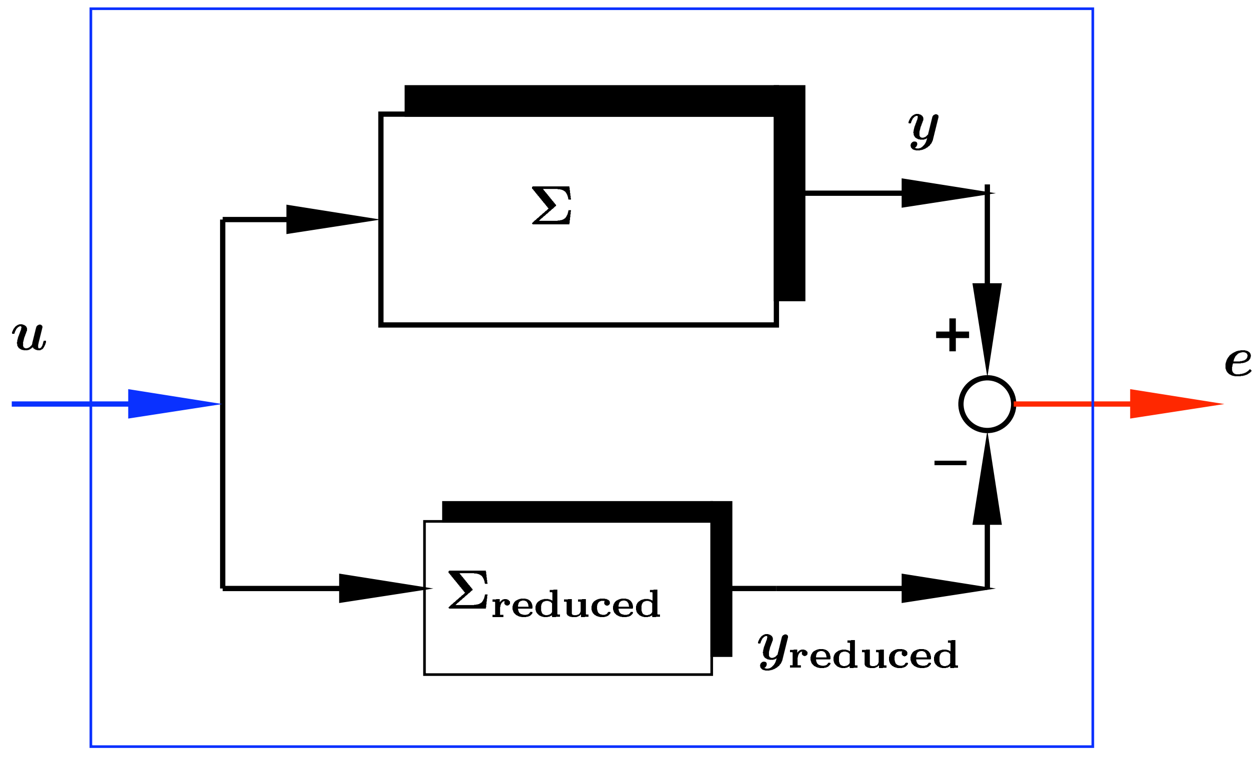
SYSTEM NORM ESTIMATION

Model reduction:

Given a (linear, time-invariant) system Σ, find a system Σ_{reduced}, with a low dimensional state space, that approximates Σ well.

'Well' $\rightsquigarrow$ **small $\mathcal{H}_\infty$-norm of tf f'n $u \mapsto e = y - y_{\text{reduced}}$:**

$$||G - G_{\text{reduced}}||_{\mathcal{H}_\infty} = \sup_{0 \neq u \in \mathcal{L}_2} \frac{||y - y_{\text{reduced}}||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}}$$



u
Σ
y
+
e
−
Σreduced
yreduced

Some beautiful results have been obtained, in particular:

Balanced reduction of linear systems:

Let $G \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}(s)$ be a strictly proper, $\mathcal{H}_\infty$ transfer function.
Then G admits a representation

$$\boxed{\Sigma: \ \frac{d}{dt}x = Ax + Bu, \quad y = Cx,}$$

with (A, B) controllable, (A, C) observable, A Hurwitz.

Moreover, Σ can be made to be balanced
(controllability grammian = observability grammian):

$$A \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} + \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} A^\top + B^\top B = 0$$

$$A^\top \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} + \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} A \ + CC^\top = 0$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ the Hankel SV's of the system.

Assume $\sigma_1, \ldots, \sigma_k$ 'large', $\sigma_k >> \sigma_{k+1}, \sigma_{k+1} \ldots, \sigma_n$ 'small'.

Neglect $x_{k+1}, \cdots, x_n$ (heuristic: these are the state components that are both most difficult to reach and most difficult to observe).

With the obvious partitioning of A, B, C as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

we obtain the k-order reduced system

$$\Sigma_{\text{reduced}}: \quad \frac{d}{dt}x_1 = A_{11}x_1 + B_1 u, \quad y_{\text{reduced}} = C_1 x_1.$$

Call its transfer f'n $G_{\text{reduced}}(s) = C_1(Is - A_{11})^{-1}B_1$.

Question:

How close is Σ_{reduced} to Σ?

!! Estimate $||G - G_{\text{reduced}}||_{\mathcal{H}_\infty} = \sup_{0 \neq u \in \mathcal{L}_2} \frac{||y - y_{\text{reduced}}||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}}$.

Theorem (Glover):

$$||G - G_{\text{reduced}}||_{\mathcal{H}_\infty} \leq 2(\sigma_{k'+1} + \sigma_{k'+2} + \cdots + \sigma_{n'})$$

with k' such that $\sigma_{k'} = \sigma_k$, where $\sigma'_1 > \sigma'_2 > \cdots > \sigma'_{n'} > 0$ are the distinct Hankel SV's of the system.

$$\begin{aligned} \sigma_1 = \sigma_2 = \cdots = \sigma_{n_1} &=: \sigma'_1 \\ > \sigma_{n_1+1} = \sigma_{n_1+2} = \cdots = \sigma_{n_1+n_2} &=: \sigma'_2 \\ > \cdots & \\ > \sigma_{n_1 \cdots + n_{n'-1}+1} = \sigma_{n_1+\cdots+n_{n'-1}+2} = \cdots = \sigma_n &=: \sigma'_{n'} \end{aligned}$$

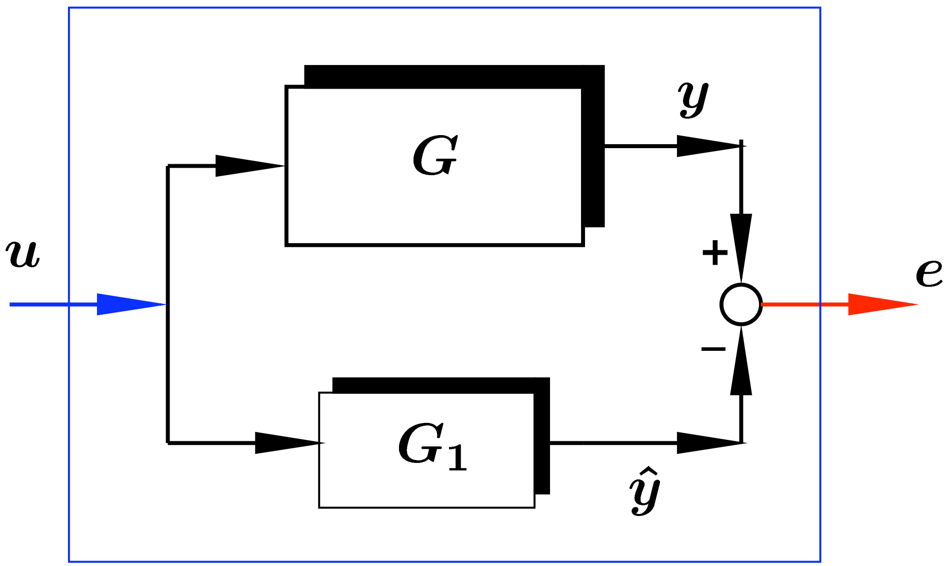
$\mathcal{H}_\infty$−bound $\leq$ 2∗ sum of the neglected SV's without repetition.

Proof using dissipative systems:

Step 1: Neglect ONE (possibly repeated) SV:

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix}.$$

Consider the 'error system':



Its dynamics:

$$
\begin{aligned}
\frac{d}{dt}x_1 &= A_{11}x_1 + A_{12}x_2 + B_1u, \\
\frac{d}{dt}x_2 &= A_{21}x_1 + A_{22}x_2 + B_2u, \\
y &= C_1x_1 + C_2x_2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\hat{x}_1 &= A_{11}\hat{x}_1 + B_1u, \\
\hat{y} &= C_1\hat{x}_1,
\end{aligned}
$$

$$
e = y - \hat{y}.
$$

Relations between system parameters due to balancing:

$$
\begin{aligned}
A_{11}\Sigma_1 + \Sigma_1 A_1^\top &= -B_1 B_1^\top, \\
A_{11}^\top \Sigma_1 + \Sigma_1 A_1 &= -C_1^\top C_1, \\
A_{12}\sigma + \Sigma_1 A_{21}^\top &= -B_1 B_2^\top, \\
A_{21}^\top \sigma + \Sigma_1 A_{12} &= -C_1^\top C_2, \\
\sigma(A_{22} + A_{22}^\top) &= -B_2 B_2^\top = -C_2^\top C_2.
\end{aligned}
$$

Now verify (straightforward, tedious):

$$\frac{d}{dt}[(x_1 - \hat{x}_1)^\top \frac{\Sigma_1}{\sigma}(x_1 - \hat{x}_1) + (x_1 + \hat{x}_1)^\top (\frac{\Sigma_1}{\sigma})^{-1}(x_1 + \hat{x}_1)$$

$$+2\, x_2^\top x_2]$$

$$= ||2\sqrt{\sigma}u||^2 - ||\sqrt{\sigma}^{-1}e||^2$$

$$-||2\sqrt{\sigma}u - \sqrt{\sigma}B_1^\top \Sigma_1^{-1}(x_1 + \hat{x}_1) - \sqrt{\sigma}^{-1}B_2^\top x_2||^2.$$

Whence $\frac{d}{dt}V(x) \leq (2\sigma)^2||u||^2 - ||e||^2.$

Conclude, using LMI-theory, $||G - G_1||_{\mathcal{H}_\infty} \leq 2\sigma.$

Step 2: Triangle inequality. In the obvious notation $G - G_{\text{reduced}}$

$$= (G_{n'} - G_{n'-1}) + (G_{n'-1} - G_{n'-2}) + \cdots + (G_{k'+1} - G_{k'}),$$

where G_ℓ= the balanced representation truncated at σ'_ℓ: neglect $\sigma'_{\ell+1}, \ldots, \sigma'{n'}$; $G_{n'} = G, G_{\text{reduced}} = G_{k'}$.

Whence

$$||G - G_{\text{reduced}}||_{\mathcal{H}_\infty} \leq$$

$$||G_{n'} - G_{n'-1}||_{\mathcal{H}_\infty} + ||G_{n'-1} - G_{n'-2}||_{\mathcal{H}_\infty} + \cdots + ||G_{k'+1} - G_{k'}||_{\mathcal{H}_\infty}$$

Combine step 1 and step 2:

$$||G - G_{\text{reduced}}||_{\mathcal{H}_\infty} \leq 2\sigma_{n'} + 2\sigma_{n'-1} + \cdots + 2\sigma_{k'+1}$$

Open problems: improve bound, find storage f'n for $G - G_{\text{reduced}}$.

Generalizations

- **Drawback 1**: requires separation of interaction variables in inputs and outputs

  $\rightsquigarrow$ **Behavioral systems.**

- **Drawback 2**: imposes storage function = state function.

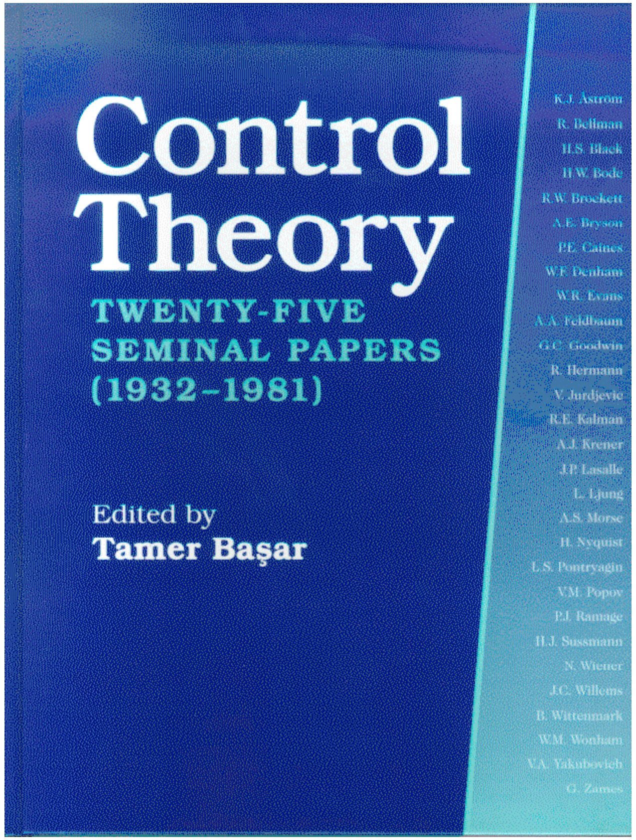
  This is something one would like to prove!

- **Drawback 3**: limited to dynamical (as opposed to distributed, PDE) systems.

Recap

The notion of a dissipative system:

- **Generalization of 'Lyapunov function' to open systems**
- **Central concept in control theory: many applications to feedback stability, robust ($\mathcal{H}_\infty$-) control, adaptive control, system identification, passivation control**
- **Other applications: system norm estimates**
- **passive electrical circuit synthesis procedures**
- **Natural systems concept for the analysis of physical systems**
- **Notable special case: second law of thermodynamics**
- **Forms a tread through modern system theory**


Control Theory
TWENTY-FIVE SEMINAL PAPERS (1932–1981)
Edited by
Tamer Başar
K.J. Åström
R. Bellman
H.S. Black
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R.W. Brockett
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THANK YOU !