STATE

and

FIRST ORDER REPRESENTATIONS

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CONJECTURE

A system with behavior described by

a linear constant coefficient PDE is Markovian

if and only if

the PDE is first order.

OUTLINE

1. **Behaviors of PDE's**
2. **Nice partitions of $\mathbb{R}^n$**
3. **Markovian systems**
4. **A precise statement of the conjecture**

n-D linear differential systems

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider the system of linear constant coefficient PDE's

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

This defines an $\mathtt{n}$-D system with behavior

$$\mathfrak{B} = \{w \in \mathfrak{D}'(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}.$$

We refer to such behaviors as 'linear differential behaviors', and to $(*)$ as a 'kernel representation' of the behavior $\mathfrak{B}$.

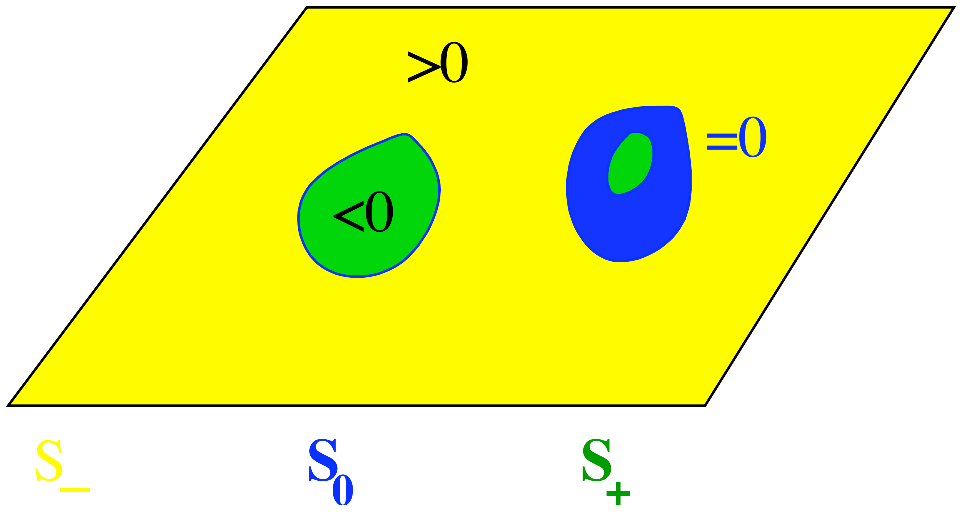
Note: $\mathfrak{B}$ has many kernel representations.

Nice partitions

(S_-, S_0, S_+) is said to be a nice partition of $\mathbb{R}^n$ if S_- and S_+ are open and S_0 is closed. I.e., (because we like to think of 'past' and 'future') if there exists a continuous function $\rho : \mathbb{R} \to \mathbb{R}^n$ such that

$$\rho(x) \quad \begin{cases} < 0 & \text{if} \quad x \in S_- \\ = 0 & \text{if} \quad x \in S_0 \\ > 0 & \text{if} \quad x \in S_+ \end{cases}$$

You may think of S_- as the past, S_0 as the present, S_+ as the future.



A nice partition

Markovian $\mathtt{n}$-D behaviors

A linear differential behavior $\mathfrak{B}$ is said to be

Markovian

if whenever (S_-, S_0, S_+) is a nice partition of $\mathbb{R}^{\mathtt{n}}$
and w_1, w_2 are elements of $\mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ such that

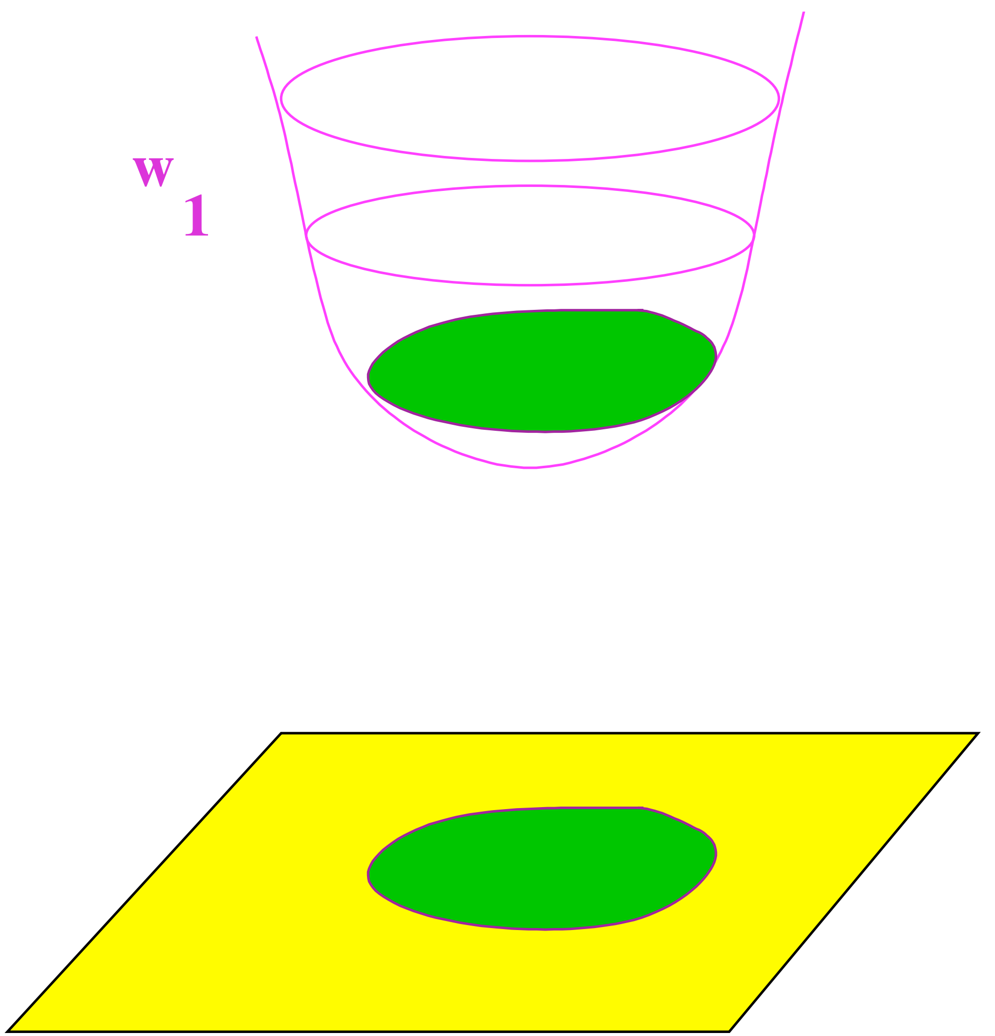
$$w_1(x) = w_2(x) \qquad \text{for } x \in S_0,$$

then

$$w_1 \underset{S_0}{\wedge} w_2 \in \mathfrak{B}.$$

$w_1 \underset{S_0}{\wedge} w_2$ **denotes the concatenation along S_0, i.e.**

$$(w_1 \underset{S_0}{\wedge} w_2)(x) := \begin{cases} w_1(x) & \text{for} \quad x \in S_- \\ w_1(x) = w_2(x) & \text{for} \quad x \in S_0 \\ w_2(x) & \text{for} \quad x \in S_+ \end{cases}$$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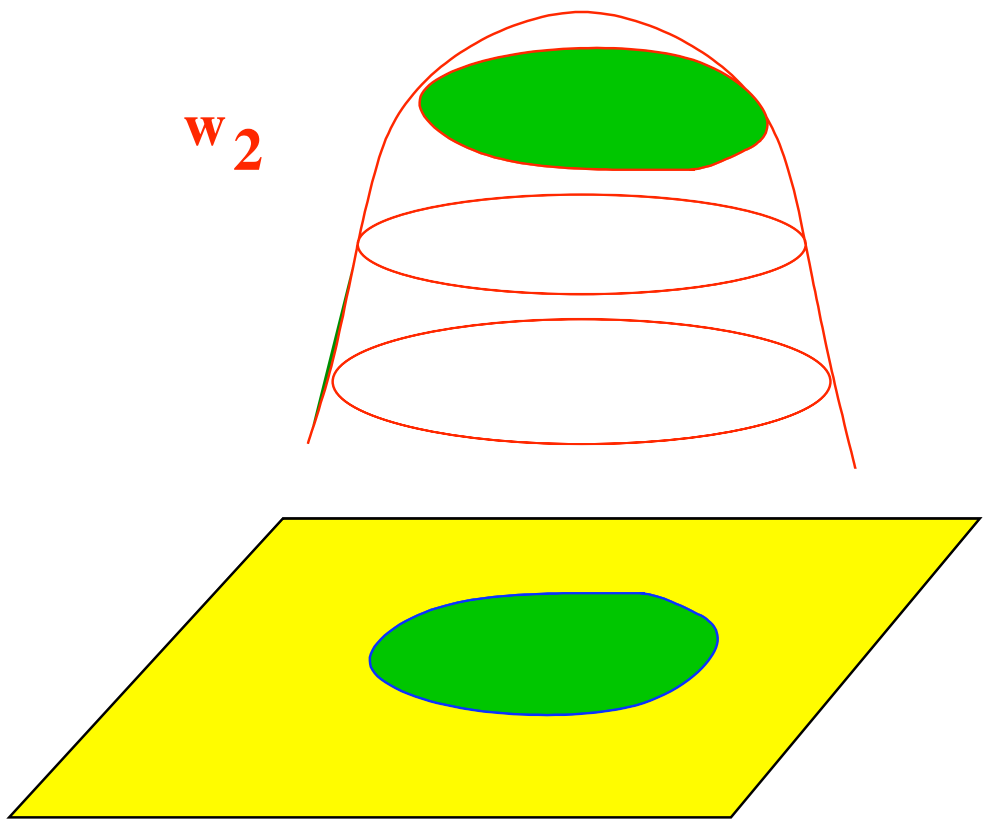
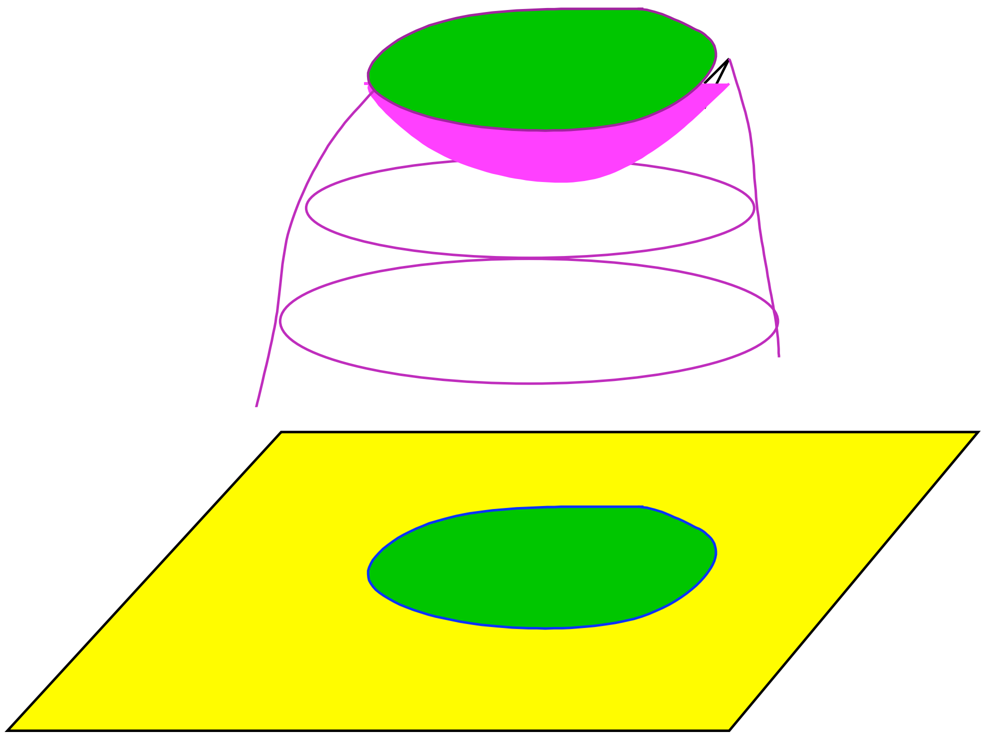

w_2

$$w_1 \underset{S_0}{\wedge} w_2$$



Precise CONJECTURE

The linear differential behavior $\mathfrak{B}$ is Markovian

if and only if

it admits a kernel representation that is

first order.

I.e. a kernel representation of the form

$$R_0 w + R_1 \frac{\partial}{\partial x_1} w + R_1 \frac{\partial}{\partial x_2} w + \cdots + R_1 \frac{\partial}{\partial x_n} w = 0.$$

Notes: The "if"-part is easy.

The conjecture is true for $\mathtt{n} = 1$, i.e. in the ODE case.

Example of a Markovian system: Maxwell's equations.

Non-examples: Diffusion equation; Wave equation.

Motivation: To understand state and state construction for $\mathtt{n}$-D systems.