BEHAVIORAL n-D SYSTEMS

Motivation& General Concepts

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Problematique:

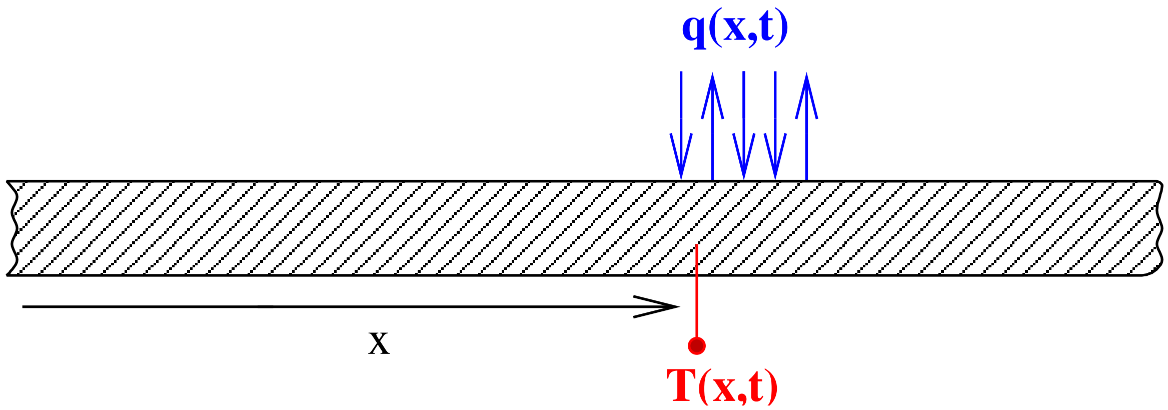
Develop a suitable *mathematical* framework for discussing dynamical / $\mathtt{n}$-D systems

aimed at modeling, analysis, and synthesis.

OUTLINE

1. Examples
2. Historical remarks
3. Examples, revisited
4. Behavioral systems
5. Linear distributed differential systems
6. Controllability & Observability
7. 3 theorems

1. *Heat diffusion*



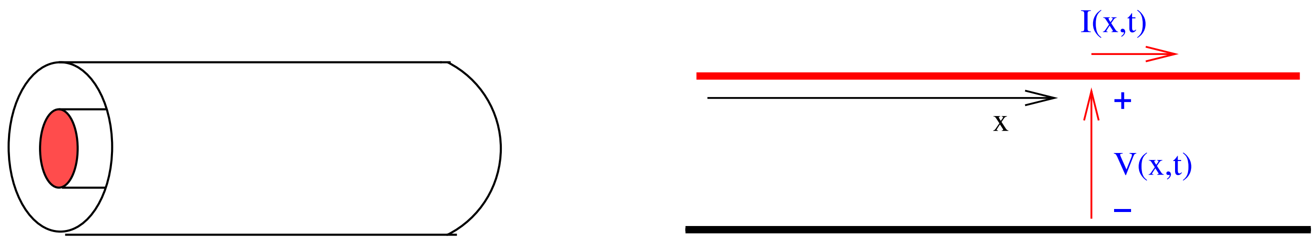
The PDE

$$\boxed{\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q}$$

describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) in a medium and the heat $q(x,T)$ supplied to / radiated away from it.

2. *Coaxial cable*

!! Model the relation between the voltage $V(x,t)$ and the current $I(x,t)$ in a coaxial cable.



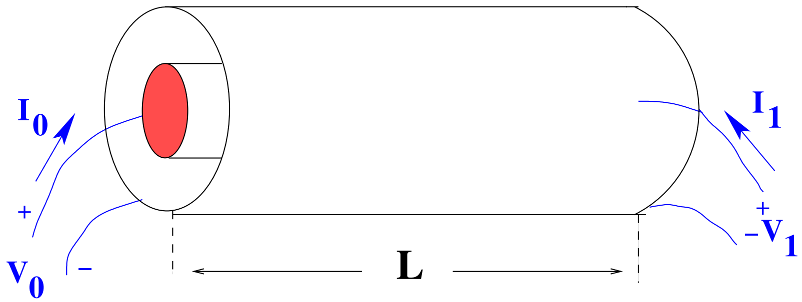
$\rightsquigarrow$ **The PDE's:**

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} V &= -L_0 \frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= -C_0 \frac{\partial}{\partial t} V. \end{aligned}}$$

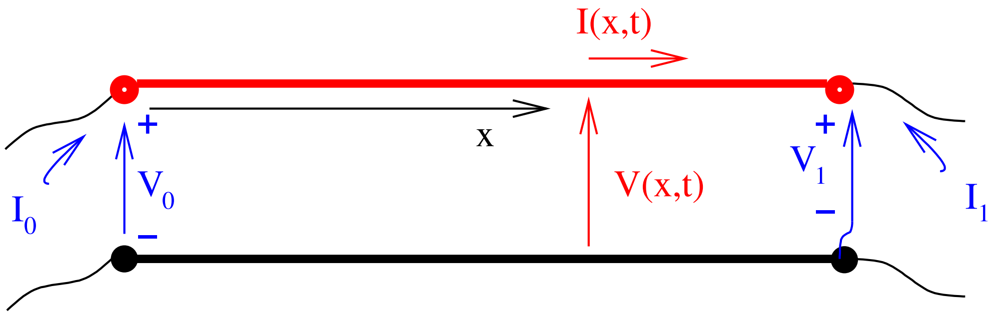
with L_0 the inductance, and C_0 the capacitance per unit length.

With boundary conditions (cable of length L):

!! Model the relation between the voltages V_0, V_1 and the currents I_0, I_1 at the ends of a uniform cable of length L.



Introduce the voltage $V(x,t)$ and the current flow $I(x,t)$ $0 \leq x \leq L$ in the cable.

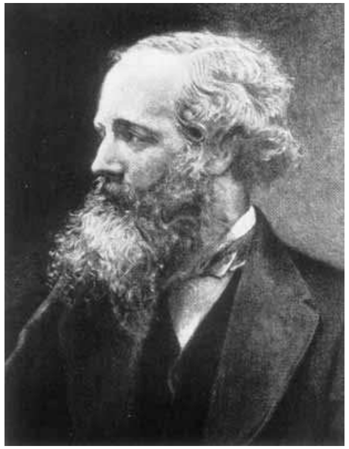


$\rightsquigarrow$ **The equations:**

$$\begin{aligned} \frac{\partial}{\partial x} V &= -L_0 \frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= -C_0 \frac{\partial}{\partial t} V, \end{aligned}$$

$$\begin{aligned} V_0(t) &= V(0,t), \\ V_1(t) &= V(L,t), \\ I_0(t) &= I(0,t), \\ I_1(t) &= -I(L,t). \end{aligned}$$

3. Maxwell's eqn's



$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0\,, \\ c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}. \end{aligned}$$

We wish to see this as an 4-D system.

Set of independent variables $= \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

dependent variables $= (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

the behavior = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ **free variables.**

OUTLINE

1. Examples
2. Historical remarks
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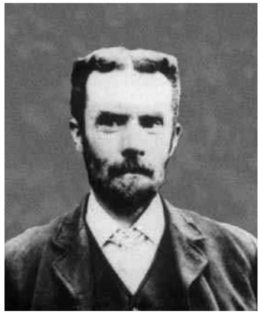
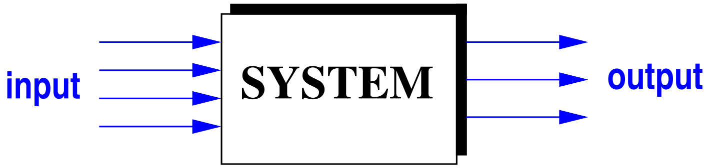
Early 20-th century: emergence of the notion of a transfer function
(Rayleigh, Heaviside).



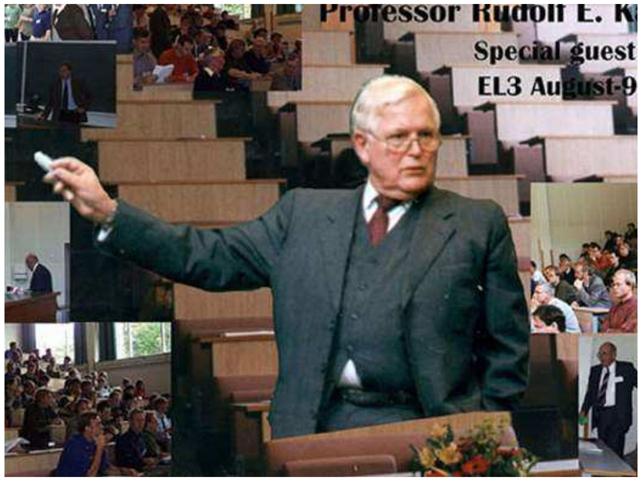


Since the 1920's: routinely used in circuit theory
$\rightsquigarrow$ impedances, admittances, scattering matrices, etc.

1930's: control embraces transfer functions
(Nyquist, Bode, $\cdots$) $\rightsquigarrow$ plots and diagrams, classical control.

__Around 1950__: Wiener sanctifies the notion of a blackbox, attempts nonlinear generalization (via Volterra series).

__1960's__: Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue



$\rightsquigarrow$ **input/state/output systems,** **and the ubiquitous**

$$\boxed{\tfrac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,}$$

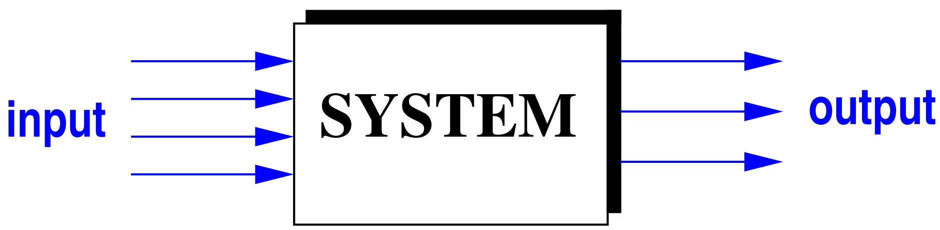
or its nonlinear counterpart

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u).$$

These are the basic models used nowadays in control and signal processing (cfr. MATLAB©).

Parallel development: **Mathematically rich generalization to ∞ dimensions with A the generator of a semigroup, etc.**

All these theories: input/output; cause ⇒ effect.



The input/state/output framework was instrumental for the energetic development of systems theory since the 1960's.

Unfortunately, for all its merits, it is simply not a good framework for modeling physical systems.

- **A physical system is not a signal processor.**
- **The idea of input-to-output (series, parallel, feedback) connection (SIMULINK©) provides a very poor, limited framework for modeling by tearing and zooming, and modularity.**
- **The structure of first pinciples models is a far distance from input/(state)/output structure.**
- **When applied to PDE's, the semi-group framework ignores the 'local' structure for the independent variables other than time.**
- **...**

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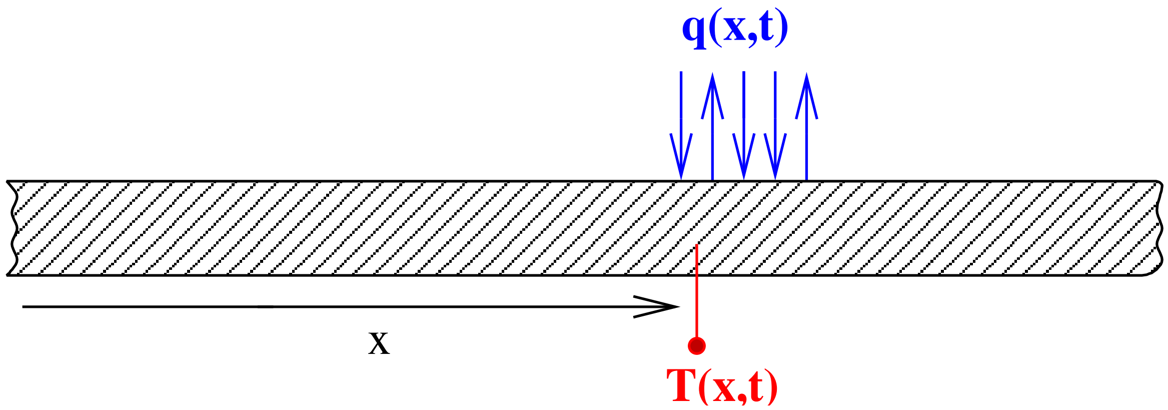
1. *Heat diffusion*



The PDE

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q$$

fits the

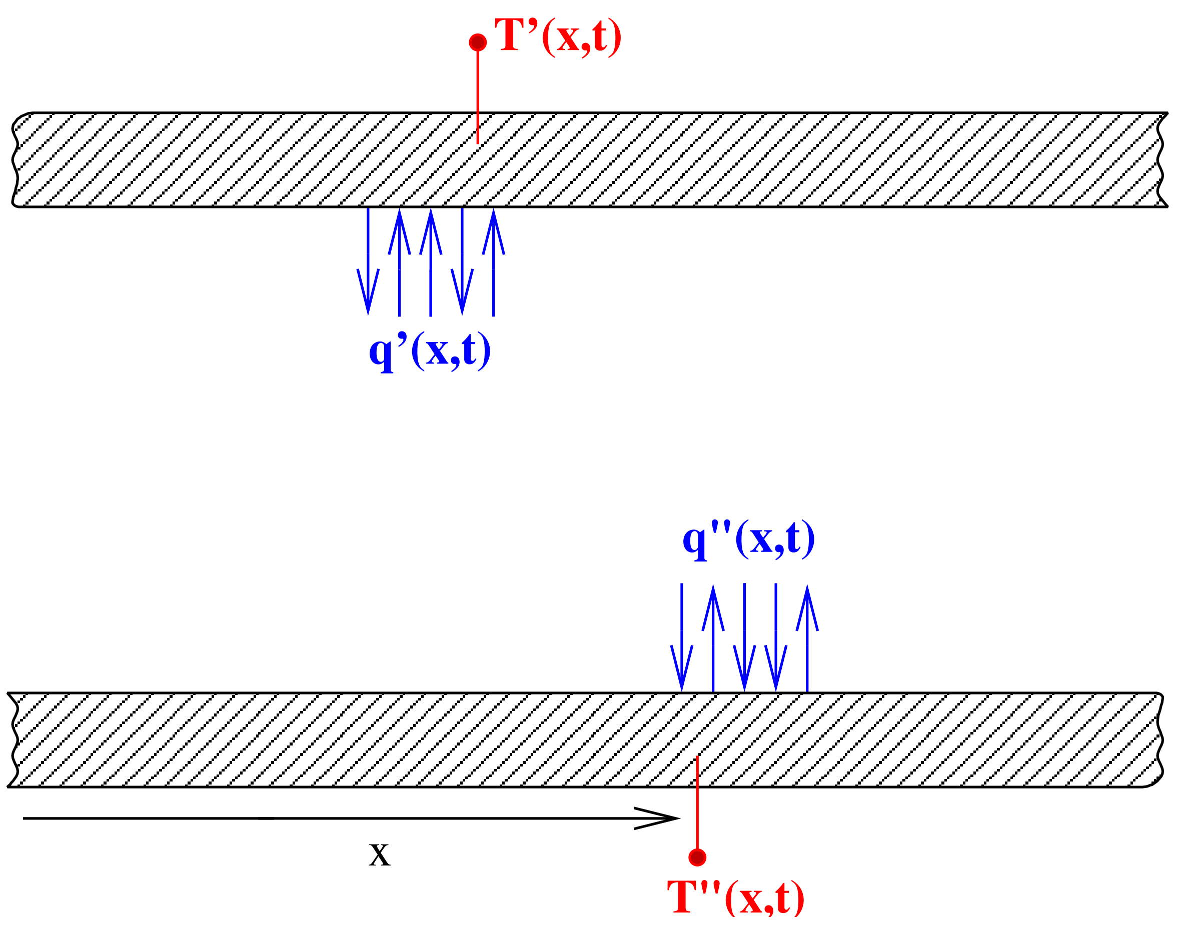
$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx$$

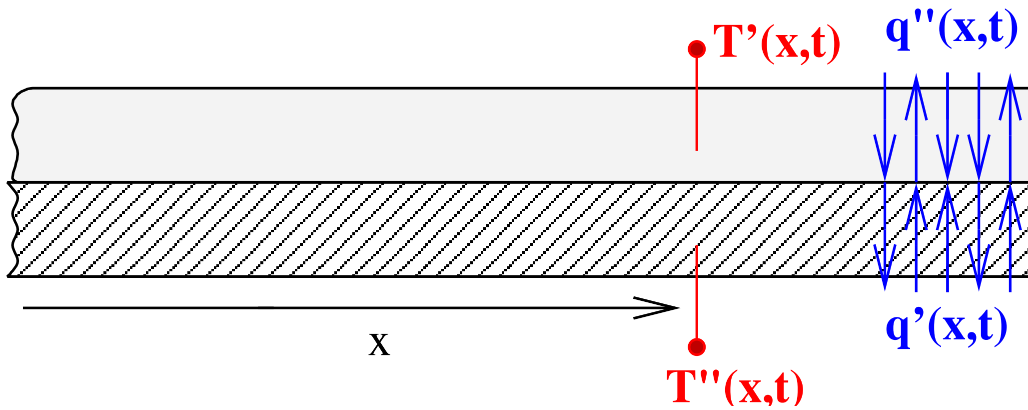
input/output framework, with

$$u(t) = q(\cdot, t); \quad y(t) = x(t) = T(\cdot, t)$$

perfectly.

Now interconnect two such systems





Interconnection:

$$\boxed{T'(x,t) = T''(x,t), \quad q'(x,t) + q''(x,t) = 0}$$

!! input'=input"; output'=output" ! $\Rightarrow\Leftarrow$ **SIMULINK©**

Interconnections contradicting SIMULINK© are in fact

normal, not exceptions,

in mechanics, fluidics, heat transfer, electrical circuits, etc.

The standard system theoretic / SIMULINK© input-to-output idea of interconnection is totally inappropriate as a paradigm for interconnecting physical systems!

Contrast this with the claim

... A third concept in control theory is the role of interconnection between subsystems. Input/output representations of systems allow us to build models of very complex systems by linking component behaviors ...

[Panel on Future Directions in Control, Dynamics, and Systems Report, 26 April 2002, page 11]

2. *Coaxial cable*

Relation between the voltage $V(x,t)$ and the current $I(x,t)$:

$$\frac{\partial}{\partial x} V = -L_0 \frac{\partial}{\partial t} I, \qquad (VI)$$

$$\frac{\partial}{\partial x} I = -C_0 \frac{\partial}{\partial t} V. \qquad (IV)$$

These imply

$$\boxed{\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V,} \qquad (V)$$

and

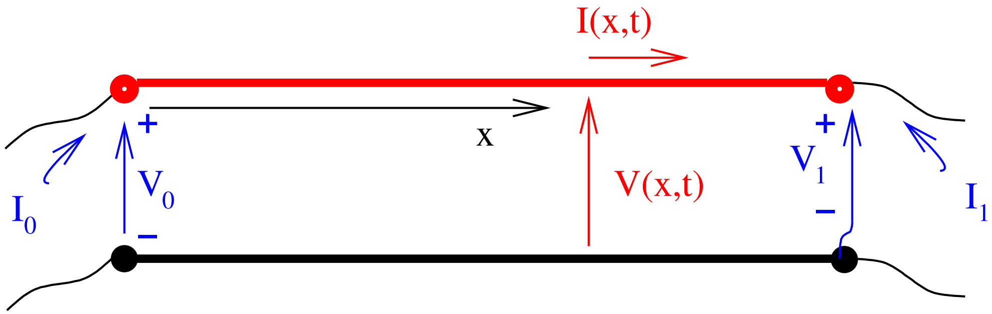
$$\boxed{\frac{\partial^2}{\partial x^2} I = L_0 C_0 \frac{\partial^2}{\partial t^2} I.} \qquad (I)$$

Wave eqn's.

Leads to the questions

- **Are (V), (I) 'consequences' of (VI) + (IV)?**
- **(V) + (I) $\Leftrightarrow$ (VI) + (IV)?**
- **(V) + (I) + (VI) $\Leftrightarrow$ (VI) + (IV)?**
- **Does (V) express all the constraints on V implied by (VI) + (IV)?**
- **Develop a calculus to obtain all consequences, to compute this elimination, to decide equivalence.**

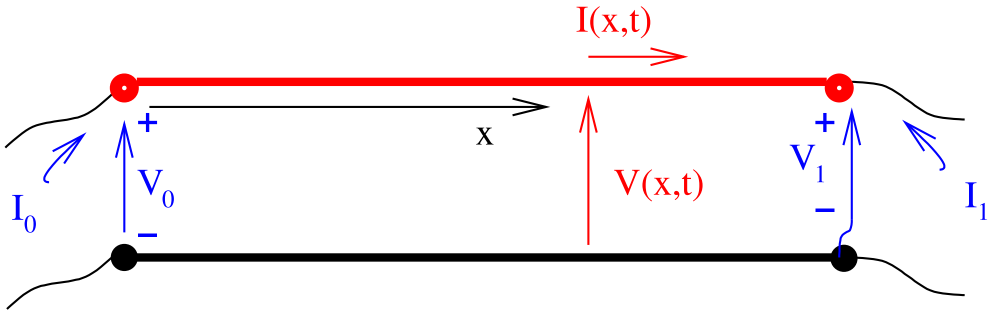
With boundary conditions:



$$\frac{\partial}{\partial x} V = -L_0 \frac{\partial}{\partial t} I,$$
$$\frac{\partial}{\partial x} I = -C_0 \frac{\partial}{\partial t} V,$$

$$V_0(t) = V(0, t),$$
$$V_1(t) = V(L, t),$$
$$I_0(t) = I(0, t),$$
$$I_1(t) = -I(L, t).$$

Viewed as a black box



Relation between V_0, V_1:

$$\frac{\partial^2}{\partial x^2}V = L_0 C_0 \frac{\partial^2}{\partial t^2}V, \quad V_0(\cdot) = V(0,\cdot), \; V_1(\cdot) = V(L,\cdot),$$

and between I_0, I_1:

$$\frac{\partial^2}{\partial x^2}I = L_0 C_0 \frac{\partial^2}{\partial t^2}I, \quad I_0(\cdot) = I(0,\cdot), \; I_1(\cdot) = I(L,\cdot).$$

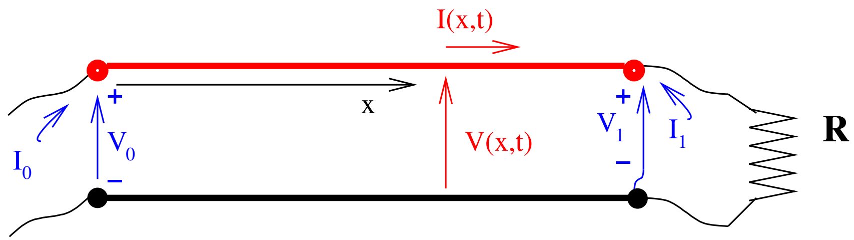
- **Two terminal variables are 'free', the other two are 'bound', (free = one voltage, one current, bound = one voltage, one current), but**

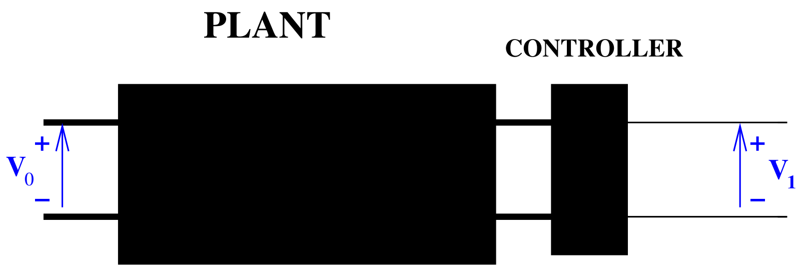
  there is no reasonable choice of inputs and outputs!

- **What is the role of $V(x,t)$ and $I(x,t)$, $0 \leq x \leq L$, in modeling the relation between V_0, I_0, V_1, I_1?**

If terminated by an impedance $\rightsquigarrow$ undesirable reflections.

characteristic impedance $$R = \sqrt{\frac{L_0}{C_0}}$$ $\Rightarrow$ **no reflections!**

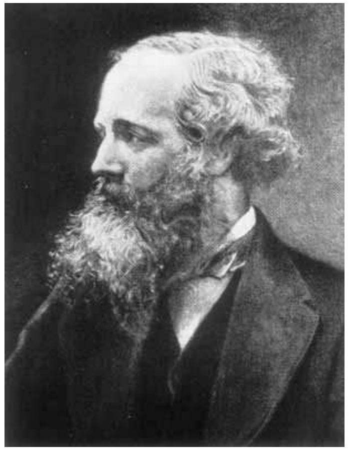




We view this termination as a behavioral controller. In this ex., the classical sensor-to-actuator feedback interpretation is an illusion.

$\exists$ very many such examples of controllers.

3. Maxwell's eqn's



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Set of independent variables $= \mathbb{R} \times \mathbb{R}^3$ (time and space),
dependent variables $= (\vec{E}, \vec{B}, \vec{j}, \rho)$
(electric field, magnetic field, current density, charge density),
$\in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,
the behavior = set of solutions to these PDE's.

Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho \,, \\
\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0.
\end{aligned}$$

Potential functions

The following equations in the

scalar potential $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$

and the

vector potential $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$,

generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Leads to the following questions:

- **Is there a fundamental reason why the behavior of $(\rho, \vec{E}, \vec{j})$ is also described by a PDE?** **'Elimination' issue.**

- **When and why is a representation in terms of a potential possible?** **'Image representation' issue.**

- **Derive algorithms for elimination, image representation.**

OUTLINE

1. Examples
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The basic concepts

Behavioral systems

A system **=** $\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T}$, **the set of *independent* variables,**

$\mathbb{W}$, **the set of *dependent* variables,**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **: *the behavior*** **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$ **: the model allows the trajectory w,**

$w \notin \mathfrak{B}$ **: the model forbids the trajectory w.**

$\mathbb{T} = \mathbb{R}$ **(in continuous-time systems),** $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$ **(in $\mathtt{n}$-D systems),**

$\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ **(in lumped systems), or a finite set (in DES).**

Emphasis today: $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$\mathfrak{B}$ **= solutions of system of linear constant coefficient PDE's.**

First principles models invariably contain auxiliary variables,
in addition to the variables the model aims at.

$\rightsquigarrow$ **Manifest and latent variables.**

Manifest = the variables the model aims at,

Latent = auxiliary variables.

We want to capture this in a mathematical definition.

A system with latent variables = $\boxed{\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\mathrm{full}})}$

$\mathbb{T}$, the set of *independent* variables.

$\mathbb{W}$, the set of *manifest* dependent variables
(= the variables that the model aims at).

$\mathbb{L}$, the set of *latent* dependent variables
(= the auxiliary modeling variables).

$\boxed{\mathfrak{B}_{\mathrm{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}}$ **: *the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).

The manifest behavior

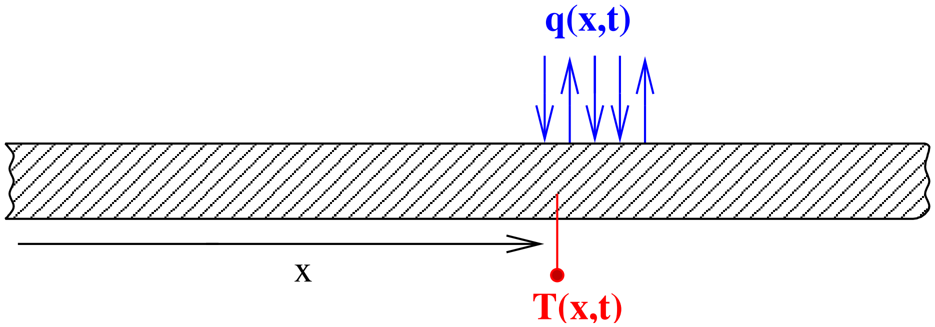
The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ induces the *manifest system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with *manifest behavior*

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

In convenient equations for $\mathfrak{B}$, the latent variables are *'eliminated'*.

Examples

1. Heat diffusion

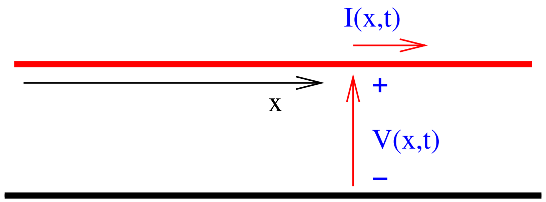


$\mathbb{T} = \mathbb{R}^2$ **(time and space);**

$\mathbb{W} = \mathbb{R}_+ \times \mathbb{R}$ **(temperature and heat);**

$\mathfrak{B}$ **= sol'ns to the PDE, the diffusion eq'n.**

2. Coaxial cable



Consider the voltage as the variable the model aims at.

$\mathbb{T} = \mathbb{R}^2$ **(time and space);**

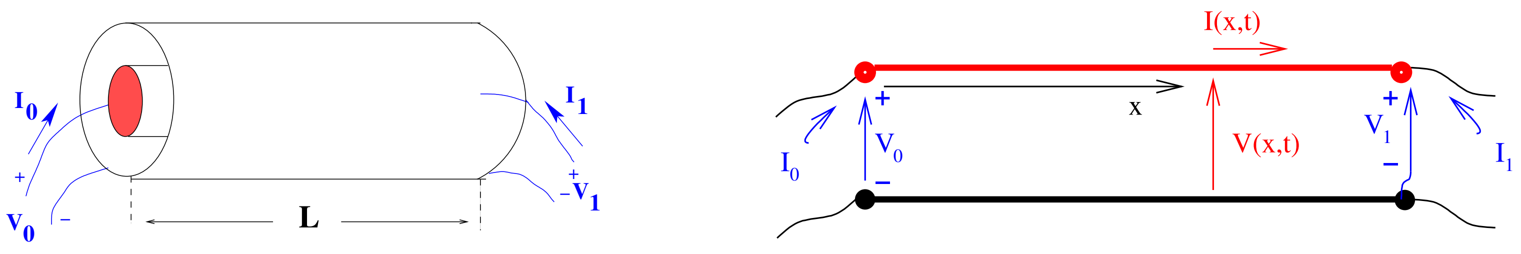
$\mathbb{W} = \mathbb{R}$ **(voltage);**

$\mathbb{L} = \mathbb{R}$ **(current);**

$\mathfrak{B}_{\text{full}}$ **= sol'ns to the PDE's;**

$\mathfrak{B}$**= sol'ns to** $\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V$?

3. Coaxial cable of length L.



Consider the terminal variables as the variables the model aims at.

$\mathbb{T} = \mathbb{R}$ **(time);**

$\mathbb{W} = \mathbb{R}^4$ **(2 voltages, 2 currents),**

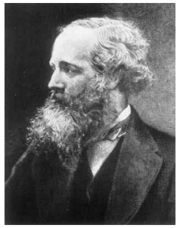
latent variables = $V(x,\cdot), I(x,\cdot); 0 \leq x \leq L$

(voltage and current in the coax)

$\mathfrak{B}_{\text{full}}$ **= sol'ns to the PDE's + boundary conditions.**

$\mathfrak{B}$**= sol'ns to ... ?**

4. Maxwell's eqn'ns



$\mathbb{T} = \mathbb{R}^4, \mathbb{W} = \mathbb{R}^{10}, \mathfrak{B} =$ **solutions to ME.**

If we view the electrical variables as manifest, and $\vec{B}$ as latent
$\mathbb{T} = \mathbb{R}^4, \mathbb{W} = \mathbb{R}^7, \mathbb{L} = \mathbb{R}^3,$
$\mathfrak{B}_{\text{full}} =$ **solutions to ME,** $\mathfrak{B} =$ **solutions to eliminated eq'ns?**

If we consider the representation in terms of the potentials ϕ, $\vec{A}$
$\mathbb{T} = \mathbb{R}^4, \mathbb{W} = \mathbb{R}^{10}, \mathbb{L} = \mathbb{R}^4,$
$\mathfrak{B}_{\text{full}} =$ **solutions to potential eqn's,** $\mathfrak{B} =$ **solutions to ME?**

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Linear differential systems

We now discuss the fundamentals of the theory of $\mathtt{n}$-D systems

$$\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$$

that are

1. ***linear,*** **meaning**
   $[(w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})] \Rightarrow [\alpha w_1 + \beta w_2 \in \mathfrak{B}]$;
2. ***shift-invariant,*** **meaning**
   $[(w \in \mathfrak{B}) \wedge (x \in \mathbb{R}^{\mathtt{n}})] \Rightarrow [\sigma^x w \in \mathfrak{B}]$,
   where σ^x denotes the $x-$shift;
3. ***differential,*** **meaning**
   $\mathfrak{B}$ consists of the solutions of a system of PDE's.

n-D systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathtt{n}$ **independent variables,**
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**
$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**, and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

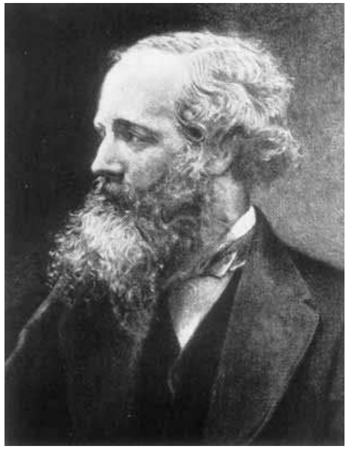
Define its behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$$

$\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**
Identical theory for $\mathfrak{D}'(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$**.**

Examples: *Diffusion eq'n, Wave eq'n*

Example: *Maxwell's equations*



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ **= set of solutions to these PDE's.**

NOMENCLATURE

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$: **the set of such systems with $\mathtt{n}$ in-, $\mathtt{w}$ dependent variables**

$\mathfrak{L}^{\bullet}$: **with any - finite - number of (in)dependent variables**

Elements of $\mathfrak{L}^{\bullet}$: ***linear differential systems***

$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$: **a *kernel representation* of the corresponding $\Sigma \in \mathfrak{L}^{\bullet}$ or $\mathfrak{B} \in \mathfrak{L}^{\bullet}$**

First principles models $\rightsquigarrow$ latent variables. **In the case of systems described by linear constant coefficient PDE's: $\rightsquigarrow$**

$$\boxed{R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\ell}$$

with $R, M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$.

For 1-D systems, the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

OUTLINE

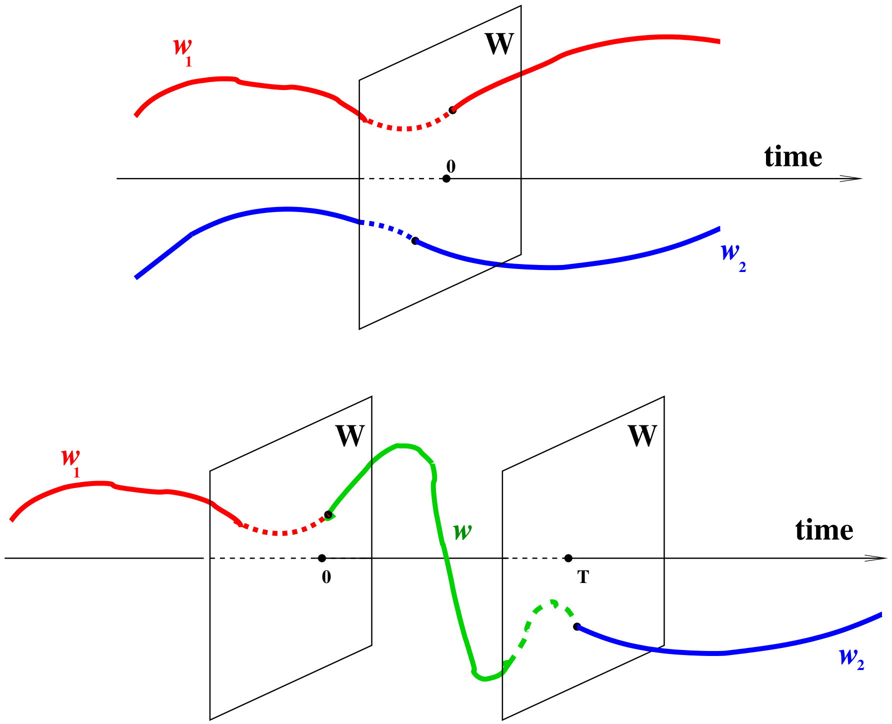
1. Examples
2. Historical remarks
3. Examples: Revisited
4. Behavioral systems
5. Linear distributed differential systems
6. Controllability & Observability
7.

Controllability

Controllability $:\Leftrightarrow$

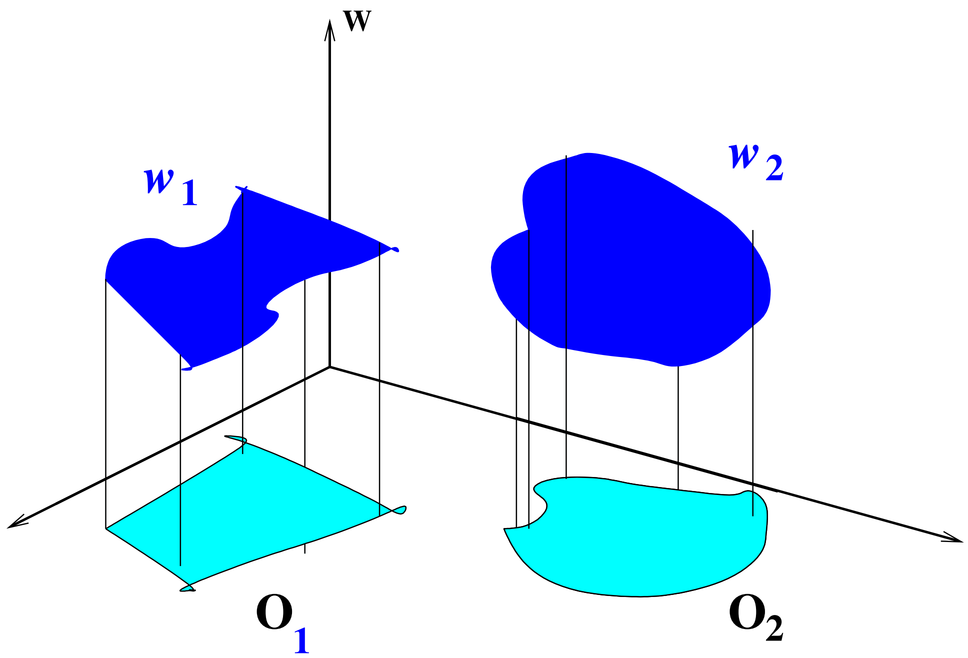
system trajectories must be ‘patch-able’, ‘concatenable’.

Case $\mathtt{n} = 1, \mathbb{T} = \mathbb{R}$, **any** $w_1, w_2 \in \mathfrak{B}$ **concatenable:**

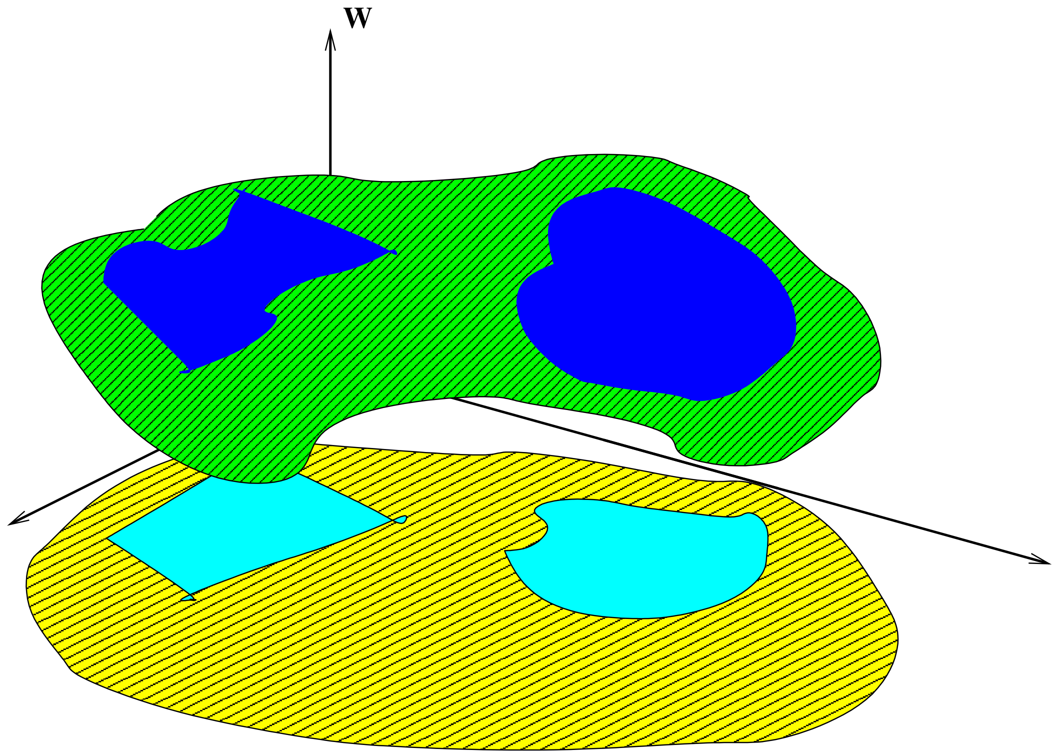


General $\mathtt{n}$, $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$.

Consider any two elements w_1, w_2 of the behavior and any two open non-overlapping $O_1, O_2 \subset \mathbb{R}^{\mathtt{n}}$:



Controllability = patchability:



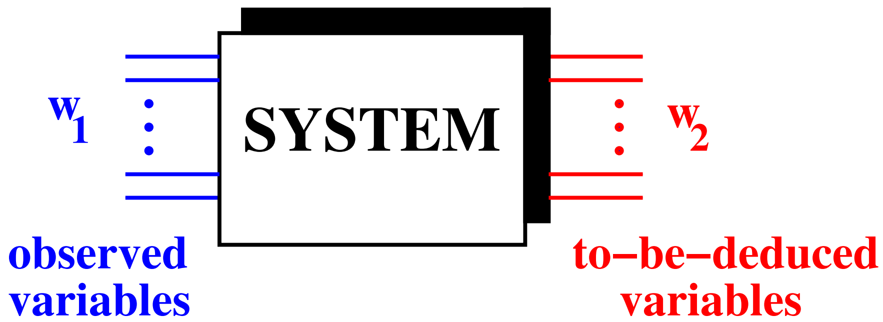
There is a sol'n that 'patches' w_1 on S_1 with w_2 on S_2.

Observability

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of

a pair of trajectories (w_1, w_2).



w_1 **: observed;** w_2 **: to-be-deduced.**

w_2 **is said to be** ***observable*** **from** w_1

if $((w_1, w_2') \in \mathfrak{B}$**, and** $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$**,**

i.e., if on $\mathfrak{B}$**, there exists a map** $w_1 \mapsto w_2$**.**

We are especially interested in the case

observed = manifest

to-be-deduced = latent

OUTLINE

1. Examples
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7. 3 theorems

Theorem 1 **Algebraization:**

$$\boxed{\mathfrak{L}_n^{\mathtt{w}} \overset{1:1}{\longleftrightarrow} \text{sub-modules of } \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_n]}$$

Theorem 2 **Elimination:**

$$\boxed{(\mathfrak{B}_{\text{full}} \in \mathfrak{L}_n^{\bullet}) \Rightarrow (\mathfrak{B} \in \mathfrak{L}_n^{\bullet})}$$

Theorem 3 **Image representation:**

$$\boxed{\text{Controllabilility} \Leftrightarrow (\exists \text{ Image representation})}$$

Algebraization of $\mathfrak{L}^\bullet$

Note that

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

and

$$U(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

have the same behavior if the polynomial matrix U is uni-modular (i.e., when $\det(U)$ is a non-zero constant).

$\Rightarrow$ R defines $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}))$, but not vice-versa!

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$??

Define the *annihilators* of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^{\top}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ is clearly an $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

Let $< R >$ denote the sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ spanned by the transposes of the rows of R. Obviously $< R > \subseteq \mathfrak{N}_{\mathfrak{B}}$. But, indeed:

$$\boxed{\mathfrak{N}_{\mathfrak{B}} = < R >!}$$

Note: Depends on $\mathfrak{C}^{\infty}$; ($\Leftarrow$) false for compact support soln's: for any $p \neq 0$, $\quad p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ has $w = 0$ as its only compact support sol'n.

Theorem 1 **(Algebraic structure of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$):**

1. $\mathfrak{N}_{\mathfrak{B}} = < R >$!

   In particular $f(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ is a consequence of $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ if and only if $f \in \; < R >$.

2. $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}} \overset{1:1}{\longleftrightarrow}$ sub-modules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$

3. $$R_1(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \text{ and } R_2(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

   define the same system iff

   $$< R_1 > \; = \; < R_2 > .$$

Elimination

The full behavior of $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$,

$$\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w+\ell}) \mid R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell\}$$

belongs to $\mathfrak{L}_n^{w+\ell}$, by definition.

Its manifest behavior equals

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \exists\ \ell \text{ such that } R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell\}.$$

Does $\mathfrak{B}$ belong to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$?

Theorem 2 (Elimination): **It does!**

Proof: **The theorem is a straightforward consequence of the 'fundamental principle': the equation**

$$A(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}) f = y$$

$A \in \mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, $y \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{n}_1})$ **given,**
$f \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{n}_2})$ **unknown, is solvable if and only if for**
$n \in \mathbb{R}^{\mathtt{n}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]$

$$(n^{\top} A = 0) \;\Rightarrow\; (n^{\top}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}) y = 0).$$

Remarks:

- **Number of equations for $\mathtt{n} = 1$ (constant coeff. lin. ODE's)**
  $\leq$ number of variables.
  Elimination $\Rightarrow$ fewer, higher order equations.
- **There exist effective computer algebra/Gröbner bases algorithms for elimination**
  $$(R, M) \mapsto R'$$
- **Not generalizable to smooth nonlinear systems.**
  Why are differential equations models so prevalent?

Examples

1.

$$\boxed{\frac{\partial^2}{\partial x^2} V = L_0 C_0 \frac{\partial^2}{\partial t^2} V,}$$

describes indeed the behavior of V in the coax.

2. Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0.
\end{aligned}
$$

Elimination theorem $\Rightarrow$

this exercise is exact & successful (+ gives algorithm).

It follows from all this that $\mathfrak{L}^\bullet_{\mathtt{n}}$ has very nice properties. It is closed under:

- Intersection: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- Addition: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- Projection: $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\mathtt{n}}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}})$.
- Action of a linear differential operator:

  $$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \\ \Rightarrow (P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}).$$

- Inverse image of a linear differential operator:

  $$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \\ \Rightarrow (P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}).$$

Image representations

Representations of $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$;

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell$$

called a *'latent variable' representation* of the manifest behavior $\mathfrak{B} = (R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1} M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\ell})$.

Missing link:

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\ell$$

called an *'image' representation* of $\mathfrak{B} = \mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\right)$.

Elimination theorem $\Rightarrow$ every image is also a kernel.

¿¿ Which kernels are also images ??

Theorem 3 (Controllability and image repr.):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

1. $\mathfrak{B}$ is controllable,

2. $\mathfrak{B}$ admits an image representation,

3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$$a^\top[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}),$$

4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,

etc.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Remarks:

- **Algorithm:** R + syzygies + Gröbner basis
  $\Rightarrow$ numerical test for on coefficients of R.
- **In the 1-D case there exists always an observable image representation $\cong$ flatness. Not so for general n-D systems: potentials are then hidden variables.**
- **$\exists$ partial results for nonlinear systems.**
- **Kalman controllability is a straightforward special case.**

Is is worth worrying about these ‘axiomatics’?

They have a deep and lasting influence! Especially in teaching.

Examples:

- **Probability and the theory of stochastic processes as an axiomatization of uncertainty.**
- **The development of input/output ideas in system theory and control - often these axiomatics are implicit, but nevertheless much very present.**
- **QM.**

Thank you for your patience & attention

Details & copies of the lecture frames are available from/at

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