DISSIPATIVE DISTRIBUTED SYSTEMS

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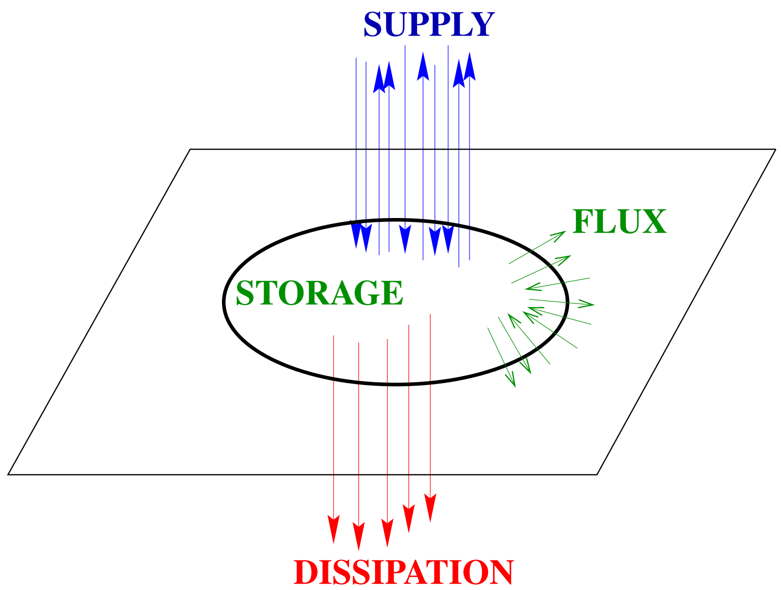
A dissipative system absorbs supply, 'globally' over time and space.

¿¿ Can this be expressed 'locally', as

rate of change in storage + spatial flux $\leq$ supply rate



rate of change in storage + spatial flux

= supply rate + (non-negative) dissipation rate ??

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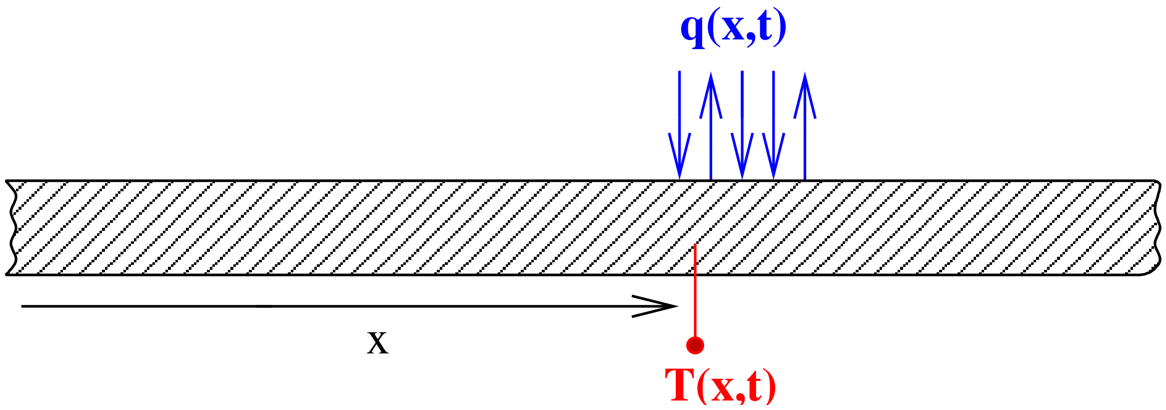
First principles motivating example: ***Heat diffusion***



The PDE

$$\boxed{\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q}$$

describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) in a medium and the heat $q(x,T)$ supplied to / radiated away from it.

For all sol'ns T, q with $T(x,t) = \text{constant} > 0$ (and therefore $q = 0$) outside a compact set, there holds:

First law:

$$\int_{\mathbb{R}^2} q(x,t)\, dx\, dt \;=\; 0,$$

Second law:

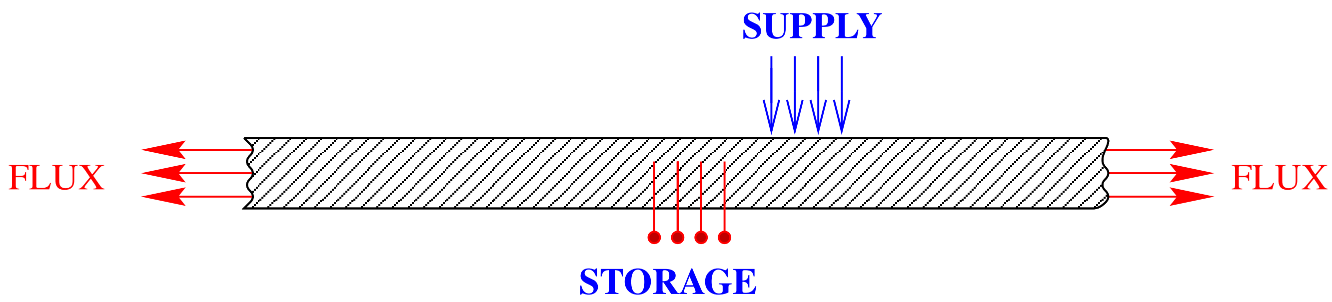
$$\int_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \;\leq\; 0.$$

$\Rightarrow$

$$\max_{x,t}\{T(x,t) \mid q(x,t) \geq 0\} \geq \min_{x,t}\{T(x,t) \mid q(x,t) \leq 0\}.$$

It is impossible to transport heat from a 'cold source' to a 'hot sink'.

Can these ‘global’ versions be expressed as ‘local’ laws?



rate of change in storage + spatial flux $\leq$ supply rate

To be invented:

an ‘extensive’ quantity for the first law: internal energy

an ‘extensive’ quantity for the second law: entropy

Define the following variables:

$$E = T \qquad : \text{ the stored energy density,}$$

$$S = \ln(T) \qquad : \text{ the entropy density,}$$

$$F_E = -\frac{\partial}{\partial x} T \qquad : \text{ the energy flux,}$$

$$F_S = -\frac{1}{T}\frac{\partial}{\partial x} T \qquad : \text{ the entropy flux,}$$

$$D_S = \left(\frac{1}{T}\frac{\partial}{\partial x} T\right)^2 \qquad : \text{ the rate of entropy production.}$$

Local versions of the first and second law:

rate of change in storage + spatial flux $\leq$ supply rate

Conservation of energy:

$$\boxed{\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} F_E = q,}$$

Entropy production:

$$\frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S = \frac{q}{T} + D_S. \qquad \text{Since} \quad (D_S \geq 0) \quad \Rightarrow$$

$$\boxed{\frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S \geq \frac{q}{T}.}$$

Our problem:

theory behind these ad hoc constructions of E, F_E and S, F_S.

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LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the ***'flow'***

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, **the** ***state space*****, and** $f : \mathbb{X} \to \mathbb{X}$.
Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ **by** $\mathfrak{B}$, **the behavior.**

$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a Lyapunov function for Σ **if along** $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalent to $\quad \overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0$

Plays a remarkably central role in the field.

Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's ‘second method’ in his Ph.D. thesis (1899).

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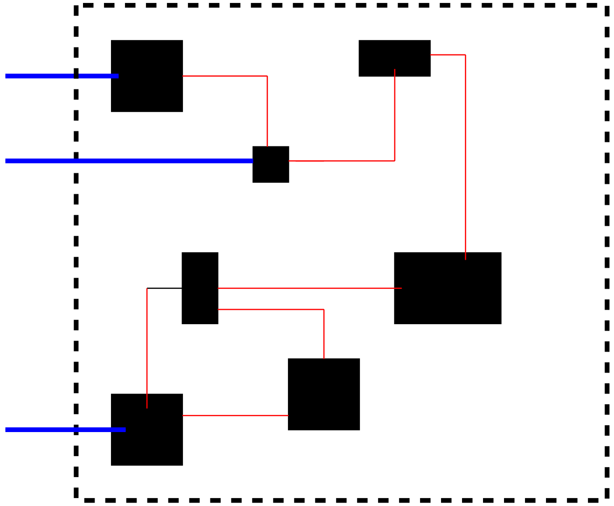
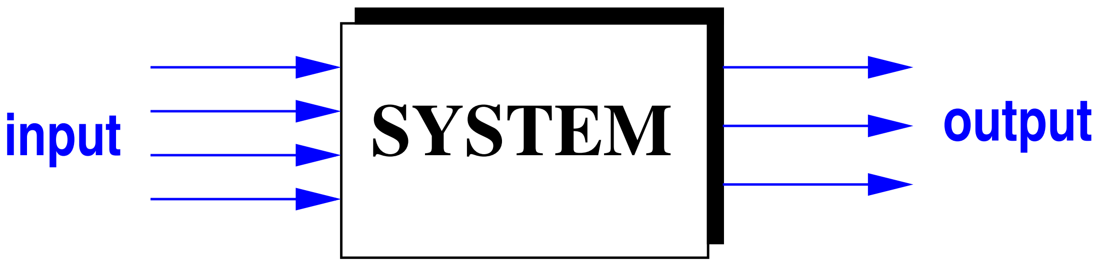
'Open' systems are a much more appropriate starting point for the study of dynamics.





Consider the 'dynamical system'

$$\Sigma : \quad \frac{d}{dt} x = f(x, u), \quad y = h(x, u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: the input, output, state.

Behavior $\mathfrak{B} =$ all sol'ns $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Let $s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ be a function, called the *supply rate*.

DISSIPATIVITY

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function* such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

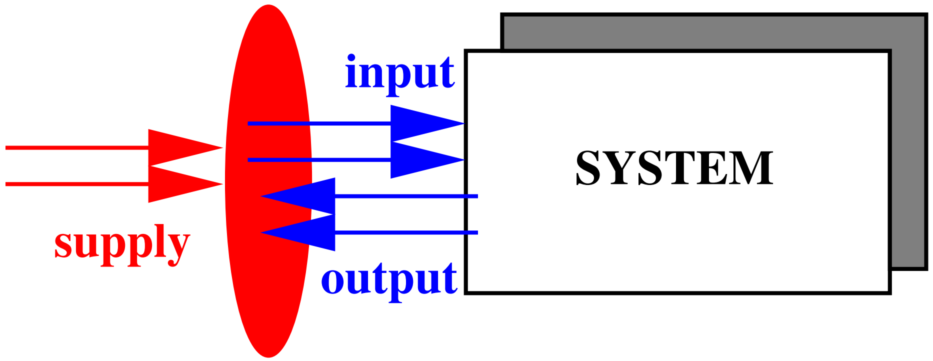
along input/output/state trajectories ($\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

This inequality is called the *dissipation inequality.*

Equivalent to $\quad \overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$
for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: 'conservative' system.

$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.



$V(x)$ then models the internally stored energy.

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ supply rate.

Special case: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

Dissipativity is the natural generalization to open systems of Lyapunov theory.

Stability for closed systems $\simeq$ Dissipativity for open systems.

THE CONSTRUCTION OF STORAGE FUNCTIONS

Basic question:

Given (a representation of) Σ, the dynamics,

and given s, the supply rate,

is the system dissipative w.r.t. s, i.e.,

does there exist a storage function V such that

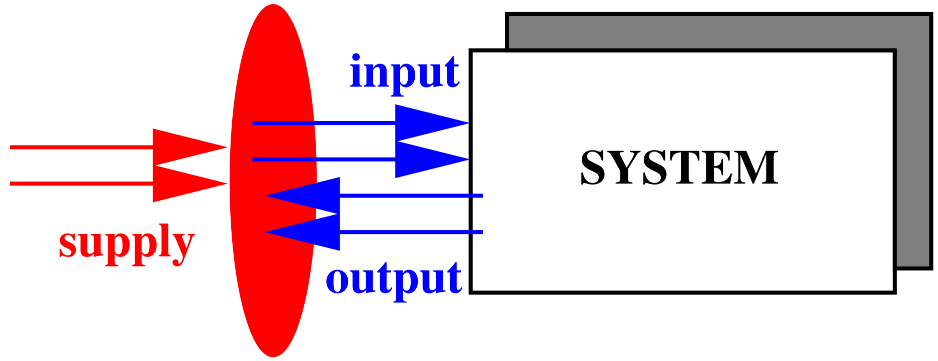
the dissipation inequality holds?

The construction of storage f'ns is very well understood, particularly for linear systems and quadratic supply rates.

Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, robust control, electrical circuit synthesis, stochastic realization theory.

V is in general far from unique. There are two 'canonical' storage functions: the available storage and the required supply.
For conservative systems, V is unique.

Plays a remarkably central role in the field.



Assume s 'power', known dynamics, what is the internal energy?

This is the question which we shall now study for systems described by PDE's.

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Polynomial matrix notation for PDE's:

PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

$$\updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1} \qquad \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}.$$

$$R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})w = 0$$

$\mathbb{T} = \mathbb{R}^{\mathrm{n}}$, **the set of independent variables,**

$\mathbb{W} = \mathbb{R}^{\mathrm{w}}$, **the set of dependent variables,**

$\mathfrak{B}$ = **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathrm{w}}[\xi_1, \cdots, \xi_{\mathrm{n}}]$**, and consider**

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{w}}) \mid (*) \text{ holds } \}.$$

$\mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{w}})$ **mainly for convenience.**

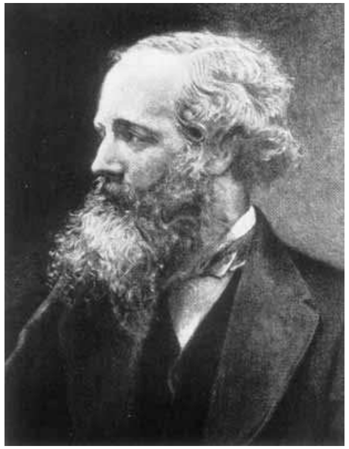
Notation for n-D linear differential systems:

$$(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{n}}.$$

Cfr. the work of Oberst, Pillai, Shankar, Wood, Zerz, ...

Examples: **Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .**

Maxwell's equations



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ **= set of solutions to these PDE's.**

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_n^w$.

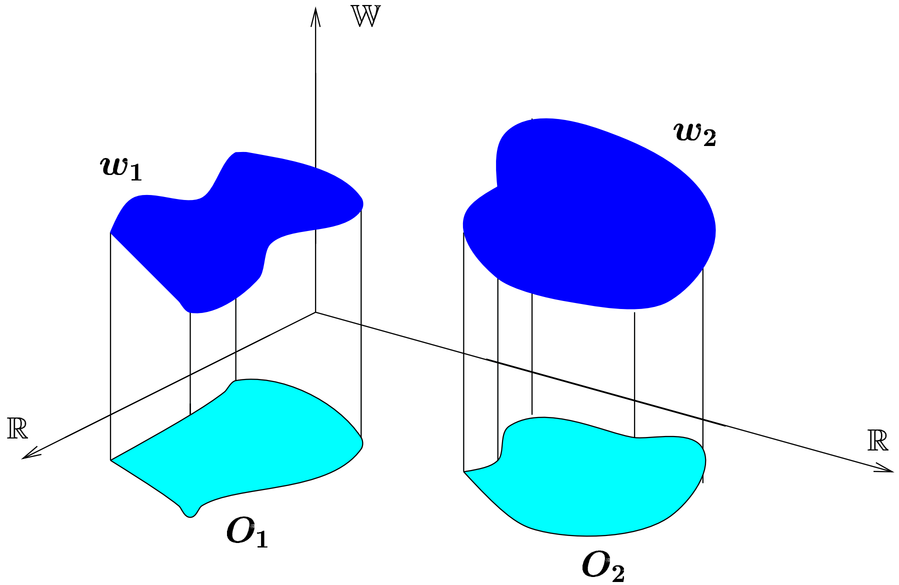
Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \ell.$$

'Elimination' thm $\Rightarrow$ $\mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\right) \in \mathfrak{L}_n^w$ **!**

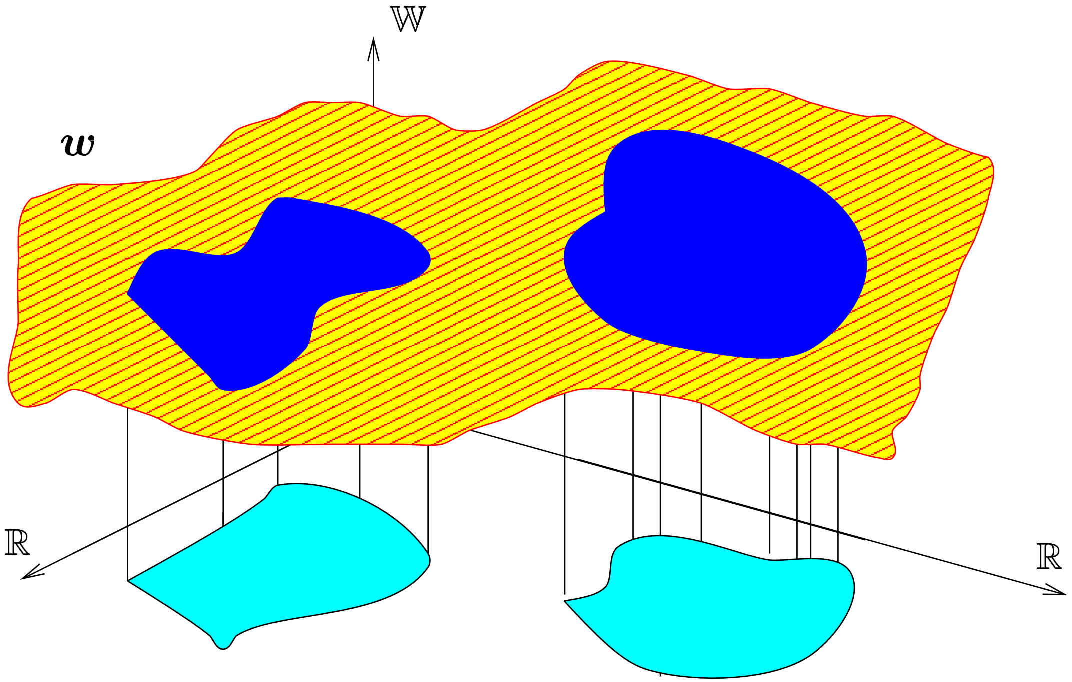
$\mathfrak{B} \in \mathfrak{L}_n^w$ admits an image representation iff it is 'controllable'.

Controllability in pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

$w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$**.**



Controllability :⇔ 'patch-ability'.

ARE MAXWELL'S EQUATIONS CONTROLLABLE ?

The following well-known equations in the ***scalar potential*** $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ **and the** ***vector potential*** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, **generate exactly the solutions to Maxwell's equations:**

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ a potential!

Not all controllable systems admit an observable image representation. For $\mathtt{n} = 1$, they do. For $\mathtt{n} > 1$, exceptionally so.

Observability means: $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})$ **is injective:**

ℓ **can be deduced from** w**.**

The latent variable in an image representation ℓ may be 'hidden'.

Example: Maxwell's equations do not allow a potential representation that is observable.

Multi-index notation:

$x = (x_1, \ldots, x_{\mathrm{n}})$,

$k = (k_1, \ldots, k_{\mathrm{n}}), \ell = (\ell_1, \ldots, \ell_{\mathrm{n}})$,

$\xi = (\xi_1, \cdots, \xi_{\mathrm{n}}), \zeta = (\zeta_1, \ldots, \zeta_{\mathrm{n}}), \eta = (\eta_1, \ldots, \eta_{\mathrm{n}})$,

$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathrm{n}}}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_{\mathrm{n}}}}{\partial x_{\mathrm{n}}^{k_{\mathrm{n}}}})$,

$dx = dx_1 dx_2 \ldots dx_{\mathrm{n}}$,

$R(\frac{d}{dx})w = 0$ for $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0$,

$w = M(\frac{d}{dx})\ell$ for $w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$,

etc.

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1. **Motivating example**
2. **Lyapunov theory**
3. **Dissipative dynamical systems**
4. **Linear differential systems**
5. **Dissipative distributed systems**
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QDF's

The quadratic map in w and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.

$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ.

DISSIPATIVE DISTRIBUTED SYSTEMS

We consider only controllable linear differential systems and QDF's.

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, **controllable, is said to be** ***dissipative*** **with respect to the supply rate** Q_{Φ} **(a QDF) if**

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\, dx \geq 0$$

for all $w \in \mathfrak{B}$ **of compact support, i.e., for all** $w \in \mathfrak{B} \cap \mathfrak{D}$**.**

Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time.

Idea: $Q_\Phi(w)(x,y,z;t)\ dxdydz\ dt$:

rate of 'energy' delivered to the system.

Dissipativity :⇔

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\ dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

A dissipative system absorbs net energy.

Example: **Maxwell's eq'ns:**

dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E}\cdot\vec{j}$ **.**

In other words, if $\vec{E},\vec{j}$ **is of compact support and satisfies**

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t}\nabla\cdot\vec{E} + \nabla\cdot\vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0, \end{aligned}$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} (-\vec{E}\cdot\vec{j}) \; dxdydz \right) dt = 0.$$

Can this be reinterpreted as: **As the system evolves,**

energy is locally stored, and redistributed over time and space?

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6. Local dissipation law
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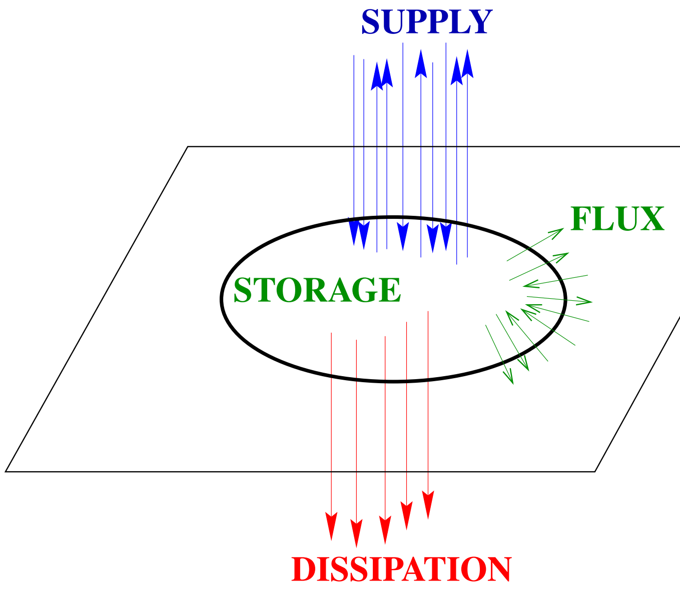
Assume that a system is ‘globally’ dissipative.

¿¿ Can this dissipativity be expressed through a ‘local’ law??

Such that in every spatial domain there holds:

$$\frac{d}{dt}\text{ Storage + Spatial flux} \leq \text{Supply.}$$



Supply = Stored + radiated + dissipated.

Main Theorem:

$\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}$, controllable, is dissipative w.r.t. the supply rate Q_Φ

iff

$\exists$ an image representation $w = M(\frac{d}{dx})\ell$ of $\mathfrak{B}$,

an n−vector of QDF's $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_{\mathrm{n}}})$

on $\mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\dim(\ell)})$, called the *flux*,

such that the *local dissipation law*

$$\boxed{\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{d}{dx})\ell$.

As usual $\nabla \cdot Q_\Psi := \frac{\partial}{\partial x_1} Q_{\Psi_1} + \cdots + \frac{\partial}{\partial x_{\mathrm{n}}} Q_{\Psi_{\mathrm{n}}}$.

Note: the local law involves

(possibly unobservable, - i.e., hidden!) latent variables (the ℓ's).

Assume $\mathtt{n} = 4$: **independent variables** $x, y, z; t$: **space and time.**
Let $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$ **be controllable. Then**

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w) \, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

if and only if

∃ an image representation $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$ **of** $\mathfrak{B}$**,**

and QDF's S**, the** ***storage,*** **and**
F_x, F_y, F_z**, the** ***spatial flux,***

such that the ***local dissipation law***

$$\boxed{\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) **that satisfy** $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$**.**

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, the rate of energy supplied.

Introduce the *stored energy density,* S, and

the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

Local version involves $\vec{B}$, unobservable from $\vec{E}$ and $\vec{j}$,

the variables in the rate of energy supplied.

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6. Local dissipation law
7. Schematic of the proof
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Using controllability and image representations, we may assume WLOG:

$$\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$\Updownarrow$ **(Parseval)**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$\Updownarrow$ **(Factorization equation)**

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

$\Updownarrow$ **(easy)**

$$\exists\ \Psi : \quad (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$

$\Updownarrow$ **(clearly)**

$$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

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8. The factorization equation

Consider

$$X^\top(-\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. *Solvable??*

$\cong$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown.

Under what conditions on Y does there exist a solution X?

Scalar case: !! write the real polynomial Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$\boxed{X^\top(\xi)X(\xi) = Y(\xi)}$$

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (for $X \in \mathbb{R}^2[\xi]$!) iff

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $\mathtt{n} = 1$, and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that this factorization equation is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $\mathtt{n} > 1$, and under this obvious positivity requirement, this equation can nevertheless in general <u>not</u> be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

This factorizability is a simple consequence of **Hilbert's 17-th pbm!**

Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general <u>not</u> be expressed as a sum of squares of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.

But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$, **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, **can be expressed as a sum of squares of** ($k = 2^n$) **rational functions, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$.

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow \quad \textbf{(Factorization equation)}$$

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi) D(\xi)$$

over the rational functions,

i.e., with D a matrix with elements in $\mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

The need to introduce

rational functions in this factorization

an image representation of $\mathfrak{B}$ to reduce the pbm to $\mathfrak{C}^{\infty}$

are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations.**
2. **The non-uniqueness of D in the factorization equation**

$$\Phi(-\xi, \xi) = D^{\top}(-\xi)D(\xi)$$

3. **The non-uniqueness (in the case $\mathtt{n} > 1$) of the solution Ψ of**

$$(\zeta + \eta)^{\top}\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^{\top}(\zeta)D(\eta)$$

For conservative systems, $\Phi(-\xi, \xi) = 0$, whence $D = 0$, but, when $\mathtt{n} = 1$, the third source of non-uniqueness remains, even when working with a specific image representation.

It seems to be a very real non-uniqueness, even for EM fields. Cfr.

The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u **[the internal energy]** ***and*** S **[the flux] ...** ***It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it*** **[our choice]** ***is probably perfectly right.***

The Feynman Lectures on Physics,
Volume II, page 27-6.

CONCLUSIONS

- **global dissipation $\Leftrightarrow$ $\exists$ local dissipation law**
- **Involves hidden latent variables (e.g. $\vec{B}$ in Maxwell's eq'ns)**
- **The proof $\cong$ Hilbert's 17-th problem**
- **Neither controllability nor observability are good generic assumptions**

Reference: H. Pillai and JCW, Dissipative distributed systems, *SIAM J. Control and Opt.*, electronically published in January 2002.

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Thank you!