THE BEHAVIORAL APPROACH

to

SYSTEMS and CONTROL

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Problematique:

Develop a suitable *mathematical* framework for discussing dynamical systems

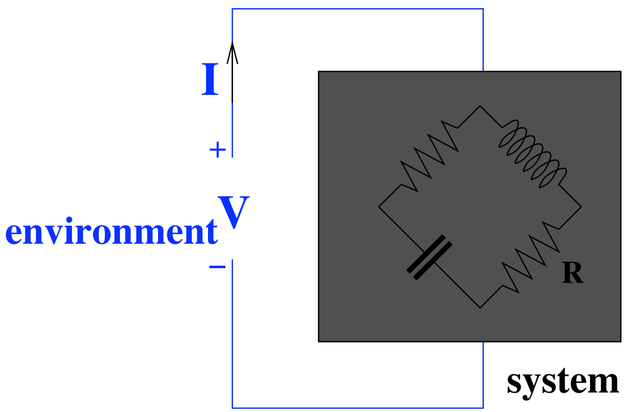
aimed at modeling, analysis, and synthesis.

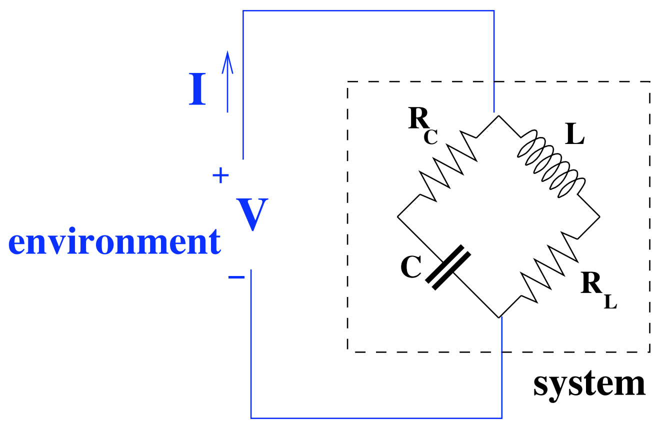
$\rightsquigarrow$ control, signal processing, system identification, . . .

$\rightsquigarrow$ engineering systems, economics, physics, . . .

Motivational examples

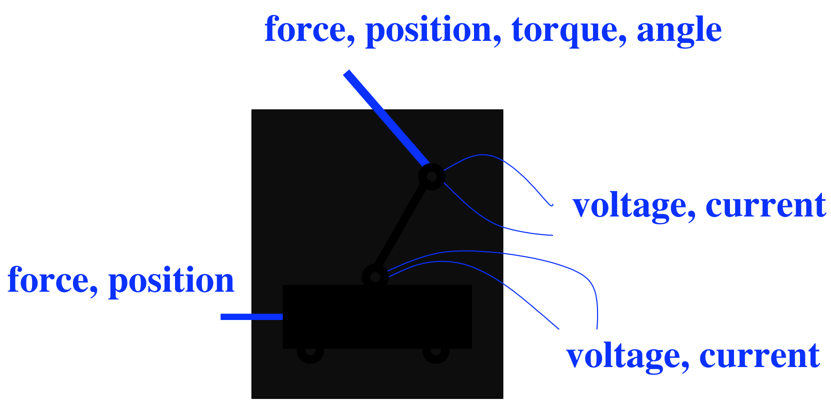
Electrical circuit

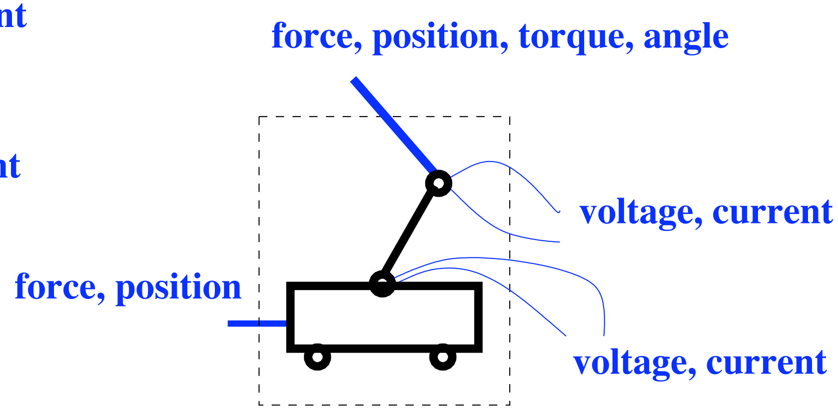




!! Model the relation between the voltage V and the current I

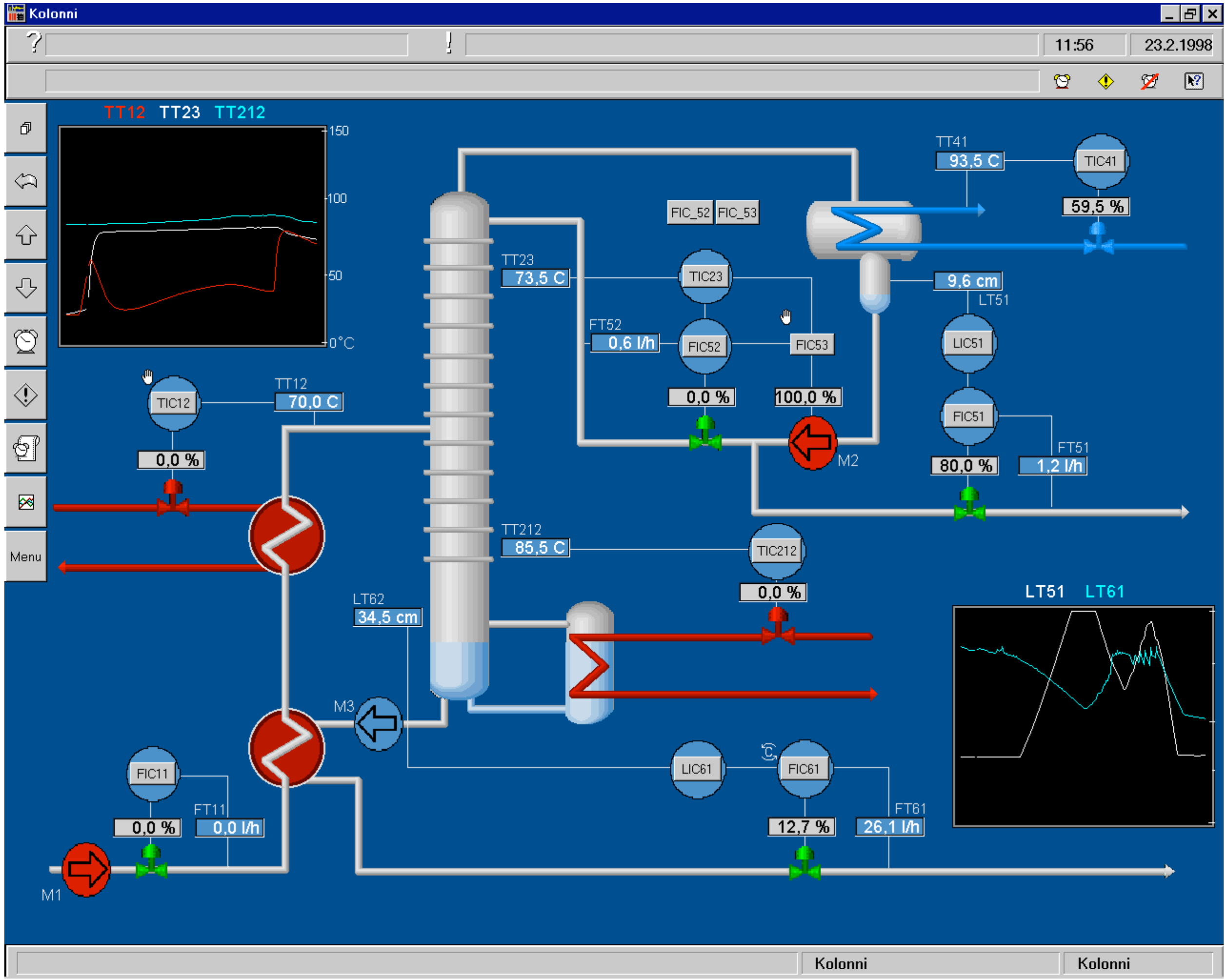
Electromechanical system





!! between the positions, forces, torque, angle, voltages, currents

Distillation column



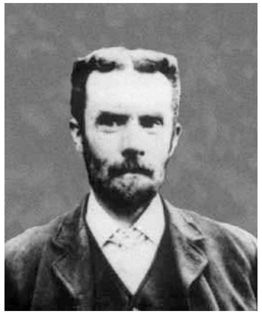
Features: Systems are typically

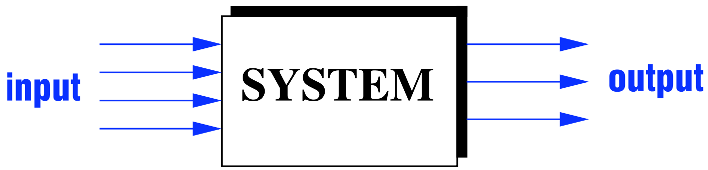
- dynamical
- open, they interact with their environment
- interconnected, with many subsystems
- modular, consisting of standard components

We are looking for a mathematical framework that is adapted to these features, and hence to computer assisted modeling.

Historical remarks

Early 20-th century: emergence of the notion of a transfer function (Rayleigh, Heaviside).





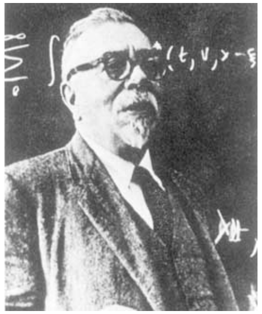
Since the 1920's: routinely used in circuit theory

$\rightsquigarrow$ impedances, admittances, scattering matrices, etc.

1930's: control embraces transfer functions

(Nyquist, Bode, $\cdots$) $\rightsquigarrow$ plots and diagrams, classical control.

__Around 1950__: **Wiener sanctifies the notion of a blackbox, attempts nonlinear generalization (via Volterra series).**



__1960's__: **Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue**

$\rightsquigarrow$ input/state/output systems, and the ubiquitous

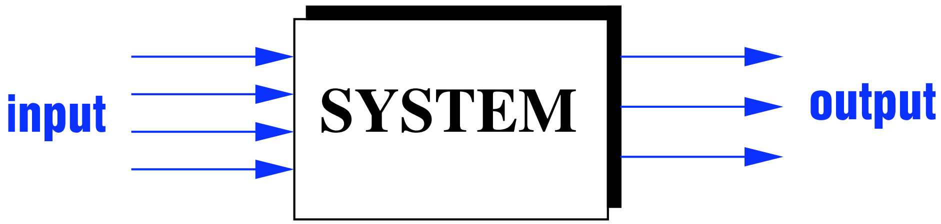
$$\boxed{\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,}$$

or its nonlinear counterpart

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u).$$

These mathematical structures, transfer functions, + their discrete-time analogs, are nowadays the basic models used in control and signal processing (cfr. MATLAB©).

All these theories: input/output; cause $\Rightarrow$ effect.

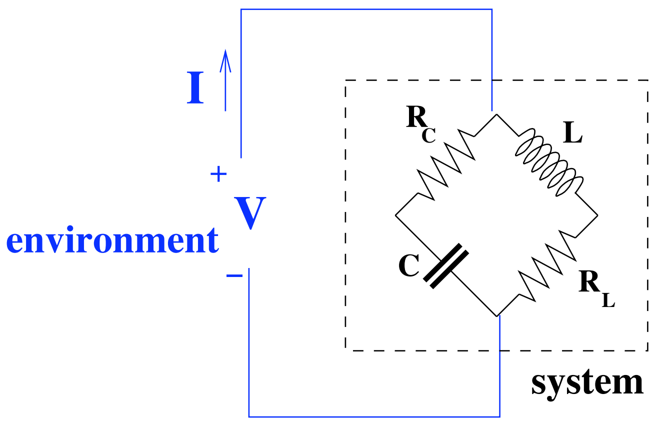


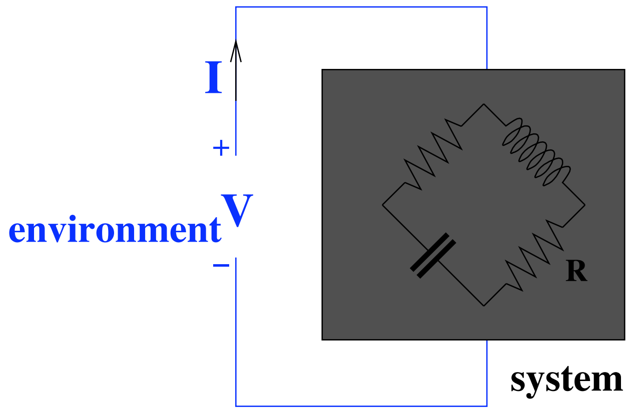
Beyond input/output

What's wrong with input/output thinking?

Let's look at examples:

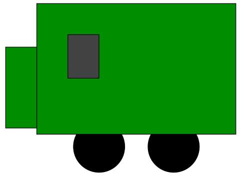
Our electrical circuit.





Is V the input? Or I ? Or both, or are they both outputs?

An automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: the force,

at the tire terminals: the forces, or, more likely, the positions?

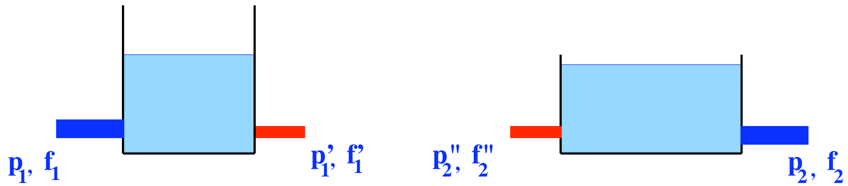
at the steering wheel: the torque or the angle?

at the gas-pedal, or the brake-pedal: the force or the position?

Difficulty: at each terminal there are many (typically paired) interconnection variables

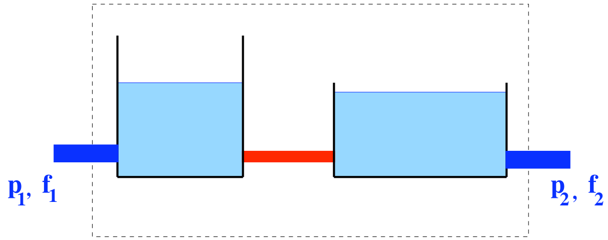
Input/output is awkward in modeling interconnections.

Consider a two-tank example.



Reasonable input choices: the pressures, output choices: the flows.

Assume that we model the interconnection of two tanks.



Interconnection: $p_1' = p_2'', \quad f_1' + f_2'' = 0$

input=input; output=output! $\Rightarrow\Leftarrow$ **SIMULINK©**

Interconnections contradicting SIMULINK© are in fact normal, not exceptions, in mechanics, fluidics, heat transfer, etc.

Mathematical difficulties:

Is a system a **map** $u(\cdot) \mapsto y(\cdot)$?

How to incorporate **'initial conditions'?**

Is it a parametrized map $(u(\cdot), \alpha) \mapsto y(\cdot)$?

All sorts of new difficulties...

Construct the state!

But from what?

From the system model!

What system?

Conclusions $*$ **for physical systems ($\Rightarrow\Leftarrow$ signal processors)** $*$

- **External variables are basic, but what 'drives' what, is not.**
- **It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.**
- **What can be the input, and what can be the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.**

Interconnection, variable sharing,

rather that input selection,

is the basic mechanism by which a system interacts with its environment.

$\Rightarrow$ We need a better framework for discussing 'open' systems!

$\rightsquigarrow$ Behavioral systems.

The basic concepts

Behavioral systems

A dynamical system = $\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis*** **(= the relevant time instances),**

$\mathbb{W}$, **the *signal space*** **(= where the variables take on their values),**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$: ***the behavior*** **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,

$w \notin \mathfrak{B}$: the model forbids the trajectory w.

Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$ (in discrete-time systems).

Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),
a function space
(in distributed systems, with time a distinguished variable),
or a finite set (in DES).

<u>Emphasis later today:</u> $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = solutions of system of linear constant coefficient ODE's.

Examples

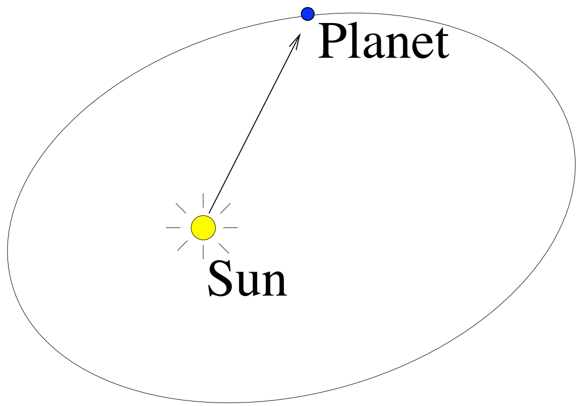
1. Planetary orbits

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{R}^3$ **(position),**

$\mathfrak{B}$ = **planetary orbits** $\cong$ **Kepler's laws:**

$$\text{ellipses, = areas in = time, } \frac{(\text{period})^2}{(\text{axis})^3} = \text{constant.}$$



2. Input / output systems

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right)$$

$$= f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**

$\mathfrak{B} =$ **all input / output pairs.**

3. Flows

$$\frac{d}{dt}x(t) = f(x(t)),$$

$\mathfrak{B}$ = **all state trajectories.**

... Of very marginal value as a paradigm for dynamics ...

Modeling closed systems by tearing and zooming

$\rightsquigarrow$ **open systems.**

4. Observed flows

$$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

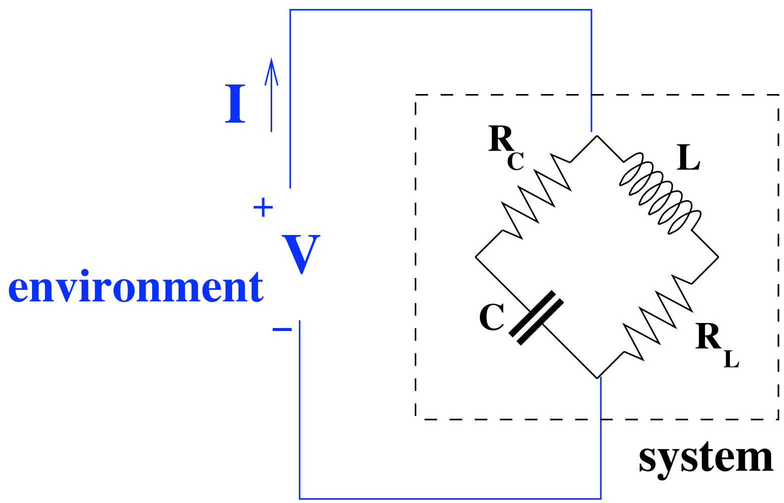
$\mathfrak{B}$ = **all possible output trajectories.**

5. Convolutional codes

6. Formal languages

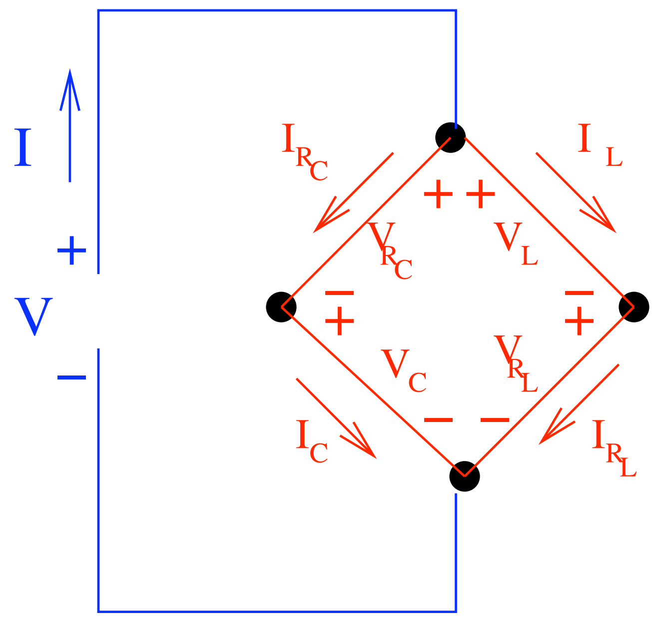
Latent variable systems

Consider our electrical RLC - circuit:



!! Model the relation between V and I !!

How does this modeling proceed?



The circuit graph

System equations

Introduce the following additional variables:

the voltage across and the current in each branch:

$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$

Constitutive equations (CE):

$$V_{R_C} = R_C I_{R_C}, \quad V_{R_L} = R_L I_{R_L}, \quad C\frac{d}{dt}V_C = I_C, \quad L\frac{d}{dt}I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{R_C} + V_C, \quad V = V_L + V_{R_L}, \quad V_{R_C} + V_C = V_L + V_{R_L}$$

Kirchhoff's current laws (KCL):

$$I = I_{R_C} + I_L, \quad I_{R_C} = I_C, \quad I_L = I_{R_L}, \quad I_C + I_{R_L} = I$$

The preceding is a complete model, but here is the

Relation between V and I.

<u>Case 1</u>: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V = (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

<u>Case 2</u>: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right)V = (1 + CR_C\frac{d}{dt})R_C I$$

Exact relations between V and I !

First principles models invariably contain auxiliary variables, in addition to the variables the model aims at.

$\leadsto$ **Manifest and latent variables.**

Manifest = the variables the model aims at,

Latent = auxiliary variables.

We want to capture this in mathematical definitions.

A dynamical system with latent variables = $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the set of relevant time instances).**

$\mathbb{W}$, **the *signal space* (= the variables that the model aims at).**

$\mathbb{L}$, **the *latent variable space* (= the auxiliary modeling variables).**

$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$: *the full behavior*

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).

The manifest behavior

Call the elements of $\mathbb{W}$ ***'manifest' variables***,

those of $\mathbb{L}$ ***'latent' variables***.

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **induces the** ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, **with** ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists \; \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

In convenient equations for $\mathfrak{B}$, **the latent variables are** ***'eliminated'.***

Examples

1. The RLC - circuit **before elimination.**

2. Models obtained by tearing and zooming

3. Input / state / output systems

$$\frac{d}{dt}x(t) = f(x(t), u(t)); \quad y(t) = h(x(t), u(t)),$$

$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{U} \times \mathbb{Y}, \mathbb{L} = \mathbb{X},$

$\mathfrak{B}_{\text{full}} =$ **all** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$ **that satisfy these equations,**

$\mathfrak{B} =$ **all (input / output)-pairs.**

4. Trellis diagrams

5. Automata

Latent variables = nodes

6. Grammars

Recapitulation

Central notions:

The behavior $\rightsquigarrow$ a model.

Manifest and latent variables $\rightsquigarrow$ specifies what the model aims at.

First principles models $\rightsquigarrow$ latent variables.

(Full) behavioral equations $\rightsquigarrow$ a specification of the (full) behavior.

Equivalent equations $:\Leftrightarrow$ the manifest behavior is the same.

Linear differential systems

We now discuss the fundamentals of the theory of systems

$$\boxed{\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})}$$

that are

1. *linear*, meaning
$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B})$;

2. *time-invariant*, meaning
$((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}))$,
where σ^t denotes the backwards $t-$shift;

3. *differential*, meaning
$\mathfrak{B}$ consists of the solutions of a system of differential equations.

Yields

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\bullet \times w}$.

Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n,$$

we obtain the short notation

$$\boxed{R(\frac{d}{dt}) w = 0.}$$

But, the theory has also been developed for PDE's.

n-D systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, n **independent variables,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, w **dependent variables,**

$\mathfrak{B}$ = **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

Define its behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$$

$\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**

Example: *Maxwell's equations*

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space),

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow$ $\exists$ free variables.

NOMENCLATURE

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **: the set of such systems with $\mathtt{n}$ in-, $\mathtt{w}$ dependent variables**

$\mathfrak{L}^{\bullet}$ **: with any - finite - number of (in)dependent variables**

Elements of $\mathfrak{L}^{\bullet}$: ***linear differential systems***

$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ **: a *kernel representation* of the corresponding $\Sigma \in \mathfrak{L}^{\bullet}$ or $\mathfrak{B} \in \mathfrak{L}^{\bullet}$**

Algebraization of $\mathfrak{L}^{\bullet}$

Note that

$$R(\frac{d}{dt})w = 0$$

and

$$U(\frac{d}{dt})R(\frac{d}{dt})w = 0$$

have the same behavior if the polynomial matrix U is uni-modular (i.e., when $\det(U)$ is a non-zero constant).

$\Rightarrow$ R defines $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, but not vice-versa!

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$??

Define the ***annihilators*** **of** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **by**

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^{\top}(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ **is clearly an** $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**.**

Let $< R >$ **denote the sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **spanned by the transposes of the rows of** R**. Obviously** $< R > \subseteq \mathfrak{N}_{\mathfrak{B}}$**. But, indeed:**

$$\boxed{\mathfrak{N}_{\mathfrak{B}} = < R >!}$$

Note: Depends on $\mathfrak{C}^{\infty}$**; (**$\Leftarrow$**) false for compact support soln's:**
for any $p \neq 0$, $\quad p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ **has only** $w = 0$
as compact support sol'n.

Conclusions:

(i) $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \overset{1:1}{\longleftrightarrow}$ **sub-modules of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$

(ii)

$$R_1(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \text{ and } R_2(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

define the same system iff

$$< R_1 > \; = \; < R_2 > .$$

Elimination

First principle models $\rightsquigarrow$ **latent variables.** **In the case of systems described by linear constant coefficient PDE's:** $\rightsquigarrow$

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell}$$

with $R, M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$**.**

This is the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

But is it(s manifest behavior) really a differential system ??

The full behavior of $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$,

$$\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w+\ell}) \mid R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell\}$$

belongs to $\mathfrak{L}_n^{w+\ell}$, **by definition. Its manifest behavior equals**

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \exists \ \ell \text{ such that } R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell.$$

Does $\mathfrak{B}$ belong to $\mathfrak{L}_n^w$?

Theorem: It does!

Proof: The 'fundamental principle'.

Example: Consider the RLC circuit.

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\rightsquigarrow$ 15 behavioral equations.

These include both the port and the branch voltages and currents.

Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.
2. The elimination theorem.

Why is there *exactly one* equation? **Passivity!**

Which PDE's describe $(\vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}, \rho$ from Maxwell's equations. Straightforward computation of the relevant left syzygy yields

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Elimination theorem $\Rightarrow$ this exercise would be exact & successful.

Remarks:

- **Number of equations for $\mathtt{n} = 1$ (constant coeff. lin. ODE's)**
  $\leq$ **number of variables.**
  Elimination $\Rightarrow$ fewer, higher order equations.
- **There exist effective computer algebra/Gröbner bases algorithms for elimination**
  $$(R, M) \mapsto R'$$
- **Not generalizable to smooth nonlinear systems.**
  Why are differential equations models so prevalent?

It follows from all this that $\mathfrak{L}^\bullet$ has very nice properties. It is closed under:

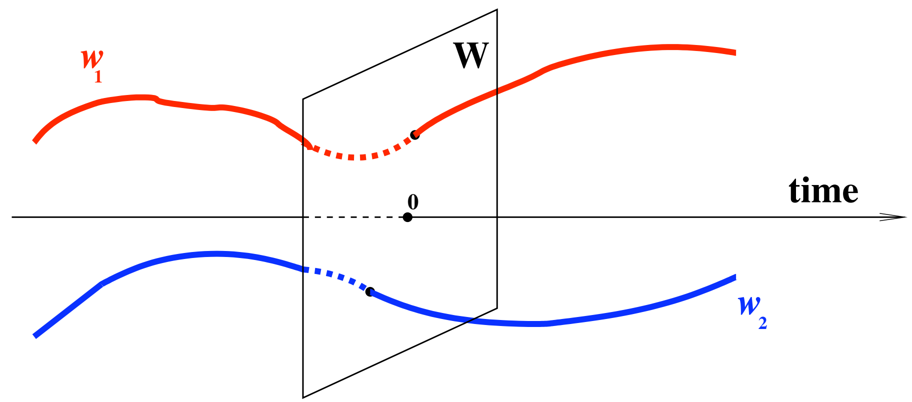
- Intersection: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- Addition: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- Projection: $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\mathtt{n}}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}})$.
- Action of a linear differential operator:
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow (P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}).$$
- Inverse image of a linear differential operator:
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow ((P(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}).$$

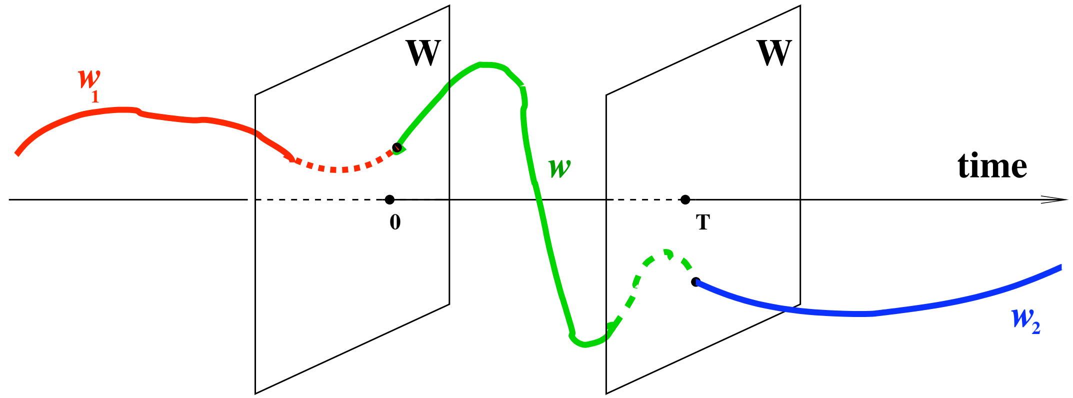
Controllability

Controllability $\Leftrightarrow$

system trajectories must be ‘patch-able’, ‘concatenable’.

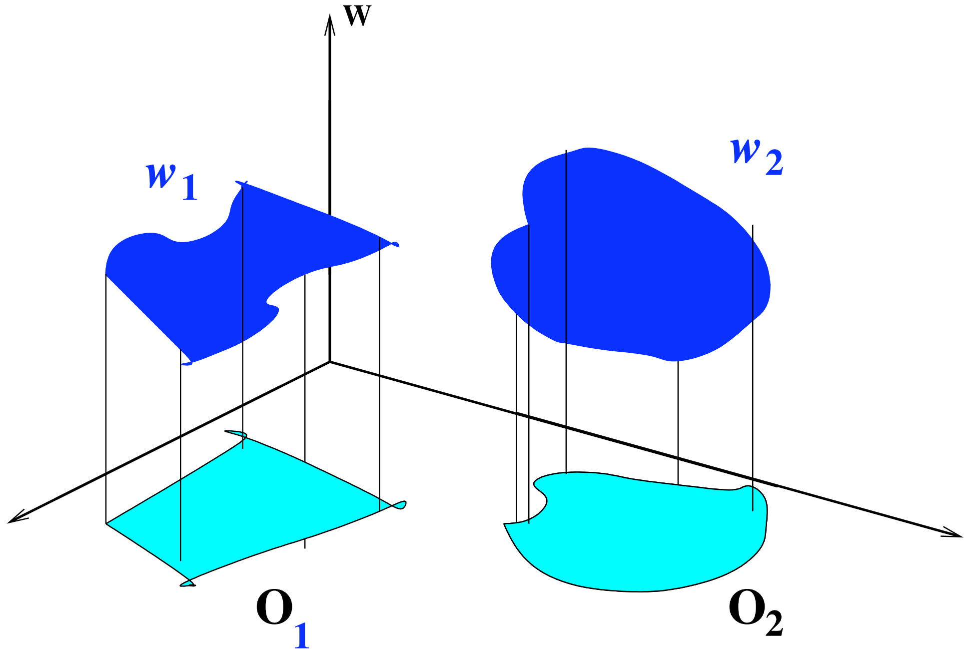
Case $\mathtt{n} = 1$



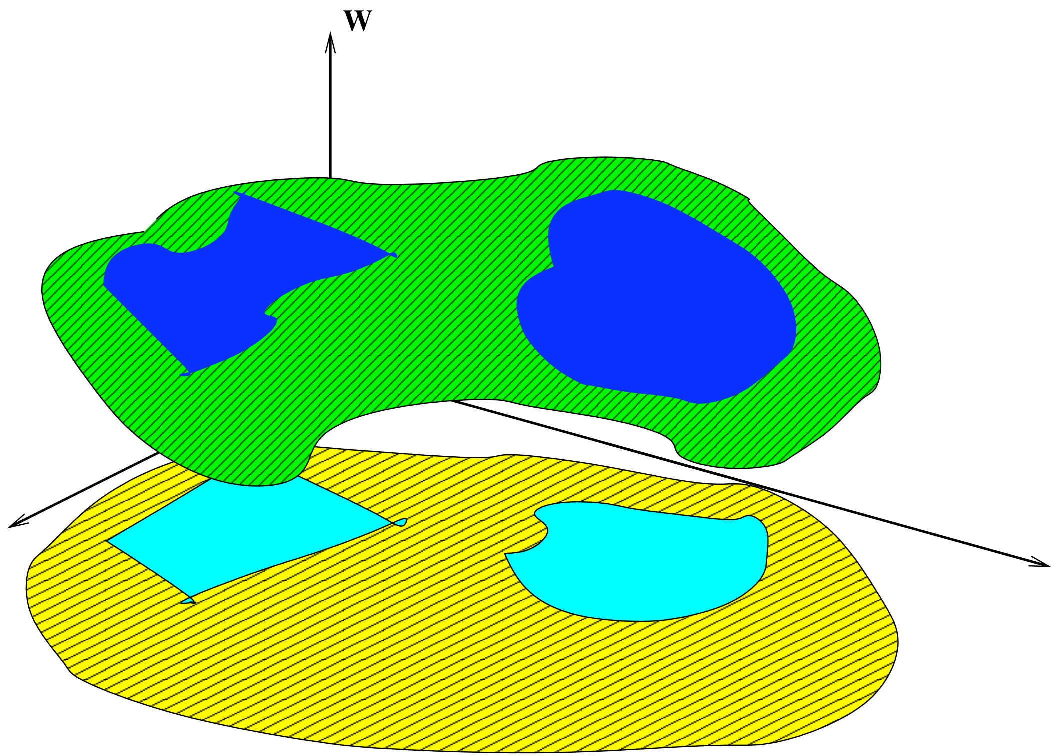


General `n`**.**

Consider two solutions:



Controllability = patchability:



Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ and $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$,

i.e., $R(\frac{d}{dt})w = 0$, controllable?

We are looking for conditions on the polynomial matrix R and algorithms in the coefficient matrices $R_0, R_1, \cdots, R_{\mathtt{n}}$.

Thm: $R(\frac{d}{dt})w = 0$ defines a controllable system if and only if

$$\boxed{\text{rank}(R(\lambda)) \text{ is independent of } \lambda \text{ for } \lambda \in \mathbb{C}.}$$

Example: $$r_1(\frac{d}{dt})w_1 = r_2(\frac{d}{dt})w_2 \quad (w_1, w_2 \text{ scalar})$$

is controllable if and only if r_1 and r_2 have no common factor.

Example: The electrical circuit is controllable unless

$$CR_C = \frac{L}{R_L} \text{ and } R_C = R_L.$$

Non-example: $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi], \quad \det(R) \neq$ constant.

Image representations

Representations of $\mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$:

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$;

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$$

called a *'latent variable' representation* of the manifest behavior
$\mathfrak{B} = (R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}))^{-1} M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\ell})$.

Missing link:

$$\boxed{w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell}$$

called an *'image' representation* of $\mathfrak{B} = \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}))$.

Elimination theorem $\Rightarrow$ **every image is also a kernel.**

¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$$a^\top[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,

etc., etc.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

Remarks:

- **Algorithm:** R + syzygies + Gröbner basis

  $\Rightarrow$ numerical test for on coefficients of R.

- $\exists$ complete generalization to PDE's

- $\exists$ partial results for nonlinear systems

- Kalman controllability is a straightforward special case

Observability

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of

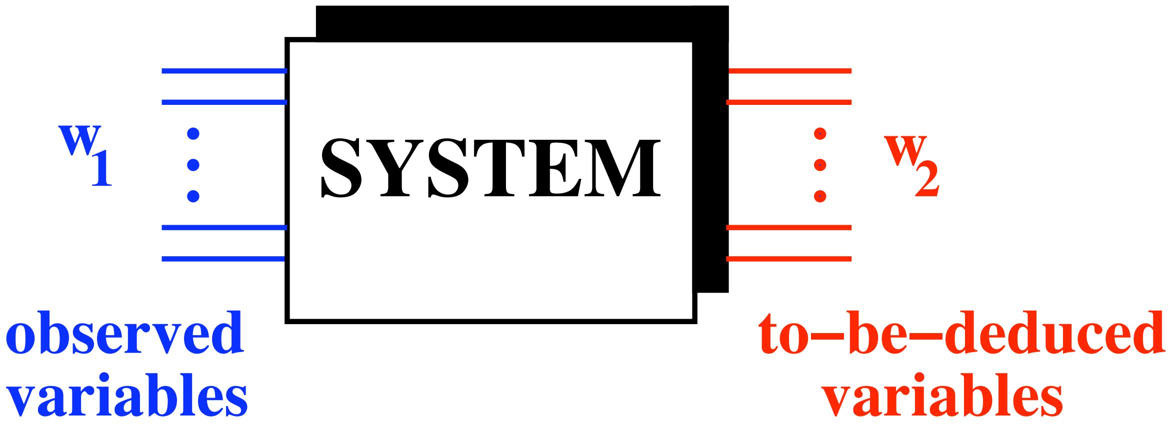
a pair of trajectories (w_1, w_2).

w_1 : observed; w_2 : to-be-deduced.

Recall: w_2 is said to be *observable* from w_1

if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,

i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.


SYSTEM
w1
w2
observed variables
to–be–deduced variables

When is in

$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$$

w_2 **observable from** w_1**?**

If and only if $\text{rank}(R_2(\lambda)) = \text{coldim}(R_2)$ **for all** $\lambda \in \mathbb{C}$**.**

In general, if and only if there exists 'consequences' (i.e. elements of $\mathfrak{N}_{\mathfrak{B}}$**) of the form** $w_2 = F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w_1$**.**

The RLC circuit is observable (branch variables observable from external port variables) iff $CR_C \neq \frac{L}{R_L}$**.**

∃ a complete theory (for constant coefficient ODE's and PDE's), including algorithms, observer design, etc.

Observability is analogous (but not 'dual') to controllability.

Further results

Many additional problem areas have been studied from the behavioral point of view.

- **System representations:** input/output representations, state representations and construction, model reduction, symmetries
- **System identification** $\Rightarrow$ the most powerful unfalsified model **(MPUM),** approximate system ID
- **Observers**
- **Control**
- **Quadratic differential forms,** dissipative systems, $\mathcal{H}_\infty$-control
- **Distributed parameter systems**

Thanks

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Is is worth worrying about these ‘axiomatics’?

They have a deep and lasting influence! Especially in teaching.

Examples:

- **Probability and the theory of stochastic processes as an axiomatization of uncertainty.**
- **The development of input/output ideas in system theory and control - often these axiomatics are implicit, but nevertheless much very present.**
- **QM.**

Thank you for your attention

Details & copies of the lecture frames are available from/at

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