THE BEHAVIORAL APPROACH

to

SYSTEMS and CONTROL

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Problematique:

Develop a suitable *mathematical* framework to discuss (dynamical) systems that interact with their environment,

aimed at modeling, analysis, and synthesis.

$\leadsto$ **control, signal processing, system identification, . . .**

OUTLINE

Part I

1. Motivational examples
2. Historical remarks
3. Basic concepts
4. Latent variables
5. Linearity, Time-invariance
6. Controllability and observability
7. Modeling by tearing and zooming

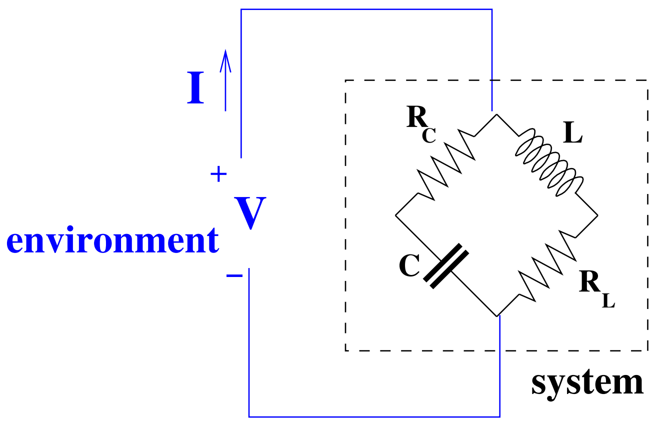
OUTLINE

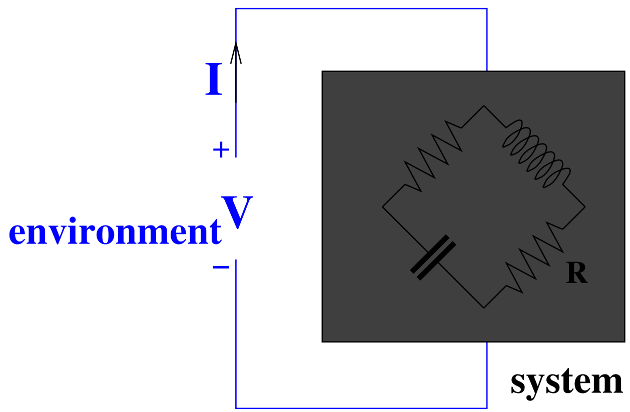
Part II

1. **Linear differential systems**
2. **Algebraization**
3. **Elimination of latent variables**
4. **Controllability**
5. **Observability**
6. **Other issues: Distributed systems**
7. **Control in a behavioral setting**

MOTIVATIONAL EXAMPLES

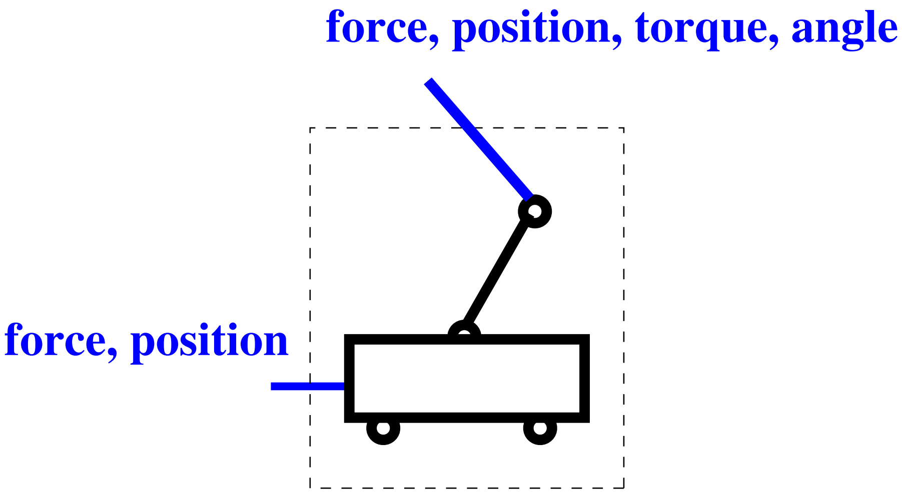
Consider the electrical circuit

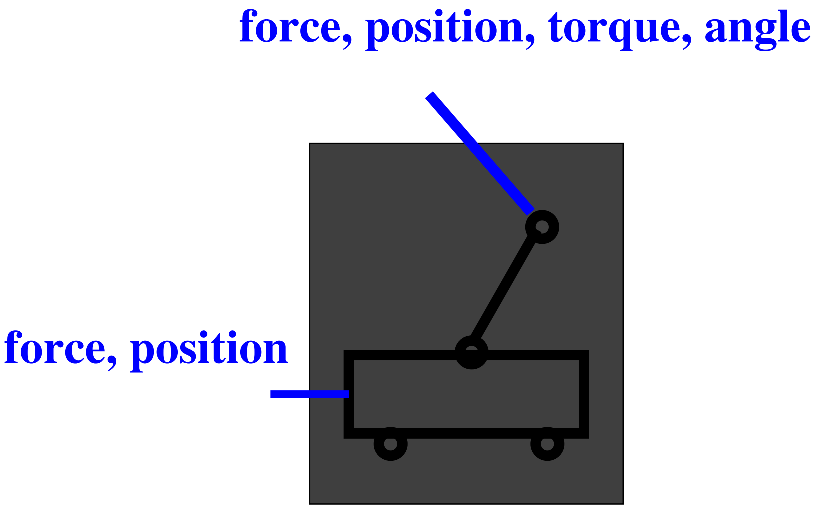




!! Model the relation between the voltage V and the current I

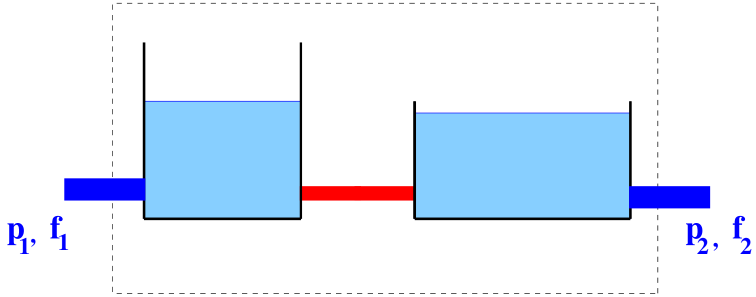
Consider the mechanical system

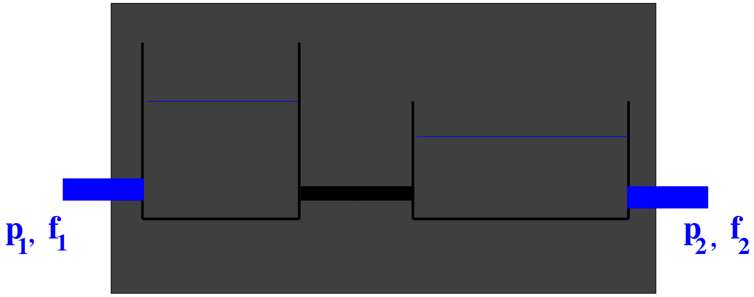




!! Model the relation between the positions, forces, torque, angle

Consider the fluidic system





!! Model the relation between the flows and the pressures

HISTORICAL REMARKS

Early 20-th century: emergence of the notion of a transfer function (Rayleigh, Heaviside).

Since the 1920's: routinely used in circuit theory (Foster, Brune, Cederbaum, $\cdots$) $\leadsto$ impedances, admittances, scattering matrices, etc.

Since the 1930's: control theory embraces transfer functions (Nyquist, Bode, $\cdots$) $\leadsto$ plots and diagrams, classical control.

Around 1950: Wiener sanctifies the notion of a blackbox, attempts nonlinear generalization (via Volterra series).

1960's: Kalman's state space ideas come in vogue

$\rightsquigarrow$ **input/state/output systems, and the ubiquitous**

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du,$$

or its nonlinear counterpart

$$\frac{d}{dt}x = f(x,u), y = h(x,u).$$

Axiomatization in the book Kalman, Falb and Arbib:

A system = a state transition function followed by a read-out map.

All these theories: <u>input/output</u>; cause $\Rightarrow$ effect.

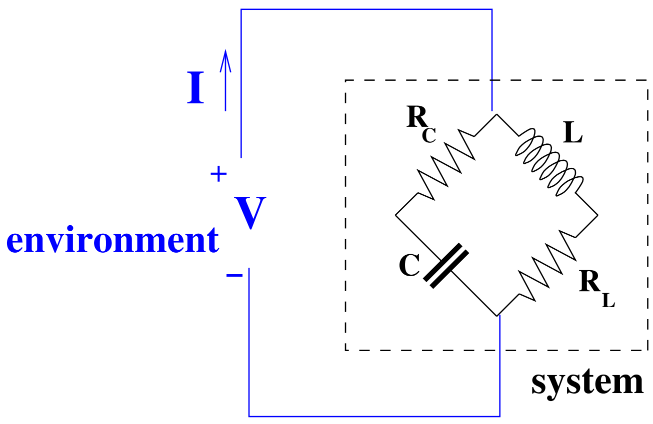
On the sidelines: sputtering

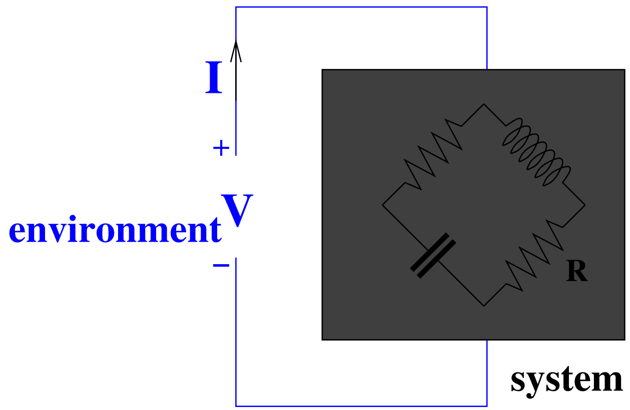
in system theory: Rosenbrock's system matrices
in circuit theory (Newcomb, Belevitch)
in CS with formal languages, automata, grammars
in DES.

What's wrong with input/output thinking?

Let's look at examples:

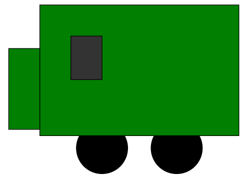
Our electrical circuit.





Is *V* the input? Or *I* ? Or both, or are they both outputs?

Consider an automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: the force,

at the tire terminals: the forces, or, more likely, the positions?

at the steering wheel: the torque or the angle?

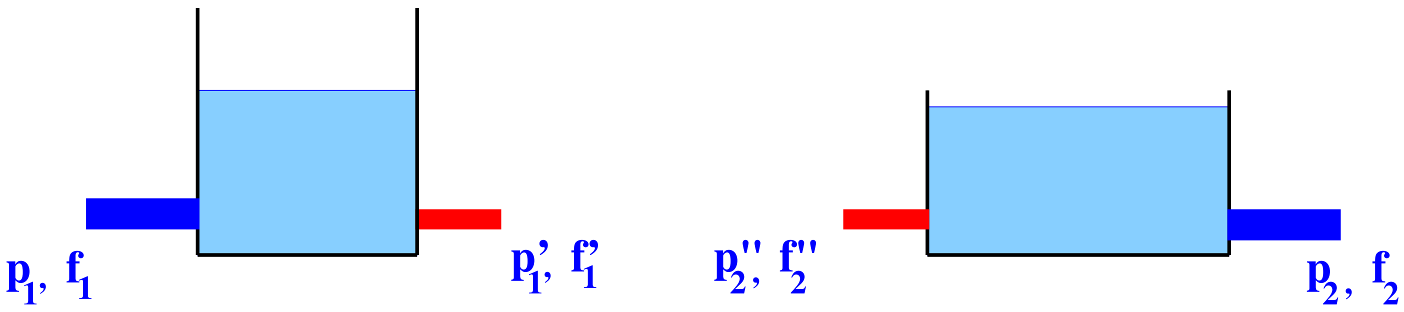
at the gas-pedal, or the brake-pedal: the force or the position?

Difficulty: at each terminal there are many (typically paired) interconnection variables

Input/output is awkward in modeling interconnections.

Consider the two-tank example.

Assume that we model the tank as an interconnection of two tanks.



Reasonable input choices: the pressures, output choices: the flows.
Now interconnect:

Interconnection: $p'_1 = p''_2, \quad f'_1 + f''_2 = 0$

input=input; output=output! $\Rightarrow\Leftarrow$ **SIMULINK©**

very many such examples (e.g. in mechanics, heat transfer, etc.)

Conclusions

- **External variables are basic, but what 'drives' what, is not.**
- **It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.**
- **What can be the input, and what can be the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'model'.**
- **Interconnection, rather that input selection, is the basic mechanism by which a system interacts with its environment.**

⇒ We need a better framework for discussing 'open' systems!

Is is worth worrying about these ‘axiomatics’?

They have a deep and lasting influence! Especially in teaching.

Examples:

- **Probability and the theory of stochastic processes as an axiomatization of uncertainty.**
- **The development of input/output ideas in system theory and control - often these axiomatics are implicit, but nevertheless much very present.**
- **QM.**

THE BASIC CONCEPTS

BEHAVIORAL SYSTEMS

A dynamical system **=** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis*** **(= the relevant time instances),**

$\mathbb{W}$, **the *signal space*** **(= where the variables take on their values),**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$: ***the behavior*** **(= the admissible trajectories).**

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

For a trajectory $w : \mathbb{T} \rightarrow \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,
$w \notin \mathfrak{B}$: the model forbids the trajectory w.

Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$ (in discrete-time systems).

Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),
a function space
(in distributed systems, with time a distinguished variable),
or a finite set (in DES).

Emphasis later today: $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = solutions of system of linear constant coefficient ODE's.

EXAMPLES

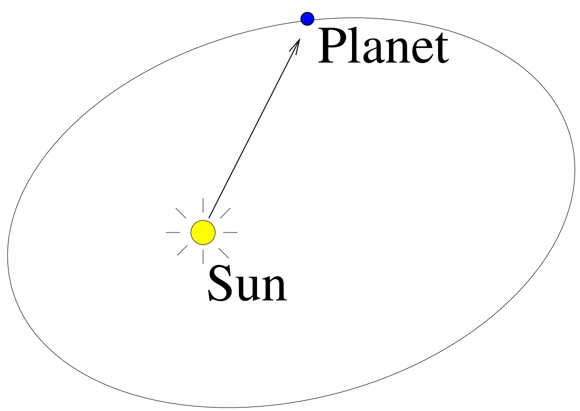
1. *Planetary orbits*

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{R}^3$ **(position),**

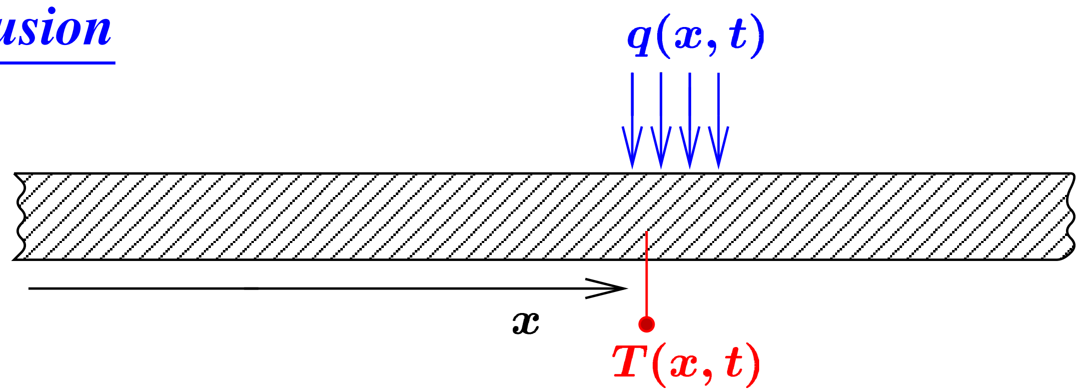
$\mathfrak{B} =$ **planetary orbits** $\cong$ **Kepler's laws:**

$$\text{ellipses, = areas in = time, } \frac{(\text{period})^2}{(\text{axis})^3} = \text{constant.}$$



Planetary orbits

2. *Heat diffusion*



A heated bar

Diffusion **describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) along a uniform bar (infinitely long), and the heat $q(x,T)$ supplied to the bar. $\rightsquigarrow$ the PDE**

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ **all (temperature, heat) distributions along a line ,**

$\mathfrak{B} =$ **all $T(\cdot,t), q(\cdot,t)$-pairs that satisfy the PDE.**

Note: We view t as a distinguished variable.

3. *Input / output systems*

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right)$$

$$= f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ (time),

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ (input $\times$ output signal spaces),

$\mathfrak{B} =$ all input / output pairs.

4'. *Flows*

$$\frac{d}{dt}x(t) = f(x(t)),$$

$\mathfrak{B}$ = **all state trajectories.**

4". *Observed flows*

$$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

$\mathfrak{B}$ = **all possible output trajectories.**

Note: It may be impossible to express $\mathfrak{B}$ as the solutions of a differential equation involving only y.

5. *Codes*

$\mathbb{A}$ = the code alphabet, say, $\mathbb{A} = \mathbb{F}^{\mathtt{w}}$, $\mathbb{F}$ a finite field,

$\mathbb{I}$ = an index set, say,

$\mathbb{I} = (1, \cdots, \mathtt{n})$ in block codes,

$\mathbb{I} = \mathbb{N}$ or $\mathbb{Z}$ in convolutional codes,

$\mathfrak{C} \subseteq \mathbb{A}^{\mathbb{I}}$ = the code; yields the system $\Sigma = (\mathbb{I}, \mathbb{A}, \mathfrak{C})$.

Redundancy structure, error correction possibilities, etc., are visible in the code behavior $\mathfrak{C}$. It is the central object of study. The encoder and decoder can be put (temporarily) into the background.

Example: The following error detecting code:

$\mathbb{I} = \mathbb{Z}$, $\mathbb{A} = \mathbb{F} = \{0, 1\}$,

$\mathfrak{B}$ = all compact support sequences $w : \mathbb{Z} \rightarrow \mathbb{F}$ such that

$$w(t) = p_0 \ell(t) + p_1 \ell(t-1) + \cdots + p_{\mathtt{n}} \ell(t-\mathtt{n})$$

for some $\ell : \mathbb{Z} \rightarrow \mathbb{F}$, with $p_0, p_1, \ldots, p_{\mathtt{n}} \in \mathbb{F}$ design parameters.

6. *Formal languages*

$\mathbb{A}$ $=$ a (finite) alphabet,

$\mathfrak{L} \subseteq \mathbb{A}^*$ $=$ the language = all ‘legal’ ‘words’ $a_1 a_2 \cdots a_k \cdots$

yields the system $\Sigma = (\mathbb{N}, \mathbb{A}, \mathfrak{L})$.

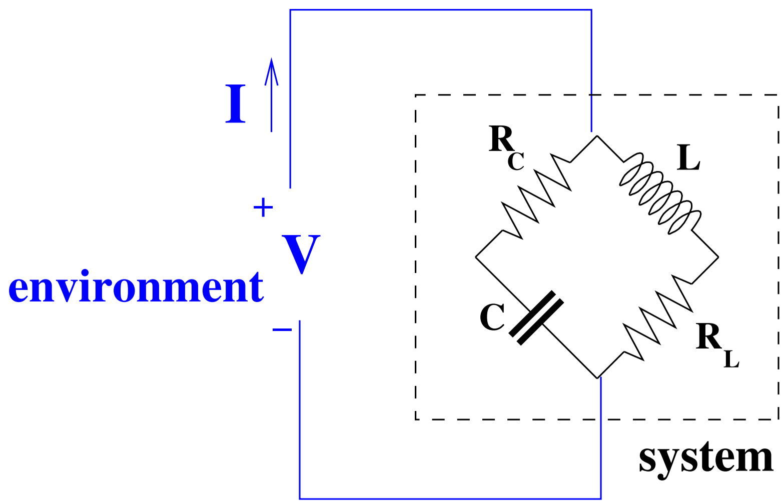
$\mathbb{A}^*$ $=$ all finite strings with symbols from $\mathbb{A}$.

Examples: All words appearing in the *van Dale*

All LaTeX documents

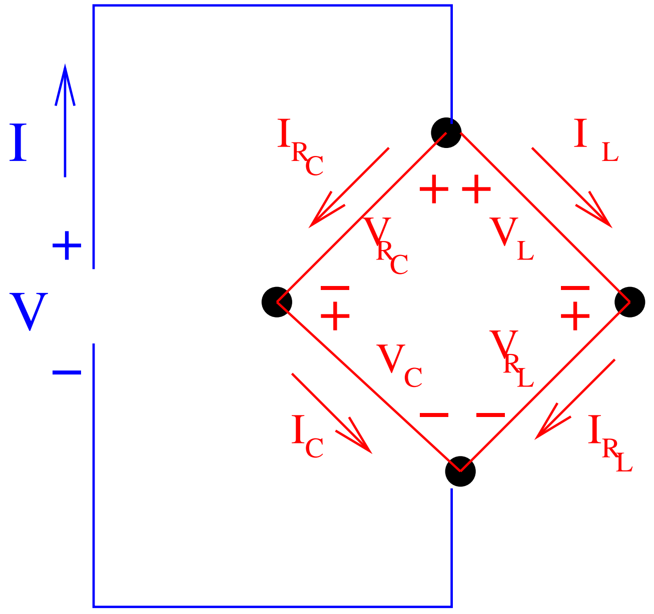
LATENT VARIABLE SYSTEMS

Consider our electrical RLC - circuit:



!! Model the relation between V and I !!

How does this modeling proceed?



The circuit graph

SYSTEM EQUATIONS

Introduce the following additional variables:

the voltage across and the current in each branch:

$$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$$

Constitutive equations (CE):

$$V_{R_C} = R_C I_{R_C}, \quad V_{R_L} = R_L I_{R_L}, \quad C\frac{d}{dt}V_C = I_C, \quad L\frac{d}{dt}I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{R_C} + V_C, \quad V = V_L + V_{R_L}, \quad V_{R_C} + V_C = V_L + V_{R_L}$$

Kirchhoff's current laws (KCL):

$$I = I_{R_C} + I_L, \quad I_{R_C} = I_C, \quad I_L = I_{R_L}, \quad I_C + I_{R_L} = I$$

RELATION BETWEEN V and I

After some calculations, we obtain the port equations:

<u>Case 1</u>: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V$$
$$= (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

<u>Case 2</u>: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right)V = (1 + CR_C\frac{d}{dt})R_C I$$

These are the exact relations between V and I !

All models of interconnected systems will have such interconnection variables.

First principles models invariably contain auxiliary variables, in addition to the variables the model aims at.

$\leadsto$ **Manifest and latent variables.**

We want to capture this is definitions.

A dynamical system with latent variables = $\boxed{\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})}$

$\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the set of relevant time instances).

$\mathbb{W}$, the *signal space* (= the variables that the model aims at).

$\mathbb{L}$, the *latent variable space* (= the auxiliary modeling variables).

$\boxed{\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}}$ **:** ***the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).

THE MANIFEST BEHAVIOR

Call the elements of $\mathbb{W}$ ***'manifest' variables*** **,**

those of $\mathbb{L}$ ***'latent' variables*** **.**

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **induces the** ***manifest system*** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$**, with** ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

In convenient equations for $\mathfrak{B}$**, the latent variables are** ***'eliminated'.***

EXAMPLES

1. *The RLC - circuit*

$\mathbb{T} = \mathbb{R}$,

$\mathbb{W} = \mathbb{R}^2$ **- manifest variables: the port voltage and current,**

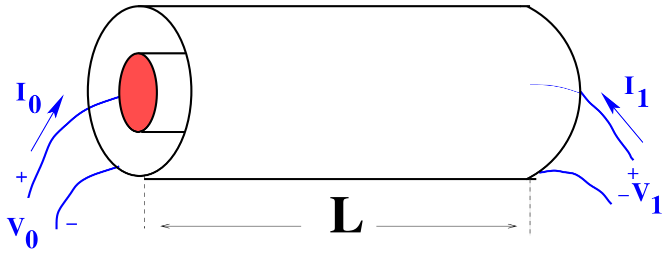
$\mathbb{L} = \mathbb{R}^8$ **- latent variables: the branch voltages and currents,**

$\mathfrak{B}_{\text{full}} =$ **all functions** $(V, I, V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L)$
that satisfy the CE's, KCL, and KVL,

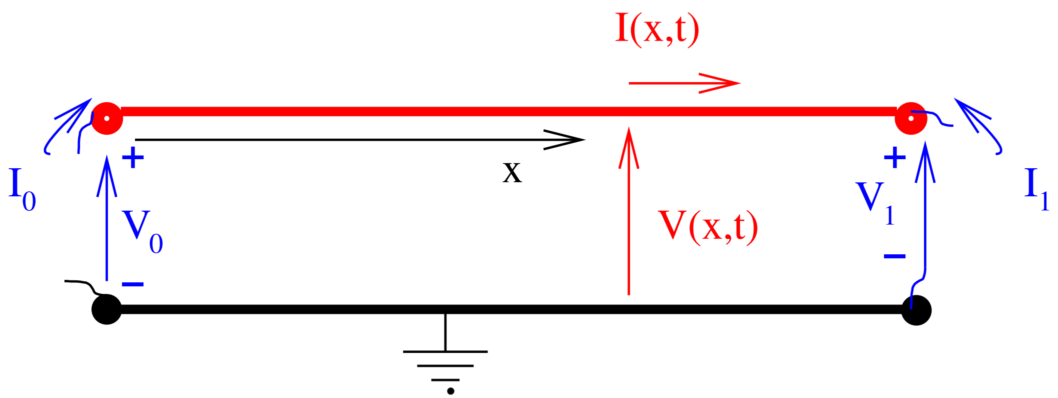
$\mathfrak{B} =$ **the functions** (V, I) **that satisfy the 'eliminated' port equations.**

2. *Coaxial cable*

!! Model the relation between the voltages V_0, V_1 and the currents I_0, I_1 at the ends of a uniform coaxial cable of length L.



Introduce the voltage $V(x,t)$ and the current flow $I(x,t)$ $0 \leq x \leq L$ in the cable.



Leads to the equations:

$$\frac{\partial}{\partial x} V = -L_0 \frac{\partial}{\partial t} I,$$
$$\frac{\partial}{\partial x} I = -C_0 \frac{\partial}{\partial t} V,$$

$$V_0(t) = V(0,t), \quad V_1(t) = V(1,t),$$

$$I_0(t) = I(0,t), \quad I_1(t) = -I(1,t).$$

with L_0 **the inductance, and** C_0 **the capacitance per unit length.**

This is a latent variable model with

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{R}^4$ **manifest variables: (voltage, current) at both ends,**

$\mathbb{L} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ **voltage and current distribution along the bar,**

$\mathfrak{B}_{\text{full}}$ **= the solutions of the above PDE's and boundary conditions,**

$\mathfrak{B}$ **= the** (V_0, I_0, V_1, I_1)**-trajectories declared possible:**

$$\mathfrak{B} = \{(V_0, I_0, V_1, I_1) : \mathbb{R} \to \mathbb{R}^4 \mid \exists \ (V, I) : [0, L] \to \mathbb{R}^2 : \text{ the above PDE's and boundary conditions are satisfied } \}$$

Note: we still view t as a distinguished variable.

3. *Input /state / output systems*

$$\frac{d}{dt}x(t) = f(x(t), u(t)); \quad y(t) = h(x(t), u(t)),$$

$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{U} \times \mathbb{Y}, \mathbb{L} = \mathbb{X},$

$\mathfrak{B}_{\text{full}} =$ **all** $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$ **that satisfy these equations,**

$\mathfrak{B} =$ **all (input / output)-pairs.**

Also,

$$f(\frac{d}{dt}x(t), x(t), w(t)) = 0$$

called 'implicit' systems.

4. *Trellis diagrams*

5. *Automata*

Latent variables = nodes

6. *Grammars*

Another way to specify a formal language whose essence is captured by latent variables.

GENERAL PROPERTIES

of

DYNAMICAL SYSTEMS

LINEARITY

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

linear

if $\mathbb{W}$ is a vector space (over a field $\mathbb{F}$),
and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$
(viewed as a vector space over $\mathbb{F}$ with respect to pointwise addition and pointwise multiplication).

Hence linearity $:\Leftrightarrow$ the *superposition principle* holds:

$$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{F})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B}).$$

TIME-INVARIANCE

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ (assume $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$) is said to be

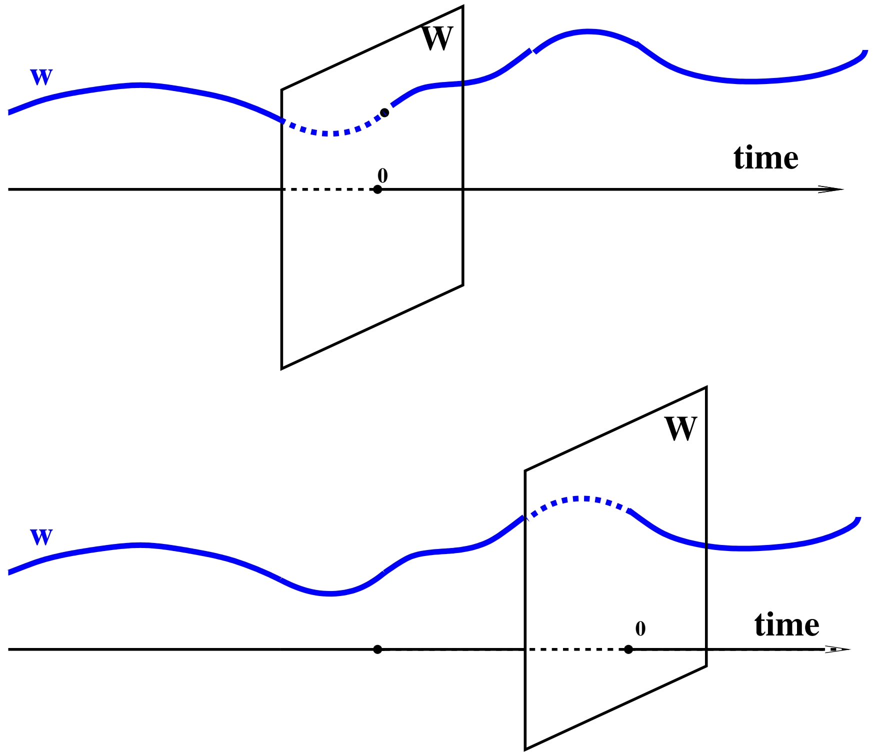
time-invariant

if

$$((w \in \mathfrak{B}) \wedge (t \in \mathbb{T})) \Rightarrow (\sigma^t w \in \mathfrak{B})),$$

where σ^t denotes the *backwards* $t-$*shift*, defined by

$$\sigma^t w(t') := w(t + t').$$



Time-invariance

DIFFERENTIAL SYSTEMS

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ (assume $\mathbb{T} = \mathbb{R}$ and $\mathbb{W}$ a differentiable manifold) is said to be a

differential system

if its behavior $\mathfrak{B}$ consists of the solutions of a system of differential equations,

$$f\left(w(t), \frac{d}{dt}w(t), \frac{d^2}{dt^2}w(t), \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}w(t), t\right) = 0.$$

These properties extend in an obvious way to latent variable systems.

CONTROLLABILITY and OBSERVABILITY

CONTROLLABILITY

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

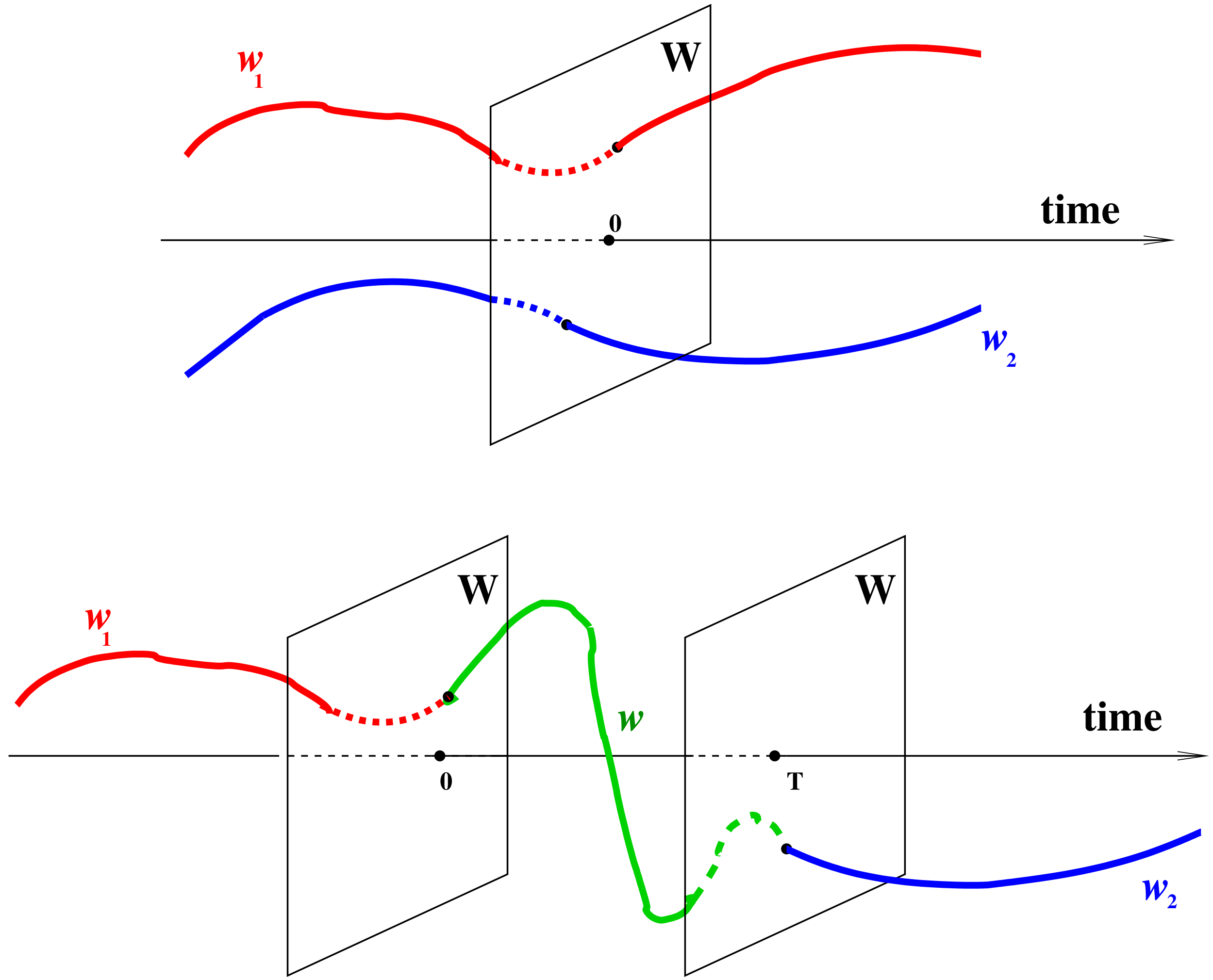
controllable

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

Controllability $\Leftrightarrow$

legal trajectories must be **'patch-able', 'concatenable'.**



Controllability

OBSERVABILITY

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of

a pair of trajectories (w_1, w_2).

w_1 : observed; w_2 : to-be-deduced.

Definition: w_2 is said to be

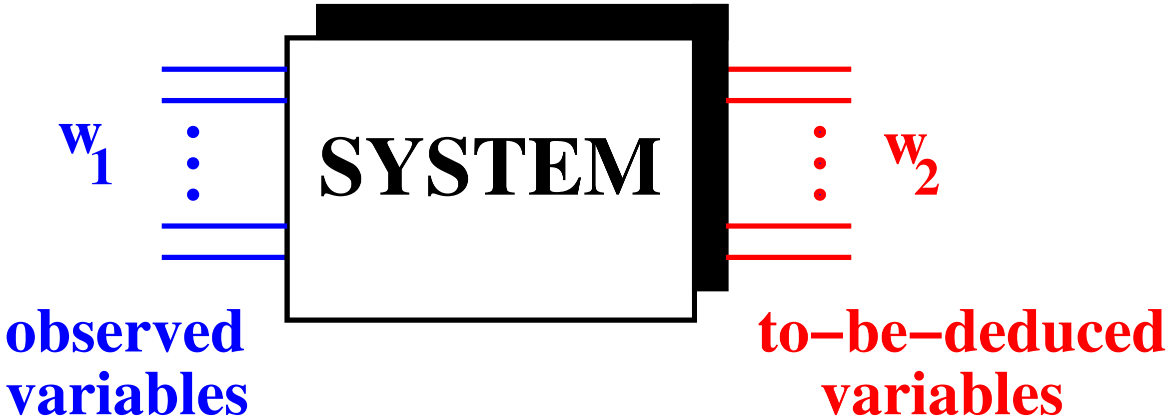
***observable* from w_1**

if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,

i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

Very often manifest = observed, latent = to-be-deduced.

We then speak of an observable latent variable system.



Observability

Special case: Kalman definitions:

controllability: variables = (input, state)

If a system is not (state) controllable, why is it?

Insufficient influence of control?

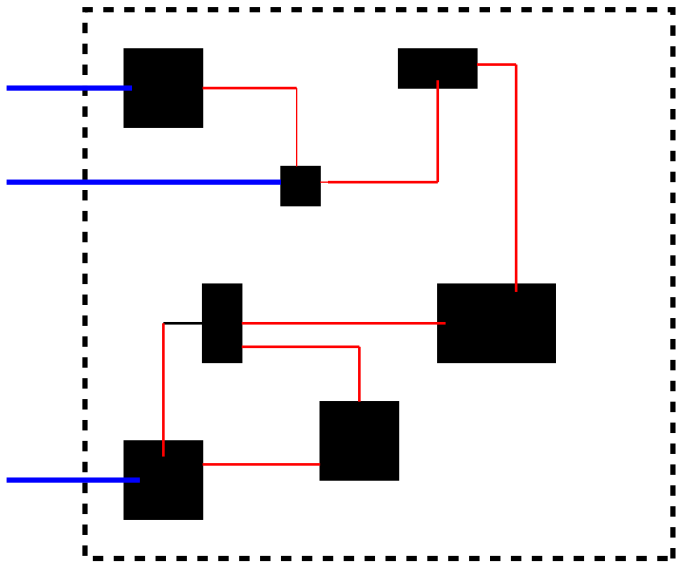
Or bad choice of state?

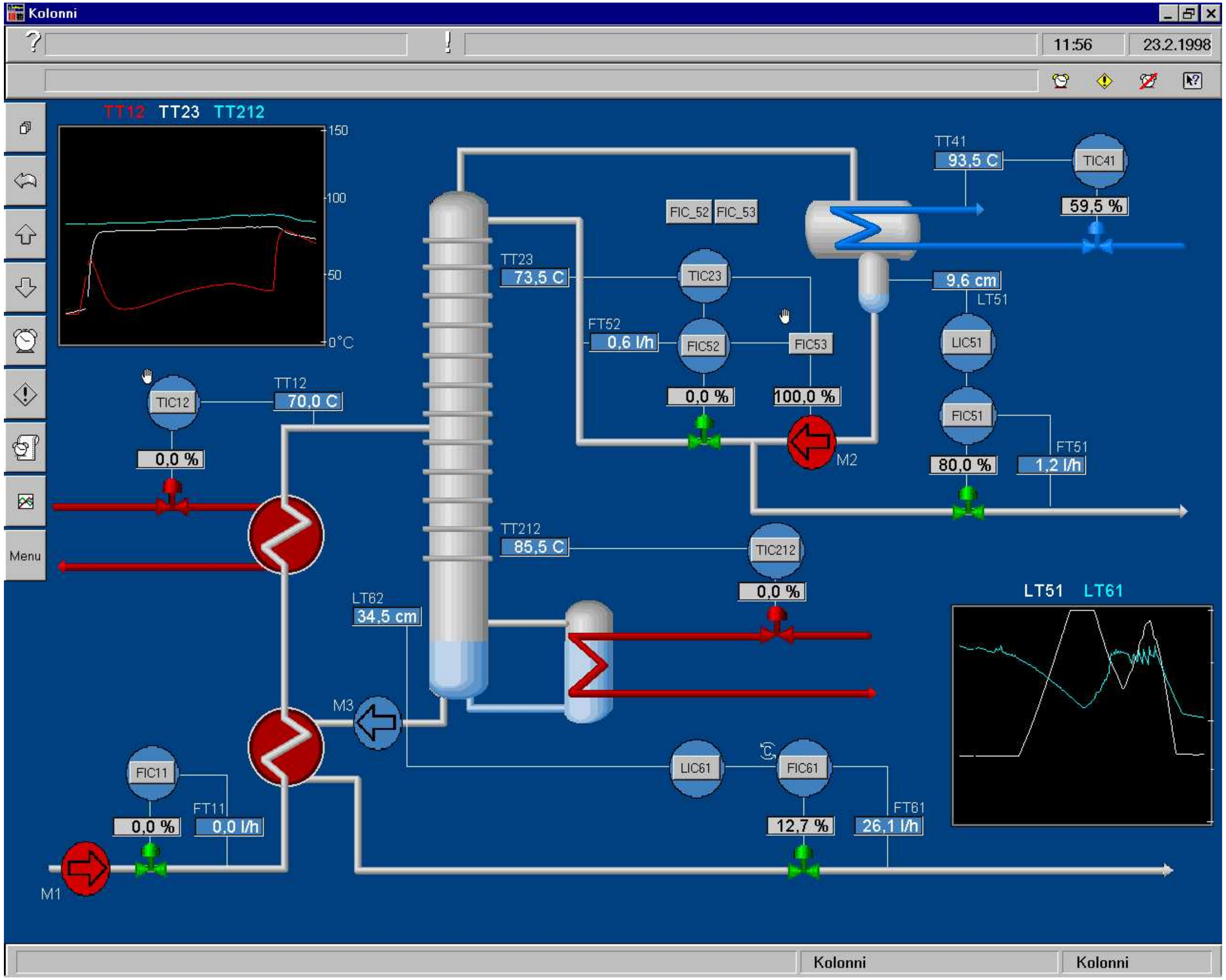
observed = (input, output), to-be-deduced = state.

Kalman definitions address rather special situations.

MODELING by TEARING and ZOOMING

Interconnected System

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TT23
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TIC23
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FIC52
FIC53
0,0 %
100,0 %
M2
9,6 cm
LT51
LIC51
FIC51
80,0 %
FT51
1,2 l/h
TIC12
TT12
70,0 C
0,0 %
TT212
85,5 C
TIC212
0,0 %
LT62
34,5 cm
LT51 LT61
M3
FIC11
0,0 %
FT11
0,0 l/h
M1
LIC61
FIC61
12,7 %
FT61
26,1 l/h
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?? How do we model such an interconnected system ??

It is not feasible to recognize the signal flow graph before we have a model (Ex.: electrical circuit).

The signal flow graph should be deduced from a model ...

Input-to-output connections, combining series, parallel, and feedback ($\Rightarrow$ SIMULINK$^{©}$) of little use.

More suitable approach $\rightsquigarrow$ **Bondgraphs:**

- **Recognize flow and effort variables, energy 'bonds'**
- **Obtain model for components**

Excellent physical motivation, much more suitable than input/output.

But

- **Does not provide a language for modeling the 'atoms'**
- **There is much more to interconnections than energy exchange via ports**
- **Does not incorporate synthesis (control, etc.) algorithms**

Behavioral ideas in modeling interconnected systems

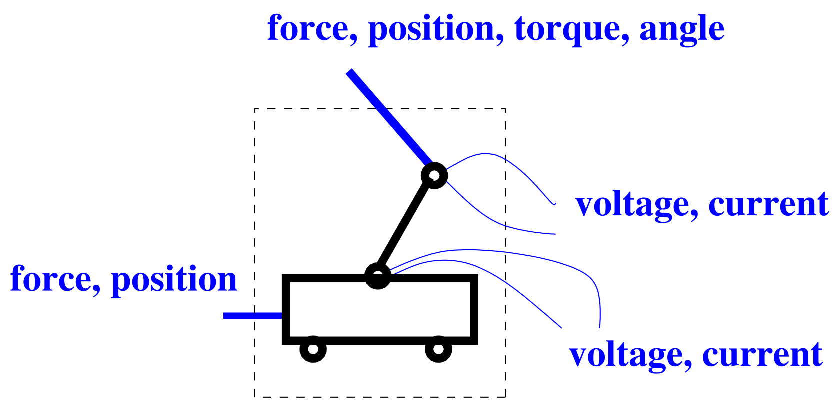
The ingredients of the language and methodology that we propose:

1. ***Modules*** **: the subsystems**
2. ***Terminals*** **: the physical links between subsystems**
3. **The *interconnection architecture* :**

   the layout of the modules and their interconnection
4. **The *manifest variable assignment* :**

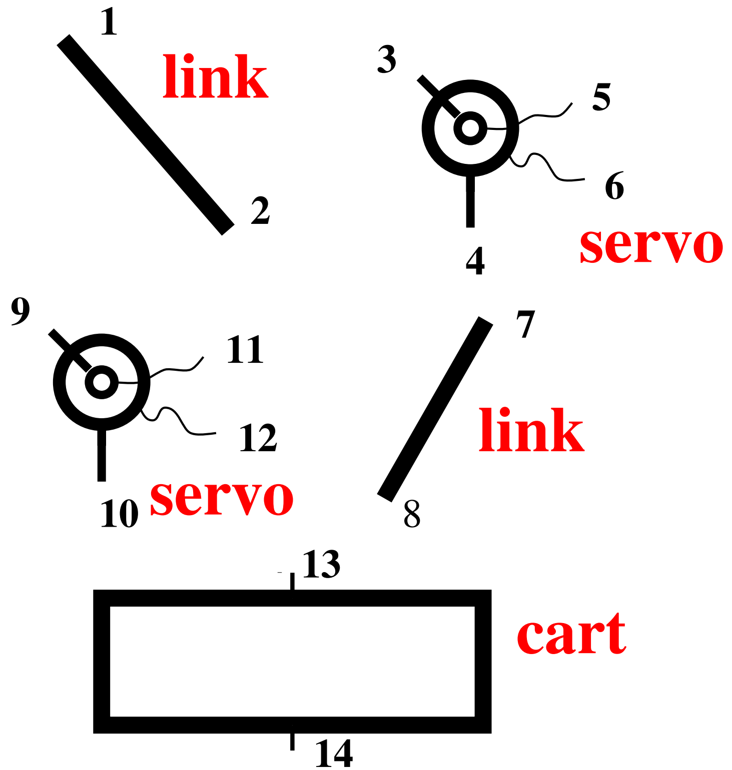
   which variables does the model aim at?

Let us look at an example...



!! Model the relation between the positions, forces, torque, angle

Tearing

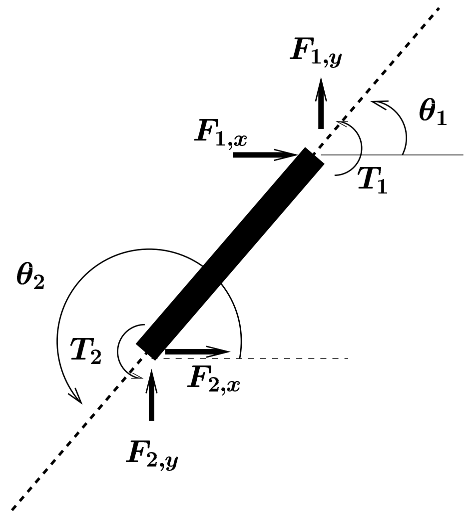


Zooming

Obtain models of the subsystems

Required modules in our example: **Solid bars, servo's.**

Solid bar



Terminals: 2 mechanical 2-D terminals.

Variables: $x_1, y_1, \theta_1, x_2, y_2, \theta_2, F_{x_1}, F_{y_1}, T_1, F_{x_2}, F_{y_2}, T_2$.

Parameters: $L \in \mathbb{R}_+$ (length),
$m \in \mathbb{R}_+$ (mass per unit length).

Behavioral equations:

$$mL\frac{d^2}{dt^2}x_c = F_{x_1} + F_{x_2},$$

$$mL\frac{d^2}{dt^2}y_c = F_{y_1} + F_{y_2} - mLg,$$

$$m\frac{L^3}{12}\frac{d^2}{dt^2}\theta_c = T_1 + T_2 - \frac{L}{2}F_{x_1}\sin(\theta_1) + \frac{L}{2}F_{y_1}\cos(\theta_1) - \frac{L}{2}F_{x_2}\sin(\theta_2) + \frac{L}{2}F_{y_2}\cos(\theta_2),$$

$$\theta_1 = \theta_c,$$

$$\theta_2 = \theta_1 + \pi,$$

$$x_1 = x_c + \frac{L}{2}\cos(\theta_c),$$

$$x_2 = x_c - \frac{L}{2}\cos(\theta_c),$$

$$y_1 = y_c + \frac{L}{2}\sin(\theta_c),$$

$$y_2 = y_c - \frac{L}{2}\sin(\theta_c).$$

Note: Contains latent variables x_c, y_c, θ_c.

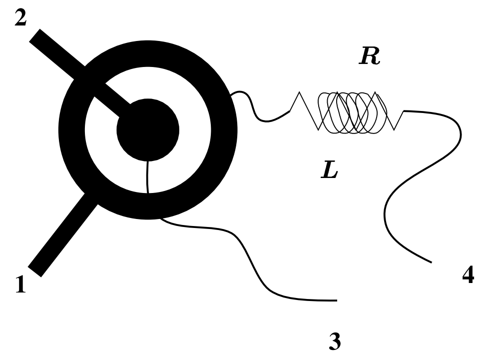
This defines a system with

$\mathbb{T} = \mathbb{R}$

$\mathbb{W} = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^*S^1) \times (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^*S^1)$

$\mathfrak{B}$ = **solutions** $(x_1, y_1, \theta_1, x_2, y_2, \theta_2, F_{x_1}, F_{y_1}, T_1, F_{x_2}, F_{y_2}, T_2)$ **of the ODE's, suitably interpreted.**

Hinge with servo



Terminals: 2 mechanical 2-D terminals, 2 electrical.

Variables: $(x_1, y_1, \theta_1, F_{x_1}, F_{y_1}, T_1,$
$x_2, y_2, \theta_2, F_{x_2}, F_{y_2}, T_2, V_3, I_3, V_4, I_4)$.

Parameters: the rotor mass m_r, the stator mass m_s,
the rotor inertia J_r, the stator inertia J_s,
the inductance L, the resistance R of the motor circuit,
the motor torque constant K.

Behavioral equations:

$$(m_r + m_s)\frac{d^2}{dt^2}x_1 = F_{x_1} + F_{x_2}$$
$$(m_r + m_s)\frac{d^2}{dt^2}y_1 = F_{x_1} + F_{x_2}$$
$$J_r\frac{d^2}{dt^2}\theta_1 = T_1 + T_m$$
$$J_s\frac{d^2}{dt^2}\theta_2 = T_2 - T_m$$
$$V_3 - V_4 = L\frac{d}{dt}I_3 + RI_3 + K\frac{d}{dt}(\theta_1 - \theta_2)$$
$$KI_3 = T_m$$
$$x_1 = x_2$$
$$y_1 = y_2$$
$$I_3 = -I_4$$

Note: The motor torque T_m is a latent variable.

This defines a system with

$\mathbb{T} = \mathbb{R}$

$\mathbb{W} = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^* S^1)^2 \times (\mathbb{R}^2)^2$

$\mathfrak{B}$ **= solutions**

$(x_1, y_1, \theta_1, F_{x_1}, F_{y_1}, T_1, x_2, y_2, \theta_2, F_{x_2}, F_{y_2}, T_2, V_3, I_3, V_4, I_4)$

of the ODE's, suitably interpreted.

The list of the modules and the associated terminals:

Module	Type	Terminals	Parameters
Link 1	bar	(7,8)	L_1, m_1
Link 2	bar	(1,2)	L_2, m_2
Cart	bar	(13,14)	L_3, m_3
Servo 1	servo	(9,10,11,12)	$m_{r_1}, m_{s_1}, J_{r_1}, J_{r_1}, L_1, R_1, K_1$
Servo 2	servo	(3,4,5,6)	$m_{r_2}, m_{s_2}, J_{r_2}, J_{r_2}, L_2, R_2, K_2$

The interconnection architecture:

Pairing
$\{2, 3\}$
$\{4, 7\}$
$\{8, 9\}$
$\{10, 13\}$

Manifest variable assignment:

the variables on the external terminals $\{1, 5, 6, 11, 12, 14\}$.

Equations for the full behavior:

Equations of the modules:

$$
\begin{aligned}
m_1 L_1 \frac{d^2}{dt^2} x_{c_1} &= F_{x_1} + F_{x_2}, \\
m_1 L_1 \frac{d^2}{dt^2} y_{c_1} &= F_{y_1} + F_{y_2} - m_1 L_1 g, \\
m_1 \frac{L_1^3}{12} \frac{d^2}{dt^2} \theta_{c_1} &= T_1 + T_2 \\
&\quad - \frac{L_1}{2} F_{x_1} \sin(\theta_1) + \frac{L_1}{2} F_{y_1} \cos(\theta_1) - \frac{L_1}{2} F_{x_2} \sin(\theta_2) + \frac{L_1}{2} F_{y_2} \cos(\theta_2),
\end{aligned}
$$

$$\theta_1 = \theta_{c_1}, \theta_2 = \theta_1 + \pi,$$

$$x_1 = x_{c_1} + \frac{L_1}{2} \cos(\theta_{c_1}), x_2 = x_{c_1} - \frac{L_1}{2} \cos(\theta_{c_1}),$$

$$y_1 = y_{c_1} + \frac{L_1}{2} \sin(\theta_{c_1}), y_2 = y_{c_1} - \frac{L_1}{2} \sin(\theta_{c_1}),$$

$$
\begin{aligned}
m_2 L_2 \frac{d^2}{dt^2} x_{c_2} &= F_{x_7} + F_{x_8}, \\
m_2 L_2 \frac{d^2}{dt^2} y_{c_2} &= F_{y_7} + F_{y_8} - m_2 L_2 g, \\
m_2 \frac{L_2^3}{12} \frac{d^2}{dt^2} \theta_{c_2} &= T_7 + T_8 \\
&\quad - \frac{L_2}{2} F_{x_7} \sin(\theta_7) + \frac{L_2}{2} F_{y_7} \cos(\theta_7) - \frac{L_2}{2} F_{x_8} \sin(\theta_8) + \frac{L_2}{2} F_{y_8} \cos(\theta_8),
\end{aligned}
$$

$$\theta_7 = \theta_{c_2}, \theta_8 = \theta_7 + \pi,$$

$$x_7 = x_{c_2} + \frac{L_1}{2} \cos(\theta_{c_2}), x_8 = x_{c_2} - \frac{L_1}{2} \cos(\theta_{c_2}),$$

$$y_7 = y_{c_2} + \frac{L_1}{2} \sin(\theta_{c_2}), y_8 = y_{c_2} - \frac{L_1}{2} \sin(\theta_{c_2}),$$

$$m_3 L_3 \frac{d^2}{dt^2} x_{c_3} = F_{x_{13}} + F_{x_{14}},$$
$$m_3 L_3 \frac{d^2}{dt^2} y_{c_3} = F_{y_{13}} + F_{y_{14}} - m_3 L_3 g,$$
$$m_3 \frac{L_3^3}{12} \frac{d^2}{dt^2} \theta_{c_3} = T_{13} + T_{14}$$
$$- \frac{L_3}{2} F_{x_{13}} \sin(\theta_{13}) + \frac{L_3}{2} F_{y_{13}} \cos(\theta_{13}) - \frac{L_3}{2} F_{x_{14}} \sin(\theta_{14}) + \frac{L_3}{2} F_{y_{14}} \cos(\theta_{14}),$$
$$\theta_{13} = \theta_{c_3}, \theta_{14} = \theta_{c_3} + \pi,$$
$$x_{13} = x_{c_3} + \frac{L_1}{2} \cos(\theta_{c_3}),$$
$$x_{14} = x_{c_3} - \frac{L_1}{2} \cos(\theta_{c_3}), y_{13} = y_{c_3} + \frac{L_1}{2} \sin(\theta_{c_3}),$$
$$y_{14} = y_{c_3} - \frac{L_1}{2} \sin(\theta_{c_3}),$$

$$(m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} x_3 = F_{x_3} + F_{x_4},$$
$$(m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} y_3 = F_{y_3} + F_{y_4},$$
$$J_{r_1} \frac{d^2}{dt^2} \theta_3 = T_3 + T_m,$$
$$J_{s_1} \frac{d^2}{dt^2} \theta_4 = T_4 - T_m,$$
$$V_5 - V_6 = L_1 \frac{d}{dt} I_5 + R_1 I_5 + K \frac{d}{dt} (\theta_3 - \theta_4),$$
$$K_1 I_5 = T_{m_1}, x_3 = x_4, y_3 = y_4, I_5 = -I_6,$$

$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}x_9 = F_{x_9} + F_{x_{10}},$$
$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}y_9 = F_{y_9} + F_{y_{10}},$$
$$J_{r_2}\frac{d^2}{dt^2}\theta_9 = T_9 + T_m,$$
$$J_{s_2}\frac{d^2}{dt^2}\theta_{10} = T_{10} - T_m,$$
$$V_{11} - V_{12} = L_2\frac{d}{dt}I_{11} + R_2 I_{11} + K\frac{d}{dt}(\theta_9 - \theta_{10}),$$
$$K_2 I_{11} = T_{m_2}, x_{10} = x_{11}, y_{10} = y_{11}, I_{11} = -I_{12},$$

Interconnection equations:

$$F_{x_2} + F_{x_3} = 0,\ F_{y_2} + F_{y_3} = 0,\ x_2 = x_3,\ y_2 = y_3,\ \theta_2 = \theta_3 + \pi,\ T_2 + T_3 = 0,$$

$$F_{x_4} + F_{x_7} = 0,\ F_{y_4} + F_{y_7} = 0,\ x_4 = x_7,\ y_4 = y_7,\ \theta_4 = \theta_7 + \pi,\ T_4 + T_7 = 0,$$

$$F_{x_8} + F_{x_9} = 0,\ F_{y_8} + F_{y_9} = 0,\ x_8 = x_9,\ y_8 = y_9,\ \theta_8 = \theta_9 + \pi,\ T_8 + T_9 = 0,$$

$$F_{x_{10}} + F_{x_{13}} = 0,\ F_{x_{10}} + F_{x_{13}} = 0, x_{10} = x_{13},\ y_{10} = y_{13},$$
$$\theta_{10} = \theta_{13} + \pi,\ T_{10} + T_{13} = 0.$$

Features:

- **Reality — 'physics' — based**
- **Mathematically precise; uses behavioral systems concepts**
- **Recognizes prevalence of latent variables**
- **More akin to bond-graphs and across/through variables, than to input/output thinking and feedback connections**
- **Not restricted to energy bonds, or ports**
- **Modular: starts from 'standard' building blocks**
- **Hierarchical: allows new systems to be build from old**
- **Models are reusable, generalizable & extend-able**
- **Assumes that accurate and detailed modeling is the aim**

RECAP

- A behavioral system = a family of trajectories
- First principle models contain latent variables
- Allows properties, as controllability, to be introduced at the system level
- Well adapted to modeling interconnected systems
- Input/output: OK for signal processing, but not for modeling physical systems

OUTLINE

Part II

1. Linear differential systems
2. Algebraization
3. Elimination of latent variables
4. Controllability
5. Observability
6. Other issues: Distributed systems
7. Control in a behavioral setting

LINEAR DIFFERENTIAL SYSTEMS

We now discuss the fundamentals of the theory of systems

$$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$$

that are

1. *linear*, meaning
   $((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B})$;

2. *time-invariant*, meaning
   $((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}))$,
   where σ^t denotes the backwards $t-$shift;

3. *differential*, meaning
   $\mathfrak{B}$ consists of the solutions of a system of differential equations.

LINEAR CONSTANT COEFFICIENT DIFFERENTIAL EQ'NS.

Variables: $w_1, w_2, \ldots w_{\mathtt{w}}$, **up to** n**-times differentiated,** g **equations.**

$$\begin{array}{rcl}
\Sigma_{j=1}^{w} R_{1,j}^{0} w_j + \Sigma_{j=1}^{w} R_{1,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{w} R_{1,j}^{n} \frac{d^n}{dt^n} w_j & = & 0 \\
\Sigma_{j=1}^{w} R_{2,j}^{0} w_j + \Sigma_{j=1}^{w} R_{2,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{w} R_{2,j}^{n} \frac{d^n}{dt^n} w_j & = & 0 \\
\vdots & & \vdots \\
\Sigma_{j=1}^{w} R_{g,j}^{0} w_j + \Sigma_{j=1}^{w} R_{g,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{w} R_{g,j}^{n} \frac{d^n}{dt^n} w_j & = & 0
\end{array}$$

Coefficients $R_{i,j}^{k}$**: 3 indices!**

- $i = 1, \ldots, g$ **: for the** i**-th differential equation,**
- $j = 1, \ldots, w$ **: for the variable** w_j **involved,**
- $k = 1, \ldots, n$ **: for the order** $\frac{d^k}{dt^k}$ **of differentiation.**

In vector/matrix notation:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix}, \quad R_{\mathrm{k}} = \begin{bmatrix} R^{\mathrm{k}}_{1,1} & R^{\mathrm{k}}_{1,2} & \cdots & R^{\mathrm{k}}_{1,\mathtt{w}} \\ R^{\mathrm{k}}_{2,1} & R^{\mathrm{k}}_{2,2} & \cdots & R^{\mathrm{k}}_{2,\mathtt{w}} \\ \vdots & \vdots & \cdots & \vdots \\ R^{\mathrm{k}}_{\mathtt{g},1} & R^{\mathrm{k}}_{\mathtt{g},2} & \cdots & R^{\mathrm{k}}_{\mathtt{g},\mathtt{w}} \end{bmatrix}.$$

Yields

$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w = 0,}$$

with $R_0, R_1, \cdots, R_{\mathrm{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$.

Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1\xi + \cdots + R_{\mathrm{n}}\xi^{\mathrm{n}},$$

we obtain the mercifully short notation

$$\boxed{R(\frac{d}{dt})w = 0.}$$

Including latent variables $\rightsquigarrow$

$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell}$$

with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$.

Examples:

1. RLC-circuit: Case 1: $CR_C \neq \frac{L}{R_L}$.

Then the relation between V and I is

$$\left(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right)V$$

$$(1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

We have $\mathtt{w} = 2$; g $=1$; $w = \begin{bmatrix} V \\ I \end{bmatrix}$;

$R(\xi) =$

$$[\ \tfrac{R_C}{R_L}\ \mid\ -1\ ] + [\ 1+\tfrac{R_C}{R_L}\ \mid\ -CR_C - \tfrac{L}{R_L}\ ]\,\xi + [\ CR_C\tfrac{L}{R_L}\ \mid -CR_C\tfrac{L}{R_L}\ ]\,\xi^2$$

2. Linear systems:

- The ubiquitous

$$P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u, \quad w = (u, y)$$

with $P, Q \in \mathbb{R}^{\bullet \times \bullet}[\xi]$, $\det(P) \neq 0$ and, perhaps, $P^{-1}Q$ proper.

- The ubiquitous

$$\tfrac{d}{dt}x = Ax + Bu; \quad y = Cx + Du, \quad w = (u, y).$$

- The descriptor systems (also called DAE's, or implicit systems)

$$\frac{d}{dt}Ex + Fx + Gw = 0.$$

representations later.

3. Linearization: Consider the system described by the systems of nonlinear differential equations

$$f\left(w(t), \frac{d}{dt}w(t), \ldots, \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}w(t)\right) = 0$$

with $f : (w_0, w_1, \ldots, w_{\mathrm{n}}) \mapsto \mathbb{R}^{\bullet}$. Assume that $\mathrm{w}^{\star} \in \mathbb{R}^{\mathtt{w}}$ is an equilibrium:

$$f(\mathrm{w}^{\star}, 0, \ldots, 0) = 0.$$

Define $R_{\mathrm{k}} = \frac{\partial}{\partial x_{\mathrm{k}}} f(\mathrm{w}^{\star}, 0, \ldots, 0)$. The system

$$R_0 w + R_1 \frac{d}{dt}w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}}w = 0,$$

is called the *linearized system* around $\mathrm{w}^{\star}$. Under reasonable conditions it describes the behavior in the neighborhood of $\mathrm{w}^{\star}$.

When shall we define $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ to be a solution of $R(\frac{d}{dt})w = 0$?

We will be 'pragmatic', and take the easy way out: $\rightsquigarrow$ **$\mathfrak{C}^\infty$ soln's!**

Transmits main ideas, easier to handle, easy theory,

sometimes (too) restrictive (step-response, etc.).

Whence, $R(\frac{d}{dt})w = 0$ defines the system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

Proposition: **This system is linear and time-invariant.**

NOTATION

$\mathfrak{L}^\bullet$: all such systems (with any - finite - number of variables)

$\mathfrak{L}^{\mathtt{w}}$: with $\mathtt{w}$ variables

$\mathfrak{B} = \ker(R(\frac{d}{dt}))$

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (no ambiguity regarding $\mathbb{T}$, $\mathbb{W}$)

NOMENCLATURE

Elements of $\mathfrak{L}^\bullet$: *linear differential systems*

$R(\frac{d}{dt})w = 0$: a *kernel representation* of the corresponding $\Sigma \in \mathfrak{L}^\bullet$ or $\mathfrak{B} \in \mathfrak{L}^\bullet$

$R(\frac{d}{dt})w = 0$ 'has' behavior $\mathfrak{B}$

Σ or $\mathfrak{B}$: the system *induced* by $R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$

ALGEBRAIZATION of $\mathfrak{L}^\bullet$

Note that

$$R(\frac{d}{dt})w = 0$$

and

$$U(\frac{d}{dt})R(\frac{d}{dt})w = 0$$

have the same behavior if the polynomial matrix U is uni-modular (i.e., when $\det(U)$ is a non-zero constant).

$\Rightarrow$ R defines $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, but not vice-versa!

¿¿ $\exists$ ‘intrinsic’ characterization of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$??

Define the *annihilators* of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi] \mid n^{\top}(\frac{d}{dt})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ is clearly an $\mathbb{R}[\xi]$ sub-module of $\mathbb{R}^{\mathtt{w}}[\xi]$.

Let $< R >$ denote the sub-module of $\mathbb{R}^{\mathtt{w}}[\xi]$ spanned by the transposes of the rows of R. Obviously $< R > \subseteq \mathfrak{N}_{\mathfrak{B}}$. But, indeed:

$$\mathfrak{N}_{\mathfrak{B}} = < R > !$$

Note: Depends on $\mathfrak{C}^{\infty}$; ($\Leftarrow$) false for compact support soln’s.

Conclusion: $\mathfrak{L}^{\mathtt{w}} \stackrel{1:1}{\longleftrightarrow}$ **sub-modules of $\mathbb{R}^{\mathtt{w}}[\xi]$**

ELIMINATION

LATENT VARIABLE SYSTEMS

First principle models $\rightsquigarrow$ latent variables. In the case of systems described by linear constant coefficient differential equations:

$$R_0 w + \cdots + R_n \frac{d^n}{dt^n} w = M_0 \ell + \cdots + M_n \frac{d^n}{dt^n} \ell .$$

In polynomial matrix notation $\rightsquigarrow$

$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell .}$$

This is the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

But is it(s manifest behavior) really a differential system ??

The full behavior of $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$, **i.e.,**

$$\mathfrak{B}_{\text{full}} = \{(w,\ell) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}+\ell}) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})\ell.\}$$

belongs to $\mathfrak{L}^{\mathtt{w}+\ell}$**, by definition. Its manifest behavior equals**

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists \ \ell \text{ such that } R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\}.$$

Does $\mathfrak{B}$ **belong to** $\mathfrak{L}^{\mathtt{w}}$ **?**

Theorem: It does!

Proof: The 'fundamental principle'.

Example: Consider the RLC circuit.

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\rightsquigarrow$ 15 behavioral equations.

These include both the port and the branch voltages and currents.

Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.
2. The elimination theorem.

Why is there *exactly one* equation? Passivity!

Remarks:

- **Number of equations (for constant coefficient linear ODE's)**

  $\leq$ **number of variables.**

  Elimination $\Rightarrow$ fewer, higher order equations.

- **Implications for DAE's**

- **There exist effective computer algebra/Gröbner bases algorithms for elimination**

  $$(R, M) \mapsto R'$$

- **Completely generalizable to constant coefficient linear PDE's (using the fundamental principle)**

- **Not generalizable to smooth nonlinear systems.**

  Why are differential equations so prevalent?

It follows from all this that $\mathfrak{L}^\bullet$ has very nice properties. It is closed under:

- Intersection: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- Addition: $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}})$.
- Projection: $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1})$.
- Action of a linear differential operator:

$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow (P(\tfrac{d}{dt})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}).$$

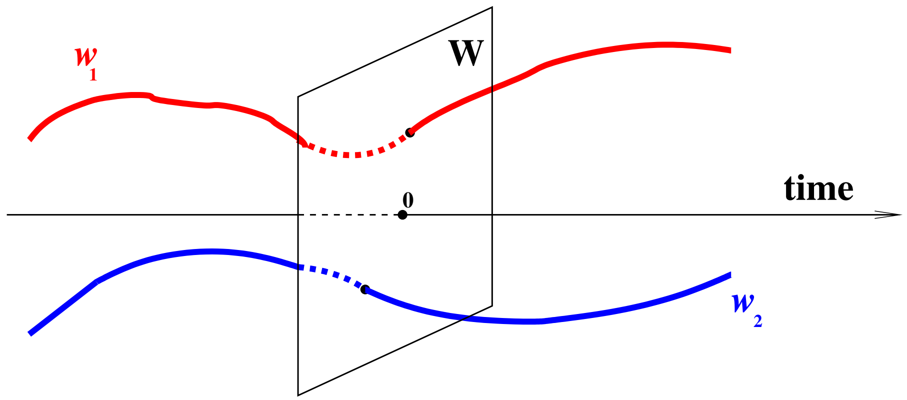
- Inverse image of a linear differential operator:

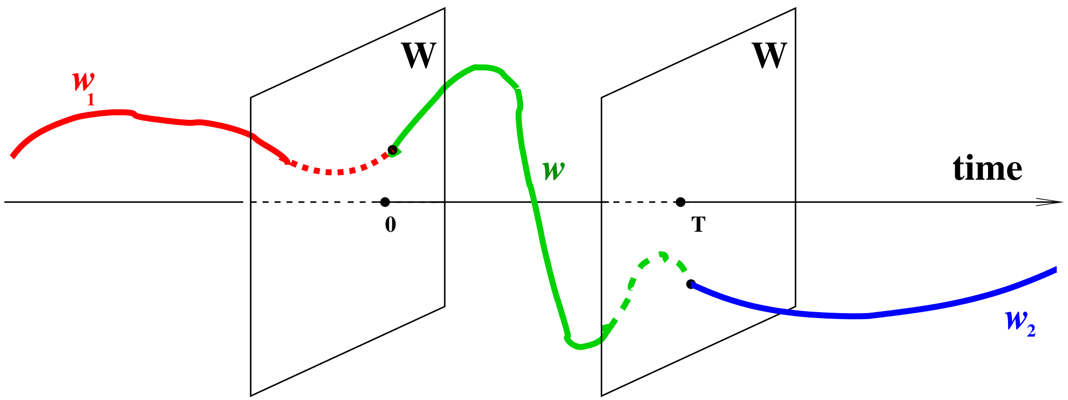
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi]) \Rightarrow ((P(\tfrac{d}{dt}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}).$$

CONTROLLABILITY

Controllability $\Leftrightarrow$

system trajectories must be 'patch-able', 'concatenable'.





Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,$$

with $w = (w_1, w_2, \cdots, w_w)$ and $R_0, R_1, \cdots, R_n \in \mathbb{R}^{g \times w}$,

i.e., $R(\frac{d}{dt})w = 0$, controllable?

We are looking for conditions on the polynomial matrix R
and algorithms in the coefficient matrices $R_0, R_1, \cdots, R_n$.

$R(\frac{d}{dt})w = 0$ defines a controllable system if and only if

rank$(R(\lambda))$ is independent of λ for $\lambda \in \mathbb{C}$.

Example: $r_1(\frac{d}{dt})w_1 = r_2(\frac{d}{dt})w_2 \quad (w_1, w_2 \text{ scalar})$

is controllable if and only if **r_1 and r_2 have no common factor.**

Example: The electrical circuit is controllable unless

$$CR_C = \frac{L}{R_L} \text{ and } R_C = R_L$$

Image representations

Representations of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$:

$$R(\tfrac{d}{dt})w = 0 \qquad (*)$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\tfrac{d}{dt}))$;

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell \quad (**)$$

called a *'latent variable' representation* of the manifest behavior $\mathfrak{B} = (R(\tfrac{d}{dt}))^{-1}M(\tfrac{d}{dt})\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\ell})$.

Missing link:

$$w = M(\tfrac{d}{dt})\ell \quad (***)$$

called an *'image' representation* of $\mathfrak{B} = \operatorname{im}(M(\tfrac{d}{dt}))$.

Elimination theorem $\Rightarrow$ **every image is also a kernel.**

¿¿ Which kernels are also images ??

Theorem: **The following are equivalent for** $\mathfrak{B} \in \mathfrak{L}^\bullet$ **:**

1. $\mathfrak{B}$ **is controllable,**
2. $\mathfrak{B}$ **admits an image representation,**
3. **for any** $a \in \mathbb{R}^{\mathtt{w}}[\xi]$,

$$a^\top[\tfrac{d}{dt}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}),$$

4. $\mathbb{R}^{\mathtt{w}}[\xi]/\mathfrak{N}_{\mathfrak{B}}$ **is torsion free,**

Remarks:

- **Algorithm:** R + syzygies + Gröbner basis
  $\Rightarrow$ numerical test for on coefficients of R.
- **$\exists$ complete generalization to PDE's**
- **$\exists$ partial results for nonlinear systems**
- **Kalman controllability is a straightforward special case**

OBSERVABILITY

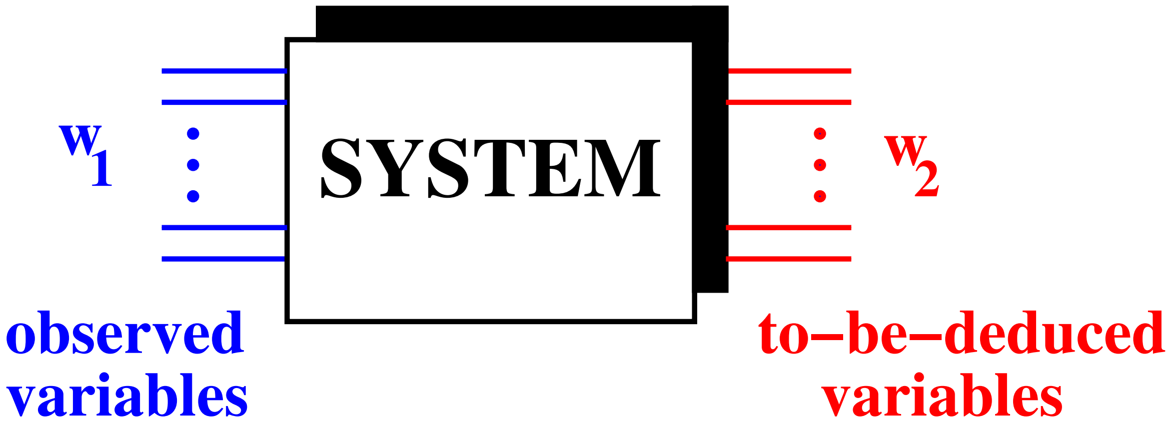
Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of
a pair of trajectories (w_1, w_2).

w_1 : observed; w_2 : to-be-deduced.

Recall: w_2 is said to be *observable* from w_1

if $((w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}) \Rightarrow (w_2' = w_2'')$,
i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.


SYSTEM
w1
w2
observed variables
to–be–deduced variables

When is in

$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$$

w_2 observable from w_1?

If and only if $\mathrm{rank}(R_2(\lambda)) = \mathrm{coldim}(R_2)$ for all $\lambda \in \mathbb{C}$.

i.e., if and only if there exists 'consequences' (i.e. elements of $\mathfrak{N}_{\mathfrak{B}}$) of the form $w_2 = F(\frac{d}{dt})w_1$.

The RLC circuit is observable (branch variables observable from external port variables) iff $CR_C \neq \frac{L}{R_L}$.

$\exists$ a complete theory (for constant coefficient ODE's and PDE's), including algorithms, observer design, etc.

Observability is analogous (but not 'dual') to controllability.

FURTHER RESULTS

Many additional problem areas have been studied from the behavioral point of view.

- **System representations:** **input/output representations, state representations, model reduction, symmetries**
- **System identification** $\Rightarrow$ **the most powerful unfalsified model (MPUM), approximate system ID**
- **Observers**
- **Control**
- **Quadratic differential forms,** **dissipative systems,** $\mathcal{H}_\infty$**-control**
- **Distributed parameter systems**

Linear differential systems (PDE's)

n-D systems)

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathtt{n}$ **independent variables,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**

$\mathfrak{B}$ = **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

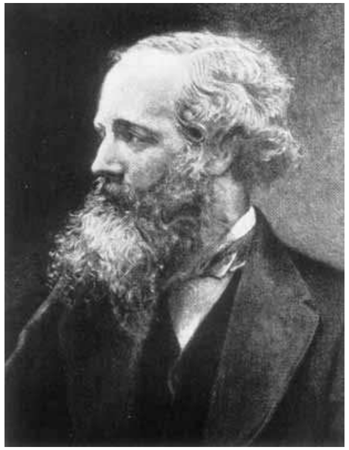
$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

Define its behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$$

$\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**

Example: *Maxwell's equations*



$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0\,, \\ c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}. \end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space),

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

Results:

1. $\mathfrak{N}_{\mathfrak{B}} = \langle R \rangle$

2. **Elimination theorem:** **The manifest behavior of**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell$$

belongs to $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$**.**

Proof uses 'fundamental principle'.

Which PDE's describe $(\vec{E}, \vec{j})$ in Maxwell's equations ?

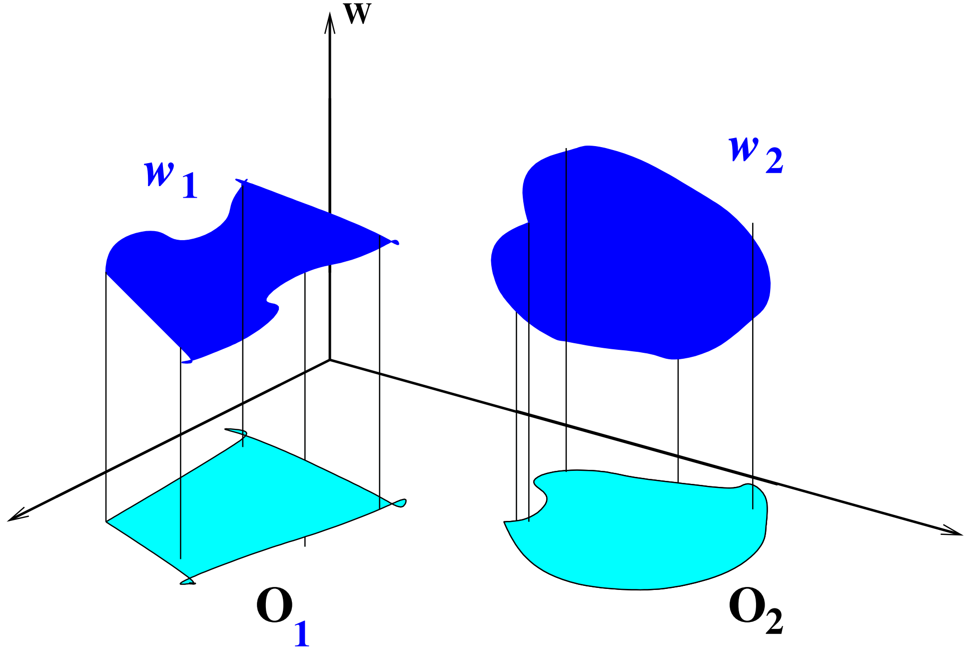
Eliminate $\vec{B}$, ρ from Maxwell's equations. Straightforward computation of the relevant left syzygy yields

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

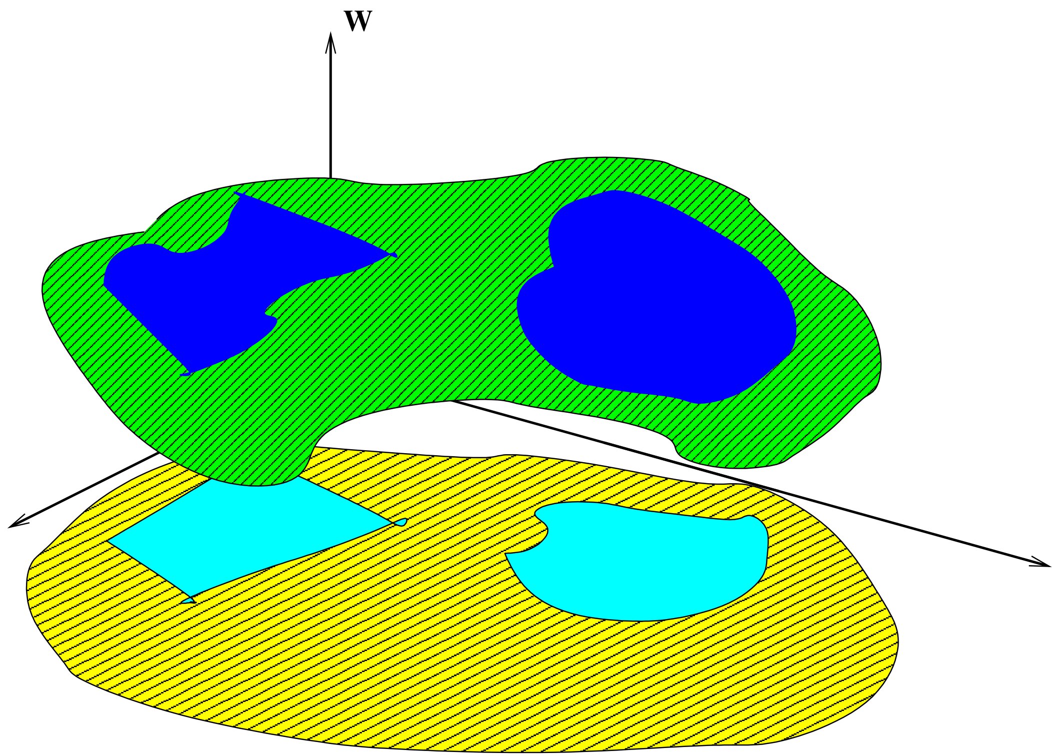
Elimination theorem $\Rightarrow$ this exercise would be exact & successful.

Controllability:

Consider two solutions:



Controllability = patchability:



Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
   $$a^\top[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,

etc.

Algorithm: R + syzygies + Gröbner basis $\Rightarrow$ numerical test on coefficients of R.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

CONTROL AS INTERCONNECTION

In the case of control, our point of view leads to

PLANT:

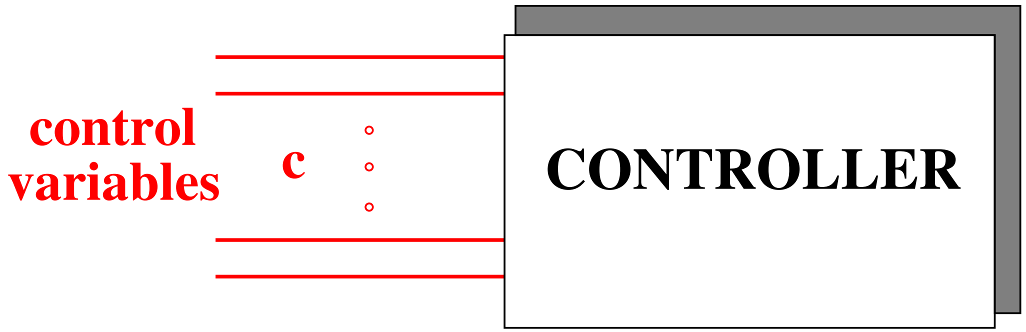
The plant has two kinds of variables

(or, often more appropriately, terminals):

- **variables to be controlled:** w**,**
- **control variables:** c**.**

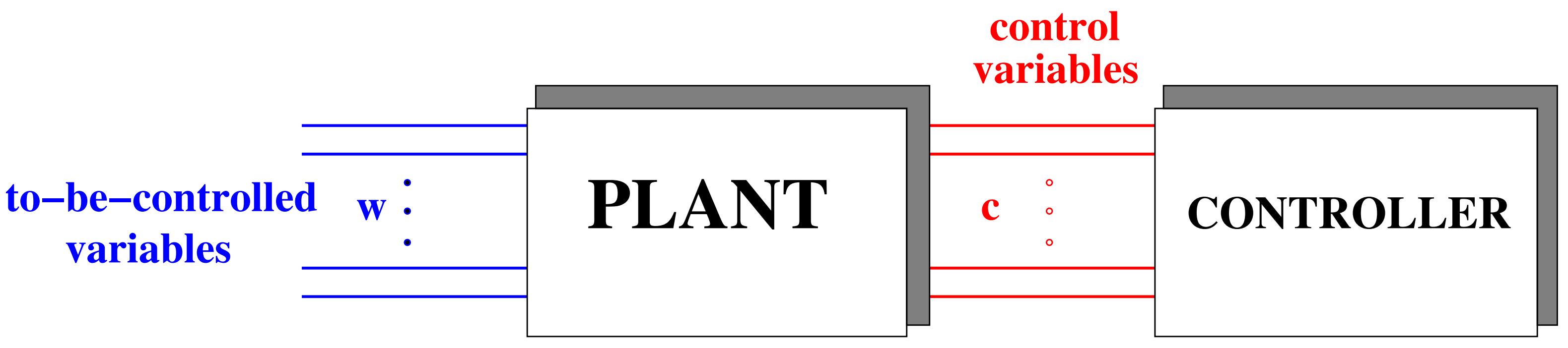
The control variables are those variables through which we interconnect the controller to the plant.

CONTROLLER:



The controller restricts the behavior of the control variables and, through these, that of the to-be-controlled variables.

CONTROLLED SYSTEM:

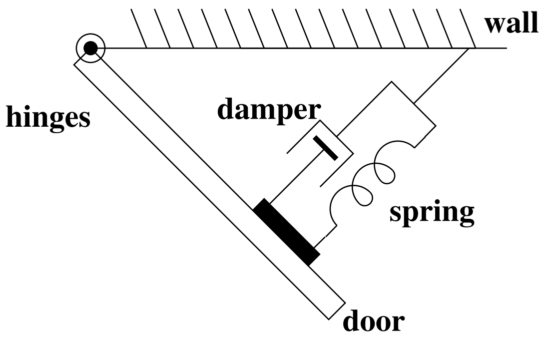


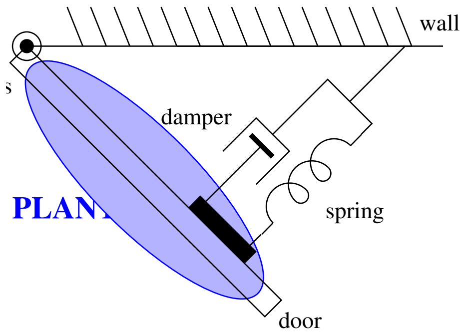
Control variables = shared variables.

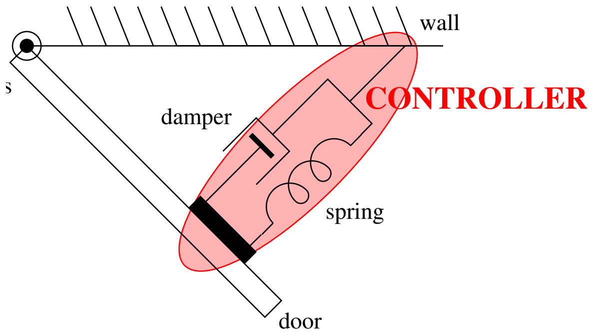
I want to discuss two items in this context:

1. **A (very low-tech) example**
2. **One general result**

Example of such a control mechanism:







Similar idea: A damper of a car, etc.

'Linearized' eq'ns $\leadsto$

Equation of motion of the door (the plant):

$$\boxed{M' \frac{d^2\theta}{dt^2} = F_c + F_e}$$

θ: opening angle,

F_c force exerted by the door closing device, F_e exogenous force.

Door closing mechanism modeled as mass-spring-damper combination (the controller):

$$\boxed{M'' \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + K\theta = -F_c.}$$

To be controlled variables: $w = (\theta, F_e)$,

Control variables: $c = (\theta, F_c)$.

Controlled behavior:

$$(M' + M'')\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = F_e$$

Specifications on the controlled system:

small overshoot, fast settling, not-to-high gain from $F_e \mapsto \theta$.

Finding a suitable controller $\rightsquigarrow$ suitable values for M', K and D.

Note: **Plant: second order;**
Controller: second order;
Controlled plant: second (not fourth) order.

A general implementability result

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be the behavior of the plant

(with $\mathtt{w}$ to-be-controlled and $\mathtt{c}$ control variables.

Let $\mathfrak{C} \in \mathfrak{L}^{\mathtt{c}}$ be the behavior of the controller

(with $\mathtt{c}$ control variables.)

This yields the controlled behavior

$$\mathfrak{K} := \{w \mid \exists \ c \in \mathfrak{C} \text{ such that } (w, c) \in \mathfrak{B}\}.$$

By the elimination theorem $\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}$.

Implementability question:

Which controlled behaviors can be obtained this way?

The answer to this question is a surprisingly simple and explicit:

Theorem: $\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}$ is **implementable if and only if**

$$\boxed{\mathfrak{N} \subset \mathfrak{K} \subset \mathfrak{P}}$$

where

$$\mathfrak{N} := \{w \mid (w, 0) \in \mathfrak{B}\},$$

is the **'hidden'** behavior, and

$$\mathfrak{P} := \{w \mid \exists \;\; c \text{ such that } (w, c) \in \mathfrak{B}\},$$

is the **'manifest plant'** behavior.

Note: pole assignment follows, many refinements,...

Remarks:

- **Many control mechanism in practice do not function as sensor output to actuator input drivers**
- **Control = Interconnection ⇒ controlled behavior is any behavior that is wedged in between hidden behavior and plant behavior**
- **Control = integrated system design; finding a suitable subsystem behavior**
- **∃ a complete theory of controller synthesis (stabilization, $\mathcal{H}_\infty$, ...) of interconnecting controllers for linear systems**
- **Functionals in optimization criteria: Quadratic Differential Forms**
- **Via (regular) implementability results, the usual feedback structures are recovered**
- **Controllability and observability: central ideas also here**

Main points

- A system = a behavior
- Importance of latent variables
- Relevance in modular modeling
- There is a complete theory for linear time-invariant differential systems
- Nice theory of controllability
- Limitation of input/output thinking
- Relevance of behaviors, even in control

Main open area:

Stochastic behaviors

Thanks

to the many colleagues, postdocs, and promovendi who have over the years contributed to this research program. In particular,

Harry Trentelman and Hans Nieuwenhuis (Groningen), Jan Willem Polderman and Arjan van der Schaft (Twente), Paulo Rapisarda (Maastricht), Paula Rocha (Aveiro), Fabio Fagnani (Torino), Christiaan Heij (Rotterdam), Siep Weiland (Eindhoven), Shiva Shankar (Chennai), Harish Pillai (Mumbai), Tommaso Cotroneo (London), Maria Elena Valcher and Sandro Zampieri (Padova), Eva Zerz (Kaiserslautern), Heide Glüssing-Lürssen (Oldenburg), Jeffrey Wood (Southampton), etc.

Thank you for your patience & attention