STATE CONSTRUCTION

in DISCRETE EVENT and CONTINUOUS SYSTEMS

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THEME

Defining a system in terms of its behavior

provides a common framework

for discrete event and continuous systems alike

A discrete event system $\cong$ a *formal language*

$\mathbb{A} \; =$ a (finite) alphabet;

$\mathbb{A}^* \; :=$ all finite strings with symbols from $\mathbb{A}$

$\mathfrak{L} \subset \mathbb{A}^* \; =:$ the language

= all 'legal' 'words' $\quad a_1 a_2 \cdots a_k \cdots$.

Examples: All words appearing in the *Webster*

All LaTeX documents

Pad the words with blanks ($\Box$'s) so as to make then 2-sided infinite.

All such words $\rightsquigarrow$ a time-invariant system $\Sigma = (\mathbb{Z}, \mathbb{A}, \mathfrak{L})$

A continuous system = ????

An I/O map ?? Does not cope with initial conditions

A parametrized family of I/O maps ??

How is this parametrization constructed?

Does not cope with initial conditions either...

Difficulties:

- **Why should there be an I/O partition in continuous systems, contrary to DES?**
- **How do we cope with initial conditions in I/O systems before the state space has been constructed?**
- **Why this difference between DES and continuous systems**

$\leadsto$ **!!! Behavioral systems !!!**

Definition: **Dynamical system** = $\boxed{\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subset \mathbb{R}$, **the *time-axis*** **(= the relevant time instances),**

$\mathbb{W}$, **the *signal space*** **(= where the variables take on their values),**

$\boxed{\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}} : \textit{the behavior}}$ **(= the admissible trajectories).**

Today: $\mathbb{T} = \mathbb{Z}$ **;**

Σ **time-invariant** $:=[w \in \mathfrak{B}] \Leftrightarrow [\sigma(w) \in \mathfrak{B}]$, $\sigma :=$ **shift.**

Examples = formal languages, DES, I/O maps, diff. eq'ns, codes,...

Definition: **Latent variable system:=** $\Sigma_{\mathbb{L}} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\mathbf{full}})$

$\mathbb{T} \subset \mathbb{R}$, **the *time-axis* (= the set of relevant time instances)**

$\mathbb{W}$, **the *signal space* (= the variables that the model aims at)**

$\mathbb{L}$, **the *latent variable space* (= the auxiliary modeling variables)**

$\mathfrak{B}_{\mathbf{full}} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ **: *the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ which the model declares possible)

Examples: models with auxiliary variables, interconnected systems, first principle models, grammars, switched systems,...

THE MANIFEST BEHAVIOR

Call the elements of $\mathbb{W}$ ***'manifest' variables*** **,**

those of $\mathbb{L}$ ***'latent' variables*** **.**

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ induces the ***manifest system*** **$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with** ***manifest behavior***

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

In convenient equations for $\mathfrak{B}$, the latent variables are ***'eliminated'*****.**

A state system = A latent variable system with a special property.

Definition: The latent variable system $\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ is said to be a *state system* if

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}}, t_0 \in \mathbb{T}, \quad \text{and} \quad x_1(t_0) = x_2(t_0)$$

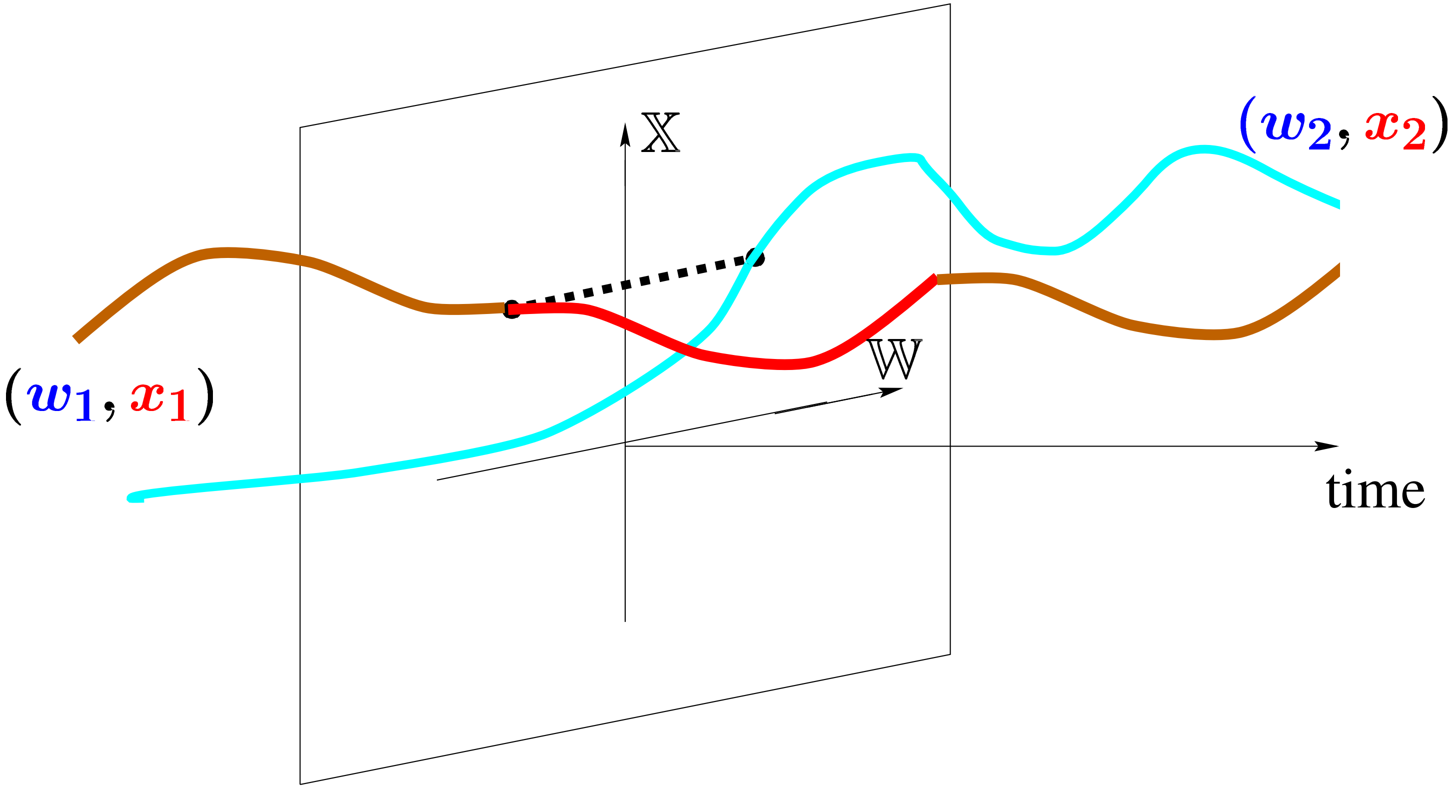
imply

$$(w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}.$$

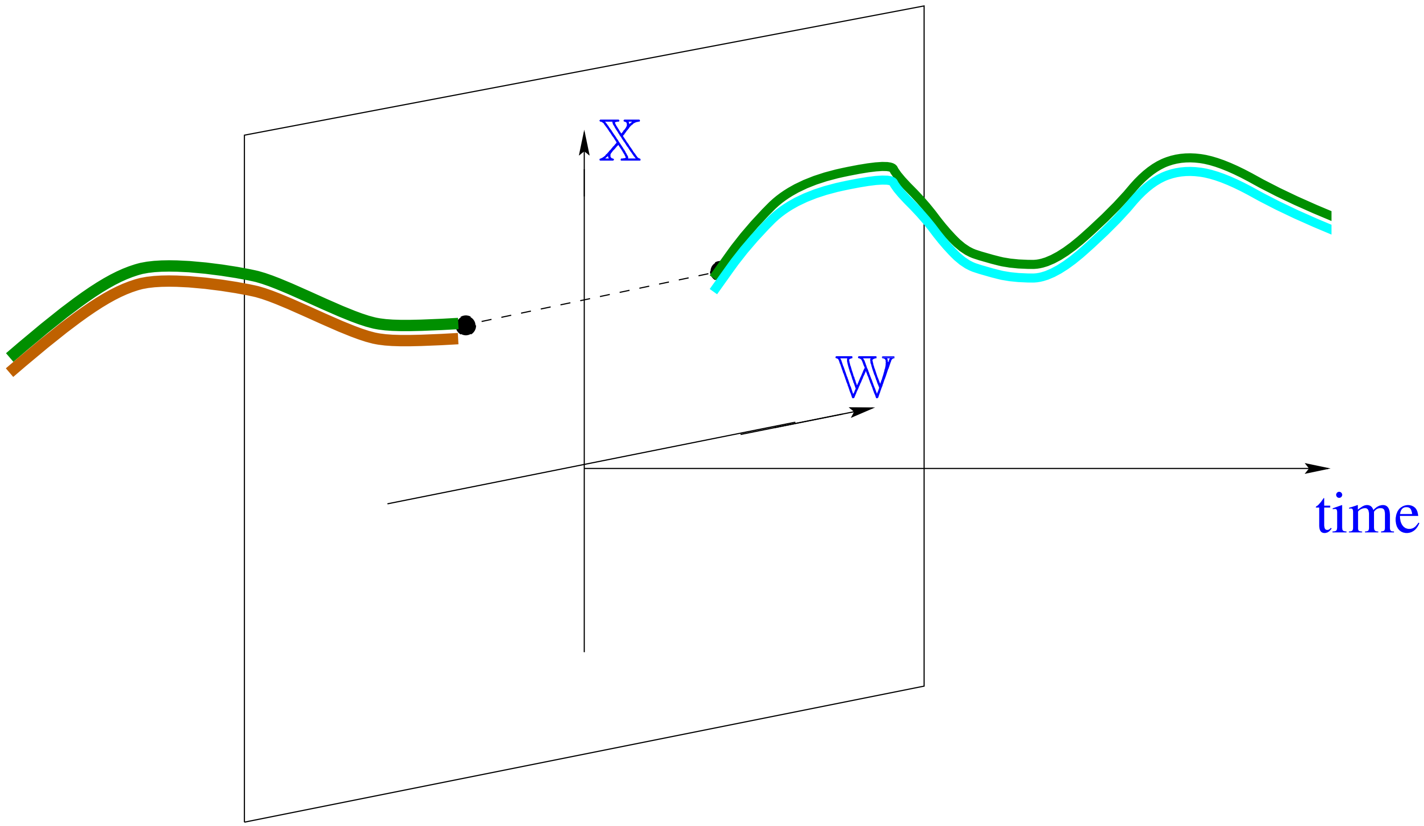
$\underset{t_0}{\wedge}$ denotes *concatenation* at t_0, defined as

$$f_1 \underset{t_0}{\wedge} f_2(t) := \begin{cases} f_1(t) & \text{for } t < t_0 \\ f_2(t) & \text{for } t \geq t_0 \end{cases}$$

In pictures:



$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}}$



State := concatenation also $\in \mathfrak{B}_{\text{full}}$!

This definition is the implementation of the idea:

> ***The state at time t, $x(t)$, contains all the information (about (w, x)!) that is relevant for the future behavior.***

The state = the memory.

> **The past and the future are 'independent', conditioned on (given) the present state.**

$\cong$ Markovianity!

Examples of state systems:

Discrete-time systems.

A latent variable system described by a difference equation that is

> ***first order*** **in the latent variable x, and**
>
> ***zero-th order*** **in the manifest variable w:**

$$\boxed{F(x(t+1), x(t), w(t)) = 0.}$$

Automata $\mathbb{W}$, $\mathbb{X}$ **finite sets, possibly initial + terminal conditions**

Trellis diagrams

QM

Definition: $\Sigma_{\mathbb{L}} = (\mathbb{Z}, \mathbb{W}, \mathbb{L}, \mathfrak{B})$ is **complete** if

$$[(w,\ell)|_{[t_0,t_1]} \in \mathfrak{B}_{\text{full}}|_{[t_0,t_1]} \forall t_0, t_1] \Rightarrow [(w,\ell) \in \mathfrak{B}_{\text{full}}].$$

Theorem: The '**complete**' latent variable system

$$\Sigma_X = (\mathbb{Z}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

is a state system *if and only if* $\mathfrak{B}_{\text{full}}$ admits a representation as a difference equation that is

first order in the latent variable x, and

zero-th order in the manifest variable w:

$$\boxed{F(x(t+1), x(t), w(t)) = 0.}$$

Otherwise (if not complete, as languages)

'initial' and/or 'terminal' conditions ...

General properties:

The state system $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ is said to be

[*state irreducible*]

$:\Leftrightarrow$ [(if f is a partial (!!) map, $f : \mathbb{X} \to \mathbb{X}'$,

such that $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}', \mathfrak{B}'_{\text{full}})$ with

$\mathfrak{B}'_{\text{full}} = \{(w, f \circ x) \mid (x, w) \in \mathfrak{B}_{\text{full}}\}$, is a state repr. of $\mathfrak{B}$),

$\Rightarrow$ (f must be a bijective map on X)].

The state systems $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ and

$\Sigma'_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}', \mathfrak{B}'_{\text{full}})$ are said to be *equivalent*

if there exists a bijection $f : \mathbb{X} \to \mathbb{X}'$ such that

$$[(w, x) \in \mathfrak{B}_{\text{full}}] \Leftrightarrow [(w, f \circ x) \in \mathfrak{B}'_{\text{full}}].$$

Clearly equivalence $\Rightarrow$ the same manifest behavior.

STATE CONSTRUCTION

!! Given a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$
find a state representation $\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ for it !!

Given $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, find a (irreducible) state space representation $\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\mathrm{full}})$ for it.

The crucial idea is to define the state space!

When do two trajectories bring the system in the same state?
When is stored memory by the two trajectories the same?

When the trajectories can be continued in the same way!

This idea of constructing an equivalence relation on the manifest behavior $\mathfrak{B}$, sometimes called 'Nerode equivalence', leads to the past canonical state construction.

Define the equivalence relation R_- on $\mathfrak{B}$ by

$$[w_1 R_- w_2] :\Leftrightarrow [(w_1 \underset{0}{\wedge} w \in \mathfrak{B}) \Leftrightarrow (w_2 \underset{0}{\wedge} w \in \mathfrak{B})].$$

Our concept of state being 'time-symmetric'

$\Rightarrow$ future canonical state representation.

In the future canonical state construction, define the equivalence relation R_+ by

$$[w_1 R_+ w_2] :\Leftrightarrow [(w \underset{0}{\wedge} w_1 \in \mathfrak{B}) \Leftrightarrow (w \underset{0}{\wedge} w_2 \in \mathfrak{B})].$$

Finally, combine both to the two-sided canonical state representation.

In the two-sided canonical state construction, define the equivalence relation $R_\pm$ by

$$[w_1 R_\pm w_2] :\Leftrightarrow [((w_1 \underset{0}{\wedge} w \in \mathfrak{B}) \Leftrightarrow (w_1 \underset{0}{\wedge} w \in \mathfrak{B})) \\ \wedge ((w \underset{0}{\wedge} w_1 \in \mathfrak{B}) \Leftrightarrow (w \underset{0}{\wedge} w_2 \in \mathfrak{B}))].$$

Obviously,

$$[w_1 R_\pm w_2] \Leftrightarrow [(w_1 R_- w_2) \wedge (w_1 R_+ w_2)].$$

We now construct the associated state representations.

For the past-canonical state construction, define the state space by $\mathbb{X}_{-} = \mathfrak{B}(\text{mod } R_{-})$, the full behavior by

$$\mathfrak{B}_{\text{full},-} = \{(w, x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

For the future-canonical state construction, define the state space by $\mathbb{X}_{+} = \mathfrak{B}(\text{mod } R_{+})$, the full behavior by

$$\mathfrak{B}_{\text{full},+} = \{(w, x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

For the two-sided-canonical state construction, define the state space by $\mathbb{X}_{\pm} = \mathfrak{B}(\text{mod } R_{\pm})$, the full behavior by

$$\mathfrak{B}_{\text{full},\pm} = \{(w, x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

The canonical state representations $\boldsymbol{\Sigma}_- := (\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_-)$ and $\boldsymbol{\Sigma}_+ := (\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_+)$ have very good properties.

In particular, they are irreducible.

The question when all irreducible state representations of a given system are equivalent has a very nice answer in terms of these canonical representations.

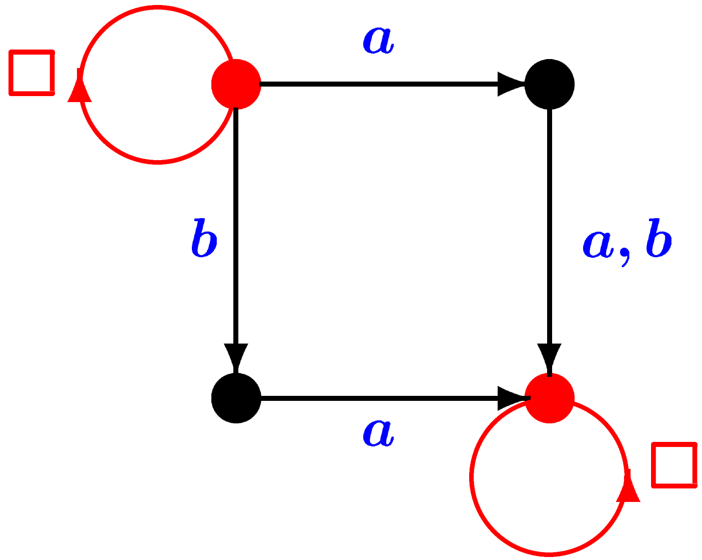
Indeed, the following conditions are equivalent:

1. **All irreducible state representations of a given system $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$ are equivalent.**
2. **$(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\mathbf{full},-})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_{\mathbf{full},+})$ are equivalent.**
3. **$(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\mathbf{full},\pm})$ is irreducible.**
4. **$(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\mathbf{full},-})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\mathbf{full},\pm})$ are equivalent.**
5. **$(\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_{\mathbf{full},+})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\mathbf{full},\pm})$ are equivalent.**

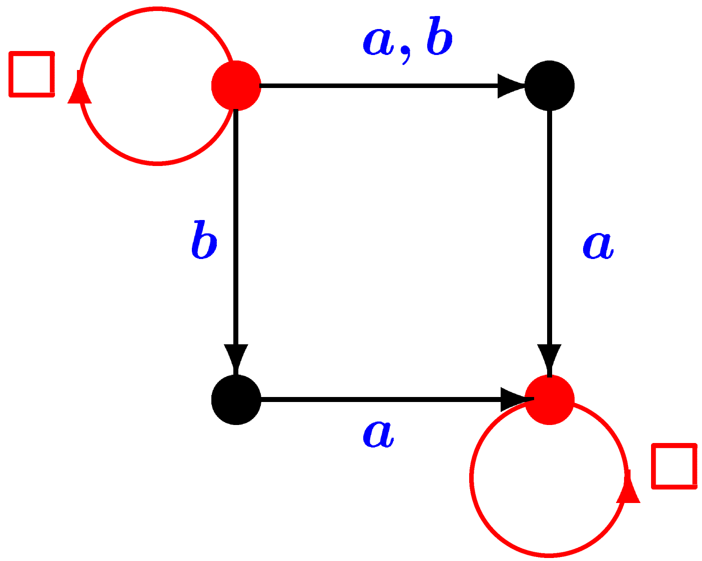
Important examples of systems for which all irreducible state representations are equivalent are linear and autonomous systems.

Example: $\mathfrak{L} = \{aa, ab, ba\}$.

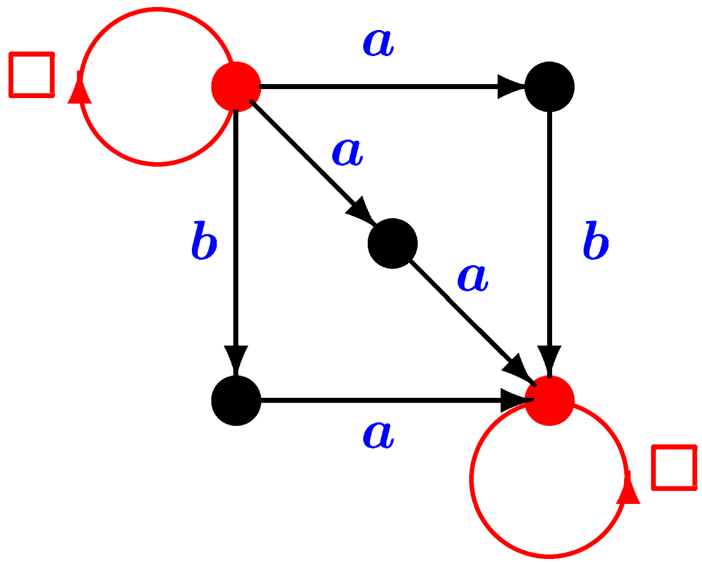
Past canonical state representation:



Future canonical representation:



Two-sided canonical representation:



Not all irreducible state representations are equivalent

Manuscript & copies of the lecture frames are available from/at

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Thank you!