THE BEHAVIORAL APPROACH

to

SYSTEMS and CONTROL

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Problematique:

Develop a suitable *mathematical* framework for discussing dynamical systems

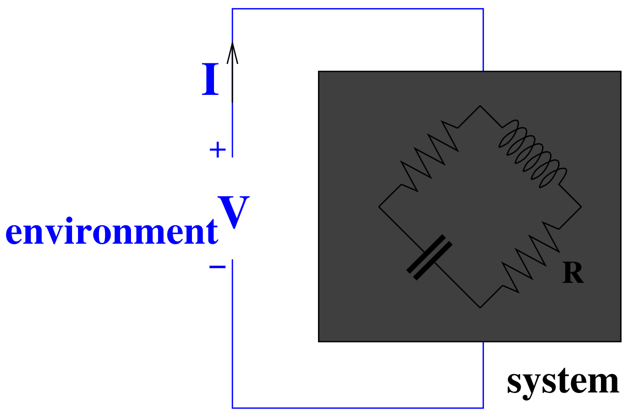
aimed at modeling, analysis, and synthesis.

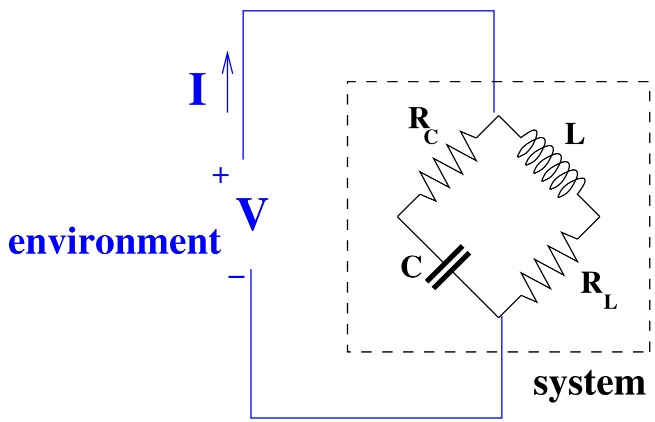
$\rightsquigarrow$ **control, signal processing, system identification, . . .**

$\rightsquigarrow$ **engineering systems, economics, physics, . . .**

Motivational examples

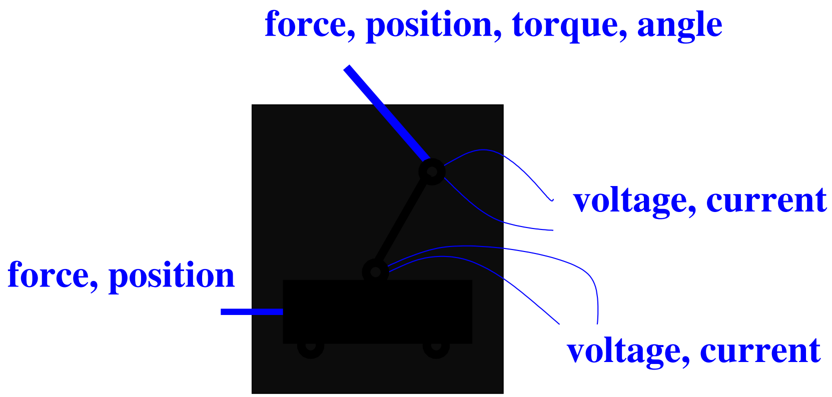
Electrical circuit

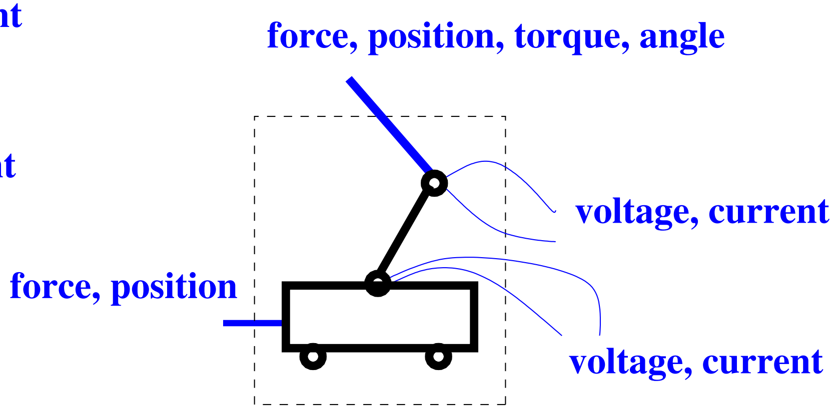




!! Model the relation between the voltage V and the current I

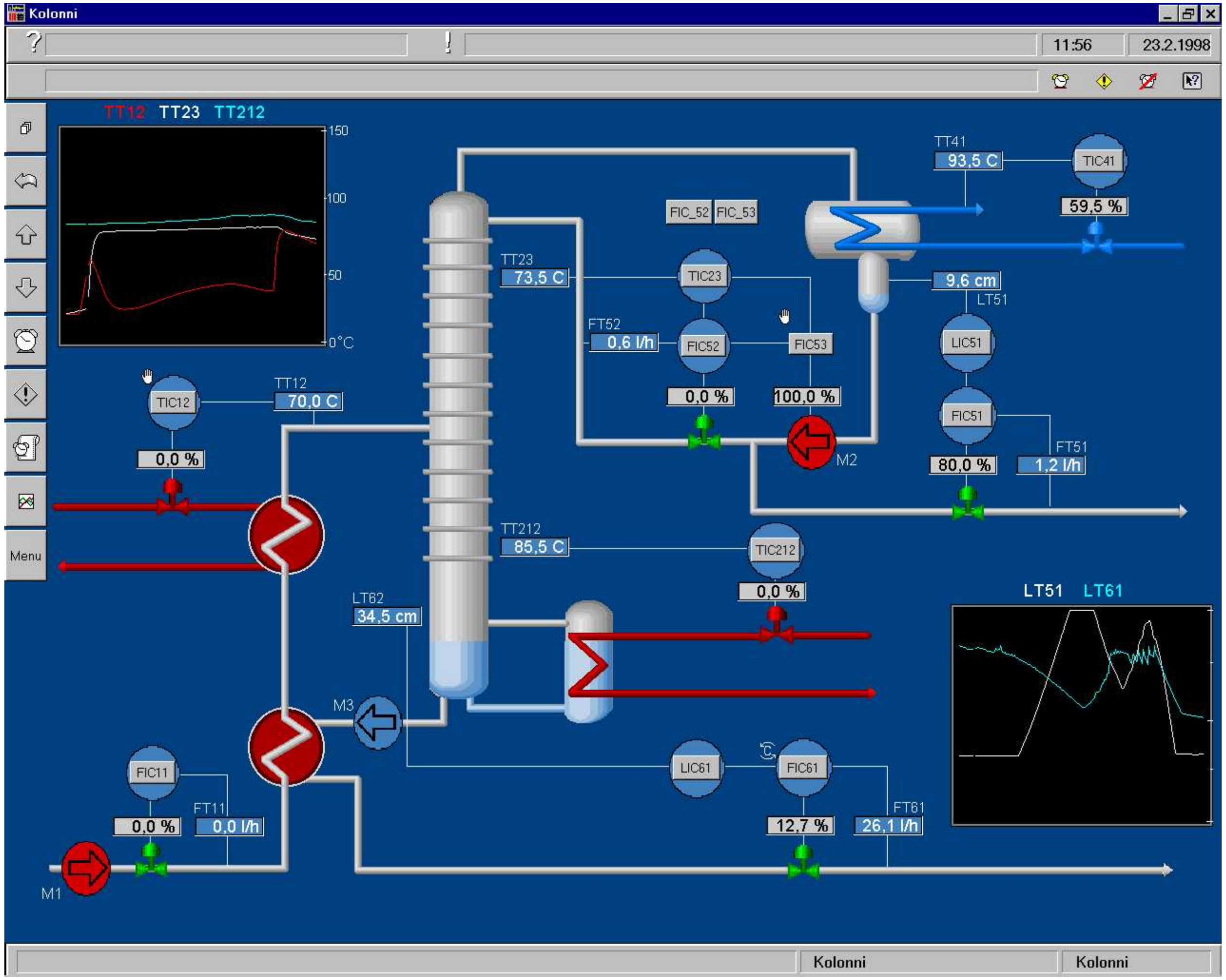
Electromechanical system





!! between the positions, forces, torque, angle, voltages, currents

Distillation column



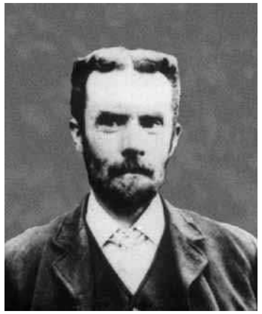
Features: Systems are typically

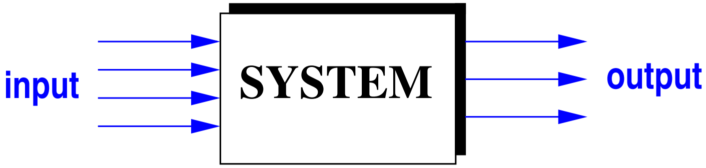
- **dynamical**
- **open,** they interact with their environment
- **interconnected,** with many subsystems
- **modular,** consisting of standard components

We are looking for a mathematical framework that is adapted to these features, and hence to computer assisted modeling.

Historical remarks

<u>Early 20-th century</u>: emergence of the notion of a transfer function
(Rayleigh, Heaviside).

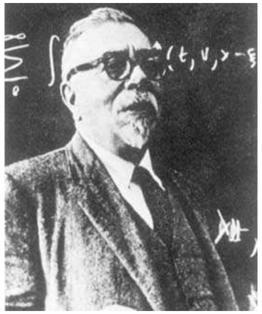




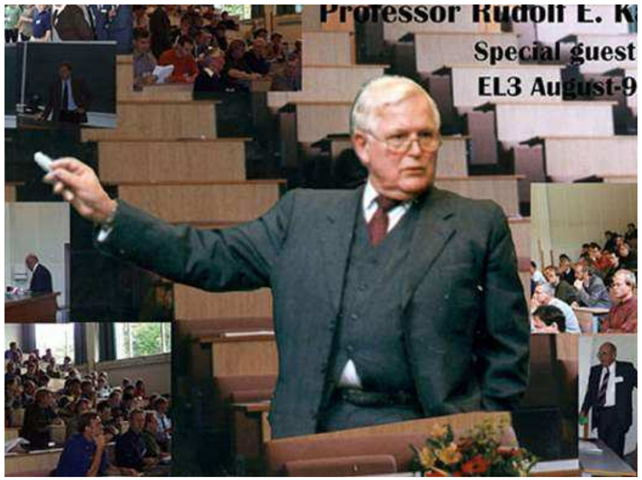
<u>Since the 1920's</u>: routinely used in circuit theory
$\rightsquigarrow$ impedances, admittances, scattering matrices, etc.

<u>1930's</u>: control embraces transfer functions
(Nyquist, Bode, $\cdots$) $\rightsquigarrow$ plots and diagrams, classical control.

Around 1950: Wiener sanctifies the notion of a blackbox, attempts nonlinear generalization (via Volterra series).



1960's: Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue



$\leadsto$ **input/state/output systems,** **and the ubiquitous**

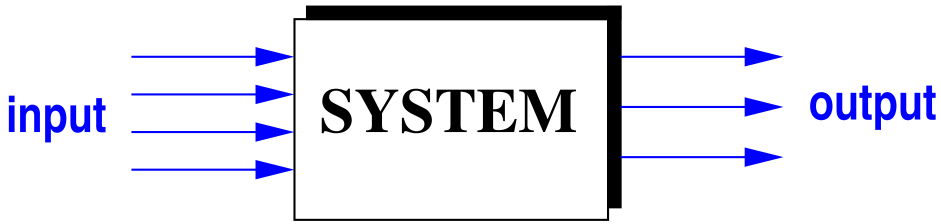
$$\boxed{\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,}$$

or its nonlinear counterpart

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u).$$

These mathematical structures, transfer functions, + their discrete-time analogs, are nowadays the basic models used in control and signal processing (cfr. MATLAB©).

All these theories: input/output; cause $\Rightarrow$ effect.

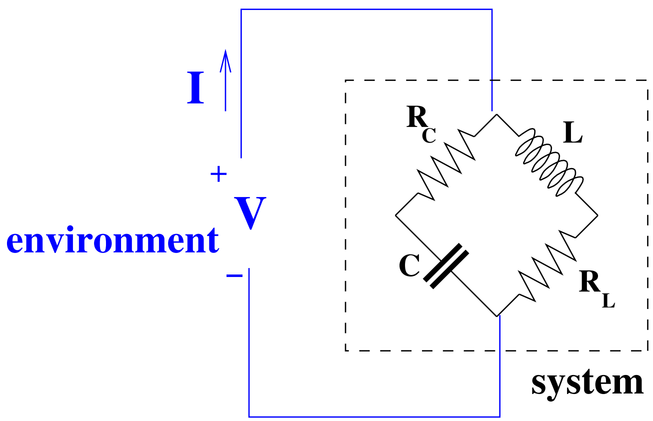


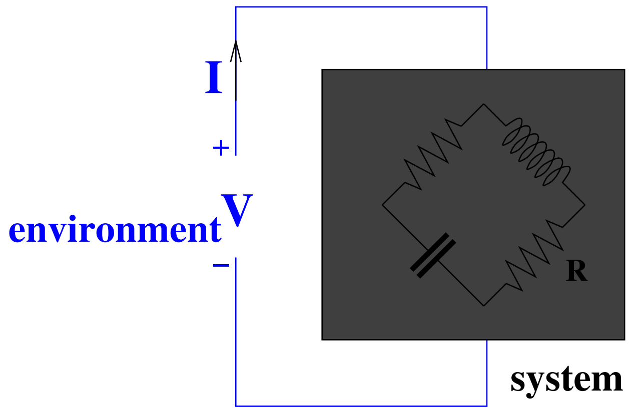
Beyond input/output

What's wrong with input/output thinking?

Let's look at examples:

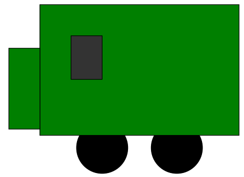
Our electrical circuit.





Is V the input? Or I ? Or both, or are they both outputs?

An automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: the force,

at the tire terminals: the forces, or, more likely, the positions?

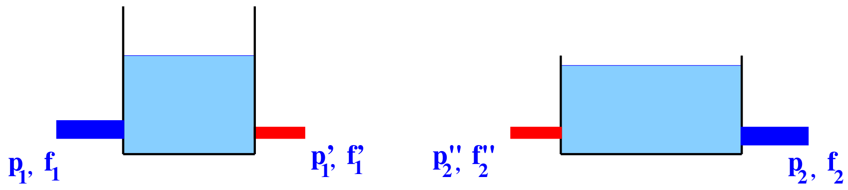
at the steering wheel: the torque or the angle?

at the gas-pedal, or the brake-pedal: the force or the position?

Difficulty: at each terminal there are many (typically paired) interconnection variables

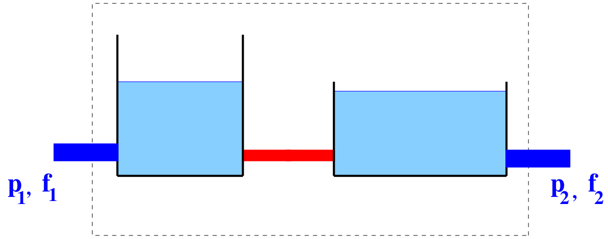
Input/output is awkward in modeling interconnections.

Consider a two-tank example.



Reasonable input choices: the pressures, output choices: the flows.

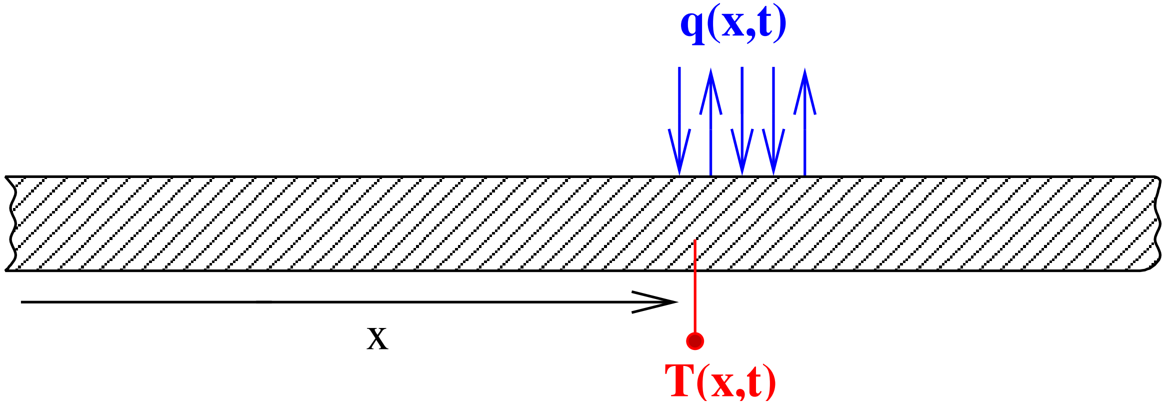
Assume that we model the interconnection of two tanks.



Interconnection: $p_1' = p_2'', \quad f_1' + f_2'' = 0$

input=input; output=output! $\Rightarrow\Leftarrow$ **SIMULINK©**

Heat diffusion



The PDE

$$\boxed{\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q}$$

fits the

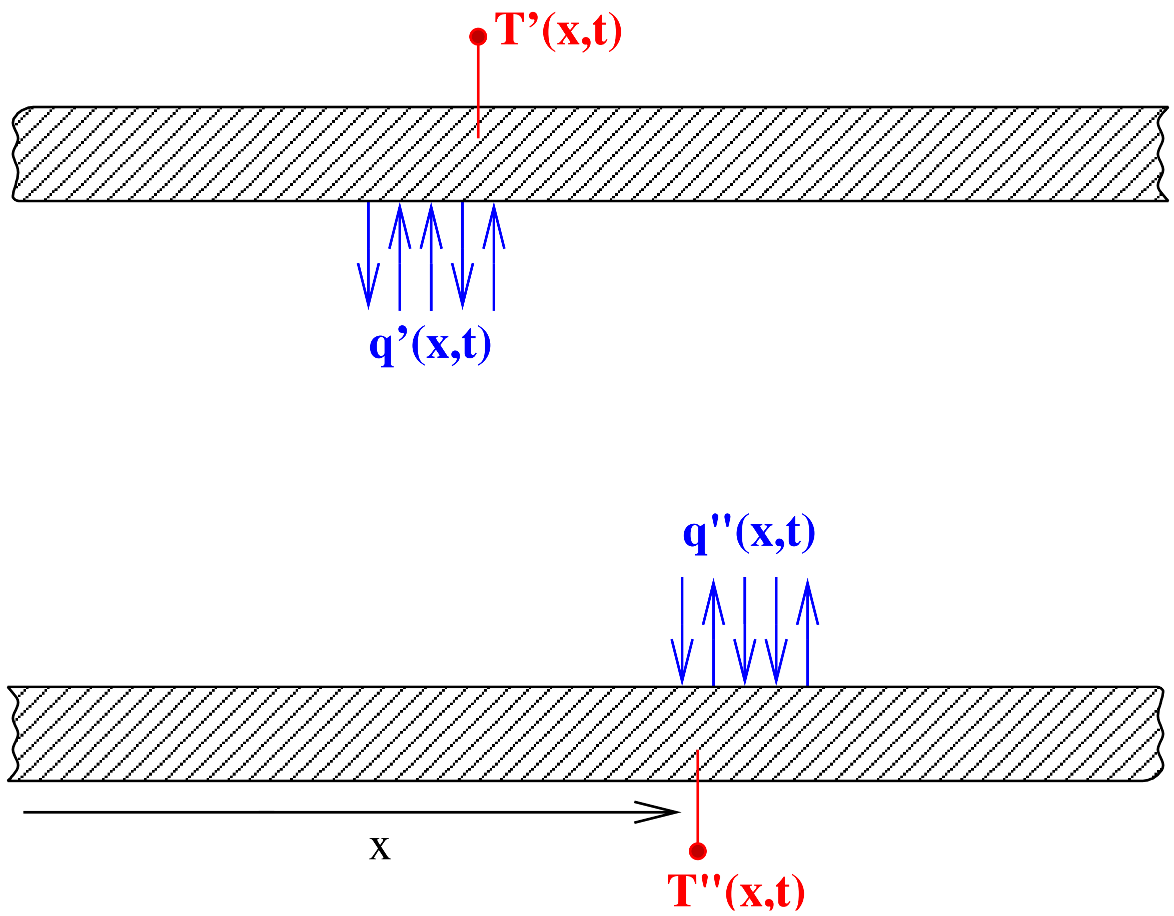
$$\frac{d}{dt} x = Ax + Bu, \quad y = Cx$$

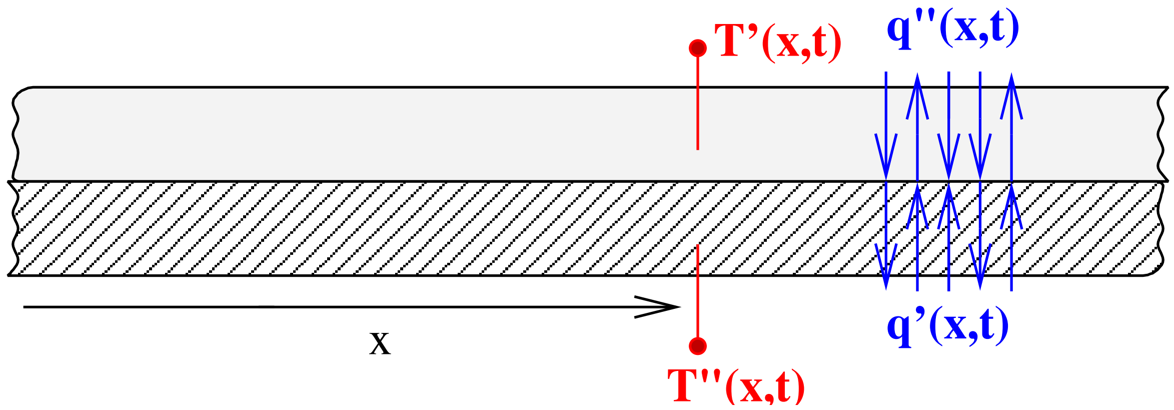
input/output framework, with

$$u(t) = q(\cdot, t); \quad y(t) = x(t) = T(\cdot, t)$$

perfectly.

Now interconnect two such systems





Interconnection:

$$T'(x,t) = T''(x,t), \quad q'(x,t) + q''(x,t) = 0$$

!! input'=input"; output'=output" ! $\Rightarrow\Leftarrow$ **SIMULINK©**

Interconnections contradicting SIMULINK© are in fact

normal, not exceptions,

in mechanics, fluidics, heat transfer, electrical circuits, etc.

The standard system theoretic / SIMULINK© input-to-output idea of interconnection is inappropriate as a paradigm for interconnecting physical systems!

Contrast this with the claim

... A third concept in control theory is the role of interconnection between subsystems. Input/output representations of systems allow us to build models of very complex systems by linking component behaviors ...

[Panel on Future Directions in Control, Dynamics, and Systems Report, 26 April 2002, page 11]

Conclusions ∗ for physical systems ($\Rightarrow\Leftarrow$ signal processors) ∗

- **External variables are basic, but what 'drives' what, is not.**
- **A physical system is not a signal processor.**
- **It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.**
- **What can be the input, and what can be the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.**

Variable sharing,

rather that input selection,

is the basic mechanism by which a system interacts with its environment.

$\Rightarrow$ We need a better framework for discussing 'open' systems!

$\rightsquigarrow$ Behavioral systems.

The basic concepts

Behavioral systems

A dynamical system = $\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the relevant time instances),**

$\mathbb{W}$, **the *signal space* (= where the variables take on their values),**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **: *the behavior*** **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

For a trajectory $w : \mathbb{T} \longrightarrow \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$ **: the model allows the trajectory w,**

$w \notin \mathfrak{B}$ **: the model forbids the trajectory w.**

Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),

or $\mathbb{Z}$, or $\mathbb{N}$ (in discrete-time systems).

Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),

a function space

(in distributed systems, with time a distinguished variable),

or a finite set (in DES).

Emphasis later today: $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$\mathfrak{B}$ **= solutions of system of linear constant coefficient ODE's.**

We now discuss the fundamentals of the theory of systems

$$\boxed{\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})}$$

that are

1. *linear*, meaning
$((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B})$;

2. *time-invariant*, meaning
$((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}))$,
where σ^t denotes the backwards $t-$shift;

3. *differential*, meaning
$\mathfrak{B}$ consists of the solutions of a system of differential equations.

Yields

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\bullet \times w}$.

Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n,$$

we obtain the short notation

$$R(\frac{d}{dt}) w = 0.$$

The theory has also been developed for n-D systems
and constant coeff. linear PDE's (as Maxwell's equations).

Associated behavior

$$\mathfrak{B} = \{\mathtt{w} : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \mid R(\frac{d}{dt})w = 0\}$$

appropriate def. of sol'n.

Note: any number of DE's, any number of variables.

Often many algebraic eqn's.

NOMENCLATURE

$\mathfrak{L}^{\mathtt{w}}$ **: the set of such systems with** $\mathtt{w}$ **dependent variables**

$\mathfrak{L}^{\bullet}$ **: with any - finite - number of dependent variables**

Elements of $\mathfrak{L}^{\bullet}$ **:** ***linear differential systems***

$R(\frac{d}{dt})w = 0$ **: a** ***kernel representation*** **of the corresponding** $\Sigma \in \mathfrak{L}^{\bullet}$ **or** $\mathfrak{B} \in \mathfrak{L}^{\bullet}$

3 basic theorems

Theorem 1 **Algebraization:**

$$\boxed{\mathfrak{L}^{\mathtt{w}} \overset{1:1}{\longleftrightarrow} \textbf{sub-modules of } \mathbb{R}^{\mathtt{w}}[\xi]}$$

Theorem 2 **Elimination:**

$$\boxed{(\mathfrak{B}_{\mathrm{full}} \in \mathfrak{L}^{\bullet}) \Rightarrow (\mathfrak{B} \in \mathfrak{L}^{\bullet})}$$

Theorem 3 **Image representation:**

$$\boxed{\textbf{Controllabilility} \Leftrightarrow (\exists \textbf{ Image representation})}$$

Elimination

First principle models $\rightsquigarrow$ latent variables.

In the case of differential eq'ns: $\rightsquigarrow$

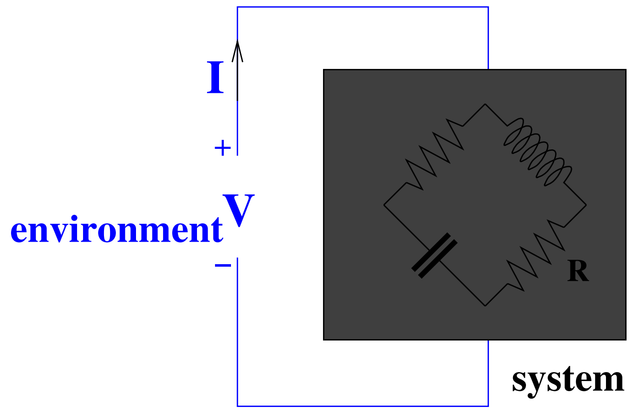
$$\boxed{R(\frac{d}{dt})w = M(\frac{d}{dt})\ell}$$

with the w's the variables that the model aims at,

with the ℓ's auxiliary variables, and

with $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ polynomials with the 'system parameters'.

Example:



$w = (V, I)$ = **the port variables,**

ℓ = **the interconnection variables, internal voltages and currents**

Differential eq'ns: Kirchhoff's laws, constitutive eq'ns .

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

is the natural model class to start a theory of finite dimensional linear time-invariant systems!

Much more so than the ubiquitous

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

But is it(s manifest behavior) really a differential system ??

The **full behavior** of $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$,

$$\mathfrak{B}_{\text{full}} = \{(w, \ell) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\}$$

belongs to $\mathfrak{L}^{\mathtt{w}+\ell}$, by definition. Its **manifest behavior** equals

$$\mathfrak{B} = \{w \mid \exists \ \ell \text{ such that } R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\}.$$

Does $\mathfrak{B}$ belong to $\mathfrak{L}^{\mathtt{w}}$?

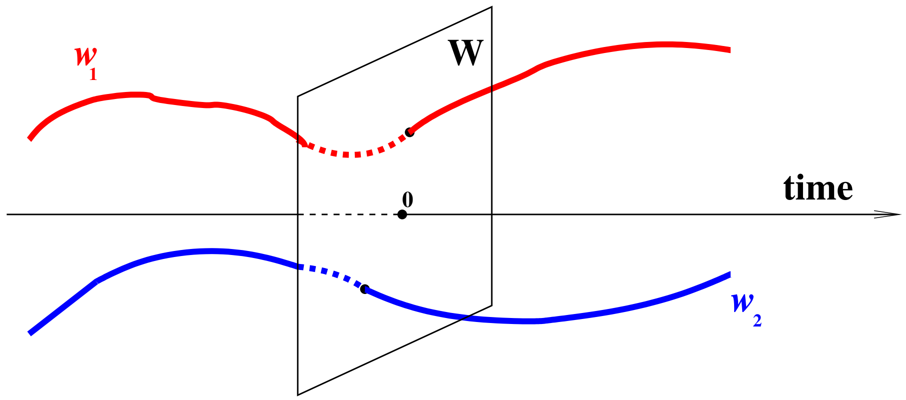
Theorem: **It does!**

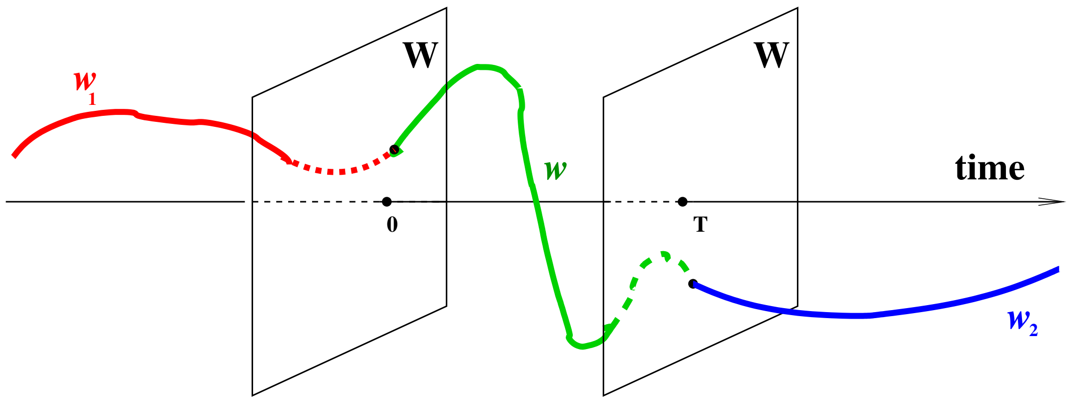
- **Number of equations (constant coeff. lin. ODE's)**
  $\leq$ **number of variables.**
  Elimination $\Rightarrow$ **fewer, higher order equations.**
- **There exist effective computer algebra/Gröbner bases algorithms for elimination**
  $$(R, M) \mapsto R'$$
- **Not generalizable to smooth nonlinear systems.**
  Why are differential equations models so prevalent?
  External behavior of interconnected nonlinear differential systems need not be a differential system.

Controllability

Controllability $\Leftrightarrow$

system trajectories must be ‘patch-able’, ‘concatenable’.

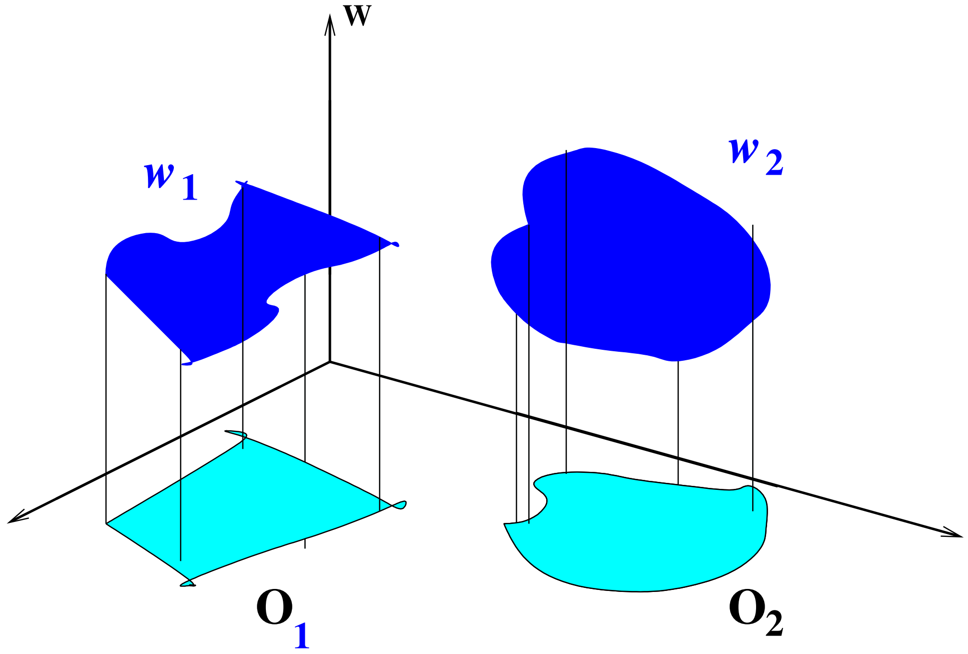




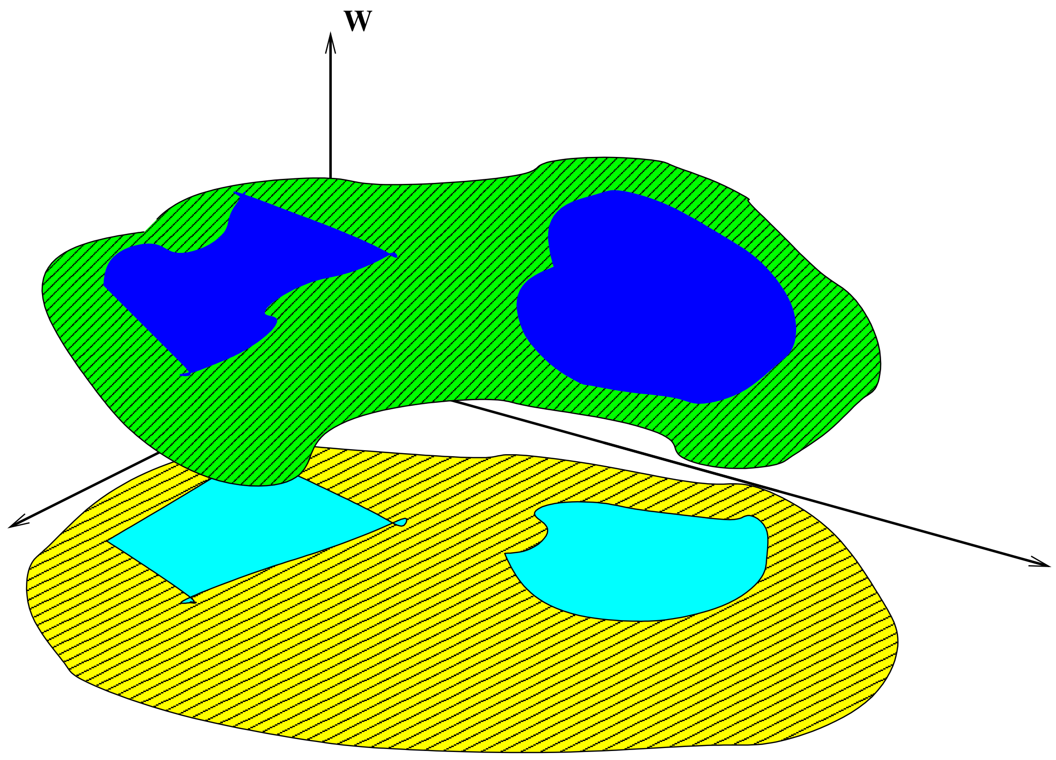
for all $w_1, w_2 \in \mathfrak{B}$, **there exists** w **such that** ...

This def. nicely generalizes to fields.

Consider two solutions:



Controllability = patchability:



Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ **and** $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$,

i.e., $R(\frac{d}{dt})w = 0$, **controllable?**

We are looking for conditions on the polynomial matrix R

and algorithms in the coefficient matrices $R_0, R_1, \cdots, R_{\mathtt{n}}$.

Thm: $R(\frac{d}{dt})w = 0$ defines a **controllable** system if and only if

$$\boxed{\text{rank}(R(\lambda)) \text{ is independent of } \lambda \text{ for } \lambda \in \mathbb{C}.}$$

Example: $$r_1(\frac{d}{dt})w_1 = r_2(\frac{d}{dt})w_2 \quad (w_1, w_2 \text{ scalar})$$

is controllable if and only if r_1 and r_2 have no common factor.

Non-example: $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi], \quad \det(R) \neq$ constant.

Image representations

Representations of $\mathfrak{L}^{\mathtt{w}}$:

$$R(\tfrac{d}{dt})w = 0$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$;

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell$$

called a *'latent variable' representation* of the manifest behavior $\mathfrak{B} = (R(\frac{d}{dt}))^{-1}M(\frac{d}{dt})\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\ell})$.

Missing link: $$w = M(\tfrac{d}{dt})\ell$$

called an ***'image' representation*** **of** $\mathfrak{B} = \mathrm{im}(M(\frac{d}{dt}))$.

Elimination theorem $\Rightarrow$ **every image is also a kernel.**

¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$:

1. $\mathfrak{B}$ is **controllable,**
2. $\mathfrak{B}$ admits an **image representation,**
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi]$, $\quad a^\top(\frac{d}{dt})\mathfrak{B}$ equals 0 or all of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$,
4. $\mathbb{R}^{\mathtt{w}}[\xi]/\mathfrak{N}_{\mathfrak{B}}$ is **torsion free,**

etc., etc.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

Remarks:

- **Algorithm:** R + syzygies + Gröbner basis

  $\Rightarrow$ numerical test for on coefficients of R.

- for the input/output system

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad w = (u, y)$$

  the **transfer f'n $P^{-1}Q$** determines (only) the **controllable part** of the behavior

- $\exists$ complete generalization to linear PDE's
- $\exists$ partial results for nonlinear systems
- Kalman controllability is a straightforward special case

Control as Interconnection

In the case of control, our point of view leads to

PLANT:

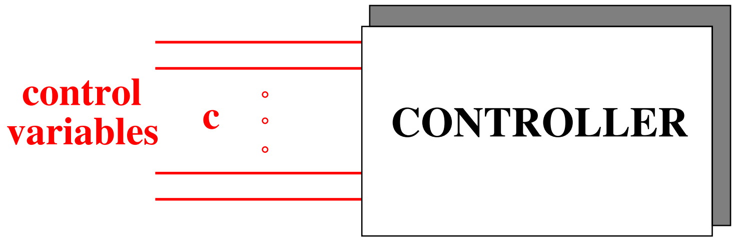
The plant has two kinds of variables

(or, often more appropriately, terminals):

- **variables to be controlled:** w**,**
- **control variables:** c**.**

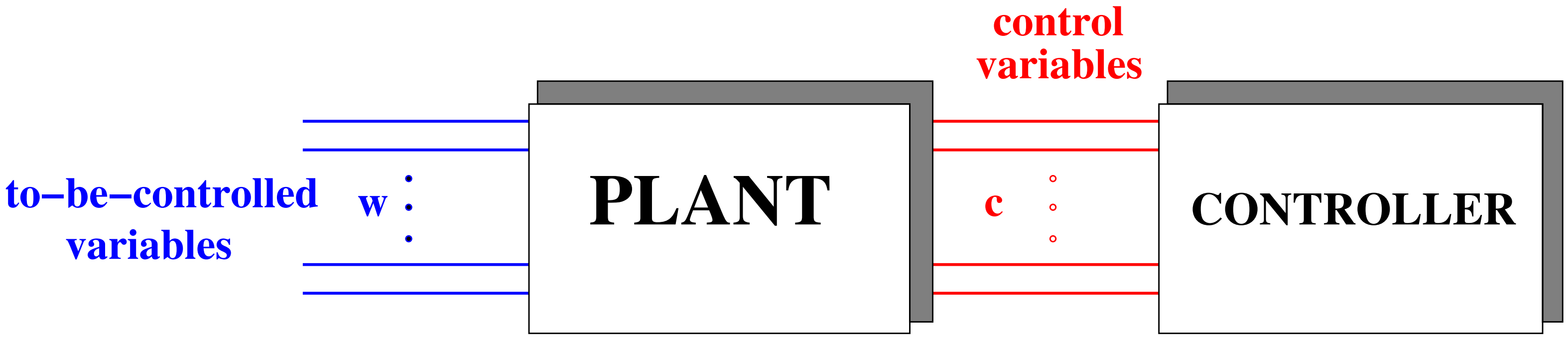
The control variables are those variables through which we interconnect the controller to the plant.

CONTROLLER:



The controller restricts the behavior of the control variables and, through these, that of the to-be-controlled variables.

CONTROLLED SYSTEM:



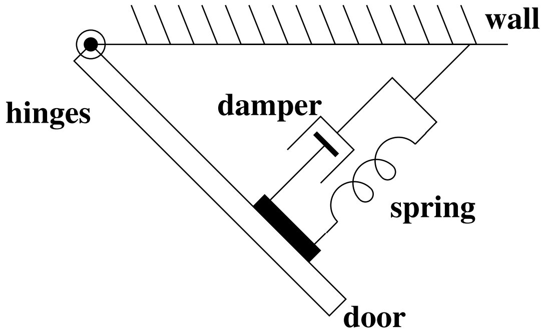
Control variables = shared variables.

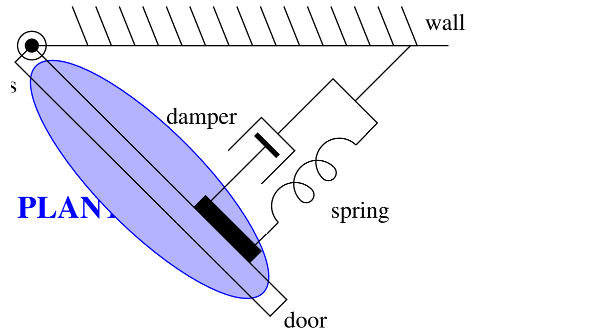
I want to discuss two items in this context:

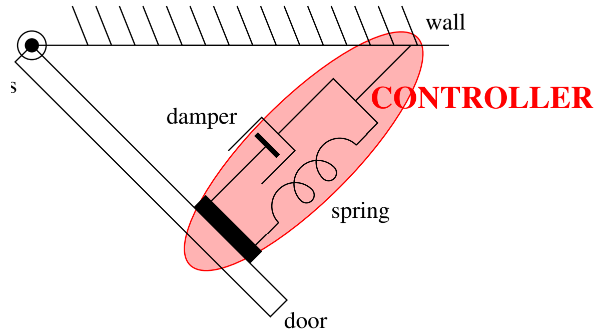
1. **A (very low-tech) example**

2. **One general result**

Example of such a control mechanism:





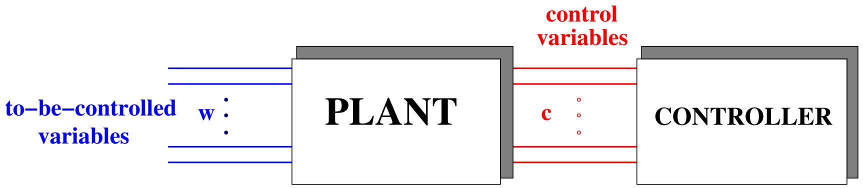


Similar idea: A damper of a car, etc.:

the very many control devices that are not

sensor-output 2 actuator-input feedback mechanisms.

A general implementability result



Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be the behavior of the plant

(with $\mathtt{w}$ to-be-controlled and $\mathtt{c}$ control variables.)

Let $\mathfrak{C} \in \mathfrak{L}^{\mathtt{c}}$ be the behavior of the controller

(with $\mathtt{c}$ control variables.)

This yields the controlled behavior

$$\mathfrak{K} := \{w \mid \exists\ c \in \mathfrak{C} \text{ such that } (w, c) \in \mathfrak{B}\}.$$

By the elimination theorem $\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}$.

Implementability question:

Which are the

controlled behaviors $\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}$

that can be obtained this way?

The answer to this question is surprisingly simple and explicit:

Theorem: $\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}$ is **implementable** **if and only if**

$$\boxed{\mathfrak{N} \subset \mathfrak{K} \subset \mathfrak{P}}$$

where

$$\mathfrak{N} := \{w \mid (w, 0) \in \mathfrak{B}\},$$

is the 'hidden' behavior, and

$$\mathfrak{P} := \{w \mid \exists \;\; c \text{ such that } (w, c) \in \mathfrak{B}\},$$

is the 'manifest plant' behavior.

Note: pole assignment follows, many refinements,...

Remarks:

- **Many control mechanism in practice do not function as sensor output to actuator input drivers**
- **Control = Interconnection $\Rightarrow$ controlled behavior can be any behavior that is wedged in between hidden behavior and plant behavior**
- **Control = integrated system design; finding a suitable subsystem**
- **$\exists$ a complete theory of controller synthesis (stabilization, $\mathcal{H}_\infty$, ...) of interconnecting controllers for linear systems**
- **Via (regular) implementability, the usual feedback structures are recovered**
- **Controllability and observability: central ideas also here**

Wrap-up

- **A system = a behavior**
- **Importance of latent variables**
- **Relevance in modular modeling**
- **There is a complete theory for linear time-invariant differential systems**
- **Nice theory of controllability**
- **Limitation of input/output thinking**
- **Relevance of behaviors, even in control**

Further results

Many additional problem areas have been studied from the behavioral point of view:

- **System representations:** **input/output representations, state representations and construction, model reduction, symmetries**
- **System identification** $\Rightarrow$ **the most powerful unfalsified model (MPUM), approximate system ID**
- **Observers**
- **Control**
- **Quadratic differential forms,** **dissipative systems,** $\mathcal{H}_\infty$**-control**
- **n-D systems (Rocha c.s.), distributed systems and PDE's**

Is is worth worrying about these ‘axiomatics’?

They have a deep and lasting influence! Especially in teaching.

Examples: Probability for uncertainty, QM, the development of input/output ideas in system theory and control - often these axiomatics are implicit, but nevertheless much very present.

Thank you for your attention

Details & copies of the lecture frames are available from/at

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