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Markovian properties for 2D behavioral systems described by PDE's: the scalar case

Paula Rocha · Jan C. Willems

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Abstract In this paper we study the characterization of deterministic Markovian properties for 2D behavioral systems in terms of their descriptions by PDE's. In particular, we consider scalar systems and show that in this case strong-Markovianity is equivalent to the existence of a first order PDE description.

Keywords 2D systems · Behavioral approach · Markovian properties

1 Introduction

Within the behavioral approach, Markov properties are defined in a deterministic setting, where the stochastic notion of conditional independence of future and past given the present is replaced by the notion of concatenability of the future and past evolution of trajectories whose present values coincide. The first definition of behavioral Markovianity was given in Willems (1989) for 1D behaviors and the extension of this definition to the 2D discrete case was proposed in Rocha and Willems (2005). This extension is general, in the sense that it allows to consider different types of Markov properties depending on what are the admissible types of partitions of the domain, and encompasses the deterministic versions of both the local and the global Markov properties for 2D stochastic processes, Goldstein (1980). The first definition of of Markovianity for nD behaviors defined over the continuous domain

We dedicate this paper to the memory of Professor Nirmal K. Bose, to whom we are thankful in many ways. His outstanding influence and efforts to develop the area of multidimensional systems contributed to attract our attention into this field, which we entered with the precious help of his books Bose (1982) and Bose (1985). Although our research has followed different paths, his work will always be a reference to us.

P. Rocha (✉)
Faculty of Engineering, University of Oporto, Rua do Dr Roberto Frias, 4200-465 Porto, Portugal
e-mail: procha@mat.ua.pt; mprocha@fe.up.pt

J. C. Willems
K.U. Leuven, ESAT/SCD (SISTA), Kasteelpark Arenberg 10, 3001 Leuven-Heverlee, Belgium
e-mail: Jan.Willems@esat.kuleuven.ac.be

$\mathbb{R}^n$, which we shall here refer to as weak-Markovianity, was given in Willems (2004) and can be considered as a deterministic version of the (stochastic) global Markov property.

Markovianity plays an important role in the set theoretic definition of the property of state. Indeed, the well known memory property implies that the state has a Markovian behavior, Willems (1989). In this context, the study of the relationship between Markovianity and the possibility of describing a behavior by means of first order ODEs or PDE's is an important issue as it allows to establish a relationship between the state/memory property (which is useful, for instance, in control) and the description via first order equations (that are more suitable for simulations). It was shown in Rapisarda and Willems (1997) that for systems given by ODE's the existence of a first order description is equivalent to the (1D) Markov property. The situation is somewhat more complicated for systems described by PDE's. In fact, in Rocha and Willems (2006) we proved that the existence of a first order description is sufficient, but not necessary for weak-Markovianity. This led us to define nD strong-Markovianity, by demanding weak-Markovianity to hold also for the restrictions of the original behavior to all the lower dimensional subspaces of $\mathbb{R}^n$.

It turns out that for systems with finite dimensional behavior first order representability is equivalent to strong-Markovianity, Rocha and Willems (2006). However what happens in the infinite dimensional case remains an open question. In Rocha and Willems (2005), we presented a preliminary analysis for the case of 2D scalar systems, and concluded that in this case strong-Markovianity is indeed equivalent to the existence of first order descriptions. The aim of this paper is to give a complete foundation for the reasonings presented there and join them with our results for finite dimensional behaviors, so as to fully treat the 2D scalar case. We hope that the study of this simple situation may shed some light into the approach to be followed in order to deal with the multivariate case.

2 2D systems described by PDE's

In this paper we consider two-dimensional (2D) behavioral systems that can be represented as the solution set of a system of linear PDE's with constant coefficients. Let $\mathbb{R}^{\bullet\times \mathtt{w}}[s_1, s_2]$ be the set of real polynomial matrices in two indeterminates with $\mathtt{w}$ columns and $R \in \mathbb{R}^{\bullet\times \mathtt{w}}[s_1, s_2]$. Associate with R the following system of PDE's

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) w = 0. \tag{1}$$

The behavior $\mathfrak{B}$ defined by this system of PDE's is its solution set over an appropriate domain. Here we consider as domain the set of all continuous functions $\mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^{\mathtt{w}})$. Hence

$$\mathfrak{B} = \{w \in \mathcal{C}^0(\mathbb{R}^2, \mathbb{R}^{\mathtt{w}}) \mid (1) \text{ holds in the distributional sense}\}.$$

As $\mathfrak{B}$ is the kernel of a partial differential operator, we refer to it as a *kernel behavior*. The PDE (1) and the matrix R are said to be a *kernel representation* and a *representation matrix* of $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right)$, respectively.

We shall restrict our attention to the scalar case, i.e., we take $\mathtt{w} = 1$; this means that we consider behaviors with one variable evolving in $\mathbb{R}^2$. Although this is a simplified situation, it illustrates some relevant aspects in the study of Markovian properties.

Thus, the kernel representations to be considered are associated with 2D polynomial columns

$$R(s_1, s_2) = \begin{bmatrix} r_1(s_1, s_2) \\ \vdots \\ r_{\mathrm{q}}(s_1, s_2) \end{bmatrix},$$

where the $r_i(s_1, s_2)$ are 2D polynomials. Factoring out the greatest common divisor $p(s_1, s_2)$ of these polynomials yields:

$$R(s_1, s_2) = F(s_1, s_2)p(s_1, s_2), \tag{2}$$

where,

$$F(s_1, s_2) = \begin{bmatrix} p_1(s_1, s_2) \\ \vdots \\ p_{\mathrm{q}}(s_1, s_2) \end{bmatrix}$$

and

$$p_i(s_1, s_2)p(s_1, s_2) = r_i(s_1, s_2), \quad i = 1, \ldots, \mathrm{q}.$$

Since the polynomials $p_i(s_1, s_2)$ have no common factors, they have at most a finite number of common zeros and hence the variety

$$\mathcal{V} := \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid p_i(\lambda_1, \lambda_2) = 0, i = 1, \ldots, \mathrm{q}\} \tag{3}$$

is finite, Zerz (1996). This means that $\ker\left(F(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\right)$ is a complex linear subspace generated by a finite number of polynomial-exponential trajectories of the form $w(x_1, x_2) = n_{(\lambda_1,\lambda_2)}(x_1, x_2)e^{\lambda_1 x_1 + \lambda_2 x_2}$, where $n(x_1, x_2)$ is a polynomial function of x_1 and x_2 and $(\lambda_1, \lambda_2) \in \mathcal{V}$, and is therefore finite dimensional.

Thus, if the polynomial $p(s_1, s_2)$ is a unit (i.e., a nonzero constant) then $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right)$ coincides with $\ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right)$ and is hence finite dimensional. In case $p(s_1, s_2)$ is not a unit, it has an infinite number of zeros and $\mathfrak{B}$ is infinite dimensional as it contains all the exponential trajectories of the form $w(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2}$ such that $p(\lambda_1, \lambda_2) = 0$.

3 Markovian 2D systems

Although we restrict to the 2D case, we shall state our definitions of Markovianity for the nD case. This in particular allows to treat both the 1D and 2D cases with the same definition, by setting $n = 1$ or $n = 2$, which will be useful in the sequel. Let Π be the set of 3-way partitions (S_-, S_0, S_+) of $\mathbb{R}^n$ such that S_- and S_+ are open and S_0 is closed; given a partition $\pi = (S_-, S_0, S_+) \in \Pi$ and a pair of trajectories (w_-, w_+) that coincide on S_0, define the *concatenation* of (w_-, w_+) along π as the trajectory $w_- \wedge_\pi w_+$ that coincides with w_- on $S_0 \cup S_-$ and with w_+ on $S_0 \cup S_+$.

Definition 1 A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathrm{w}})^{\mathbb{R}^n}$ is said to be *weak-Markovian* if for any partition $\pi \in \Pi$ and any pair of trajectories $w_-, w_+ \in \mathfrak{B}$ such that $w_{-|_{S_0}} = w_{+|_{S_0}}$, the trajectory $w_- \wedge_\pi w_+$ is also an element of $\mathfrak{B}$.

The definition of strong-Markovianity requires the introduction of other preliminary concepts.

Given a kernel behavior $\mathfrak{B}$ defined over $\mathbb{R}^n$ and a linear subspace $\mathcal{S}$ of $\mathbb{R}^n$, define the behavior $\mathcal{K}(\mathfrak{B}|_{\mathcal{S}})$ as the smallest kernel behavior containing the restriction $\mathfrak{B}|_{\mathcal{S}}$ of $\mathfrak{B}$ to $\mathcal{S}$. Moreover, let Π_S be the set of 3-way partitions (S_-, S_0, S_+) of $\mathcal{S}$ such that S_- and S_+ are open (in $\mathcal{S}$) and S_0 is closed.

Definition 2 A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathrm{w}})^{\mathbb{R}^n}$ is said to be *strong-Markovian* if for any subspace S of $\mathbb{R}^n$, any partition $\pi_S \in \Pi_S$, and any pair of trajectories $w_-, w_+ \in \mathcal{K}(\mathfrak{B}|_S)$ such that $w_{-|_{S_0}} = w_{+|_{S_0}}$, the trajectory $w_- \wedge_\pi w_+$ is an element of $\mathcal{K}(\mathfrak{B}|_S)$.

Clearly, strong-Markovianity implies weak-Markovianity and, moreover, these two properties coincide for one-dimensional behaviors.

2D behaviors described by a system of first order PDE's

$$\left(\sum_{i=1}^{2} R_i \frac{\partial}{\partial x_i} + R_0\right) w = 0 \tag{4}$$

can be shown to be weak-Markovian. However, not every weak-Markovian 2D behavior admits a first order description (Rocha and Willems 2006). In the next section we shall prove that, for the particular case of scalar 2D behaviors, strong-Markovianity is enough to guarantee first order representability.

4 First order PDE's and Markovianity

Let $\mathfrak{B} \subset \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ be a 2D behavior with kernel representation associated to a matrix $R(s_1, s_2) = F(s_1, s_2)p(s_1, s_2)$ as in (2). We shall assume that $\mathfrak{B}$ is non trivial, i.e., $\{0\} \neq \mathfrak{B} \neq \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$, which means that $R(s_1, s_2)$ is a nonconstant polynomial column.

Assume first that $p(s_1, s_2) = K \in \mathbb{R} \setminus \{0\}$. Then, $\mathfrak{B} = \ker F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ is finite dimensional and, by Rocha and Willems (2006), it is strong-Markovian if and only if it can be represented as

$$\mathfrak{B} = \ker \begin{bmatrix} g_1\left(\frac{\partial}{\partial x_1}\right) \\ g_2\left(\frac{\partial}{\partial x_2}\right) \end{bmatrix},$$

with $g(s_1) = s_1 + a_0$ and $g(s_2) = s_2 + b_0$. Assume now that $p(s_1, s_2)$ is a nonconstant 2D polynomial.

Given $\alpha \in \mathbb{R}$, define the following behaviors:

$$\mathfrak{B}^\alpha_{2\mathrm{D}} := \{w \in \mathfrak{B} \mid \forall t \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall x_2 \in \mathbb{R}\ \ w(t - \alpha x_2, x_2) = c\}$$

and

$$\mathfrak{B}^\alpha_{1\mathrm{D}} := \{v \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \mid v(t) = w(t - \alpha x_2, x_2),\ t \in \mathbb{R}, w \in \mathfrak{B}^\alpha_{2D}\}.$$

The behavior $\mathfrak{B}^{\alpha}_{2\mathrm{D}}$ consists of all the trajectories in $\mathfrak{B}$ that are constant along all the lines $\mathcal{L}^{\alpha}_t := \{(x_1, x_2) \mid x_1 + \alpha x_2 = t\}$, $t \in \mathbb{R}$. It is not difficult to check that

$$\begin{aligned}\mathfrak{B}^{\alpha}_{2\mathrm{D}} &= \mathfrak{B} \cap \ker\left(\frac{\partial}{\partial x_2} - \alpha\frac{\partial}{\partial x_1}\right)\\ &= \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) \cap \ker\left(\frac{\partial}{\partial x_2} - \alpha\frac{\partial}{\partial x_1}\right)\\ &= \ker\left(R\left(\frac{\partial}{\partial x_1}, \alpha\frac{\partial}{\partial x_1}\right)\right)\\ &= \ker\left(\rho_\alpha\left(\frac{\partial}{\partial x_1}\right)\tilde{p}_\alpha\left(\frac{\partial}{\partial x_1}\right)\right),\end{aligned} \tag{5}$$

where $\rho_\alpha(s)$ is the greatest common divisor of the univariate polynomials $p_1(s, \alpha s), \ldots, p_{\mathrm{q}}(s, \alpha s) \in \mathbb{R}[s]$, and

$$\tilde{p}_\alpha(s) := p(s, \alpha s). \tag{6}$$

As for $\mathfrak{B}^{\alpha}_{1\mathrm{D}}$, this is a 1D behavior whose trajectories correspond to the restriction of the trajectories in $\mathfrak{B}^{\alpha}_{2\mathrm{D}}$ to the x_1-axis and can alternatively be given by

$$\mathfrak{B}^{\alpha}_{1\mathrm{D}} = \{v \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \mid v(t) = w(t, 0), t \in \mathbb{R}, w \in \mathfrak{B}^{\alpha}_{2\mathrm{D}}\}.$$

Thus, due to (5),

$$\mathfrak{B}^{\alpha}_{1D} = \ker\left(\rho_\alpha\left(\frac{d}{\mathrm{d}t}\right)\tilde{p}_\alpha\left(\frac{d}{\mathrm{d}t}\right)\right). \tag{7}$$

Note that, since the variety $\mathcal{V}$ defined in (3) is finite, the polynomials $p_i(s, \alpha s), i = 1, \ldots, \mathrm{q}$ are not all zero polynomials. This implies that $\rho_\alpha(s)$ is a nonzero polynomial. Moreover, there is only a finite number of values $\alpha \in \mathbb{R}$ for which the polynomial $\tilde{p}_\alpha(s)$ is the zero polynomial. Indeed, let $p(s_1, s_2) = \sum_{i,j} p_{ij} s_1^i s_2^j$, then

$$\tilde{p}_\alpha(s) = \sum_k \left(\sum_{i+j=k} p_{ij}\alpha^j\right) s^k$$

and for this to be the zero polynomial we must have that

$$\sum_{i+j=k} p_{ij}\alpha^j = 0, \quad k \geq 0.$$

But this can only happen for a finite number of values $\alpha \in \mathbb{R}$, otherwise all the coefficients p_{ij} should be zero and $p(s_1, s_2)$ would be the zero polynomial, contradicting our assumption that $p(s_1, s_2)$ is nonconstant. Thus we conclude that the set $\mathcal{N}$ of values $\alpha \in \mathbb{R}$ for which $\rho_\alpha(s)\tilde{p}_\alpha(s)$ is the zero polynomial (and consequently for $\mathfrak{B}^{\alpha}_{1D} = \ker\ 0 = \mathcal{C}^0(\mathbb{R}, \mathbb{R})$) is finite.

In order to show that strong-Markovianity implies first order representability, we start by proving that weak-Markovianity alone already implies that $p(s_1, s_2)$ in (2) is a first order 2D polynomial.

Lemma 1 *Let* $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) \subset \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ *be an infinite-dimensional 2D weak-Markovian kernel behavior and let* $\alpha \in \mathbb{R}$. *Then the behavior* $\mathfrak{B}^{\alpha}_{1D}$ *is a 1D Markovian behavior.*

Proof In order to prove this result it suffices to show that every trajectory v of $\mathfrak{B}^{\alpha}_{1D}$ such that $v(0) = 0$ is concatenable with the zero trajectory, i.e., if $\pi = ((-\infty, 0), \{0\}, (0, +\infty))$ then $v \wedge_{\pi} 0 \in \mathfrak{B}^{\alpha}_{1D}$. Let then $v \in \mathfrak{B}^{\alpha}_{1D}$ be a trajectory such that $v(0) = 0$. Take $w \in \mathfrak{B}^{\alpha}_{2D}$ such that $v(t) = w(t, 0)$. Then, $w(-\alpha x_2, x_2) = w(0, 0) = v(0) = 0$, for all $x_2 \in \mathbb{R}$, i.e, w is zero on the line $\mathcal{L}^{\alpha}_0$. By the weak-Markovianity of $\mathfrak{B}$, this implies that w is concatenable with the zero trajectory along the obvious partition $\pi_0 = (\mathcal{S}_-, \mathcal{L}^{\alpha}_0, \mathcal{S}_+)$ of $\mathbb{R}^2$ determined by the line $\mathcal{L}^{\alpha}_0$. In other words, $w^* := w \wedge_{\pi_0} 0 \in \mathfrak{B}$. But w^* is also a trajectory of $\mathfrak{B}^{\alpha}_{2D}$. Moreover, its corresponding trajectory in $\mathfrak{B}^{\alpha}_{1D}$, $v^*(t) := w^*(t, 0)$, coincides with $v \wedge_{\pi} 0$. This shows that v is concatenable with the zero trajectory as desired. □

An important consequence of this fact is that either the scalar 1D behavior $\mathfrak{B}^{\alpha}_{1D}$ coincides with $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ or then it is represented by a first order ODE and has hence dimension not higher than 1. Another consequence of this lemma is given by the following result.

Corollary 1 *Let* $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) \subset \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ *be an infinite-dimensional 2D weak-Markovian kernel behavior and* $p(s_1, s_2)$ *be the corresponding right factor in factorization (2). Then* $p(s_1, s_2)$ *is a 2D first order polynomial, i.e.,* $p(s_1, s_2) = a_1 s_1 + a_2 s_2 + a_0$*, for suitable coefficients* $a_0, a_1, a_2 \in \mathbb{R}$.

Proof Assume that $\mathfrak{B}$ is weak-Markovian and let $\alpha \in \mathbb{R} \setminus \mathcal{N}$. Then, since 1D-Markovianity is equivalent to first order representability Rapisarda and Willems (1997), by (7) and Lemma 1, the polynomial $\rho_{\alpha}(s)\tilde{p}_{\alpha}(s)$, that we know to be nonzero, must have degree not higher than 1. Therefore the degree of $\tilde{p}_{\alpha}(s)$ cannot be higher than 1. This means that the coefficients p_{ij} of $p(s_1, s_2)$ must satisfy

$$\sum_{i+j=k} p_{ij}\alpha^j = 0, \quad k \geq 2.$$

Since this is valid for all the values $\alpha \in \mathbb{R} \setminus \mathcal{N}$, we conclude that $p_{ij} = 0$, for $i + j \geq 2$ and $p(s_1, s_2) = a_1 s_1 + a_2 s_2 + a_0$, with $a_1 = p_{10}, a_2 = p_{01}, a_0 = p_{00}$. □

We next show that, in case $\mathfrak{B}$ is strong-Markovian, $\ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) = \{0\}$.

Lemma 2 *Let* $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) \subset \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ *be a 2D strong-Markovian behavior, such that* $R(s_1, s_2) = F(s_1, s_2)p(s_1, s_2)$ *as in (2). If* $p(s_1, s_2)$ *is a nonconstant polynomial, then*

$$\ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) = \{0\}$$

.

Proof (i) Assume that $\mathfrak{B}$ is strong-Markovian and that the polynomials $p_j(s_1, s_2)$, $j = 1, \ldots, \mathrm{q}$, have a common zero $(\lambda_1^*, \lambda_2^*)$ with $\lambda_1^* \neq 0$. Then λ_1^* is a zero of $\rho_{\alpha^*}(s)$ with $\alpha^* = \frac{\lambda_2^*}{\lambda_1^*}$, which means that $s - \lambda_1^*$ is a factor of $\rho_{\alpha^*}(s)$. Hence, $(s - \lambda_1^*)\tilde{p}_{\alpha^*}(s)$ is a factor of $\rho_{\alpha^*}(s)\tilde{p}_{\alpha^*}(s)$ and therefore $\ker\left(\left(\frac{d}{dt} - \lambda_1^*\right)\tilde{p}_{\alpha^*}\left(\frac{d}{dt}\right)\right) \subset \mathfrak{B}^{\alpha^*}_{1D}$. But this implies that $\tilde{p}_{\alpha^*}(s)$ is the zero polynomial, since otherwise $\mathfrak{B}^{\alpha^*}_{1D}$ would be described by a higher order ODE and would therefore not be 1D Markovian. Consequently $p(s_1, s_2)$ must be of the form $p(s_1, s_2) = a_2 s_2 - a_2\alpha^* s_1$, with $a_2 \neq 0$. Thus, without loss of generality, we may take $a_2 = 1$ and $p(s_1, s_2) = s_2 - \alpha^* s_1$. Now, considering new independent variables $(\tilde{x}_1, \tilde{x}_2)$

such that $(x_1, x_2) = (b_1\tilde{x}_1 - \alpha^*\tilde{x}_2, b_2\tilde{x}_1 + \tilde{x}_2)$, with $\alpha^* b_2 + b_1 \neq 0$, but renaming them again as (x_1, x_2) in order to avoid an excess of notation, we obtain that $p(s_1, s_2) = s_2$ and the polynomials $p_j(s_1, s_2)$ have a common zero $(\lambda_1^*, \lambda_2^*)$ with $\lambda_2^* = 0$. This implies that $\ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right)$ contains in particular all constant trajectories $z(x_1, x_2) \equiv k_0$, and it is easy to check that every trajectory of the form $w(x_1, x_2) = k_0 + k_1 x_2$ is an element of $\mathfrak{B}$. As a consequence, the restriction of $\mathfrak{B}$ to the subspace $\mathcal{S}_0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ contains $\{y \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \mid y(x_2) = w(0, x_2) = k_0 + k_1 x_2,\ k_0, k_1 \in \mathbb{R}\} = \ker\left(\frac{d^2}{\mathrm{d}x_2^2}\right)$ and the same applies to $\mathcal{K}\left(\mathfrak{B}|_{\mathcal{S}_0}\right)$. Note now that $w \in \mathfrak{B}$ if and only if $w(x_1, x_2) = k(x_1) + f(x_1, x_2)$, with $k(x_1) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ and $f(x_1, x_2) \in \mathcal{F}$, where $\mathcal{F}$ is a kernel behavior generated by a finite number of polynomial-exponential trajectories. Thus $\mathcal{K}\left(\mathfrak{B}|_{\mathcal{S}_0}\right)$ is finite dimensional. Moreover, its dimension higher than 1 since it contains $\ker\left(\frac{d^2}{\mathrm{d}x_2^2}\right)$. This means that $\mathcal{K}\left(\mathfrak{B}|_{\mathcal{S}_0}\right)$ is not 1D Markovian, contradicting the fact that $\mathfrak{B}$ is strong-Markovian. Going back to original variables (x_1, x_2), we conclude that the polynomials $p_j(s_1, s_2)$, $j = 1, \ldots, \mathtt{q}$ cannot have a common zero $(\lambda_1^*, \lambda_2^*)$ with $\lambda_1^* \neq 0$.

(ii) Interchanging the roles of x_1 and x_2 we also conclude that the p_j's cannot have a common zero $(\lambda_1^*, \lambda_2^*)$ with $\lambda_2^* \neq 0$.

(iii) Assume finally that the polynomials $p_j(s_1, s_2)$, $j = 1, \ldots, \mathtt{q}$ have as only common zero $(\lambda_1^*, \lambda_2^*) = (0, 0)$. Making a suitable linear change of the independent variables (x_1, x_2) it is possible to transform the original polynomial $p(s_1, s_2)$ into $p(s_1, s_2) = s_2 + a_0$, whereas the only common zero of the transformed polynomials $p_j(s_1, s_2)$ remains $(0, 0)$. Similar to what happened before, it is now easy to see that $w \in \mathfrak{B}$ if and only if $w(x_1, x_2) = k(x_1) + f(x_1, x_2)$, with $k(x_1) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ and $f(x_1, x_2) \in \mathcal{F}$, where $\mathcal{F}$ is a kernel behavior generated by a finite number of polynomial trajectories. This again leads to a contradiction since it implies that $\mathcal{K}\left(\mathfrak{B}|_{\mathcal{S}_0}\right)$ is finite dimensional with dimension higher than 1 and is therefore not 1D Markovian. □

Note that if $\ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) = \{0\}$ then $\ker\left(R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) = \ker\left(F\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right) = \ker\left(p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right)$. Thus, Corollary 1 and Lemma 2 clearly imply that every strong-Markovian infinite dimensional kernel behavior $\mathfrak{B} \subset C^0(\mathbb{R}^2, \mathbb{R})$ can be described by a first order PDE, i.e.,

$$\mathfrak{B} = \ker\left(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_0\right).$$

Conversely, it is not difficult to prove that $\mathfrak{B} = \ker\left(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_0\right)$ is strong-Markovian. Indeed, assume that $(a_1, a_2) \neq (0, 0)$, otherwise $\mathfrak{B}$ would be trivial and therefore obviously strong-Markovian. Without loss of generality, we may suppose that $a_1 = 0$ and $a_2 = 1$, i.e., $p(s_1, s_2) = a_0 + a_1 s_1 + a_2 s_2 = s_2 + a_0$, otherwise we would perform a change in the independent variables to bring $p(s_1, s_2)$ into this form. Then $w \in \mathfrak{B}$ if and only if $w(x_1, x_2) = k(x_1)e^{-a_0 x_2}$, with $k \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$. This easily allows to check that $\mathfrak{B}$ is weak-Markovian. Moreover, it implies that the restriction of $\mathfrak{B}$ to the subspace $\mathcal{S}_0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ coincides with $\mathcal{K}\left(\mathfrak{B}|_{\mathcal{S}_0}\right)$ and is equal to $\ker\left(\frac{d}{dx_2} + a_0\right)$, which is clearly 1D Markovian, whereas the restrictions of $\mathfrak{B}$ to the other 1-dimensional subspaces $\mathcal{S}$ of $\mathbb{R}^2$ coincide with $\mathcal{K}(\mathfrak{B}|_{\mathcal{S}})$, are equal to $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$, and are therefore also 1D Markovian. This yields the following result.

Proposition 1 *Let* $\mathfrak{B} \subset \mathcal{C}^0\left(\mathbb{R}^2, \mathbb{R}\right)$ *be an infinite dimensional 2D kernel behavior. Then the following are equivalent:*

1. $\mathfrak{B}$ *is strong-Markovian*
2. $\mathfrak{B}$ *is described by one first order PDE, i.e.,* $\mathfrak{B} = \ker\left(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_0\right)$, *for suitable real coefficients* a_0, a_1, a_2.

The previous results on both the finite and the infinite dimensional cases can be summarized as follows.

Theorem 1 *Let* $\mathfrak{B} \subset \mathcal{C}^0\left(\mathbb{R}^2, \mathbb{R}\right)$ *be a scalar 2D kernel behavior. Then the following are equivalent:*

1. $\mathfrak{B}$ *is strong-Markovian*
2. $\mathfrak{B}$ *is described by a set of at most two first order PDE, i.e.,* $\mathfrak{B} = \ker(A_1 \frac{\partial}{\partial x_1} + A_2 \frac{\partial}{\partial x_2} + A_0)$, *with* $A_0, A_1, A_2 \in \mathbb{R}^{\mathtt{q}\times 1}$, $\mathtt{q} \leq 2$.

5 Conclusion

In this paper we proved that representability by means of first order PDE's is equivalent to strong-Markovianity for scalar 2D systems. Although this is a very particular situation, the results obtained here suggest that a straight connection between Markovianity and first order PDE's might also exist in the multivariate case and encourage future research in this direction.

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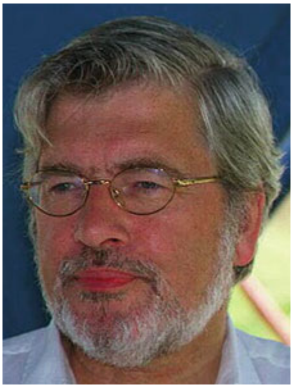
Author Biographies

Paula Rocha graduated in Applied Mathematics at the University of Porto and obtained her Ph.D. degree in Systems and Control from the University of Groningen, The Netherlands, under the supervision of Jan C. Willems. After having worked as an Assistant Professor at the Department of Technical Mathematics of the Delft University of Technology, she moved to the Department of Mathematics of the University of Aveiro, Portugal, where she has been a Professor till the end of 2008. She is currently a Professor at the Department of Electrical and Computer Engineering of the Faculty of Engineering, University of Oporto, Portugal. Her interests are mainly in the area of Systems and Control, namely in the fields of the behavioral approach to one- and multi-dimensional (nD) systems, and biomedical systems.



Jan C. Willems was born in Bruges in Flanders, Belgium. He studied engineering at the University of Ghent and obtained the M.Sc. degree from the University of Rhode Island in 1965, and the Ph.D. degree from the Massachusetts Institute of Technology in 1968, both in electrical engineering. He was an assistant professor in the Department of electrical engineering at MIT from 1968 to 1973. On February 1, 1973, he was appointed Professor of Systems and Control in the Mathematics Department of the University of Groningen. In the mid 80's, started to develop what is called the behavioral approach and, in 1988, he was awarded the Automatica Prize Paper Award for a series of 3 articles in which this framework was put forward. In 2003, Professor Willems became emeritus professor from the University of Groningen and became guest professor at the Department of Electrical Engineering, with the research group on Signals, Identification, System Theory and Automation (SISTA), at the K.U. Leuven, Belgium. During the academic year 2003–2004, he held the Chaire Francqui at the Faculty of Applied Sciences of the UniversitT Catholique de Louvain. Professor Willems is a life fellow of the IEEE. He has been on the editorial board of a number of journals, in particular, as managing editor of the SIAM Journal of Control and Optimization as and founding and managing editor of Systems & Control Letters. In 1998, he received the IEEE Control Systems award.