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A balancing approach to the realization of systems with internal passivity and reciprocity

T. Reis [a,*], J.C. Willems [b]

[a] Institut für Numerische Simulation, Technische Universität Hamburg-Harburg, Schwarzenbergstraße 95 E, 21073 Hamburg, Germany
[b] ESAT/SCD (SISTA), K.U. Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium

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ABSTRACT

This paper addresses the realization of positive real transfer functions which are symmetric with respect to some signature matrix. We show that a realization that is jointly internally reciprocal and internally passive can be achieved by positive real balancing.

1. Introduction

The basis of nearly all state space methods in network synthesis [1,2] are some particular realizations of linear time-invariant dynamical systems

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, \qquad y = Cx + Du \tag{1}$$

with equal input and output dimensions. The essential property that has to be fulfilled is *internal passivity*, that is

$$\begin{bmatrix} A^\top + A & B - C^\top \\ B^\top - C & -D - D^\top \end{bmatrix} \le 0. \tag{2}$$

On the other hand, in the reciprocal case, the matrices have to have the special block structure

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^\top & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \\ C &= \begin{bmatrix} B_{11}^\top & -B_{21}^\top \\ -B_{12}^\top & B_{22}^\top \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & D_{12} \\ -D_{12}^\top & D_{22} \end{bmatrix}, \end{aligned} \tag{3}$$

which is called *internal reciprocity*. Based on a realization with the properties (2) and (3), a circuit with ideal transformers and positive resistances, capacitances and inductances that captures the input–output behavior of the dynamical system (1) can be readily constructed [1]. Hereby, there is the following correspondence between the numbers of circuit elements and properties of the matrices A, B, C, D as in (2) and (3). The partition of the state yields the numbers of capacitances and inductances, the input (output) partition corresponds to the numbers of the ports driven by voltages and currents, and, finally, the rank of the matrix (2) is the number of required resistances.

Whereas the construction of either internally passive or internally reciprocal realizations can be done without any great difficulties, the realization of a jointly internally passive and internally reciprocal system is a challenging task and was first treated in a state space setting in [1,3]. It is shown in these works that a realization with these properties exists if and only if its transfer function is positive real and symmetric with respect to the signature matrix corresponding to the partition of the D matrix in (3).

In this article we give a novel and constructive approach to the construction of realizations fulfilling (2) and (3). Starting with a dynamical system (1) with sign symmetric and positive real transfer function, our approach is based on *positive real balancing* [4,5]. We show that all positive real balanced realizations are internally passive. On the other hand, we prove that among all positive real balanced realizations there exists at least one that is internally reciprocal. This realization can be constructed by

* Corresponding author.
E-mail addresses: timo.reis@tu-harburg.de (T. Reis), Jan.Willems@esat.kuleuven.be (J.C. Willems).

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a rather simple state space transformation of the system by a block-diagonal orthogonal symmetric matrix.

This article is organized as follows. After introducing the notation and required fundamentals of linear systems theory in Section 2, we consider each reciprocal and passive systems in Sections 3 and 4. In particular, we consider both properties from a time- and frequency-domain point of view. In Section 5, we collect the basic facts of positive real balanced realizations and present results about their connection to reciprocal and passive systems. In the main result, we show that each sign symmetric and positive real transfer function has a realization which is jointly positive real balanced, internally reciprocal and internally passive. This article concludes with the illustration of the results by means of two examples.

2. Preliminaries

Throughout the paper, $\mathbb{R}^{n,m}$ and $\mathbb{R}(s)^{n,m}$ denote the spaces of $n \times m$ matrices with entries consisting of real numbers and, respectively, rational functions in the indeterminate s. We use the symbol $\mathbf{G}\ell_n(\mathbb{R})$ for the group of invertible real $n \times n$ matrices and δ_{ij} is the Kronecker delta. The open complex right half-plane is denoted by $\mathbb{C}_+$ and the complex conjugate of a number $s \in \mathbb{C}$ by $\bar{s}$. The matrix $A^\top$ stands for the transpose of A and we write $A^{-\top} = (A^{-1})^\top$. An identity matrix of order n is denoted by I_n or simply by I. The zero $n\times m$ matrix is denoted by $0_{n,m}$ or simply by 0. The symbol $\|\cdot\|$ stands for the Euclidean vector norm in the vector case and for the maximal singular value if the argument is a matrix. For Hermitian matrices $P, Q \in \mathbb{C}^{n,n}$ we write $P > Q$ ($P \geq Q$) if $P - Q$ is positive (semi)definite. We call $S \in \mathbb{R}^{n,n}$ a *signature matrix* if it is diagonal and involutive, i.e., $S^2 = I_n$.

In the following we introduce some basics from linear systems theory.

Definition 1. The *behavior* of the system (1) by

$$\mathcal{B} = \left\{ \begin{array}{l} (u(\cdot), x(\cdot), y(\cdot)) \mid u(\cdot) : \mathbb{R} \to \mathbb{R}^m, y(\cdot) : \\ \mathbb{R} \to \mathbb{R}^p \text{are continuous}, x(\cdot) : \mathbb{R} \to \mathbb{R}^n \\ \text{is differentiable and } \frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, y = Cx + Du \end{array} \right\}.$$

Definition 2. The dynamical system (1) is said to be *minimal* if it is both controllable and observable.

The *transfer function* of the dynamical system (1) is given by $\mathbf{G}(s) = D + C(sI - A)^{-1}B \in \mathbb{R}^{p,m}(s)$. We also refer to (1) as a *realization of* $\mathbf{G}(s)$. Instead of (1) we also use the notation $[A, B, C, D]$.

Lemma 1 ([1,6]). *Let* $[A_1, B_1, C_1, D_1]$ *and* $[A_2, B_2, C_2, D_2]$ *be two minimal realizations of the transfer function* $\mathbf{G}(s)$. *Then* $D_1 = D_2$ *and there exists a matrix* $Q \in \mathbb{R}^{n,n}$ *such that*

$$A_1Q = QA_2, \qquad C_1Q = C_2, \qquad B_1 = QB_2.$$

The matrix Q with these properties is unique and invertible.

3. Passivity

In this section, we consider passivity which is a special property of square systems, i.e., the input and output dimensions coincide. By modeling the net flow of energy out of the system by the L_2 inner product of input and output, the concept of passivity means that the system cannot produce energy.

Definition 3. Let a square system (1) be given and let $\mathcal{B}$ be its behavior. Then (1) is called *passive* if for all $t \in [0, \infty)$ and $(u(\cdot), x(\cdot), y(\cdot)) \in \mathcal{B}$ with $x(0) = 0$, there holds

$$\int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau \geq 0.$$

Theorem 2 ([3,6]). *A necessary and sufficient condition for passivity of (1) is the positive realness of the transfer function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$, *i.e.,* $\mathbf{G}(s)$ *has no poles in* $\mathbb{C}^+$ *and* $\mathbf{G}(\lambda) + \mathbf{G}^\top(\overline{\lambda}) \geq 0$ *for all* $\lambda \in \mathbb{C}^+$.

A sufficient criterion for passivity is the existence of a *storage* [3] which is a function $V : \mathbb{R}^n \to [0, \infty)$ with the properties $V(0) = 0$ and

$$V(x(t)) - V(x(0)) \leq \int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau \quad \forall t \in [0, \infty), (u(\cdot), x(\cdot), y(\cdot)) \in \mathcal{B}.$$

If the function $V(x) = \frac{1}{2}\|x\|^2$ is a storage, then (1) is called *internally passive.*

The set of quadratic storage functions can be nicely characterized in terms of the *Kalman–Yacubovich–Popov lemma* (or also called *positive real lemma*) [1,6,7]. This result states that for a minimal system, passivity is equivalent to the solvability of the linear matrix inequality (LMI)

$$\begin{bmatrix} A^\top X + XA & XB - C^\top \\ B^\top X - C & -D - D^\top \end{bmatrix} \leq 0, \quad X = X^\top > 0. \tag{4}$$

It can be verified that (4) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}x^\top(t)Xx(t) \leq u^\top(t)y(t) \quad \forall t \in \mathbb{R}, (u(\cdot), x(\cdot), y(\cdot)) \in \mathcal{B}.$$

There is a one-to-one correspondence between the solutions of (4) and the quadratic storage functions via the relation $V(x) = \frac{1}{2}x^\top Xx$. In particular, we can conclude that (1) is internally passive if and only if (4) is fulfilled for $X = I_n$. Since X is a solution of (4) if and only if $T^\top XT$ solves the LMI corresponding to the realization $[T^{-1}AT, T^{-1}B, CT, D]$, we can set up the Algorithm 1 for computing an internally passive realization. We now briefly review the properties of the solution set of the LMI (4).

Algorithm 1 Construction of an internally passive realization

Given a minimal realization $[A, B, C, D]$ of the transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$ that is sign symmetric with respect to the signature matrix S_{ext}, compute an internally reciprocal realization $[A_p, B_p, C_p, D]$ of $\mathbf{G}(s)$.

1. Solve the LMI (4) for some symmetric $X \in \mathbb{R}^{n,n}$.
2. Perform a factorization $X^{-1} = TT^\top$ for some $T \in \mathbb{R}^{n,n}$.
3. Define $A_p = T^{-1}AT$, $B_p = T^{-1}B$ and $C_p = CT$.

Proposition 3 ([7,1,3]). *Let a passive minimal system (1) be given. Then the set*

$$\mathcal{S}_{\text{PRL}} = \{X \in \mathbb{R}^{n,n} : X = X^\top \text{and (4) holds true}\}$$

is convex and compact. Moreover, there exist some $X_{\min}, X_{\max} \in \mathcal{S}_{\text{PRL}}$ *such that for all* $X \in \mathcal{S}_{\text{PRL}}$, *there holds*

$$0 < X_{\min} \leq X \leq X_{\max}.$$

The extremal solutions $X_{\min}$, $X_{\max}$ *are characterized by*

$$S_a(x_0) = x_0^\top X_{\min}x_0, \qquad S_r(x_0) = x_0^\top X_{\max}x_0 \tag{5}$$

where

$$S_a(x_0) = \lim_{t\to\infty} \sup \left\{ -\int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau, \textit{ where } \atop (u(\cdot),x(\cdot),y(\cdot)) \in \mathcal{B},\ x(0)=x_0 \right\},$$

$$S_r(x_0) = \lim_{t\to-\infty} \inf \left\{ \int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau, \textit{ where } \atop (u(\cdot),x(\cdot),y(\cdot)) \in \mathcal{B},\ x(0)=x_0 \right\}.$$

The quadratic functionals $S_a(\cdot), S_r(\cdot)$ are called the *available storage* and *required supply*, respectively [3]. The first functional expresses the maximal energy that can be extracted from the system (1) initialized at x_0. The latter one stands for the minimal energy that has to be put into the system to steer from 0 to the final state x_0. It is known that the extremal solutions minimize the rank of (4) [8].

Since positive realness of $\mathbf{G}(s)$ is equivalent to the positive realness of $\mathbf{G}(s)^\top$, an analogous statement holds true for the LMI

$$\begin{bmatrix} AY+YA^\top & YC^\top - B \\ CY - B^\top & -D-D^\top \end{bmatrix} \le 0. \tag{6}$$

It immediately follows that $X \in \mathcal{S}_{\mathrm{PRL}}$ if and only if $Y = X^{-1}$ solves (6). As a consequence, the extremal solutions of (6) satisfy

$$X_{\min} = Y_{\max}^{-1}, \qquad X_{\max} = Y_{\min}^{-1}.$$

4. Reciprocity

In this part we collect some facts about reciprocal systems. Reciprocity of a system is equivalently characterized by the symmetry of the product of the transfer function with some signature matrix.

Definition 4. Let $S_{\mathrm{ext}} = \mathrm{diag}(s_1,\ldots,s_m)$ be a signature matrix. Then (1) is said to be *reciprocal with (external) signature* S_{ext}, if for all $i,j \in \{1,\ldots,m\}$, the inputs with respective components $\tilde{u}_k(t) = \delta_{ki}v(t)$ and $\tilde{\tilde{u}}_k(t) = \delta_{kj}v(t)$ of the system with zero initial condition results in outputs $\tilde{y}(t)$ and $\tilde{\tilde{y}}(t)$ whose components fulfill $s_j\tilde{y}_j = s_i\tilde{\tilde{y}}_i$. The components of u (y) corresponding to the $+1$ entries in S are called inputs (outputs) with *even parity* and those corresponding to the -1 entries are called inputs (outputs) with *odd parity*.

For further characterizations of reciprocity in terms of adjoints and time reversals of linear systems, we refer to [9].

Theorem 4 ([6])**.** *A square system* (1) *is reciprocal with respect to the signature matrix* S_{ext} *if and only if its transfer function* $\mathbf{G}(s)$ *is sign symmetric with respect to* S_{ext}*, i.e., the transfer function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$ *fulfills* $S_{\mathrm{ext}}\mathbf{G}(s) = \mathbf{G}(s)^\top S_{\mathrm{ext}}$.

Having a minimal realization $[A,B,C,D]$ of $\mathbf{G}(s)$, it can be easily verified from $S_{\mathrm{ext}}\mathbf{G}(s) = \mathbf{G}(s)^\top S_{\mathrm{ext}}$ that $[A^\top, C^\top S_{\mathrm{ext}}, S_{\mathrm{ext}}B^\top, S_{\mathrm{ext}}D^\top S_{\mathrm{ext}}]$ is an alternative minimal realization of $\mathbf{G}(s)$. Lemma 1 then implies the existence of $Q \in \mathbb{R}^{n,n}$, such that

$$AQ = QA^\top, \qquad B = QC^\top S_{\mathrm{ext}}, \qquad CQ = S_{\mathrm{ext}}B^\top. \tag{7}$$

Performing a transpose of the above equations, we see that (7) is also fulfilled if Q is replaced by $Q^\top$. The uniqueness of the solution of (7) then implies the symmetry of Q. This leads us to the equivalence of (7) to the slightly simpler equations

$$AQ = QA^\top, \qquad B = QC^\top S_{\mathrm{ext}}, \quad Q = Q^\top. \tag{8}$$

Of particular interest are realizations in which $Q = S$ for some signature matrix S. These systems are called *internally reciprocal* and S is called *internal signature matrix*. In the case where the diagonal elements of the internal signature matrix are ordered, the matrices A, B, C and D are structured as in (3).

An internally reciprocal realization of $\mathbf{G}(s)$ can be constructed from a minimal realization $[A,B,C,D]$ of a sign symmetric transfer function via Algorithm 2. The second step of Algorithm 2 consists

Algorithm 2 Construction of an internally reciprocal realization

Given a minimal realization $[A,B,C,D]$ of the transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$ that is sign symmetric with respect to the signature matrix S_{ext}, compute an internally reciprocal realization $[A_r,B_r,C_r,D]$ of $\mathbf{G}(s)$.

1. Solve equation (8) for some $Q \in \mathbb{R}^{n,n}$.
2. Compute $T \in \mathbb{R}^{n,n}$ such that $Q = TST^\top$ for some signature matrix $S \in \mathbb{R}^{n,n}$.
3. Define $A_r = T^{-1}AT$, $B_r = T^{-1}B$ and $C_r = CT$.

of an application of Sylvester's law of inertia. It is straightforward to verify that $A_r^\top S = SA_r$ and $SB_r = C_r^\top S_{\mathrm{ext}}$, i.e., the realization $[A_r,B_r,C_r,D]$ is internally reciprocal. As a consequence, we have that any transfer function $\mathbf{G}(s)$ which is symmetric with respect to some external signature matrix S has a minimal realization which is internally reciprocal.

5. Positive real balanced realizations

Definition 5. A dynamical system (1) is called *positive real balanced* if the minimal solutions of (4) and (6) satisfy $X_{\min} = Y_{\min} = \Sigma = \mathrm{diag}(\sigma_1 I_{n_1},\ldots,\sigma_k I_{n_k})$ with $\sigma_1 > \cdots > \sigma_k > 0$. The numbers σ_j are called the *passivity characteristic values* and the numbers n_j are the respective *multiplicities*.

It can be seen that, by a state space transformation $[T^{-1}AT, T^{-1}B, CT, D]$, the minimal solutions $X_{\min}$, $Y_{\min}$ of (4) and (6) transform to $T^\top X_{\min}T$, $T^{-1}Y_{\min}T^{-\top}$. By $(T^\top X_{\min})(T^{-1}Y_{\min}T^{-T}) = T^\top(X_{\min}Y_{\min})(T^\top)^{-1}$, we see that the spectrum of $X_{\min}Y_{\min}$ is invariant with respect to state space transformations. In particular, the squares of passivity characteristic values are the eigenvalues of $X_{\min}Y_{\min}$. Hence, passivity characteristic values as well as their respective multiplicities are *input–output-invariants* of the system, i.e., they do not depend on the particular realization of a given positive real $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$.

To constructively obtain a positive real balanced realization from another realization of a positive real transfer function, we can apply *square-root balancing*. In order to see that the realization

Algorithm 3 Square-root balancing [4]

Given a minimal realization $[A,B,C,D]$ of the positive real transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$, compute a balanced realization $[A_b,B_b,C_b,D]$ of $\mathbf{G}(s)$.

1. Solve the LMIs (4) and (6) for minimal solutions $X_{\min}$ and $Y_{\min}$.
2. Compute matrices $L, R \in \mathbb{R}^{n,n}$ with $X_{\min} = L^\top L$ and $Y_{\min} = R^\top R$.
3. Perform a singular value decomposition
$$LR^\top = U\Sigma V^\top \tag{9}$$
for some orthogonal matrices $U, V \in \mathbb{R}^{n,n}$ and
$$\Sigma = \mathrm{diag}(\sigma_1 I_{n_1},\ldots,\sigma_k I_{n_k})$$
with decreasing and disjoint numbers $\sigma_1,\ldots,\sigma_k$.
4. For $T = R^\top V\Sigma^{-1/2}$, define $A_b = T^{-1}AT$, $B_b = T^{-1}B$ and $C_b = CT$.

$[A_b,B_b,C_b,D]$ constructed by Algorithm 3 is really positive real balanced, we make use of the relation $T^{-1} = \Sigma^{-1/2}U^\top L$ and

$$\begin{aligned} T^\top X_{\min} T &= T^\top L^\top L T \\ &= \Sigma^{-1/2} V^\top R L^\top L R^\top V \Sigma^{-1/2} \\ &= \Sigma^{-1/2} V^\top V \Sigma U^\top U \Sigma V^\top V \Sigma^{-1/2} \\ &= \Sigma, \\ T^{-1} Y_{\min} T^{-\top} &= T^{-1} R^\top R T^{-\top} \\ &= \Sigma^{-1/2} U^\top L R^\top R L^\top U \Sigma^{-1/2} \\ &= \Sigma^{-1/2} U^\top U \Sigma V^\top V \Sigma U^\top U \Sigma^{-1/2} \\ &= \Sigma. \end{aligned}$$

The most popular application of balanced realizations is in model order reduction [4].

Definition 6. Let $[A, B, C, D]$ be a positive real balanced realization. Let $\sigma_1 > \cdots > \sigma_k > 0$ be the passivity characteristic values, n_j be the respective multiplicities. Let $\ell < k$, $r = \sum_{j=1}^{\ell} n_j$ and consider the partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [C_1\ C_2]$$

with $A_{11} \in \mathbb{R}^{r,r}$, $B_1 \in \mathbb{R}^{r,m}$, $C_1 \in \mathbb{R}^{m,r}$. Then $[A_{11}, B_1, C_1, D]$ is called a positive real truncated balanced realization.

Passivity of the realization $[A_{11}, B_1, C_1, D]$ directly follows from $\Sigma_1 = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_\ell I_{n_\ell})$ being the minimal solution of the LMI (4) corresponding to the truncated model.

Taking a closer look to the energetic interpretation (5) of the minimal solutions, balanced truncation means an elimination of states $x \in \mathbb{R}^n$ that have the following two properties:

- a large amount of energy is required to steer to x;
- only a small amount of energy can be extracted from the system which is initialized with x.

One can readily infer that states with these two properties do not contribute significantly to the input–output behavior of the system and can therefore be eliminated from the system without significant change in the system behavior. For error bounds of positive real balanced truncation, we refer to [10] and its bibliography.

6. Passive and reciprocal realizations via positive real balancing

Each of the constructions of an internally passive and an internally reciprocal realization can be simply performed by Algorithms 1 and 2, respectively. However, neither of these procedures produces in general a realization that is jointly internally passive and reciprocal. We now show that realizations with both these properties can be constructed by positive real balancing. More precisely, we prove the existence of realizations which are positive real balanced, internally passive and internally reciprocal. By further characterizing how two positive real balanced realizations are related, we derive a constructive way to obtain realizations with the desired properties.

First we present two auxiliary results required for the proof of the main theorems.

Lemma 5. *Let $M, R \in \mathbb{R}^{n,n}$ be symmetric with $R > 0$ and $MR + RM \le 0$. Then $M \le 0$.*

Proof. Let $v \in \mathbb{R}^n \setminus \{0\}$ be such that $Mv = \lambda v$. Then $v^\top (MR + RM) v \le 0$ and $v^\top (MR + RM) v = 2\lambda v^\top R v$. Hence, for all eigenvalues of M, there holds $\lambda \le 0$ which implies $M \le 0$. □

Lemma 6. *Let $[A_1, B_1, C_1, D_1]$, $[A_2, B_2, C_2, D_2]$ be two minimal and positive real balanced realizations of the positive real transfer function $\mathbf{G}(s)$. Let $\sigma_1 > \cdots > \sigma_k > 0$ be the passivity characteristic values and let n_j be the respective multiplicities. Then $D_1 = D_2$ and there exist orthogonal matrices $U_j \in \mathbb{R}^{n_j, n_j}$ for $j = 1, \ldots, k$ such that for $U = \operatorname{diag}(U_1, \ldots, U_k)$, there holds $A_1 U = U A_2$, $B_1 = U B_2$ and $C_2 = C_1 U$.*

Proof. Assume that for some $U \in \mathbf{G}\ell_n(\mathbb{R})$ holds $[A_2, B_2, C_2, D_2] = [U^{-1} A_1 T, U^{-1} B_1, C_1 U, D_1]$ and $\Sigma = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$. Since the minimal solutions of the LMIs (4) and (6) corresponding to these two systems are related by $X_{\min,2} = U^\top X_{\min,1} U$, $Y_{\min,2} = U^{-1} Y_{\min,1} U^{-\top}$, the assumption that both systems are positive real balanced leads to $U^\top \Sigma U = \Sigma$, $U^{-1} \Sigma U^{-\top} = \Sigma$. Thus, $U\Sigma = \Sigma U$, and partitioning $U = (U_{ij})_{i,j=1,\ldots,k}$ for $U_{ij} \in \mathbb{R}^{n_i, n_j}$ gives rise to

$$\sigma_i U_{ij} = \sigma_j U_{ij} \quad \forall i, j \in \{1, \ldots, k\}.$$

Since $\sigma_1, \ldots, \sigma_k$ are distinct, we have $U_{ij} = 0$ whenever $i \ne j$. The orthogonality of the matrices on the block-diagonal is now a consequence of $U^\top \Sigma U = \Sigma$. □

Theorem 7. *Assume that the dynamical system (1) is positive real balanced. Then (1) is internally passive.*

Proof. Since (1) is positive real balanced, we have

$$\begin{bmatrix} A^\top \Sigma + \Sigma A & \Sigma B - C^\top \\ B^\top \Sigma - C & -D - D^\top \end{bmatrix} \le 0, \tag{10}$$

$$\begin{bmatrix} A\Sigma + \Sigma A^\top & \Sigma C^\top - B \\ C\Sigma - B^\top & -D - D^\top \end{bmatrix} \le 0.$$

The latter linear matrix inequality is equivalent to

$$\begin{bmatrix} A\Sigma + \Sigma A^\top & -\Sigma C^\top + B \\ -C\Sigma + B^\top & -D - D^\top \end{bmatrix} \le 0. \tag{11}$$

Taking the sum of (10) and (11), we obtain

$$\begin{aligned} 0 &\ge \begin{bmatrix} (A + A^\top)\Sigma + \Sigma(A + A^\top) & \Sigma(B - C^\top) + (B - C^\top) \\ (B^\top - C)\Sigma + (B^\top - C) & -2(D + D^\top) \end{bmatrix} \\ &= \begin{bmatrix} A^\top + A & B - C^\top \\ B^\top - C & -D - D^\top \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \\ &\quad + \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^\top + A & B - C^\top \\ B^\top - C & -D - D^\top \end{bmatrix}. \end{aligned}$$

Then Lemma 5 implies that

$$\begin{bmatrix} A^\top + A & B - C^\top \\ B^\top - C & -D - D^\top \end{bmatrix} \le 0. \quad \square$$

Theorem 8. *Let a passive and reciprocal dynamical system (1) with transfer function $\mathbf{G}(s)$ be given. Then there exists a positive real balanced realization of $\mathbf{G}(s)$ which is internally symmetric.*

Proof. Since Algorithm 2 produces an internally reciprocal realization, we may assume to have a minimal realization $[A, B, C, D]$ with the property that for some signature matrix S holds $SA^\top = AS$, $SC^\top = BS_{\text{ext}}$ and $S_{\text{ext}} D^\top = DS_{\text{ext}}$. We are step-by-step using Algorithm 3 to construct a balanced realization that is internally reciprocal. Internal reciprocity implies that for $Y = SXS$, there holds

$$\begin{aligned} 0 &\ge \begin{bmatrix} S & 0 \\ 0 & S_{\text{ext}} \end{bmatrix} \begin{bmatrix} A^\top X + XA & XB - C^\top \\ B^\top X - C & -D - D^\top \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S_{\text{ext}} \end{bmatrix} \\ &= \begin{bmatrix} AY + YA^\top & YC^\top - B \\ CY - B & -D - D^\top \end{bmatrix}. \end{aligned}$$

Hence, the minimal solutions of (4) and (6) are related by $X_{\min} = SY_{\min}S$ and, consequently, $L \in \mathbb{R}^{n,n}$ satisfies $X_{\min} = L^\top L$ if and only if $R = LS$ fulfills $Y_{\min} = R^\top R$. Since $R = LS$, the singular value decomposition (9) of the symmetric matrix $LR^\top = LSL^\top$ has to be performed. However, symmetry implies that an eigendecomposition $LSL^\top = U\Lambda U^\top$ with some orthogonal matrix $U \in \mathbb{R}^{n,n}$ and a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $|\lambda_1| \ge \cdots \ge |\lambda_n|$ is related to a singular value decomposition (9) via $\Sigma = |\Lambda|$ and

$V = U\mathrm{sign}(\Lambda)$, where $|\Lambda|$ and $\mathrm{sign}(\Lambda)$ denote the entry-wise modulus and, respectively, the sign function of Λ. Knowing that the state space transformation with

$$T = R^\top V\Sigma^{-1/2} = SL^\top U\mathrm{sign}(\Lambda)|\Lambda|^{-1/2}$$

yields a positive real balanced realization $[A_b, B_b, C_b, D]$ with $A_b = T^{-1}AT$, $B_b = T^{-1}B$, $C_b = CT$, we now show that $\mathrm{sign}(\Lambda)$ is a signature matrix. Taking into account that $T^{-1} = |\Lambda|^{-1/2}U^\top L$, we compute

$$\begin{aligned}\mathrm{sign}(\Lambda)A_b &= \mathrm{sign}(\Lambda)|\Lambda|^{-1/2}U^\top LASL^\top U\mathrm{sign}(\Lambda)|\Lambda|^{-1/2}\\ &= \mathrm{sign}(\Lambda)|\Lambda|^{-1/2}U^\top LSA^\top L^\top U\mathrm{sign}(\Lambda)|\Lambda|^{-1/2}\\ &= |\Lambda|^{-1/2}\mathrm{sign}(\Lambda)U^\top LSA^\top L^\top U|\Lambda|^{-1/2}\mathrm{sign}(\Lambda)\\ &= T^\top A^\top T^{-\top}\mathrm{sign}(\Lambda) = A_b^\top\mathrm{sign}(\Lambda),\\ \mathrm{sign}(\Lambda)B_b &= \mathrm{sign}(\Lambda)|\Lambda|^{-1/2}U^\top LB\\ &= |\Lambda|^{-1/2}\mathrm{sign}(\Lambda)U^\top LSC^\top S_{\mathrm{ext}}\\ &= T^\top C^\top S_{\mathrm{ext}}\\ &= C_b^\top S_{\mathrm{ext}}. \quad\square\end{aligned}$$

As a direct conclusion of Lemma 6, Theorems 7 and 8, we can formulate the following result.

Corollary 9. *Let a reciprocal dynamical system (1) be given that is positive real balanced with characteristic values $\sigma_1, \sigma_2, \dots, \sigma_k$ and multiplicities $n_1, n_2, \dots, n_k$. Then there exist orthogonal matrices $U_j \in \mathbb{R}^{n_j,n_j}$ for $j = 1, \dots, k$ such that for $U = \mathrm{diag}(U_1, \dots, U_k)$, the realization $[U^{-1}AU, U^{-1}B, CU, D]$ is internally reciprocal and internally passive.*

To make the above result more constructive, we observe that for a positive real balanced realization, reciprocity of the system implies that there exists some $Q \in \mathbb{R}^{n,n}$ such that (8) is fulfilled. On the other hand, a closer look to the LMIs (4) and (6) yields that the realization

$$[Q^{-1}AQ, Q^{-1}B, CQ, D] = [A^\top, C^\top S_{\mathrm{ext}}, S_{\mathrm{ext}}B^\top, D]$$

is balanced as well. Lemma 6 then implies that $Q = \mathrm{diag}(Q_1, \dots, Q_k)$ for some symmetric and orthogonal $Q_j \in \mathbb{R}^{n_j,n_j}$. From the joint symmetry and orthogonality of Q_j we get the existence of an orthogonal matrix $T_j \in \mathbb{R}^{n_j,n_j}$ such that $Q_j = T_jS_jT_j^\top$ for some signature matrix $S_j \in \mathbb{R}^{n_j,n_j}$. A transformation with $T = \mathrm{diag}(T_1, \dots, T_k)$ then finally leads to an internally reciprocal balanced realization. This approach is summarized in Algorithm 4.

Algorithm 4 Construction of an internally passive and internally reciprocal balanced realization

Given a minimal realization $[A, B, C, D]$ of the positive real transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$, compute a realization $[A_{br}, B_{br}, C_{br}, D]$ of $\mathbf{G}(s)$ that is internally reciprocal and internally passive.

1. Run Algorithm 3 to obtain a balanced realization $[A_b, B_b, C_b, D]$ of $\mathbf{G}(s)$.
2. Partition $A_b = (A_{ij})_{i,j=1,\dots,k}$, $B_b = (B_i)_{i=1,\dots,k}$, $C_b = (C_j)_{j=1,\dots,k}$ according to the multiplicities of the passivity characteristic values and, for $j = 1, \dots, k$, solve the equations
$$A_{jj}Q_j = Q_jA_{jj}^\top, \qquad B_jS_{\mathrm{ext}} = Q_jC_j^\top \tag{12}$$
for some symmetric $Q_j \in \mathbb{R}^{n_j,n_j}$.
3. For $j = 1, \dots, k$, compute the eigenvalue decomposition $Q_j = T_jS_jT_j^\top$ for some signature matrix $S_j \in \mathbb{R}^{n_j,n_j}$.
4. For $T = \mathrm{diag}(T_1, \dots, T_k)$, define $A_{br} = T^{-1}A_bT$, $B_{br} = T^{-1}B_b$ and $C_{br} = C_bT$.

In the following, we present two conclusions of the results presented so far. In the first result, we specialize to the case where all passivity characteristic values have single multiplicity. Since, by Lemma 6, two balanced realizations of a system of this type are related by a state space transformation with a signature matrix and, on the other hand, such a transformation does not destroy internal reciprocity, we can infer that the following holds true.

Corollary 10. *Let $[A, B, C, D]$ be a positive real balanced realization of the positive real transfer function $\mathbf{G}(s) \in \mathbb{R}(s)^{m,m}$ that is sign symmetric with respect to the signature matrix S_{ext}. Moreover, assume that all passivity characteristic values have single multiplicity. Then $[A, B, C, D]$ is internally passive and externally reciprocal.*

Our second corollary concerns truncated balanced realizations. Since, by Corollary 9, a certain block-diagonal orthogonal transformation leads to an internally reciprocal realization, we can deduce that even reciprocity is not lost after positive real balanced truncation.

Corollary 11. *Let $[A_{11}, B_1, C_1, D]$ be a truncated positive real balanced realization of the positive real transfer function that is sign symmetric with respect to the signature matrix S_{ext}. Then $[A_{11}, B_1, C_1, D]$ is internally passive and externally reciprocal. Furthermore, there exists some block-diagonal orthogonal matrix $T \in \mathbb{R}^{r,r}$ such that $[T^{-1}A_{11}T, T^{-1}B_1, C_1T, D]$ is internally reciprocal.*

7. Example

Consider the transfer function

$$\mathbf{G}(s) = \frac{s^3 + 4s^2 + s + 2}{s^3 + 2s^2 + 1}.$$

We can easily construct a realization of $\mathbf{G}(s)$ in *controller form* [11, p. 288], that is

$$[A, B, C, D] = \left[\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & -2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, [1 \quad 1 \quad 2], 1\right].$$

Positive realness of $\mathbf{G}(s)$ follows, since it admits a real partial fraction decomposition

$$\mathbf{G}(s) = 1 + \frac{1}{s+1} + \frac{s}{s^2+1},$$

whereas sign symmetry of $\mathbf{G}(s)$ is a trivial consequence of the one-dimensionality of input and output.

Solving the positive real lemma equations for $X_{\min}$, $Y_{\min}$ with Matlab®, we obtain

$$X_{\min} = \begin{bmatrix} 1.3726 & 1.0000 & 0.0294\\ 1.0000 & 4.0000 & 1.0000\\ 0.0294 & 1.0000 & 1.3726\end{bmatrix},$$

$$Y_{\min} = \begin{bmatrix} 0.5429 & -0.0429 & -0.4571\\ -0.0429 & 0.5429 & -0.0429\\ -0.4571 & -0.0429 & 0.5429\end{bmatrix}.$$

Performing Cholesky factorizations $X_{\min} = L^TL$, $Y_{\min} = R^TR$ and a singular value decomposition $U\Sigma V^T = LR^T$, we obtain passivity characteristic values $\sigma_1 = 1$, $\sigma_2 = 0.1716$ with respective multiplicities $n_1 = 2$, $n_2 = 1$. Performing a state space transformation with T as in step 4 of Algorithm 3, we obtain a realization

$$[A_b, B_b, C_b, D] = \left[\begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}0.9392\\0.3435\\-1\end{bmatrix},\ [0.9392 \quad 0.3435 \quad -1], 1\right].$$

This system is internally passive but not internally reciprocal. To additionally achieve internal reciprocity, we perform step 2 of

Algorithm 4, i.e., we numerically solve (12). This gives us

$$Q_1 = \begin{bmatrix} 0.3435 & -0.9392 \\ -0.9392 & -0.3435 \end{bmatrix}, \qquad Q_2 = 1.$$

Indeed, both matrices are symmetric and orthogonal. Now performing step 3 and step 4 of Algorithm 4, i.e., eigendecompositions $Q_1 = T_1 S_1 T_1^T$, $Q_2 = T_1 S_1 T_1^T$ and another state space transformation with $T = \operatorname{diag}(T_1, T_2)$, we obtain the realization

$$[A_{br}, B_{br}, C_{br}, D] = \left[\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, [0 \quad -1 \quad -1], 1 \right].$$

It can be readily seen that this realization is reciprocal with signature matrix $S = \operatorname{diag}(1, -1, -1)$. Internal passivity simply follows from

$$\begin{bmatrix} A + A^\top & B - C^\top \\ B^\top - C & -D - D^\top \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \leq 0.$$

8. Conclusion

We have presented an alternative approach to the construction of jointly internally reciprocal and internally passive realizations of positive real and sign symmetric matrices. It is shown that positive real balanced realizations are internally passive and, furthermore, there exists some positive real balanced realization that is internally reciprocal. As a consequence, we could derive a novel method that delivers jointly internally passive and internally reciprocal realizations. Another conclusion from the presented results is that positive real balanced truncation not only preserves passivity but also reciprocity of the system.

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