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Behaviors defined by rational functions

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Abstract

In this article behaviors defined by 'differential equations' involving matrices of rational functions are introduced. Conditions in terms of controllability, observability, and stabilizability for the existence of rational representations that are prime over various subrings of the field of rational functions are derived. Elimination of latent variables, (observable) image-like representations of controllable systems, and the structure of the rational annihilators of a behavior are discussed.

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1. Introduction

It is a pleasure to contribute this article to this special issue in honor of Paul Fuhrmann on the occasion of his 70th birthday. Throughout his career issues of system representation have played a central role in his research. The aim of this paper is to combine rational representations with behaviors. This article deals with topics which lay close to Paul's heart.

In the behavioral approach, a system is viewed as a family of time trajectories, called the *behavior* of the system. Usually, a behavior is specified as the set of solutions of a system of

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differential equations. However, system equations involving integral equations (as convolutions) and transfer functions are also common. In these situations it is not always clear how the behavior is actually defined. The present article deals with representations of behaviors in terms of matrices of rational functions.

Until now, the behavioral theory of linear time-invariant differential systems has been dominated by polynomial matrix representations, and a rather complete theory, including control, $\mathscr{H}_\infty$-theory, etc. has been developed starting from such representations. Unfortunately, contrary to more conventional approaches, representations using rational functions have been neglected. In fact, the basic idea of how to define a behavior in terms of rational functions has been introduced only recently in [7] for discrete-time systems. In this paper, we deal with continuous-time systems.

A few words about the notation and nomenclature used. We use standard symbols for the sets $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, etc. $\mathbb{C}$ denotes the complex numbers and $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} | \mathrm{Re}(s) \geqslant 0\}$ the closed right half of the complex plane. We use $\mathbb{R}^\mathtt{n}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc. for vectors and matrices. When the number of rows or columns is immaterial (but finite), we use the notation $\bullet$, $\bullet \times \bullet$, etc. Of course, when we then add or multiply vectors or matrices, we assume that the dimensions are compatible. Matrices of polynomials and rational functions play an important role in this paper. Some of the properties which we use are collected in Appendix A for easy reference. $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\mathtt{n})$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^\mathtt{n}$. The notation rank, dim, rowdim, coldim, det, ker, im, degree, etc. is self-explanatory; $\mathrm{diag}(M_1, M_2, \ldots, M_\mathtt{n})$ denotes the block matrix with the matrices $M_1, M_2, \ldots, M_\mathtt{n}$ on the diagonal, and zeros elsewhere, and $\mathrm{row}(M_1, M_2, \ldots, M_\mathtt{n})$ denotes the block matrix obtained by stacking them next to each other; col is defined analogously. I denotes the identity matrix, and 0 the zero matrix. When we want to emphasize the dimension, we write $I_\mathtt{n}$ and $0_{\mathtt{n}_1\times\mathtt{n}_2}$. More notation is introduced in Appendix A.

2. Review: Polynomial representations

A *dynamical system* is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the *time-set*, $\mathbb{W}$ the *signal space*, and $\mathscr{B} \subseteq \mathbb{W}^\mathbb{T}$ the *behavior*. Hence a behavior is just a family of functions of time, mappings from $\mathbb{T}$ to $\mathbb{W}$. In this article, we deal exclusively with continuous-time systems, $\mathbb{T} = \mathbb{R}$, with a finite dimensional signal space, $\mathbb{W} = \mathbb{R}^\bullet$. Moreover, we assume throughout that our systems are (i) *linear*, meaning that $\mathscr{B}$ is a linear subspace of $(\mathbb{R}^\bullet)^\mathbb{R}$, (ii) *time-invariant*, meaning that $\mathscr{B} = \sigma^t(\mathscr{B})$ for all $t \in \mathbb{R}$, where σ^t is defined by $\sigma^t(f)(t') := f(t'+t)$, and (iii) *differential*, meaning that the behavior consists of the set of solutions of a system of differential equations. We now describe property (iii) more precisely in the linear time-invariant case.

We consider behaviors $\mathscr{B} \subseteq (\mathbb{R}^\bullet)^\mathbb{R}$ that are solution set of a system of linear constant coefficient differential equations. In other words, there exists a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $\mathscr{B}$ is the solution set of

$$\boxed{R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0.} \qquad (\mathscr{R})$$

We need to make precise when we want to call $w : \mathbb{R} \to \mathbb{R}^\bullet$ a solution of $(\mathscr{R})$. We shall deal with infinite differentiable solutions only. By considering weak solutions, we could have used locally integrable solutions, or we could also go to distributions. But this generality is no issue in this paper. Hence $(\mathscr{R})$ defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathscr{B})$ with

$$\mathscr{B} = \{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) | (\mathscr{R}) \text{ is satisfied}\}.$$

Note that we may as well denote this as $\mathscr{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, since $\mathscr{B}$ is actually the kernel of the differential operator $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{coldim}(R)}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{rowdim}(R)})$.

We denote the set of linear time-invariant differential systems or their behaviors by $\mathscr{L}^\bullet$, and by $\mathscr{L}^{\mathtt{w}}$ when the number of variables is $\mathtt{w}$. Note that a behavior $\mathscr{B} \in \mathscr{L}^\bullet$ is *defined* in terms of the representation ($\mathscr{R}$), as $\mathscr{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, with R some polynomial matrix in $\mathbb{R}[\xi]^{\bullet\times\bullet}$. The analogous discrete-time system can be defined without involving a representation. Indeed, $\mathscr{B} \subseteq (\mathbb{R}^\bullet)^{\mathbb{Z}}$ linear, shift-invariant, and closed in the topology of pointwise convergence *implies* the existence of an $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $\mathscr{B} = \ker(R(\sigma))$. Hence, in this case, the representation as the kernel of a difference operator can be deduced from properties of the behavior. Unfortunately, we know of no simple continuous-time analogue of this result (see [6, p. 279] for some remarks concerning this point, and [3] for a recent paper dealing with this matter).

3. Rational representations

The aim of this article is to discuss representations of $\mathscr{L}^\bullet$, more general than by differential equations, namely representations by means of matrices of rational functions. These play a very important role in the field, in the context of robust stability, system topologies, the parametrization of all stabilizable controllers, model reduction, etc.

Let $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and consider the system of 'differential equations'

$$\boxed{G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0.} \tag{$\mathscr{G}$}$$

Since G is a matrix of rational functions, it is not clear when $w : \mathbb{R} \to \mathbb{R}^\bullet$ is a solution of equation ($\mathscr{G}$). This is not a question of smoothness, but a matter of giving a meaning to the equality, since $G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is not a differential operator (and not even a map). We do this as follows (see Appendix A for the nomenclature used).

Definition 1. Let (P, Q) be a left coprime matrix factorization over $\mathbb{R}[\xi]$ of $G = P^{-1}Q$. Define

$$[\![w : \mathbb{R} \to \mathbb{R}^\bullet \text{ is a solution of } (\mathscr{G})]\!] :\Leftrightarrow \left[\!\!\left[Q\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)w = 0\right]\!\!\right].$$

Whence ($\mathscr{G}$) defines the system $\Sigma = \left(\mathbb{R}, \mathbb{R}^\bullet, \ker\left(Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)\right) \in \mathscr{L}^\bullet$.

In this definition, it is left implicit when $w : \mathbb{R} \to \mathbb{R}^\bullet$ is considered to be a solution of $Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$. As mentioned before, in the present paper, we assume, for simplicity of exposition, that $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$.

There are some immediate consequences, comments and caveats which need to be made regarding this definition. Note, first of all, that using the above definition, it now makes sense to ask if for a given $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ and a given $w_1 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$, $w_2 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ satisfies $w_2 = G(\frac{\mathrm{d}}{\mathrm{d}t})w_1$. View this as a special case of ($\mathscr{G}$), by writing it as $\left[I \;\; -G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right]\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = 0$. A left coprime factorization over $\mathbb{R}[\xi]$ of $G = P^{-1}Q$ yields a left coprime matrix factorization over $\mathbb{R}[\xi]$ of $[I \;\; -G] = P^{-1}\left[I \;\; -P^{-1}Q\right]$. Hence (w_1, w_2) is a solution of $w_2 = G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_1$ iff $P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_2 = Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_1$.

It follows from this that $G\left(\frac{d}{dt}\right)$ is *not* a map on $\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$. Rather, $w_1 \mapsto G\left(\frac{d}{dt}\right)w_1$ is the point-to-set map that associates with $w_1 \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$, the set $w_2' + v$, with $w_2' \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$ a particular solution of $P\left(\frac{d}{dt}\right)w_2 = Q\left(\frac{d}{dt}\right)w_1$, and $v \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$ any function that satisfies $P\left(\frac{d}{dt}\right)v = 0$. This is a finite dimensional linear subspace of $\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$ of dimension equal to the degree of $\det(P)$. Hence, if P is not unimodular, equivalently, if G is not a polynomial matrix, $G\left(\frac{d}{dt}\right)$ is not a point-to-point map. In particular, $G\left(\frac{d}{dt}\right)0 = \ker\left(P\left(\frac{d}{dt}\right)\right)$. More generally, for any $w_1 \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$, $G\left(\frac{d}{dt}\right)w_1$ is the residue class $w_2' + \ker\left(P\left(\frac{d}{dt}\right)\right)$, with w_2' any particular solution of $P\left(\frac{d}{dt}\right)w_2 = Q\left(\frac{d}{dt}\right)w_1$.

Viewing $G\left(\frac{d}{dt}\right)$ as a point-to set map leads to the definition of its kernel as

$$\ker\left(G\left(\frac{d}{dt}\right)\right) := \left\{w \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet}) \,\middle|\, 0 \in G\left(\frac{d}{dt}\right)w\right\},$$

i.e. $\ker\left(G\left(\frac{d}{dt}\right)\right)$ consists of the set of solutions of ($\mathscr{G}$), and of its image as

$$\operatorname{im}\left(G\left(\frac{d}{dt}\right)\right) := \left\{w_2 \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet}) \,\middle|\, w_2 \in G\left(\frac{d}{dt}\right)w_1,\ \text{for some } w_1 \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})\right\}.$$

Whence equation ($\mathscr{G}$) defines the system $\Sigma = \left(\mathbb{R},\mathbb{R}^{\bullet},\ker\left(G\left(\frac{d}{dt}\right)\right)\right) := \left(\mathbb{R},\mathbb{R}^{\bullet},\ker\left(Q\left(\frac{d}{dt}\right)\right)\right) \in \mathscr{L}^{\bullet}$.

For $G \in \mathbb{R}(\xi)$, $G = \frac{q}{p}$, with $p, q \in \mathbb{R}[\xi]$ coprime, the set of solutions of $\frac{q}{p}\left(\frac{d}{dt}\right)w = 0$ is defined to be equal to that of the differential equation $q\left(\frac{d}{dt}\right)w = 0$. In this case $\ker\left(G\left(\frac{d}{dt}\right)\right)$ is finite dimensional, with dimension equal to degree(q). Our interest is mainly in the case G 'wide': more columns than rows. For example, the behavior of $\frac{q_1}{p_1}\left(\frac{d}{dt}\right)w_1 + \frac{q_2}{p_2}\left(\frac{d}{dt}\right)w_2 = 0$, with $p_1, q_1 \in \mathbb{R}[\xi]$ and $p_2, q_2 \in \mathbb{R}[\xi]$ both coprime, is equal to the set of solutions of $p_2'q_1\left(\frac{d}{dt}\right)w_1 + p_1'q_2\left(\frac{d}{dt}\right)w_2 = 0$, where $p_1 = dp_1'$, $p_2 = dp_2'$, $d \in \mathbb{R}[\xi]$, and p_1', p_2' coprime, and d a greatest common divisor of p_1 and p_2. This implies that, because of common factors, the behavior of $G_1\left(\frac{d}{dt}\right)w_1 = G_2\left(\frac{d}{dt}\right)w_2$ with $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and with $\det(G_1) \neq 0$, is not necessarily equal to the behavior of $w_1 = (G_1^{-1}G_2)\left(\frac{d}{dt}\right)w_2$. Note, more generally, that $(G_1G_2)\left(\frac{d}{dt}\right)$ need not be equal to $G_1\left(\frac{d}{dt}\right)G_2\left(\frac{d}{dt}\right)$. Inequality holds if, for example, $G_1(\xi) = \frac{1}{\xi}$ and $G_2(\xi) = \xi$. This shows a form on non-associativity. This should not be surprising. In fact, even in classical systems theory, series connection of the systems is not 'associative' and may not 'commute'. The series connection of the system with transfer function ξ followed by the system with transfer function $\frac{1}{\xi}$ has any constant output corresponding to the zero input, while the output is necessarily zero if we take the transfer function to be the product $\xi\frac{1}{\xi}$, or if the series connection is reversed.

Since the representations ($\mathscr{R}$) are merely a subset of the representations ($\mathscr{G}$), matrices of rational functions form a representation class of $\mathscr{L}^{\bullet}$ that is more redundant, and hence richer, than the polynomial matrices. This redundancy can be used to obtain rational representations with properties that cannot be obtained using polynomial representations.

Definition 1 may evoke some scepticism, since the denominator P of the coprime factorization over $\mathbb{R}[\xi]$ of $G = P^{-1}Q$ does not enter into the specification of the solution set, other than through the coprimeness requirement on P, Q. We now mention other views which support Definition 1.

1. Decompose G as $G = R + F$ with $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ and $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ strictly proper. Let $F(s) = C(Is - A)^{-1}B$ be a state controllable (in the usual sense) realization of F. Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw, \quad 0 = R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + Cx. \tag{LS}$$

This defines a set of w-trajectories $w : \mathbb{R} \to \mathbb{R}^{\bullet}$. It equals $\ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. More precisely, the w-behavior of (LS), i.e.

$$\{w \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\bullet}) \mid \exists x \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\bullet}) \text{ such that (LS) holds}\}$$

is equal to $\ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. This may be seen as follows. Let (F_1, F_2) be a left coprime factorization over $\mathbb{R}[\xi]$ of $F = F_1^{-1}F_2$. From the state space theory of systems, it is well known that $(w, y) \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\bullet})$ satisfies $F_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = F_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w$ iff there exists $x \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\bullet})$ such that $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw$, $y = Cx$. Now, there exists a y such that $y + R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ and $F_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = F_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w$ iff $F_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + F_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$. Therefore, (LS) yields the w's that satisfy $F_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + F_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$. Equivalently, those that satisfy $G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$, since $(F_1, F_2 + F_1R)$ is a left coprime factorization over $\mathbb{R}[\xi]$ of $G = F_1^{-1}F_2 + R$.

2. Consider the (unique) controllable input/output system $w \mapsto y$ with transfer function G. Now consider its *zero dynamics*, i.e. the 'inputs' corresponding to 'output' $y = 0$. This set of 'inputs' equals $\ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$.

3. It is tempting to interpret ($\mathscr{G}$) in terms of Laplace transforms. However, this is not particularly enlightening, since, as is well-known, Laplace transforms are an awkward implementation of symbolic calculus, which inspired our definition of $\ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Laplace transforms methods need to add restrictions to the growth of the functions considered, and worry about one-sidedness and domains of convergence. No such issues occur in our definition. The connection of our definition with what one would obtain using Laplace transforms can be explained as follows. Consider the system with transfer function G. View it as mapping taking the one-sided Laplace transformable inputs with bounded support on the left, into one-sided Laplace transformable outputs also with bounded support on the left, by $\hat{u}(s) \mapsto \hat{y}(s) = G(s)\hat{u}(s)$. This yields a family of input–output pairs. Now consider the outputs that are zero on the half-line $[0, \infty)$. Denote the corresponding inputs by $\mathscr{B}' \subseteq \mathscr{C}^{\infty}([0, \infty), \mathbb{R}^{\bullet})$. It turns out that the smallest linear time-invariant differential system that contains these inputs is precisely $\ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$. Note that this characterization does little more than what was explained in point 1 and 2 above.

That $\mathscr{B} \in \mathscr{L}^{\bullet}$ admits a representation as the kernel of a polynomial matrix in $\frac{\mathrm{d}}{\mathrm{d}t}$ is a matter of definition. However, representations using the ring of proper (stable) rational functions (see Appendix A for definitions and notation) play a very important role in control theoretic applications. We state this representability in the next proposition.

Proposition 2. *Let $\mathscr{B} \in \mathscr{L}^{\bullet}$. There exists $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ such that $\mathscr{B} = \ker\left(G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$.*

Proof. Assume that $\mathscr{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ with $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ of full row rank (such a representation always exists—see [4, Theorem 2.5.25]). Let $\lambda \in \mathbb{R}$, $\lambda > 0$ be such that $\text{rank}(R(-\lambda)) = \text{rank}(R)$ and let $\mathtt{n} \in \mathbb{N}$ be such that $\frac{R(\xi)}{(\xi+\lambda)^{\mathtt{n}}}$ is proper. Now take $G(\xi) = \frac{R(\xi)}{(\xi+\lambda)^{\mathtt{n}}}$. The factorization $G = P^{-1}R$ with $P(\xi) = (\xi+\lambda)^{\mathtt{n}} I_{\text{rowdim}(R)}$ is left coprime over $\mathbb{R}[\xi]$. Hence $\mathscr{B} = \ker\left(R\left(\frac{d}{dt}\right)\right) = \ker\left(G\left(\frac{d}{dt}\right)\right)$. □

Obviously, this proposition is readily generalized to any 'stability' domain $\subset \mathbb{C}$ that is symmetric with respect to the real axis and is not contained in the set of zeros of $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ with R of full row rank and such that $\mathscr{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$. These zeros actually correspond to the uncontrollable modes of $\mathscr{B}$. This possibility of refining the stability domain also holds for many results further in the paper, in particular for Theorem 5.

4. Controllability and stabilizability

In this section, we relate controllability and stabilizability of a system to properties of their rational representations. We first recall the behavioral definitions of these notions.

Definition 3. The time-invariant system $\Sigma = (\mathbb{R}, \mathbb{R}^{\bullet}, \mathscr{B})$ is said to be *controllable* if for all $w_1, w_2 \in \mathscr{B}$, there exists $T \geqslant 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geqslant T$ (see Fig. 1).

It is said to be *stabilizable* if for all $w \in \mathscr{B}$, there exists $w' \in \mathscr{B}$, such that $w'(t) = w(t)$ for $t < 0$ and $w'(t) \to 0$ for $t \to \infty$ (see Fig. 2).

Observe that for $\mathscr{B} \in \mathscr{L}^{\bullet}$, controllability implies stabilizability. Denote the controllable elements of $\mathscr{L}^{\bullet}$ by $\mathscr{L}^{\bullet}_{\text{contr}}$, and of $\mathscr{L}^{\mathtt{w}}$ by $\mathscr{L}^{\mathtt{w}}_{\text{contr}}$, and the stabilizable elements of $\mathscr{L}^{\bullet}$ by $\mathscr{L}^{\bullet}_{\text{stab}}$, and of $\mathscr{L}^{\mathtt{w}}$ by $\mathscr{L}^{\mathtt{w}}_{\text{stab}}$. It is easy to derive tests for controllability and stabilizability in terms of kernel representations.

Proposition 4. *($\mathscr{G}$) defines a controllable system iff G has no zeros, and a stabilizable one iff G has no zeros in $\overline{\mathbb{C}}_+$.*

Proof. Factor G in terms of the Smith–McMillan form (see Appendix A for the notation) as $G = (\Pi U^{-1})^{-1} Z V$. By the definition of $\ker\left(G\left(\frac{d}{dt}\right)\right)$, $\ker\left(G\left(\frac{d}{dt}\right)\right) = \ker\left(ZV\left(\frac{d}{dt}\right)\right)$. The

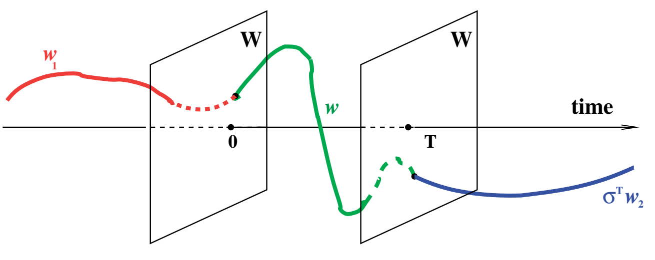

Fig. 1. Controllability.

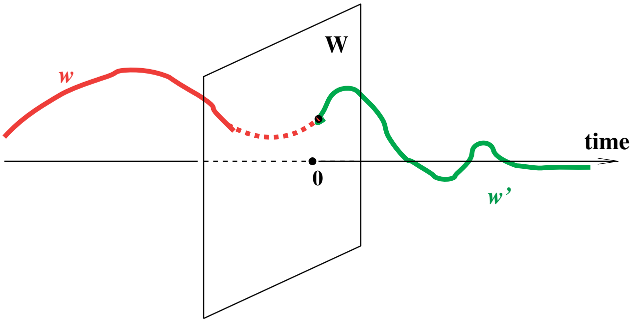

Fig. 2. Stabilizability.

system $ZV\left(\frac{d}{dt}\right)w = 0$ is known to be controllable iff all the ζ_k's are equal to 1 [4, Theorem 5.2.10], and stabilizable iff all the ζ_k's are Hurwitz [4, Theorem 5.2.30]. □

The following result links controllability and stabilizability of systems in $\mathscr{L}^\bullet$ to the existence of left prime representations over various rings.

Theorem 5
1. *$\mathscr{B} \in \mathscr{L}^\bullet$ is controllable iff it admits a representation ($\mathscr{R}$) with $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ left prime over $\mathbb{R}[\xi]$.*
2. *$\mathscr{B} \in \mathscr{L}^\bullet$ iff it admits a representation ($\mathscr{G}$) with G left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$.*
3. *$\mathscr{B} \in \mathscr{L}^\bullet$ is stabilizable iff it admits a representation ($\mathscr{G}$) with $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.*

Proof. (1) Each $\mathscr{B} \in \mathscr{L}^\bullet$ admits a representation ($\mathscr{R}$) with R of full row rank [4, Theorem 2.5.25]. This representation is controllable iff $R(\lambda) \in \mathbb{C}^{\bullet\times\bullet}$ has full row rank for all $\lambda \in \mathbb{C}$ (see [4, Theorem 5.2.10]), equivalently, iff R is left prime over $\mathbb{R}[\xi]$.

(2) 'if': by definition. The proof of the 'only if' part is analogous to the proof of the 'only if' part of (3). Just take $\mathbb{S}$ in that proof such that it does not contain any zeros of R.

(3) 'if': G left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$ implies that it has no zeros in $\overline{\mathbb{C}}_+$. Now apply Proposition 4.

(3) 'only if': the proof of this part is a little more involved. As a preamble to the general case, assume first that $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is described by a scalar equation

$$r_1\left(\frac{d}{dt}\right)w_1 + r_2\left(\frac{d}{dt}\right)w_2 + \cdots + r_{\mathtt{w}}\left(\frac{d}{dt}\right)w_{\mathtt{w}} = 0$$

with $r_1, r_2, \ldots, r_{\mathtt{w}} \in \mathbb{R}[\xi]$. Since $\mathscr{B}$ is stabilizable, $r_1, r_2, \ldots, r_{\mathtt{w}}$ have no common roots in $\overline{\mathbb{C}}_+$. Take $p \in \mathbb{R}[\xi]$ Hurwitz, left coprime with $[r_1\ \ r_2\ \ \cdots\ \ r_{\mathtt{w}}]$, and with

$$\text{degree}(p) = \max(\{\text{degree}(r_1), \text{degree}(r_2), \ldots, \text{degree}(r_{\mathtt{w}})\}).$$

Then

$$\frac{r_1}{p}\left(\frac{d}{dt}\right)w_1 + \frac{r_2}{p}\left(\frac{d}{dt}\right)w_2 + \cdots + \frac{r_{\mathtt{w}}}{p}\left(\frac{d}{dt}\right)w_{\mathtt{w}} = 0$$

is a representation of $\mathscr{B}$ that is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.

In order to prove the general case, we first establish the following lemma.

Lemma 6. *Consider* $F \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ *with* $\det(F) \neq 0$. *Let* $\mathbb{S} \subset \mathbb{C}$ *have a non-empty intersection with the real axis. There exists* $P \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ *such that*

1. $\det(P) \neq 0$,
2. $\det(P)$ *has all its roots in* $\mathbb{S}$,
3. $P^{-1}F \in \mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{n}}$ *is bi-proper.*

Proof. The proof goes by induction on $\mathtt{n}$. The case $\mathtt{n} = 1$ is straightforward. Assume that $\mathtt{n} \geqslant 2$. Note that by taking $(F, P) \mapsto (UF, UP)$, we can depart from a suitable form for F obtained by pre-multiplying by a $U \in \mathscr{U}_{\mathbb{R}[\xi]}$. Assume therefore (e.g. the Hermite form) that F is of the form

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

with F_{11} and F_{22} square of dimension $< \mathtt{n}$. Assume, by the induction hypothesis, that P_{11} satisfies the conditions of the lemma with respect to F_{11} and P_{22} with respect to F_{22}.

We now prove the lemma by taking P conformably,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}.$$

We will choose P_{12} such that P satisfies the conditions of the lemma with respect to F. Note that

$$P^{-1} = \begin{bmatrix} P_{11}^{-1} & -P_{11}^{-1}P_{12}P_{22}^{-1} \\ 0 & P_{22}^{-1} \end{bmatrix}.$$

Hence

$$P^{-1}F = \begin{bmatrix} P_{11}^{-1}F_{11} & P_{11}^{-1}F_{12} - P_{11}^{-1}P_{12}P_{22}^{-1}F_{22} \\ 0 & P_{22}^{-1}F_{22} \end{bmatrix}.$$

Rewrite this as

$$\begin{bmatrix} P_{11}^{-1}F_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & F_{11}^{-1}F_{12}F_{22}^{-1}P_{22} - F_{11}^{-1}P_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P_{22}^{-1}F_{22} \end{bmatrix}.$$

Let $F_{11}^{-1}F_{12}F_{22}^{-1}P_{22} = M + N$, with $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ the polynomial part and $N \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ the strictly proper part. Choose $P_{12} = F_{11}M$. Then

$$P^{-1}F = \begin{bmatrix} P_{11}^{-1}F_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & N \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P_{22}^{-1}F_{22} \end{bmatrix}.$$

$P^{-1}F$ satisfies the conditions of the lemma. □

We now return to the proof of Theorem 5. Assume that $\mathscr{B} \in \mathscr{L}^{\bullet}$ is stabilizable. Let $\mathscr{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$ be a kernel representation of it with $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ of full row rank. Whence R has no zeros in $\overline{\mathbb{C}}_{+}$. It is well-known [4, Theorem 3.3.22] that, up to a permutation of the columns, we can assume that R is of the form $R = [R_1 \;\; R_2]$, with R_1 square, $\det(R_1) \neq 0$, and $R_1^{-1}R_2$ proper. Assume, for ease of exposition, that this permutation has been carried out.

Choose $\mathbb{S} \subset \mathbb{C}$ with a non-empty intersection with the real axis, with $\mathbb{S} \cap \overline{\mathbb{C}}_{+} = \varnothing$, and such that it contains none of the zeros of R. Now, choose P as in Lemma 6, with R_1 playing the role of F. Then $P^{-1}R = [P^{-1}R_1 \;\; P^{-1}R_2]$. Note that $P^{-1}R_2$ is proper, since $P^{-1}R_2 = P^{-1}R_1R_1^{-1}R_2$.

Observe that

(i) $P^{-1}R$ is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$, and
(ii) $(P^{-1}R)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ is a rational representation of $\mathscr{B}$.

To prove that $P^{-1}R$ satisfies (i), note that $P^{-1}R \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ is proper, has no poles (the zeros of P) and no zeros (the zeros of R) in $\overline{\mathbb{C}}_+$, and has a bi-proper submatrix ($P^{-1}R_1$). This implies, by Proposition 17, that $P^{-1}R$ is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.

To prove that it satisfies (ii), note that P and R are left coprime, since they both have full row rank, and the $\lambda \in \mathbb{C}$ where $P(\lambda)$ drops rank are disjoint from those where $R(\lambda)$ does.

The proof of Theorem 5 is complete. □

The above theorem spells out exactly what the condition is for the existence of a kernel representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$: *stabilizability*. It is of interest to compare Theorem 5, point 3, with the classical results obtained by Vidyasagar in his book [5] (this builds on a series of earlier results, for example [2,8,1]). In these publications, the aim is to obtain a representation of a system that is given as a transfer function to start with,

$$y = F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \qquad (\mathscr{F})$$

where $F \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$. This is a special case of ($\mathscr{G}$), and, since $[I_{\mathtt{p}} \;\; -F]$ has no zeros, this system is controllable (by Proposition 4), and therefore stabilizable. Thus, by Theorem 5, it also admits a representation $G_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y = G_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u$ with $G_1, G_2 \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$, and left coprime over $\mathbb{R}(\xi)_{\mathscr{S}}$. This is an important, classical, result. However, Theorem 5 implies that, if we are in the controllable case, there exists a representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$, such that $[G_1 \;\; G_2]$ has no zeros at all.

The main difference of our result from the classical left coprime factorization results over $\mathbb{R}(\xi)_{\mathscr{S}}$ is that we faithfully preserve controllability, or, more generally, the non-controllable part, whereas in the classical approach all stabilizable systems with the same transfer function are identified. By taking a trajectory based definition, rather than a transfer function based definition, the behavioral point of view is able to carefully keep track of all trajectories, also of the non-controllable ones. Loosely speaking, left coprime factorizations over $\mathbb{R}(\xi)_{\mathscr{S}}$ manage to avoid unstable pole-zero cancellations. Our approach avoids altogether introducing common poles and zeros as well as pole-zero cancellations. Since the whole issue of coprime factorizations over the ring of stable rational functions started from a need to deal carefully with pole-zero cancellations [8], we feel that our trajectory based mode of thinking offers a useful point of view.

At this point, we can go through the whole theory of behaviors and cast the results and algorithms in the context of rational representations, or cast the theory of coprime factorizations over $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ in the behavioral setting. We give only some salient facts, with very brief proofs, concerning three further topics: elimination of latent variables, image representations, and the structure of the rational annihilators of a behavior.

5. Latent variables

Until now, we have dealt with representations involving the variables w only. However, many models, e.g. first principles models obtained by interconnection and state models, include auxiliary variables in addition to the variables the model aims at. We call the latter *manifest variables*, and the auxiliary variables *latent variables*. In the context of rational models, this leads to the model class

$$\boxed{R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell} \qquad (\mathscr{R}\mathscr{M})$$

with $R, M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. Since we have reduced the behavior of the system of 'differential equations' ($\mathscr{R}\mathscr{M}$), involving rational functions, to one involving only polynomials, the *elimination theorem* [4, Theorem 6.2.2] remains valid. Consequently, the *manifest behavior* of ($\mathscr{R}\mathscr{M}$), defined as

$$\{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid \exists \ell \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \text{ such that } (\mathscr{R}\mathscr{M}) \text{ holds}\}$$

belongs to $\mathscr{L}^\bullet$.

Definition 7. The latent variable representation ($\mathscr{R}\mathscr{M}$) is said to be *observable* if, whenever (w, ℓ_1) and (w, ℓ_2) satisfy ($\mathscr{R}\mathscr{M}$), then $\ell_1 = \ell_2$. It is said to be *detectable* if, whenever (w, ℓ_1) and (w, ℓ_2) satisfy ($\mathscr{R}\mathscr{M}$), then $\ell_1(t) - \ell_2(t) \to 0$ as $t \to \infty$.

The following proposition follows immediately from the definitions.

Proposition 8. *($\mathscr{R}\mathscr{M}$) is observable iff M has full column rank and has no zeros. It is detectable iff M has full column rank, and has no zeros in $\overline{\mathbb{C}}_+$.*

6. Image-like representations

Consider now the following special cases of ($\mathscr{R}$), ($\mathscr{G}$), and ($\mathscr{R}\mathscr{M}$):

$$\boxed{w = M\left(\frac{d}{dt}\right) \ell} \qquad (\mathscr{M})$$

with $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, and

$$\boxed{w = H\left(\frac{d}{dt}\right) \ell} \qquad (\mathscr{H})$$

with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. Of course, ($\mathscr{H}$) should be interpreted as $\left[I \ \ -H\left(\frac{d}{dt}\right)\right]\begin{bmatrix} w \\ \ell \end{bmatrix} = 0$, and so becomes a special case of ($\mathscr{G}$). Note that the manifest behavior of ($\mathscr{M}$) is the image of the differential operator $M\left(\frac{d}{dt}\right)$. This representation is hence called an *image representation* of its manifest behavior. $M\left(\frac{d}{dt}\right)$ is a point-to-point map. As explained earlier, it is appropriate, however, to call also ($\mathscr{H}$) an image representation of its manifest behavior, by viewing $H\left(\frac{d}{dt}\right)$ as a point-to-set map. In the observable case, hence if H is of full column rank and has no zeros, H has a polynomial left inverse, and ($\mathscr{H}$) defines a map from w to ℓ. The well known relation between controllability and image representations remains valid in the rational case.

Theorem 9. *The following are equivalent for $\mathscr{B} \in \mathscr{L}^\bullet$.*

1. *$\mathscr{B}$ is controllable,*
2. *$\mathscr{B}$ admits an image representation ($\mathscr{M}$),*

3. *$\mathscr{B}$ admits an observable image representation ($\mathscr{M}$),*
4. *$\mathscr{B}$ admits an image representation ($\mathscr{M}$) with $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ right prime over $\mathbb{R}[\xi]$,*
5. *$\mathscr{B}$ admits a representation ($\mathscr{H}$) with $H \in \mathbb{R}(\xi)^{\bullet\times\bullet}$,*
6. *$\mathscr{B}$ admits a representation ($\mathscr{H}$) with $H \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ right prime over $\mathbb{R}(\xi)_{\mathscr{S}}$,*
7. *$\mathscr{B}$ admits an observable representation ($\mathscr{H}$) with $H \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ right prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.*

Proof. The equivalence of (1), (2), (3), and (4) is classical (see [4]). Obviously, since (1) ⇒ (2), (1) ⇒ (5). To prove that (5) implies controllability, let (P, Q) be a left coprime factorization over $\mathbb{R}[\xi]$ of $H = P^{-1}Q$. Then $w = H\left(\frac{d}{dt}\right)\ell$ is equivalent to $P\left(\frac{d}{dt}\right)w = Q\left(\frac{d}{dt}\right)\ell$. From left coprimeness, it follows that this system, viewed with variables (w, ℓ), is controllable. But this implies, from the very definition of controllability, that the w-behavior is controllable as well. It follows that also (6) or (7) implies controllability. It remains to be proven that controllability implies the proof of the existence of an observable representation ($\mathscr{H}$) with $H \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ right prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. In order to see this, start with an image representation ($\mathscr{M}$) with M right prime over $\mathbb{R}[\xi]$, and follow the line of the proof of Theorem 5, point 3. □

7. The annihilators

In this section, we study the polynomial vectors or vectors of rational functions that annihilate an element of $\mathscr{L}^\bullet$. We shall see that the polynomial annihilators form a module over $\mathbb{R}[\xi]$, and that the rational annihilators of a controllable system form a vector space over $\mathbb{R}(\xi)$.

Obviously, for $n \in \mathbb{R}[\xi]^\bullet$ and $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$, the statements $n^\top\left(\frac{d}{dt}\right)w = 0$, and, hence, for $\mathscr{B} \in \mathscr{L}^\bullet$, $n^\top\left(\frac{d}{dt}\right)\mathscr{B} = 0$, meaning $n^\top\left(\frac{d}{dt}\right)w = 0$ for all $w \in \mathscr{B}$, are well-defined. However, since we have given a meaning to ($\mathscr{G}$), these statements are also well-defined for $n \in \mathbb{R}(\xi)^\bullet$.

Definition 10. (i) $[\![n \in \mathbb{R}[\xi]^\bullet$ is a *polynomial annihilator* of $\mathscr{B} \in \mathscr{L}^\bullet]\!] :\Leftrightarrow [\![n^\top\left(\frac{d}{dt}\right)\mathscr{B} = 0]\!]$.
(ii) $[\![n \in \mathbb{R}(\xi)^\bullet$ is a *rational annihilator* of $\mathscr{B} \in \mathscr{L}^\bullet]\!] :\Leftrightarrow [\![n^\top\left(\frac{d}{dt}\right)\mathscr{B} = 0]\!]$.

Denote the set of polynomial and of rational annihilators of $\mathscr{B} \in \mathscr{L}^\bullet$ by $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ and $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$, respectively. It is well known that for $\mathscr{B} \in \mathscr{L}^\bullet$, $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ is an $\mathbb{R}[\xi]$-module, indeed, a finitely generated one, since all $\mathbb{R}[\xi]$-submodules of $\mathbb{R}(\xi)^{\mathtt{w}}$ are finitely generated. However, also $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is an $\mathbb{R}[\xi]$-module, but a submodule of $\mathbb{R}(\xi)^{\mathtt{w}}$ viewed as an $\mathbb{R}[\xi]$ module (rather than as an $\mathbb{R}(\xi)$-vector space). These $\mathbb{R}[\xi]$-modules are not finitely generated. The elements of $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ are given by $q_1r_1 + q_2r_2 + \cdots + q_{\mathtt{n}}r_{\mathtt{n}}$ with $q_1, q_2, \ldots, q_{\mathtt{n}}$ free elements of $\mathbb{R}[\xi]$ and $R = \begin{bmatrix} r_1 & r_2 & \cdots & r_{\mathtt{n}} \end{bmatrix}^\top \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $\mathscr{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$. The elements of $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ on the other hand are given by $\frac{1}{p}(q_1r_1 + q_2r_2 + \cdots + q_{\mathtt{n}}r_{\mathtt{n}})$ with the q's and R as before, and $p \in \mathbb{R}[\xi]$ such that $\begin{bmatrix} p & (q_1r_1 + q_2r_2 + \cdots + q_{\mathtt{n}}r_{\mathtt{n}})^\top \end{bmatrix}$ is left prime. The question occurs when $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is a vector space. This has a very nice answer, given in the following theorem.

Theorem 11. *Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.*

1. *$\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ is an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{\mathtt{w}}$.*
2. *$\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is an $\mathbb{R}(\xi)$-vector subspace of $\mathbb{R}(\xi)^{\mathtt{w}}$ iff $\mathscr{B}$ is controllable.*

Proof. The first part is again classical from the theory of polynomial matrix representations.

To prove the second part, observe that $\mathscr{B}$ admits a kernel representation $\left[\Lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ 0_{\mathtt{p},\mathtt{w}-\mathtt{p}}\right] V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{p}})$, $\lambda_{\mathtt{k}} \in \mathbb{R}[\xi]$ monic, and $V \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ unimodular over $\mathbb{R}[\xi]$. Define $\tilde{\mathscr{B}} = \ker\left(\left[\Lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ 0\right]\right)$. Then $V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathscr{B} = \tilde{\mathscr{B}}$, and therefore, $V^{\top}\tilde{\mathscr{B}}^{\perp_{\mathbb{R}(\xi)}} = \mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$. This simple bijection between $\tilde{\mathscr{B}}^{\perp_{\mathbb{R}(\xi)}}$ and $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ implies that it suffices to prove the second statement of the theorem for $\tilde{\mathscr{B}}^{\perp_{\mathbb{R}(\xi)}}$ instead of for $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$. $\tilde{\mathscr{B}}^{\perp_{\mathbb{R}(\xi)}}$ is actually readily determined: it consists of all vectors of rational functions $\mathrm{col}(g_1, \ldots, g_{\mathtt{p}}, g_{\mathtt{p}+1}, \ldots, g_{\mathtt{w}})$, with $g_{\mathtt{k}} = \frac{\zeta_{\mathtt{k}}}{\pi_{\mathtt{k}}}$, $\zeta_{\mathtt{k}}, \pi_{\mathtt{k}} \in \mathbb{R}[\xi]$ coprime, and $\lambda_{\mathtt{k}}$ a factor of $\zeta_{\mathtt{k}}$ for $\mathtt{k} = \mathtt{p}+1, \ldots, \mathtt{w}$, and with $g_{\mathtt{k}} = 0$ for $\mathtt{k} = \mathtt{p}+1, \ldots, \mathtt{w}$. This is obviously an $\mathbb{R}(\xi)$ vector space iff all the $\lambda_{\mathtt{k}}$'s are equal to 1. □

The above theorem also settles the question what the relation is between the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$ and $\mathscr{L}^{\mathtt{w}}$, and between the $\mathbb{R}(\xi)$-vector subspaces of $\mathbb{R}(\xi)^{\mathtt{w}}$ and $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\text{contr}}$.

Theorem 12

1. *Denote the* $\mathbb{R}[\xi]$*-submodules of* $\mathbb{R}[\xi]^{\mathtt{w}}$ *by* $\mathfrak{M}^{\mathtt{w}}$. *There is a bijective relation between* $\mathscr{L}^{\mathtt{w}}$ *and* $\mathfrak{M}_{\mathtt{w}}$, *given by*

$$\mathscr{B} \in \mathscr{L}^{\mathtt{w}} \mapsto \mathscr{B}^{\perp_{\mathbb{R}[\xi]}} \in \mathfrak{M}^{\mathtt{w}},$$
$$\mathbb{M} \in \mathfrak{M}^{\mathtt{w}} \mapsto \left\{ w \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\top} w = 0\ \forall n \in \mathbb{M} \right\}.$$

2. *Denote the linear* $\mathbb{R}(\xi)$*-subspaces of* $\mathbb{R}(\xi)^{\mathtt{w}}$ *by* $\mathfrak{L}^{\mathtt{w}}$. *There is a bijective relation between* $\mathscr{L}^{\mathtt{w}}_{\text{contr}}$ *and* $\mathfrak{L}^{\mathtt{w}}$ *given by*

$$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\text{contr}} \mapsto \mathscr{B}^{\perp_{\mathbb{R}(\xi)}} \in \mathfrak{L}^{\mathtt{w}},$$
$$\mathbb{L} \in \mathfrak{L}^{\mathtt{w}} \mapsto \left\{ w \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\top} w = 0\ \forall n \in \mathbb{L} \right\}.$$

This theorem shows a precise sense in which a controllable linear system (an infinite dimensional subspace of $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ whenever $\mathscr{B} \neq \{0\}$) can be identified with a *finite dimensional* vector space. Indeed, through the polynomial annihilators $\mathscr{L}^{\mathtt{w}}$ is in one-to-one correspondence with the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$, and, through the rational annihilators, $\mathscr{L}^{\mathtt{w}}_{\text{contr}}$ is in one-to-one correspondence with the $\mathbb{R}(\xi)$-subspaces of $\mathbb{R}(\xi)^{\mathtt{w}}$.

We now briefly consider the controllable part of a system, and relate it to the annihilators.

Definition 13. Let $\mathscr{B} \in \mathscr{L}^{\bullet}$. The *controllable part* of $\mathscr{B}$ is defined as

$$\mathscr{B}_{\text{contr}} := \{w \in \mathscr{B} | \forall t_0, t_1 \in \mathbb{R}, t_0 \leqslant t_1, \exists w' \in \mathscr{B}, \text{ of compact support, such that } w(t) = w'(t) \text{ for } t_0 \leqslant t \leqslant t_1\}.$$

It is easy to see that $\mathscr{B}_{\text{contr}} \in \mathscr{L}^{\bullet}_{\text{contr}}$.

Consider the system $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, and its rational annihilators $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$. In general, this is an $\mathbb{R}[\xi]$-submodule, but not $\mathbb{R}(\xi)$-vector subspace of $\mathbb{R}(\xi)^{\mathtt{w}}$. But its polynomial elements, $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ always form an $\mathbb{R}[\xi]$-submodule over $\mathbb{R}[\xi]^{\mathtt{w}}$, and this module determines $\mathscr{B}$ uniquely. Therefore, $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ determines $\mathscr{B}$ uniquely. Moreover, $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ forms an $\mathbb{R}(\xi)$-vector space iff $\mathscr{B}$ is controllable. More generally, the $\mathbb{R}(\xi)$-span of $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is exactly $\mathscr{B}_{\text{contr}}^{\perp_{\mathbb{R}(\xi)}}$. Therefore the $\mathbb{R}(\xi)$-span of the rational annihilators of two systems are the same iff they have the same controllable part. Of course, other properties of systems can be deduced from these annihilators. For instance, stabilizability (see Theorem 5).

8. Conclusions

The set of solutions of the system of 'differential equations' $G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ with G a matrix of rational functions can be defined very concretely in terms of a left coprime factorization over $\mathbb{R}[\xi]$ of G. This implies that $G\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ defines a linear time-invariant differential behavior. This definition bring the behavioral theory of systems and the theory of representations using proper stable rational functions in line with each other.

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Appendix A

$\mathbb{R}[\xi]$ denotes the set of polynomials with real coefficients in the indeterminate ξ, and $\mathbb{R}(\xi)$ denotes the set of real rational functions in the indeterminate ξ. $\mathbb{R}[\xi]$ is a ring, and $\mathbb{R}[\xi]^{\mathtt{n}}$ a finitely generated module over $\mathbb{R}[\xi]$. $\mathbb{R}(\xi)$ is a field, and $\mathbb{R}(\xi)^{\mathtt{n}}$ is an $\mathtt{n}$ dimensional $\mathbb{R}(\xi)$-vector space.

The polynomials $p_1, p_2 \in \mathbb{R}[\xi]$ are said to be *coprime* if they have no common roots. A polynomial $p \in \mathbb{R}[\xi]$ is said to be *Hurwitz* if it has no roots in $\overline{\mathbb{C}}_+$.

We now review some salient facts regarding coprime factorizations. For general rings, see [5]. In this appendix, we deal concretely with three rings that each have $\mathbb{R}(\xi)$ as their field of fractions:

1. the ring $\mathbb{R}[\xi]$ of polynomials,
2. the ring $\mathbb{R}(\xi)_{\mathscr{P}}$ of proper rational functions, and
3. the ring $\mathbb{R}(\xi)_{\mathscr{S}}$ of stable proper rational functions.

Informally, this means: 1. all poles at ∞, 2. no poles at ∞, 3. only finite stable poles. We now give formal definitions, and review some salient facts regarding (matrices over) these rings.

A.1. $\mathbb{R}[\xi]$

An element $U \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}[\xi]$ if it has an inverse in $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$. This is the case iff $\det(U)$ is equal to a non-zero constant. We denote the $\mathbb{R}[\xi]$-unimodular elements of $\mathbb{R}[\xi]^{\bullet\times\bullet}$ by $\mathscr{U}_{\mathbb{R}[\xi]}$.

$M \in \mathbb{R}(\xi)^{n_1 \times n_2}$ can be brought into a simple canonical form, called the *Smith–McMillan form*, using pre- and post-multiplication by elements from $\mathscr{U}_{\mathbb{R}[\xi]}$, so by pre- and post-multiplication by polynomial matrices.

Proposition 14. *Let $M \in \mathbb{R}(\xi)^{n_1 \times n_2}$. There exist $U \in \mathbb{R}[\xi]^{n_1 \times n_1}$, $V \in \mathbb{R}[\xi]^{n_2 \times n_2}$, both unimodular, $\Pi \in \mathbb{R}[\xi]^{n_1 \times n_1}$, and $Z \in \mathbb{R}[\xi]^{n_1 \times n_2}$ such that*

$$M = U\Pi^{-1}ZV, \quad \Pi = \operatorname{diag}\left(\pi_1, \pi_2, \ldots, \pi_{n_1}\right),$$
$$Z = \begin{bmatrix} \operatorname{diag}(\zeta_1, \zeta_2, \ldots, \zeta_r) & 0_{r \times (n_2 - r)} \\ 0_{(n_1 - r) \times r} & 0_{(n_1 - r) \times (n_2 - r)} \end{bmatrix}$$

with $\zeta_1, \zeta_2, \ldots, \zeta_r, \pi_1, \pi_2, \ldots, \pi_{n_1}$ non-zero monic elements of $\mathbb{R}[\xi]$, the pairs ζ_k, π_k coprime for $k = 1, 2, \ldots, r$, $\pi_k = 1$ coprime for $k = r + 1, r + 2, \ldots, n_1$, and with ζ_{k-1} a factor of ζ_k and with π_k a factor of π_{k-1}, for $k = 2, \ldots, r$. Of course, $r = \operatorname{rank}(M)$.

The roots of the π_k's (hence of π_1, disregarding multiplicity issues) are called the *poles* of M, and those of the ζ_k's (hence of ζ_r, disregarding multiplicity issues) the *zeros* of M. When $M \in \mathbb{R}[\xi]^{\bullet \times \bullet}$, the π_k's are absent (they are equal to 1). We then speak of the *Smith form*.

$M \in \mathbb{R}[\xi]^{n_1 \times n_2}$ is said to be *left prime over* $\mathbb{R}[\xi]$ if for every factorization $M = FM'$ with $F \in \mathbb{R}[\xi]^{n_1 \times n_1}$ and $M' \in \mathbb{R}[\xi]^{n_1 \times n_2}$, F is unimodular over $\mathbb{R}[\xi]$.

Proposition 15. *Let $M \in \mathbb{R}[\xi]^{n_1 \times n_2}$. The following are equivalent.*

1. *M is left prime over $\mathbb{R}[\xi]$,*
2. $\operatorname{rank}(M(\lambda)) = n_1 \quad \forall \lambda \in \mathbb{C}$,
3. $\exists N \in \mathbb{R}[\xi]^{n_2 \times n_1}$ *such that* $MN = I_{n_1}$,
4. *M is of full row rank and it has no zeros.*

The polynomial matrices $M_1, M_2, \ldots, M_n \in \mathbb{R}[\xi]^{n \times \bullet}$ are said to be *left coprime* over $\mathbb{R}[\xi]$ if the matrix M formed by them, $M = \operatorname{row}(M_1, M_2, \ldots, M_n)$, is left prime over $\mathbb{R}[\xi]$.

The pair (P, Q) is said to be a *left factorization over* $\mathbb{R}[\xi]$ of the matrix of rational functions $M \in \mathbb{R}(\xi)^{n_1 \times n_2}$ if

(i) $P \in \mathbb{R}[\xi]^{p \times p}$ and $Q \in \mathbb{R}[\xi]^{p \times m}$,
(ii) $\det(P) \neq 0$, and
(iii) $M = P^{-1}Q$.

It is said to be a *left coprime factorization over* $\mathbb{R}[\xi]$ of M if, in addition,

(iv) P and Q are left coprime over $\mathbb{R}[\xi]$.

The existence of a left coprime factorization of $M \in \mathbb{R}(\xi)^{n_1 \times n_2}$ over $\mathbb{R}[\xi]$ is readily deduced from the Smith–McMillan form. Take $P = \Pi U^{-1}$ and $Q = ZV$. It is easy to see that a left coprime factorization over $\mathbb{R}[\xi]$ of $M \in \mathbb{R}(\xi)^{n_1 \times n_2}$ is unique up to pre-multiplication of P and Q by a unimodular $U \in \mathscr{U}_{\mathbb{R}[\xi]}$.

A.2. $\mathbb{R}(\xi)_{\mathscr{P}}$

The *relative degree* of $f \in \mathbb{R}(\xi)$, $f = \frac{n}{d}$, with $n, d \in \mathbb{R}[\xi]$, is defined as the degree of the denominator d minus the degree of the numerator n. The rational function $f \in \mathbb{R}(\xi)$ is said to be *proper* if the relative degree is $\geqslant 0$, *strictly proper* if it is > 0, and *bi-proper* if it is equal to 0. Denote

$$\mathbb{R}(\xi)_{\mathscr{P}} := \{f \in \mathbb{R}(\xi) | f \text{ is proper}\}.$$

$\mathbb{R}(\xi)_{\mathscr{P}}$ is a ring, in fact, a proper Euclidean domain.

$M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ is said to be *proper* if each of its elements is proper. $M \in \mathbb{R}(\xi)^{\mathtt{n} \times \mathtt{n}}$ is said to be *bi-proper* if $\det(M) \neq 0$ and M, M^{-1} are both proper. $U \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n} \times \mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}(\xi)_{\mathscr{P}}$ if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n} \times \mathtt{n}}$. There holds: [[$U \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n} \times \mathtt{n}}$ is unimodular over $\mathbb{R}(\xi)_{\mathscr{P}}$]] $\Leftrightarrow$ [[it is bi-proper]] $\Leftrightarrow$ [[$\det(U)$ is bi-proper]]. We denote the unimodular elements of $\mathbb{R}(\xi)_{\mathscr{P}}^{\bullet \times \bullet}$ by $\mathscr{U}_{\mathbb{R}(\xi)_{\mathscr{P}}}$.

$M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$ is said to be *left prime* over $\mathbb{R}(\xi)_{\mathscr{P}}$ if for every factorization over $\mathbb{R}(\xi)_{\mathscr{P}}$ $M = FM'$ with $F \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_1}$ and $M' \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$, F is unimodular over $\mathbb{R}(\xi)_{\mathscr{P}}$. The algebraic structure of $\mathbb{R}(\xi)_{\mathscr{P}}$ leads to the following proposition.

Proposition 16. *Let $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$. The following are equivalent*:

1. *$M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ and is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$.*
2. *M is proper, and it has an $\mathtt{n}_1 \times \mathtt{n}_1$ submatrix that is bi-proper.*
3. *$M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ and $\exists N \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_2 \times \mathtt{n}_1}$ such that $MN = I_{\mathtt{n}_1}$.*

The matrices of rational functions $M_1, M_2, \ldots, M_{\mathtt{n}} \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n} \times \bullet}$ are said to be *left coprime over* $\mathbb{R}(\xi)_{\mathscr{P}}$ if the matrix M formed by them, $M = \text{row}(M_1, M_2, \ldots, M_{\mathtt{n}})$, is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$.

A.3. $\mathbb{R}(\xi)_{\mathscr{S}}$

Define

$$\mathbb{R}(\xi)_{\mathscr{S}} := \{f \in \mathbb{R}(\xi) | f \text{ proper, and has no poles in } \overline{\mathbb{C}}_+\}.$$

Other stability domains are of interest, but we stick with the usual ‘Hurwitz’ domain for the sake of concreteness.

It is easy to see that $\mathbb{R}(\xi)_{\mathscr{S}}$ is a ring. $\mathbb{R}(\xi)$ is its field of fractions. In [5, p. 10], it is proven that $\mathbb{R}(\xi)_{\mathscr{S}}$ is a proper Euclidean domain.

An element $U \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n} \times \mathtt{n}}$ is said to be *unimodular over* $\mathbb{R}(\xi)_{\mathscr{S}}$ if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n} \times \mathtt{n}}$. This is the case iff $\det(U)$ is bi-proper and miniphase (*miniphase* : $\Leftrightarrow$ no poles and no zeros in $\overline{\mathbb{C}}_+$). We denote the unimodular elements of $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \bullet}$ by $\mathscr{U}_{\mathbb{R}(\xi)_{\mathscr{S}}}$.

$M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ is said to be *left prime* over $\mathbb{R}(\xi)_{\mathscr{S}}$ if for every factorization over $\mathbb{R}(\xi)_{\mathscr{S}}$, $M = FM'$ with $F \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_1}$ *and* $M' \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$, F is unimodular over $\mathbb{R}(\xi)_{\mathscr{S}}$. The algebraic structure of $\mathbb{R}(\xi)_{\mathscr{S}}$ leads to the following proposition.

Proposition 17. *Let $M \in \mathbb{R}(\xi)^{\mathtt{n}_1 \times \mathtt{n}_2}$. The following are equivalent.*

1. *$M \in \mathbb{R}(\xi)_{\mathscr{S}}$ and is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.*
2. *M has no poles and no zeros in $\overline{\mathbb{C}}_+$, it is proper, and it has an $\mathtt{n}_1 \times \mathtt{n}_1$ submatrix that is bi-proper.*
3. *$M \in \mathbb{R}(\xi)_{\mathscr{S}}$ and $\exists N \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_2 \times \mathtt{n}_1}$ such that $MN = I_{\mathtt{n}_1}$.*

The matrices of rational functions $M_1, M_2, \ldots, M_{\mathtt{n}} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}\times\bullet}$ are said to be *left coprime over* $\mathbb{R}(\xi)_{\mathscr{S}}$ if the matrix M formed by them, $M = \text{row}(M_1, M_2, \ldots, M_{\mathtt{n}})$, is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$.

Right (co-)prime, right (co-)prime factorizations, etc., are defined in complete analogy with their left counterparts.

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