European Journal of Control (2007)13:134–151

DOI:10.3166/EJC.13.134–151

European Journal of Control

Dissipative Dynamical Systems

Jan C. Willems

ESAT-SISTA, K.U. Leuven, B-3001 Leuven, Belgium

Dissipative systems provide a strong link between physics, system theory, and control engineering. Dissipativity is first explained in the classical setting of input/state/output systems. In the context of linear systems with quadratic supply rates, the construction of a storage leads to a linear matrix inequality (LMI). It is in this context that LMI's first emerged in the field. Next, we phrase dissipativity in the setting of behavioral systems, and present the construction of two canonical storages, the available storage and the required supply. This leads to a new notion of dissipativity, purely in terms of boundedness of the free supply that can be extracted from a system. The storage is then introduced as a latent variable associated with the supply rate as the manifest variable. The equivalence of dissipativity with the existence of a non-negative storage is proven. Finally, we deal with supply rates that are given as quadratic differential forms and state several results that relate the existence of a (non-negative) storage to the two-variable polynomial matrix that defines the quadratic differential form. In the ECC presentation, we mainly discuss distributed dissipative systems described by constant coefficient linear PDE's. In this setting, the construction of storage functions leads to Hilbert's 17-th problem on the representation of non-negative polynomials as a sum of squares.

Keywords: Dissipativity; supply rate; storage; dissipation inequality; LMI; quadratic differential forms

E-mail: Jan.Willems@esat.kuleuven.be

1. Introduction

The purpose of this paper is to present a tutorial and somewhat informal introduction to the notion of dissipativity of dynamical systems. In the classical setting, the dissipation inequality involves an input/state/output system, with a supply rate (a function of the input and output variables), and with a storage (a state function). Together these are required to satisfy the dissipation inequality, which states that the increase in storage over a time interval cannot exceed the supply delivered to the system during this time-interval. This classical definition is reviewed in Section 4. For closed systems (flows on manifolds), isolated from their environment, it is natural to assume that the supply rate is zero. In this case the dissipation inequality reduces to the requirement that the storage is a Lyapunov function. Motivated by this, we start this paper in Section 3 by a brief introduction to Lyapunov theory. Given the central importance of Lyapunov theory in systems and control, one should expect dissipativity to play also an important role in the field.

Given a system in input/state/output form and a supply rate, the question emerges if there exists a storage such that the dissipation inequality is satisfied. If a non-negative storage exists, we call the system dissipative with respect to the supply rate. The problem of constructing a (non-negative) storage has been studied very extensively, both for general systems and, especially, for linear systems with a supply rate that is a quadratic function of the input and output variables.

Received 20 December 2006; Accepted 20 January 2007.
Recommended by S.G. Tzafestas and P.J. Antsaklis

Under suitable conditions (controllability, etc.), there are two "canonical" storage functions, the available storage and the required supply (see Section 6). The set of storages is convex, and is bounded from below by the available storage and from above by the required supply. Hence under the appropriate conditions, the set of storages, obviously a partially ordered set, attains its infimum and its supremum. In the case of linear systems with quadratic supply rates, discussed in Section 5 in the input/state/output setting, there exists a storage that is a quadratic function of the state, if there exists a storage at all. In fact, both the available storage and the required supply are then quadratic in the state. In this case, the dissipation inequality becomes a linear matrix inequality (LMI). It is this problem that brought LMI's to central stage in the field.

The concept of dissipative system and the dissipation inequality was introduced as a concept of its own in [8]. During its short history, it has been applied to many areas in the field, for example, to stability of interconnected systems, stabilization by adding dissipation, robustness and model reduction, information and entropy flow, to oscillator design and synchronization, etc. Dissipativity is a system theoretic concept that aims directly at the analysis and synthesis of physical systems. It is one of the rare concepts in the field which by its very nature also applies to and aims at physical reality. There are indeed immediate applications to electrical circuit theory, to the analysis of viscoelastic materials, to the theory of mechanical systems, to thermodynamics, etc. A nice example illustrating this relevance is the recent book [6].

Dissipativity is a property of open systems that is relevant for analysis as well as synthesis. On the level of analysis, we can ask under what conditions on the model parameters and in what sense a system is dissipative. Or deduce stability robustness by viewing a system as an interconnection of dissipative subsystems. On the synthesis level, it can be used to design controllers that add dissipation or that achieve robustness by making the controlled system dissipative when viewed from the terminals of the uncertain part of the system. The interplay of analysis and synthesis through the dissipation inequality is perhaps most apparent in the context of electrical circuits. This has been developed in [1], for example, and will be discussed a bit throughout this paper.

We will discuss three running examples in this paper. The first are general electrical one-ports. The second example consists of a vessel that exchanges heat with its environment. The third is a heated rod. We set up the dynamical equations of these examples in Section 2.

Input/state/output systems form often an artificial approach to the modeling of physical systems. It is easy to generalize the classical notion of dissipativity to behavioral state systems. We explain this in Section 6. Also the a priori assumption that the storage is a state function is another, more subtle, shortcoming of the classical definition of dissipativity. Indeed, the fact that the storage is a state function is something that one wishes to prove, not assume. This is discussed in Sections 7 and 11. We illustrate these issues further in the context of circuit synthesis in Section 12.

These drawbacks lead to a new definition of dissipativity, in which the behavior simply consists of the possible supply trajectories. This is discussed in Section 8, following the recent paper [13]. The storage is now viewed as a latent variable that is associated with the supply rate as the manifest variable, such that they jointly satisfy the dissipation inequality. We prove that a system is dissipative if and only if there exists an associated storage that is non-negative. The proof relies on the fact that a system is dissipative if and only if the 'free' supply is bounded.

In Section 9, we study a special, but very useful family of supply rates, namely supply rates that are given as the image of a quadratic differential form (QDF) acting on a free variable. It can be shown that controllable linear systems with quadratic supply rates can be represented this way. For such supply rates, we derive several results on dissipativity and on the existence of a storage in terms of the two-variable polynomial matrix that parametrizes the QDF. It turns out that the storage function is itself often also a QDF. In this case, the construction of the storage function becomes again an LMI in the space of two-variable polynomial matrices. The question emerges whether this storage is a state function. This is discussed in Section 11.

Finally, in Section 13, we mention the generalization to distributed parameter systems. In this case, the construction of the storage function leads to Hilbert's 17-th problem on the representation of positive polynomials as a sum-of-squares. Of interest in this context of PDE's is the non-observability of the storage function. Because of space limitations, we refer to [8] for a detailed exposition of the generalization to PDE's. In the ECC presentation, PDE's with quadratic supply rates will be discussed in some detail.

A few words about notation. We use standard symbols for the sets $\mathbb{R}, \mathbb{C}$, etc. We use $\mathbb{R}^{n}$, $\mathbb{R}^{n \times m}$, etc. for vectors and matrices over $\mathbb{R}$, and analogously over other sets. When the number of rows and/or columns is immaterial, we use the notation $\mathbb{R}^{\bullet}$, $\mathbb{R}^{\bullet \times \bullet}$, etc. Of course, when we then add or multiply vectors or matrices, we assume that the dimensions are compatible. $\mathbb{R}[\xi]$ denotes the set of polynomials with real coefficients in the indeterminate ξ, and $\mathbb{R}(\xi)$ denotes the set of real rational functions in the indeterminate ξ. $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$. $\mathcal{D}^\infty(\mathbb{R}, \mathbb{R}^n)$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$ with compact support.

2. Examples

We will frequently return to the following three examples of dissipative systems.

2.1. Electrical Circuits

Electrical circuits are the paradigmatic examples of interconnected dynamical systems: the constitutive equations of the subsystems are clearly defined (at least for lumped linear time-invariant elements), and so are the interconnection constraints by which these elements are interconnected (Kirchhoff's current and voltage laws). We view an electrical circuit as a device with terminals (wires) connecting it to its environment (see Fig. 1). Assume that the circuit contains the classical circuit elements: resistors, inductors, capacitors, transformers, and gyrators. The most appropriate way of describing the external dynamic behavior of electrical circuits is in terms of the potentials and currents on the external terminals. In this paper, we only deal with 2-terminal circuits. In this case, it can be shown that the terminals behave as a port. In other words, only the difference of the terminal potentials enters in the behavioral equations, and the sum of the currents going into the terminals equals zero. We may therefore as well consider the port voltage and current as the external variables (see the middle part of Fig. 1).

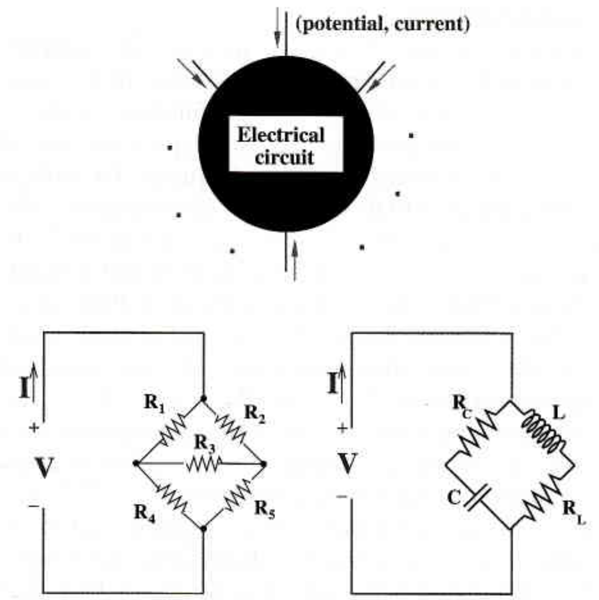

Fig. 1. Electrical circuit.

In fact, we will mainly deal with the specific circuit shown on the right hand side of this figure. It can be shown (see [7, pages 11–12]) that the following differential equation describes exactly the behavior of the port variables of this circuit. In other words, $(V(\cdot), I(\cdot)) : \mathbb{R} \to \mathbb{R}^2$ can occur as port voltage and current history if and only if it satisfies this ODE. For $CR_C \neq L/R_L$, the behavioral equation is

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{\mathrm{d}}{\mathrm{d}t} + CR_C \frac{L}{R_L} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) V = \left(1 + CR_C \frac{\mathrm{d}}{\mathrm{d}t}\right)\left(1 + \frac{L}{R_L}\frac{\mathrm{d}}{\mathrm{d}t}\right) R_C I,$$

while for $CR_C = L/R_L$, it becomes instead

$$\left(\frac{R_C}{R_L} + CR_C \frac{\mathrm{d}}{\mathrm{d}t}\right) V = \left(1 + CR_C \frac{\mathrm{d}}{\mathrm{d}t}\right) R_C I.$$

These behavioral equations are equivalent (in the sense that after elimination of I_L and V_C, we obtain the above equations) to

$$\begin{aligned} R_L I_L + L\frac{\mathrm{d}}{\mathrm{d}t} I_L &= V, \\ V_C + CR_C \frac{\mathrm{d}}{\mathrm{d}t} V_C &= V, \\ \frac{V - V_C}{R_C} + I_L &= I. \end{aligned} \tag{1}$$

2.2. Thermal Vessel

The second example is extensively studied in [14]. Consider the vessel shown in Fig. 2. It contains a material at temperature T and heat is brought into to the vessel at rate Q' and temperature T'. Assume that the relations among these variables are as follows (ρ accounts for the specific heat of the material).

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} T = Q', \tag{2}$$

combined with

$$[\![(Q' \geq 0) \text{ and } (T' \geq T)]\!]$$

or

$$[\![(Q' \leq 0) \text{ and } (T' \leq T)]\!] \tag{3}$$

The inequalities express that it is impossible to transport heat from cold to hot, a consequence of the second law of thermodynamics. Assume that the units are chosen such that $\rho = 1$.

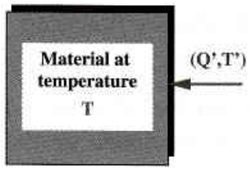

Fig. 2. Thermal vessel.

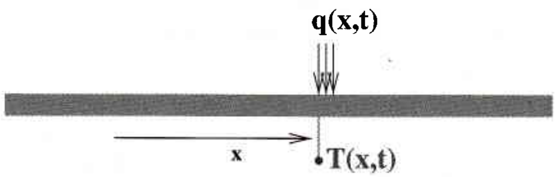

Fig. 3. Heated bar.

2.3. Heated Bar

The third example is also extensively studied in [14]. Consider the heated bar shown in Fig. 3. Assume that the length of the bar is L and that there is no heat transport at the ends. Let $T(x,t), q(x,t), t \in \mathbb{R}$, $0 \leq x \leq L$ denote the temperature of the bar and the rate of heat absorbed by the bar. Fourier's law of heat conduction leads to the behavioral equations

$$\rho \frac{\partial}{\partial t} T = \gamma \frac{\partial^2}{\partial x^2} T + q,$$

with boundary conditions

$$\frac{\partial}{\partial x} T(\cdot, 0) = \frac{\partial}{\partial x} T(\cdot, L) = 0.$$

The coefficient ρ accounts for the specific heat of the material of the bar, and γ for the heat diffusion coefficient. We assume that the units are chosen so that $\rho = 1, \gamma = 1, L = 1$.

3. Lyapunov Functions

Since we view dissipativity and the dissipation inequality as a natural generalization to open systems of the notion of Lyapunov functions, we first discuss these briefly.

One of the most effective ways of obtaining stability results is by means of Lyapunov functions. Consider the 'classical' dynamical system, the *flow*,

$$\dot{\mathrm{x}} = f(\mathrm{x}), \qquad (\mathcal{F})$$

with x the state, $\mathbb{X}$ the state space, $\mathrm{x} \in \mathbb{X}$, and f the *vectorfield*. For simplicity of exposition, we assume that $\mathbb{X} \subseteq \mathbb{R}^{\bullet}$. Then $f : \mathbb{X} \to \mathbb{R}^{\bullet}$. We view f as a map which assigns the 'velocity' by $\dot{\mathrm{x}} = f(\mathrm{x}) \in \mathbb{R}^{\bullet}$ when the state is at $\mathrm{x} \in \mathbb{X}$. The vectorfield governs the motion.

The *behavior* $\mathcal{B}^{\mathcal{F}}$ of $(\mathcal{F})$ is defined as the set of solutions,

$$\mathcal{B}^{\mathcal{F}} := \{x : \mathbb{R} \to \mathbb{R}^{\bullet} | \; x \text{ is absolutely continuous, and } \frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(x(t)) \text{ for almost all } t \in \mathbb{R}\}$$

Assume that for all $\mathrm{x} \in \mathbb{X}$, there exists a unique $x \in \mathcal{B}^{\mathcal{F}}$ such that $x(0) = \mathrm{x}$. In many applications, it suffices to consider solutions x on $[0, \infty)$, but we do not aim at generality in this respect.

The real-valued function $V : \mathbb{X} \to \mathbb{R}$ is said to be a *Lyapunov function* for $(\mathcal{F})$ if it is non-increasing along solutions. Hence if

$$V(x(t_2)) \leq V(x(t_1))$$

$$\forall x \in \mathcal{B}^{\mathcal{F}} \text{ and } \forall t_1, t_2 \in \mathbb{R}, \text{ with } t_2 \geq t_1.$$

This condition on V can be checked without explicit knowledge of $\mathcal{B}^{\mathcal{F}}$. It can be verified directly from the vectorfield f and the function V. Indeed, V, assumed differentiable, is a Lyapunov function for $(\mathcal{F})$ if and only if

$$\dot{V}^{\mathcal{F}} := \nabla V \cdot f \text{ satisfies } \dot{V}^{\mathcal{F}}(\mathrm{x}) \leq 0 \; \forall \mathrm{x} \in \mathbb{X},$$

where

$$\nabla V := \mathrm{col}\left(\frac{\partial V}{\partial \mathrm{x}_1}, \frac{\partial V}{\partial \mathrm{x}_2}, \ldots\right)$$

denotes the gradient of V.

Lyapunov functions have found numerous applications in the theory of differential equations and in applied mathematics. For example, under reasonable smoothness conditions on $\mathbb{X}, f$, and V, it can be shown that all bounded $x \in \mathcal{B}^{\mathcal{F}}$ approach, as $t \to \infty$, the largest $\mathcal{F}$-invariant set contained in $\{\mathrm{x} \in \mathbb{X} | \dot{V}^{\mathcal{F}}(\mathrm{x}) = 0\}$. Under appropriate positivity and growth conditions on V and definiteness conditions on $\dot{V}^{\mathcal{F}}$, this often allows to conclude global stability: $[\![x \in \mathcal{B}^{\mathcal{F}}]\!] \Rightarrow [\![x(t) \to 0 \text{ as } t \to \infty]\!]$. Note that we do not require a Lyapunov function to be non-negative. In fact, there are applications where this is not useful, for example when applying Lyapunov methods to obtain instability results one needs Lyapunov functions that are unbounded from below.

An important problem that emerges is the construction of Lyapunov functions. Usually, this refers to the construction of a function $V : \mathbb{X} \to \mathbb{R}$ from which global stability may be deduced. More generally, for a given $\mathcal{F}$, one may want to classify all Lyapunov functions V, possibly with non-negativity of V added as an additional requirement. This theory is very well established for linear flows:

$$\dot{\mathrm{x}} = A\mathrm{x}, \quad \text{with } \mathrm{x} \in \mathbb{R}^{\bullet} \text{ and } A \in \mathbb{R}^{\bullet\times\bullet},$$

and quadratic Lyapunov functions

$$V(\mathrm{x}) = \mathrm{x}^{\top} Q \mathrm{x}, \quad \text{with } Q = Q^{\top} \in \mathbb{R}^{\bullet\times\bullet}.$$

The condition that V is a Lyapunov function leads to the *Lyapunov equation*

$$A^{\top} Q + QA \preccurlyeq 0.$$

If $Q \succ 0$, this proves stability. If, in addition, $(A, A^{\top} Q + QA)$ is observable, we obtain asymptotic stability. If $Q \not\succcurlyeq 0$, there is no asymptotic stability. Combined with the observability condition, instability follows. This equation has been studied in great detail in linear algebra and in the control and systems literature.

The examples given in Section 2 readily lead to Lyapunov functions. Consider the electrical circuit. Of course, this is not a flow. It becomes a flow when subjected to suitable terminations. The short circuit ($V = 0$) equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} I_L = -\frac{R_L}{L} I_L, \quad \frac{\mathrm{d}}{\mathrm{d}t} V_C = -\frac{1}{CR_C} V_C.$$

The energy stored in the circuit, $\frac{1}{2} C V_C^2 + \frac{1}{2} L I_L^2$ is a Lyapunov function. Its derivative equals $-V_C^2/R_C - R_L I_L^2$, the heat dissipated in the resistors. The open circuit ($I = 0$) equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} I_L = -\frac{R_C + R_L}{L} I_L + \frac{1}{L} V_C,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_C = -\frac{1}{C} I_L.$$

Again the energy stored in the circuit, $\frac{1}{2} C V_C^2 + \frac{1}{2} L I_L^2$ is a Lyapunov equation. Its derivative equals $-V_C^2/R_C - R_L I_L^2$. In the case that all the elements are positive($C > 0, L > 0, R_C > 0, R_L > 0$), this proves asymptotic stability of both the closed and open circuit behavior. But when C and/or L are negative and $R_C > 0$, $R_L > 0$, this leads to instability.

In the case of the thermal vessel, we obtain a flow by isolating it from its environment and taking $Q' = 0$, leading to $\frac{\mathrm{d}}{\mathrm{d}t} T = 0$ for the dynamic equations. Obviously T is a Lyapunov function, leading to 'neutral' stability. Note that every function of T, in particular the negative of the entropy $-\ln T$, is a Lyapunov function. The heated bar becomes an infinite-dimensional flow by assuming $q = 0$, leading to the equation

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T.$$

This yields the Lyapunov functions $\int_0^1 T(x, \cdot)\, \mathrm{d}x$, the energy, and $-\int_0^1 \ln T(x, \cdot)\, \mathrm{d}x$, the negative of the entropy. The derivative of $\int_0^1 T(x, \cdot)\, \mathrm{d}x$ is zero, and no stability can be concluded on the basis of it. The derivative of $-\int_0^1 \ln T(x, \cdot)\, \mathrm{d}x$ equals $-\int_0^1 \frac{1}{T^2(x, \cdot)} \left(\frac{\partial}{\partial x} T(x, \cdot)\right)^2 \mathrm{d}x$. From here, it can be shown that T converges to a uniform temperature $T_\infty = \int_0^1 T(x, 0)\, \mathrm{d}x$.

Lyapunov functions were first introduced by Aleksandr Mikhailovich Lyapunov (1857–1918) in his doctoral dissertation in 1892. They play a remarkably central role in applied mathematics in general, and in systems and control in particular.

4. Dissipative Systems in an Input/State/Output Setting

Flows are examples of 'closed' dynamical systems. Each trajectory is determined by the initial conditions. The trajectory is autonomous and driven purely by the vectorfield, by the internal dynamics of the system. The environment has no influence on the motion.

'Open' dynamical systems, on the other hand, take the influence of the environment explicitly into consideration. They are a much more logical and richer starting point for a theory of dynamics. In the state space models that have become in vogue in systems and control since the 1960s, this interaction with the environment is formalized through inputs and outputs. The environment acts on the system by imposing inputs, and the system reacts through the outputs. This leads to models of the form

$$\dot{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}), \; y = h(\mathrm{x}, \mathrm{u}), \tag{Σ}$$

with u the input value, $\mathbb{U}$ the input space, $\mathrm{u} \in \mathbb{U}$, y the output value, $\mathbb{Y}$ the output space, $\mathrm{y} \in \mathbb{Y}$, and x the state, $\mathbb{X}$ the state space, $\mathrm{x} \in \mathbb{X}$. For simplicity of exposition, we assume again $\mathbb{X} \subseteq \mathbb{R}^{\bullet}$. The map $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{\bullet}$ is called the (controlled) *vectorfield*, and $h : \mathbb{X} \times \mathbb{U} \to \mathbb{Y}$ is called the *read-out*. Thus the vectorfield assigns to $(\mathrm{x}, \mathrm{u}) \in \mathbb{X} \times \mathbb{U}$ the state 'velocity' $\dot{\mathrm{x}} = f(\mathrm{x}, \mathrm{u}) \in \mathbb{R}^{\bullet}$, and the read-out assigns to $(\mathrm{x}, \mathrm{u}) \in \mathbb{X} \times \mathbb{U}$, the output value $\mathrm{y} = h(\mathrm{x}, \mathrm{u}) \in \mathbb{Y}$. As part of the system specification, there is also a space of admissible inputs, $\mathcal{U} \subseteq \mathbb{U}^{\mathbb{R}}$. Assume that $\mathcal{U}$ is shift-invariant and closed under concatenation. Shift-invariant means $\sigma^t \mathcal{U} = \mathcal{U}$ for all $t \in \mathbb{R}$, with σ^t the *shift operator*; σ^t acting on $f : \mathbb{R} \to \mathbb{F}$ is defined as the map from $\mathbb{R}$ to $\mathcal{F}$ defined by $\sigma^t f(t') := f(t' + t)$. Closed under concatenation means $[\![u_1, u_2 \in \mathcal{U}$ and $t \in \mathbb{R}]\!]$

$\Rightarrow [\![u_1 \wedge_t u_2 \in \mathcal{U}]\!]$, with $\wedge_t$ concatenation at t. For $f_1, f_2 : \mathbb{T} \to \mathbb{F}$, and $t \in \mathbb{T}$, the *concatenation* $f_1 \wedge_t f_2$, is defined as the map $f_1 \wedge_t f_2 : \mathbb{T} \to \mathbb{F}$, with

$$f_1 \wedge_t f_2(t') := \begin{cases} f_1(t') & \text{for } t' < t \\ f_2(t') & \text{for } t' \geq t \end{cases}$$

The *behavior* $\mathcal{B}^\Sigma$ of (Σ) is

$$\mathcal{B}^\Sigma := \{(u, y, x) : \mathbb{R} \to (\mathbb{U} \times \mathbb{Y} \times \mathbb{X})|$$
$$u \in \mathcal{U}, x \text{ absolutely continuous, and}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = f(x(t), u(t)), \text{ for almost all } t \in \mathbb{R},$$
$$y(t) = h(x(t), u(t)) \forall t \in \mathbb{R}.\}$$

It is easy to see that shift-invariance of $\mathcal{U}$ and the fact that the controlled vectorfield and read-out do not depend on time explicitly, imply that $\mathcal{B}^\Sigma$ is also shift-invariant.

The notion of a dissipative system involves

(i) a dynamical system Σ,
(ii) a real-valued functions: $\mathbb{U} \times \mathbb{Y} \to \mathbb{R}$, called the *supply rate*, and
(iii) a real-valued function $V : \mathbb{X} \to \mathbb{R}$, called the *storage*.

Definition 1. The system Σ is said to satisfy the dissipation inequality with respect to the supply rate s and the storage V if

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} s(u(t), y(t))\ \mathrm{d}t \qquad \text{(DissIneq)}$$

holds for all $(u, y, x) \in \mathcal{B}^\Sigma$ and $t_1, t_2 \in \mathbb{R}$, with $t_2 \geq t_1$. □

As in the case of a Lyapunov function, the dissipation inequality can be checked without explicit knowledge of $\mathcal{B}^\Sigma$. It can be verified directly from the vectorfield f, the supply rate s, and the storage V. Assume that $\forall \mathrm{u} \in \mathbb{U}, \exists u \in \mathcal{U}$ such that $u(0) = \mathrm{u}$, and that for all $\mathrm{x} \in \mathbb{X}$ and $u \in \mathcal{U}$, there exists a $(u, y, x) \in \mathcal{B}^\Sigma$ such that $x(0) = \mathrm{x}$. Assume also that V is differentiable. Then (DissIneq) holds if and only if

$$\dot{V}^\Sigma := \nabla V \cdot f$$
$$\text{satisfies } \dot{V}^\Sigma(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{u}, h(\mathrm{x}, \mathrm{u}))$$
$$\forall \mathrm{x} \in \mathbb{X} \text{ and } \forall \mathrm{u} \in \mathbb{U}.$$

The dissipation inequality expresses the following. We have an open dynamical system Σ. It interacts with its environment through the input and output variables. A certain function of these variables, $s(\mathrm{u}, \mathrm{y})$, has the meaning of the rate at which a relevant quantity (mass flow, power, entropy flow, heat flow) flows in and out of the system (s is counted positive when it flows into the system). Some of the supply is stored, some of it is dissipated. It is assumed that the amount stored is a function, $V(\mathrm{x})$, of the state of the system. The difference of what is supplied and what is stored, is dissipated. The dissipation inequality states that the dissipation is non-negative.

Let us apply this definition to our examples of Section 2. Our definition requires considering state equations as (1) for the circut. It is readily seen that there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} C V_C^2 + \frac{1}{2} L I_L^2\right) = VI - \frac{(V - V_c)^2}{R_C} - R_L I_L^2.$$

Whence the dissipation inequality holds with storage $\frac{1}{2} C V_C^2 + \frac{1}{2} L I_L^2$ (the stored energy) and supply rate VI, the electrical power delivered to the circuit by the environment. The difference of the increase of the storage and the supply equals $-R_L I_L^2 - (V - V_c)^2/R_C$, the negative of the heat dissipated in the resistors.

For the thermal vessel, we obtain $\frac{\mathrm{d}}{\mathrm{d}t} T = Q'$. Hence the dissipation inequality holds with equality with storage T and supply rate Q'. This corresponds to conservation of energy. We also have $\frac{\mathrm{d}}{\mathrm{d}t} \ln T \geq Q'/T'$, leading to the dissipation inequality with storage $-\ln T$, the negative of the entropy, and supply rate $-Q'/T'$. Their difference, $Q'(1/T - 1/T')$ corresponds to entropy production. We will later return to the fact that the combination of Eq. (2) and the inequality (3) does not define an input/state/output system.

For the heated bar, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 T(x, \cdot) \mathrm{d}x = \int_0^1 q(x, \cdot) \mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \ln T(x, \cdot) \mathrm{d}x$$
$$= \int_0^1 \frac{q(x, \cdot)}{T(x, \cdot)} \mathrm{d}x + \int_0^1 \left(\frac{1}{T(x, \cdot)} \frac{\partial}{\partial x} T(x, \cdot)\right)^2 \mathrm{d}x$$
$$\geq \int_0^1 \frac{q(x, \cdot)}{T(x, \cdot)} \mathrm{d}x$$

with similar interpretations.

Note that in the dissipation inequality, we did not require the storage to be non-negative. It is to some extent a matter of taste whether one wants to add this requirement in the definition of dissipativity, and there are arguments for and against adding non-negativity as a universal requirement. But in this paper, we will reserve the term 'dissipative system' to systems with a non-negative storage.

Definition 2. (Σ) is said to be *dissipative* with respect to the supply rate $s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}$ if there exists a non-negative storage $V : \mathbb{X} \rightarrow \mathbb{R}$ such that the dissipation inequality (DissIneq) holds. Σ is said to be *cyclo-dissipative* with respect to the supply rate $s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}$ if there exists any storage $V : \mathbb{X} \rightarrow \mathbb{R}$ such that the dissipation inequality (DissIneq) holds.

Both dissipativity and cyclo-dissipativity are relevant in physical applications. For electrical circuits and ordinary mechanical systems, with the supply delivered electrical or mechanical power and the storage internal energy, it is natural to assume that the storage is non-negative (or, what basically amounts to be same thing, bounded from below, since the dissipation inequality remains satisfied after we add a constant to the storage). Indeed, if we want to be able to conclude that the future integral of the supply that can be extracted from a system is bounded, then we need non-negativity of the storage. We do not consider the nomenclature 'dissipative' appropriate in situations in which the storage that can be extracted is infinite. This implies in particular that for dissipativity, we need $C > 0, L > 0, R_C > 0, R_L > 0$ in the circuit example. Moreover, for the thermal vessel and the heated bar, it is only the conservation of energy that leads to a dissipative system. In other applications, for example in thermodynamics or in the mechanics of a planet orbiting the sun, cyclo-dissipativity is the more relevant concept, since in this case the stored energy is neither bounded from below nor from above. Similarly, as we have seen, the entropy often contains a logarithms, leading to a function that is also neither bounded from below nor from above. The nomenclature 'cyclo-dissipative' stems from the fact that $(u, y, x) \in \mathcal{B}^{\Sigma}$ and x periodic imply

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T s(u(t), y(t)) \mathrm{d}t \geq 0 :$$

the system dissipates supply when operated in a periodic regime. So, in particular the entropy production in the thermal vessel and the heated bar lead to cyclo-dissipativity.

Definition 2 was introduced as a concept of its own in 1973 [11], building on earlier work of Brockett, Kalman, Yacubovich, Popov, and others. When a system is isolated from its environment, then it is natural to assume that the supply rate is zero: $s = 0$. In this case, the dissipation inequality reduces to the requirement that the storage V is a Lyapunov function. The notion of a dissipative system is hence a natural generalization to open systems of the notion of a Lyapunov function. Dissipativity has Lyapunov theory as a special case, but it can be used to analyze issues that have no analogue for closed systems (for example, the minimum phase property, see [4]). In view of the central importance of Lyapunov functions, and the fact that open systems form a much more logical starting point for a theory of dynamics than flows are, one should expect dissipative systems to play a central role in the field.

5. LQ Dissipative Systems

The question emerges whether, for a given system Σ and a given supply rate $s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}$, there exists a (non-negative) storage $V : \mathbb{X} \rightarrow \mathbb{R}$ such that the dissipation inequality holds. And, if a storage exists, how the family of storages looks like. These issues have been studied very extensively, both for general systems, but especially for linear systems with a quadratic supply rate. One of the salient facts is the following. Under reasonable conditions, having to do with the existence of an equilibrium state and controllability, one can define two functions, the *available storage*, $V_{\text{av}} : \mathbb{X} \rightarrow \mathbb{R}$, and the *required supply*, $V_{\text{req}} : \mathbb{X} \rightarrow \mathbb{R}$. We will review their construction in Section 6. The system is cyclo-dissipative if and only if V_{av} and V_{req} are bounded, in which case both are storages themselves. Moreover, the set of storages is convex, and each storage (suitably normalized by an additive constant) is bounded from below by V_{av} and from above by V_{req}. The somewhat surprising fact is that under mild conditions the set of storages, obviously a partially ordered set, thus attains its infimum and supremum.

In the linear-quadratic case, the system is assumed to be linear, and the supply rate a quadratic form. However, since the exact expression of the output in terms of the input and the state is immaterial, we may as well assume that the supply rate is quadratic in (u, x), yielding

$$\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u},$$
$$s(\mathrm{u}, \mathrm{x}) = \mathrm{u}^{\top} R \mathrm{u} + 2\mathrm{u}^{\top} S \mathrm{x} + \mathrm{x}^{\top} L \mathrm{x},$$

with $R = R^{\top}, L = L^{\top}$. In this LQ-case it can be shown that there exists a storage if and only if there exists one that is a quadratic functional of the state. In fact, V_{av} and V_{req}, if bounded, are quadratic functionals of the state. The dissipation inequality with $V(x) = x^\top Qx$ becomes

$$\left[\left[\frac{d}{dt}x = Ax + Bu\right]\right]$$
$$\Rightarrow \left[\left[\frac{d}{dt}x^\top Qx \le u^\top Ru + 2u^\top Sx + x^\top Lx\right]\right].$$

This is easier to comprehend, but equivalent to the following matrix inequality which is explicit in the system and supply rate parameters (A, B, R, S, L)

$$\left[\begin{array}{c|c} A^\top Q + QA - L & QB - S^\top \\ \hline B^\top Q - S & -R \end{array}\right] \preccurlyeq 0,$$

$$Q = Q^\top.(\text{LMI})$$

If we are looking for a non-negative storage, we should augment (LMI) with $Q = Q^\top \succcurlyeq 0$. Hence, in the LQ-case cyclo-dissipativity requires the solvability of (LMI) for $Q = Q^\top$, with $Q = Q^\top \succcurlyeq 0$ added for dissipativity. Under suitable conditions (controllability, etc.), a storage exists if and only if V_{av} and V_{req} are bounded, and these extreme storages are quadratic. Hence a non-negative storage exists if and only if V_{req}, the supremal storage function, is non-negative. This implies that under appropriate conditions the set of solutions Q to (LMI) is convex and attains its supremum and its infimum (in the partial ordering of symmetric matrices by non-negative definiteness of the difference).

The inequalities (LMI) are special cases of inequalities of the type

$$\alpha_1 M_1 + \alpha_2 M_2 + \cdots + \alpha_n M_n \succcurlyeq 0,$$

with the M_k's real symmetric matrices, deduced from the system and supply rate parameter matrices (A, B, R, S, L), and the α_k's real numbers that lead to Q. This type of inequality (with the M's given, and the α's unknown) is called a *linear matrix inequality* (LMI). LMI's are very much like linear programming inequalities, and have been studied very deeply. In fact, it is the problem of the existence of a storage in the LQ-case that brought us to the *acronym* LMI. With its relation to the algebraic Riccati equation and inequality, and to semi-definite programming, and to robustness, the applications became seemingly unbounded.

Let us apply this to the electrical circuit example introduced in Section 2.1. The Eq. (1) describe the port behavior, with the current through the inductor and the voltage across the capacitor as state variables. We have already seen that this system is dissipative if all the elements are positive: $C > 0, L > 0, R_C > 0, R_L > 0$. If, in addition, $CR_C \neq L/R_L$, then this state system is state controllable and state observable. In this case the associated (*LMI*) has a convex compact set of solutions $Q = Q^\top \succ 0$, that moreover attains its infimum and supremum. In the case $R_C = R_L = 1, C = 1, L = 1$, the port equations become

$$\frac{d}{dt}(I_L - V_C) = -(I_L - V_C),$$
$$I = V + (I_L - V_C).$$

The system is uncontrollable (both in the state and behavioral sense – we explain later what we mean by this). In this case $Q(I_L - V_C)^2$ is a quadratic storage function if and only if $Q \ge 1/2$. In particular, the set of quadratic storages does not attain its supremum.

6. Behavioral Systems

As we have argued extensively elsewhere, the partition of external variables into inputs and outputs is often very awkward, especially in the context of physical systems. For example, in the case of the electrical circuits, it cannot be decided beforehand if a circuit viewed from a port is voltage or current driven, and this may very well depend on the specific system parameters. Consider the thermal vessel discussed in Section 2.2. The behavior of the external variables (Q', T') is described by the combination of (2, 3). Obviously, in these equations neither Q' nor T' are free variables, and hence, viewing this as an input/state/output system is not appropriate. Later, we shall discuss why the input/output setting is problematic for general thermodynamic systems.

These considerations motivated the development of the behavioral approach, in which a dynamical system is characterized by its behavior. The behavior is the set of trajectories which meet the dynamical laws of the system. Formally, a *dynamical system* is defined by $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the time-set, $\mathbb{W}$ the signal space, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior.* Σ is said to be *linear* if $\mathbb{W}$ is a vector space, and $\mathcal{B}$ a linear subspace of $\mathbb{W}^{\mathbb{R}}$. It is said to be *time-invariant* if $\mathbb{T}$ is closed under addition, and $\sigma^t\mathcal{B} \subseteq \mathcal{B}\ \forall t \in \mathbb{T}$. In the continuous-time setting $\mathbb{T} = \mathbb{R}$, pursued here, the behavior of a dynamical system is typically defined as the set of all solutions to a system of differential(-algebraic) equations. Note that the notion of behavior involves open systems, but avoids the input/output partition of the variables. Input/output systems are covered by taking

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$, but usually the use of the input/output nomenclature implies in addition issues of non-anticipation and causality. These concepts are often tenuous and irrelevant, and are avoided in the behavioral approach.

A *linear time-invariant differential* dynamical system $(\mathbb{R}, \mathbb{R}^\bullet, \mathcal{B})$ is a system with behavior

$$\mathcal{B} = \{w : \mathbb{R} \to \mathbb{R}^\bullet | R\left(\frac{d}{dt}\right) w = 0\}$$

for some $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$. The precise definition of when $w : \mathbb{R} \to \mathbb{R}^\bullet$ is a solution of this differential equation is often of secondary importance. For the purposes of the present paper, it is convenient to consider solutions in $C^\infty(\mathbb{R}, \mathbb{R}^\bullet)$. Since $\mathcal{B}$ is the kernel of the differential operator $R(\frac{d}{dt}) : C^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(R)}) \to C^\infty(\mathbb{R}, \mathbb{R}^{\text{rowdim}(R)})$, we often write $\mathcal{B} = \text{kernel}(R(\frac{d}{dt}))$, and call $R(\frac{d}{dt})w = 0$ a *kernel representation* of the associated linear time-invariant differential system. We denote the set of differential systems $(\mathbb{R}, \mathbb{R}, \text{kernel}(R(\frac{d}{dt})))$ for some $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, or their behaviors, by $\mathcal{L}^\bullet$, or by $\mathcal{L}^{\text{w}}$ when the number of variables is w. While linear time-invariant differential systems are *defined* as kernels of linear constant coefficient differential operators, they are often represented in other ways. State space systems, or, more generally, systems described by constant coefficient linear differential equations with latent variables, systems defined by transfer functions, etc. are all representations of elements of $\mathcal{L}^\bullet$.

An important property of a dynamical system is controllability. In the behavioral setting, this notion takes the following appealing form. $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and assumed time-invariant, is said to be *controllable* if for all $w_1, w_2 \in \mathcal{B}$, there exists $T \geq 0$ and $w \in \mathcal{B}$, such that

$$w(t) = \begin{cases} w_1(t) & \text{for } t < 0 \\ w_2(t - T) & \text{for } t \geq T. \end{cases}$$

Informally, controllability means 'patchability' of elements of the behavior.

Observability pertains to systems in which the variables form a product space, with w_1, an 'observed' variable, and w_2, to be deduced from the observations and the laws of the system. Consider $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathcal{B})$. We call w_2 *observable from* w_1 in Σ if $[[(w_1, w_2'), (w_1, w_2'') \in \mathcal{B}]] \Rightarrow [[w_1' = w_2'']]$, i.e. if there exists a map $F : (\mathbb{W}_1)^{\mathbb{T}} \to (\mathbb{W}_2)^{\mathbb{T}}$ such that $(w_1, w_2) \in \mathcal{B}$ implies $w_2 = F(w_1)$.

Details and conditions for controllability and observability may be found in [7].

A *latent variable dynamical system* is a refinement of the notion of a dynamical system, in which the behavior is represented with the aid of auxiliary variables, called *latent variables*. Formally, a *latent variable dynamical system* is defined by $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathcal{L}, \mathcal{B}_{\text{full}})$ with $\mathbb{T} \subseteq \mathbb{R}$ the time-set, $\mathbb{W}$ the signal space, $\mathcal{L}$ the space of latent variables, and $\mathcal{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathcal{L})^{\mathbb{T}}$ the *full behavior*. $\mathcal{B}_{\text{full}}$ consists of the trajectories $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathcal{L}$ which are compatible with the laws of the system. These involve both the manifest variables w and the latent variables ℓ. Σ_L induces the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ with *manifest behavior*

$$\mathcal{B} = \{w : \mathbb{T} \to \mathbb{W} | \exists \ell : \mathbb{T} \to \mathcal{L} \text{ such that } (w, \ell) \in \mathcal{B}_{\text{full}}\}.$$

The motivation for latent variable systems is that in first principles models, the behavioral equations invariably contain auxiliary ('latent') variables (state variables being the best known examples, but interconnection variables the most prevalent ones) in addition to the ('manifest') variables the model aims at. Latent variables should be an essential part of any theory of dynamical systems.

A state system is a special case of a latent variable system. $\Sigma_X = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ is said to be a *state system* if the full behavior has a concatenability property, requiring that

$$[[(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}}, t \in \mathbb{T}, x_1(t) = x_2(t)]]$$

$$\Rightarrow [[(w_1, x_1) \wedge_t (w_2, x_2) \in \mathcal{B}_{\text{full}}]].$$

An example of a state system is provided by (Σ), under the assumption that $\mathcal{U}$ is closed under concatenation. It is easy to prove that $(\mathbb{R}, \mathbb{U} \times \mathbb{Y}, \mathbb{X}, \mathcal{B}^{\Sigma})$ defines a state system. More generally, assume $\mathbb{X} \subseteq \mathbb{R}^\bullet$. Then any system defined by behavioral equations of the form

$$(\text{w}, \text{x}, \dot{\text{x}}) \in \mathbb{B}$$

with $\mathbb{B}$ a subset of $\mathbb{W} \times \mathbb{X} \times \mathbb{R}^\bullet$ defines a state system. In words: a system is a state system if it is described by differential equations that are zero-th order in the manifest and first order in the latent variables. Of course, Σ is such an example, and so are the relations (2, 3).

It is easy to see that for an input/state/output system, controllability of the (u, x)-behavior of Σ, is equivalent to the classical notion of state controllability. Similarly, the classical notion of state observability corresponds to observability of x from (u, y) in Σ. Informally, we call a latent variable system observable if in $\mathcal{B}_{\text{full}}$ the latent variable is observable from the manifest one.

The input/output structure turns out to be completely unimportant in the definition of the dissipation inequality, and hence for the notion of a (cyclo-)dissipative system. The analogue definition of the dissipation inequality involves

(i) a state system $(\mathbb{R}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$, with $\mathcal{B}_{\text{full}}$ shift-invariant,
(ii) the supply rate $s : \mathbb{W} \to \mathbb{R}$, and
(iii) the storage $V : \mathbb{X} \to \mathbb{R}$.

The *dissipation inequality* becomes

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} s(w(t))\, dt \qquad (\text{DissIneq}_{\mathcal{B}})$$

for all $(w, x) \in \mathcal{B}_{\text{full}}$ and $t_1, t_2 \in \mathbb{R}$, with $t_2 \geq t_1$.

The main results regarding the construction of storage are readily generalized to the behavioral setting. This generalization is straightforward, but nevertheless very meaningful since the input/output partition is almost always artificial when applied to physical systems and their interconnections, and in view of the fact that the theory of dissipative systems offers important insights for the analysis of physical systems.

Consider the system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathcal{B})$, time-invariant ($\sigma^t \mathcal{B} = \mathcal{B}\ \forall t \in \mathbb{R}$), and assume that there exists $\mathrm{w}^* \in \mathbb{W}$ such that $w^* \in \mathcal{B}$, with w^* defined by $w^*(t) = \mathrm{w}^*\ \forall t \in \mathbb{R}$. w^* can be viewed as an equilibrium. Let $\Sigma_{\mathbb{X}} = (\mathbb{R}, \mathbb{W}, \mathbb{X}, \mathcal{B}_{\text{full}})$ be a state representation of Σ. Assume that $\Sigma_{\mathbb{X}}$ is state observable, meaning that for all $w \in \mathcal{B}$, there exists a unique x such that $(w, x) \in \mathcal{B}_{\text{full}}$. This implies in particular that there exists $\mathrm{x}^* \in \mathbb{X}$ such that $(w^*, x^*) \in \mathcal{B}_{\text{full}}$, with x^* defined by $x^*(t) = \mathrm{x}^*\ \forall t \in \mathbb{R}$. x^* can be viewed as the equilibrium state. Assume also the following reachability assumption. For all $\mathrm{x} \in \mathbb{X}$, there exists $(w, x) \in \mathcal{B}_{\text{full}}$ with $x(0) = \mathrm{x}$, and $x(t) = \mathrm{x}^*$ for $|t|$ sufficiently large, i.e. each state can be reached from and steered to the equilibrium state.

Let $s : \mathbb{W} \to \mathbb{R}$ be the supply rate. Define $V_{\text{req}}, V_{\text{av}} : \mathbb{X} \to \mathbb{R}$ as

$$V_{\text{req}}(\mathrm{x}) := \inf \int_t^0 s(w(t'))\, \mathrm{d}t',$$

with the infimum taken over all $t \leq 0$ and $(w, x) \in \mathcal{B}_{\text{full}}$ such that $(w, x)(t) = (\mathrm{w}^*, \mathrm{x}^*)$ for t sufficiently small, and

$$V_{\text{av}}(\mathrm{x}) := \sup - \int_0^t s(w(t'))\, \mathrm{d}t',$$

with the supremum taken over all $t \geq 0$ and $(w, x) \in \mathcal{B}_{\text{full}}$ such that $(w, x)(t) = (\mathrm{w}^*, \mathrm{x}^*)$ for t sufficiently large. It can be shown that the following are equivalent:

(i) $\exists V : \mathbb{X} \to \mathbb{R}$ such that $(\text{DissIneq}_{\mathcal{B}})$ holds,
(ii) $V_{\text{req}} > -\infty$,
(iii) $V_{\text{av}} < \infty$.

Moreover, if $V : \mathbb{X} \to \mathbb{R}$ satisfies (DissIneq$\mathcal{B}$), then $V_{\text{av}} \leq V - V(\mathrm{x}^*) \leq V_{\text{req}}$, and if V_1, V_2 are both storages, so is $\alpha V_1 + (1 - \alpha) V_2$ for $0 \leq \alpha \leq 1$.

The fact that the input/output framework is irrelevant and undesirable for the main results the theory of dissipative systems shows that we may as well use the behavioral setting, and take the supply s itself as the manifest variable. The issue remains how to deal with the storage. Should it be taken to be a state function, or can one just postulate its existence and *prove* that it is a state function instead of *postulating* that it is? We deal with this approach in the remainder of this paper.

7. Shortcomings of the Classical Notion of Dissipativity

The classical theory of dissipative systems as discussed in the previous sections has a number of shortcomings. Some main ones are the following.

One of the important applications of the theory of dissipative systems is to the stability of interconnected systems. Under suitable conditions, the interconnection of dissipative systems is stable, with the sum of the storages of the components functioning as a Lyapunov function. Often, this methodology is used to prove robust stability, with one of the components the plant, and the other component, the uncertain system. The theory of dissipative systems however requires a state representation of both the plant and the uncertain system. It is very awkward to assume knowledge of the state space and the state dynamics of an uncertain system.

In the classical definition of a dissipative system, the storage is assumed to be a state function. But this is something one would like to prove, rather than assume. Also, a state representation is never unique, and the question occurs if non-minimal state representations are relevant in the theory of dissipative systems. Indeed, they are. Consider for example the system $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax,\ y = Cx$, with u free. Is there a state function such that the dissipation inequality holds with respect to the supply rate $\mathrm{u}^\top \mathrm{y}$? In the standard notation, this system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + 0u,\ y = Cx,\ s\colon (\mathrm{u}, \mathrm{y}) \rightsquigarrow \mathrm{u}^\top \mathrm{y}.$$

It is easy to see that there does *not* exist a $V(\mathrm{x})$ such that the dissipation inequality holds, i.e. such that $\nabla V(\mathrm{x}) \cdot A\mathrm{x} \leq \mathrm{u}^\top C\mathrm{x}$, $\forall \mathrm{x}$ and u. However, if we realize this system non-minimally as

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax, \ \frac{\mathrm{d}}{\mathrm{d}t}z = -A^\top z + C^\top u, \ y = Cx,$$

then it is easy to verify that $\mathrm{V}: (\mathrm{z},\mathrm{x}) \rightsquigarrow \mathrm{z}^\top \mathrm{x}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} z^\top x = u^\top y$ along solutions. So a storage such that the dissipation inequality holds does *not* exist if we require it to be a function of a given (minimal) state representation. We must introduce an unobservable state. There are electrical circuits where the physical state, and hence the storage, if it is assumed to be a state function, is unobservable from the external port behavior. We discuss this further in the next section.

The paradigmatic examples of laws that are best formulated in the language of dissipative systems are the first and second law of thermodynamics. A thermodynamic engine (see Fig. 4) is a system that interacts with its environment by means of work, heat flow, and temperature. Assume that the thermodynamic engine has a work 'terminal', where work is delivered to the environment, and several, but a finite number, n heat 'terminals', along which heat is delivered to the engine at a particular temperature. A typical thermodynamic engine usually has many work terminals, where work is done in the form of mechanical or electrical work, etc. However, in order to formulate the first and second law of thermodynamics, there is no need to distinguish between the different work terminals, and so they can be lumped into one. This lumping cannot be done for the thermal terminals, because of the required pairing of heat-flow with temperature. The variables of interest are hence

$$W(\cdot), Q_1(\cdot), T_1(\cdot), Q_2(\cdot), T_2(\cdot), \ldots, Q_n(\cdot), T_n(\cdot),$$

all real-valued. The first law of thermodynamics states that every thermodynamic engine is conservative with respect to the supply rate $\sum_{k=1}^n Q_k(\cdot) - W(\cdot)$ and dissipative with respect to $-\sum_{k=1}^n Q_k(\cdot)/T_k(\cdot)$. We do not dwell of the precise formulation, but we only discuss the appropriateness of the classical input/output setting.

A thermodynamic engine is a prime example where input/output thinking is misplaced. It makes no sense physically to declare some of the variables $W(\cdot), Q_1(\cdot), T_1(\cdot), Q_2(\cdot), T_2(\cdot), \ldots, Q_n(\cdot), T_n(\cdot)$ input variables, and the other output variables. A cause/effect level of description, if useful at all, requires a model that deals with a more detail and much lower level of aggregation. Heat flow may happen by pressures that lead to mass transport, work flow may be realized by electrical voltage differences leading to electrical power flow, or by forces leading to mechanical work, etc. Input/output thinking is hopeless in this example. Formulating general laws, as the the first and second law, pertaining to any thermodynamic system, in terms of input/state/output models is unrealistic. Also the assumption that the internal energy and entropy are state functions is awkward. In an abstract sense, every signal can be viewed as a function of a (non-minimal) state. But to require these to be functions of a the state of a minimal state representation of the behavior of $(W(\cdot), Q_1(\cdot), T_1(\cdot), Q_2(\cdot), T_2(\cdot), \ldots, Q_n(\cdot), T_n(\cdot))$ also presents problems, since usually the internal energy and entropy are functions state variables on a lower level of aggregation. They are obtained from viewing the thermodynamic system as an interconnection of thermodynamic systems, and treating the internal energy and entropy as extensive quantities.

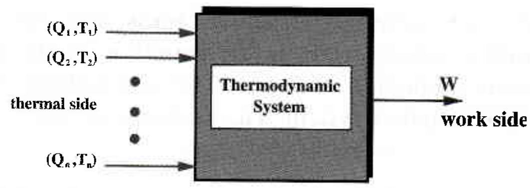

Fig. 4. Thermodynamic engine.

8. An Intrinsic Definition of a Dissipativity

We now give a 'no frills' definition of dissipativity. It is, of course, stated in the language of behaviors, and it is very direct. The idea is the following. We have a dynamical system that exchanges supply (of energy, or mass, or whatever is relevant for the situation at hand) with its environment, expressed by a real-valued supply rate, s, taken to be positive when supply flows into the system. Modeling the dynamics leads to a family of trajectories $s: \mathbb{R} \to \mathbb{R}$ that express the possible supply histories, and to a dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$, and $s: \mathbb{R} \to \mathbb{R}$ belongs to the behavior $\mathcal{B}$ if it is a possible history of the way supply flows in and out of the system. Dissipativity simply states that the maximum amount of supply that is ever extracted along a particular trajectory is bounded. More precisely, if for any trajectory, and starting at a particular time, the net amount of supply that flows out of the system cannot be arbitrarily large. In other words, the 'free' supply is bounded, supply cannot be produced in infinite amount by the system. Everything that can be extracted more than is being supplied must have been stored at the initial time, and is therefore bounded.

Definition 3. Let $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ be a dynamical system. A trajectory $s : \mathbb{R} \to \mathbb{R}, s \in \mathcal{B}$, models the rate of supply absorbed by the system. Σ is said to be dissipative: $\Leftrightarrow$

$$\forall s \in \mathcal{B} \text{ and } t_0 \in \mathbb{R}, \exists K \in \mathbb{R}, \text{such that}$$
$$-\int_{t_0}^{T} s(t)\mathrm{d}t \leq K \text{ for } T \geq t_0.$$

A special case that leads to dissipativity is when $[\![s \in \mathcal{B}]\!] \Rightarrow [\![\int_{-\infty}^{t} s(t')\,\mathrm{d}t' \geq 0 \;\forall t \in \mathbb{R}]\!]$. This is relevant when all trajectories $s \in \mathcal{B}$ have bounded support on the left (this can be viewed as systems that start 'at rest'). More generally, dissipativity follows if for all $s \in \mathcal{B}$ there exists $s' \in \mathcal{B}$ such that $s(t) = s'(t)$ for $t \geq 0$, and with $\int_{-\infty}^{t} s'(t')\,\mathrm{d}t' \geq 0$ for all $t \in \mathbb{R}$.

We now connect this definition with the storage. The storage is viewed as a latent variable V that is coupled to the supply rate s. This leads to a latent variable system $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathcal{B}_{\text{full}})$, such that (s, V) belongs to the full behavior $\mathcal{B}_{\text{full}}$ if the pair $V : \mathbb{R} \to \mathbb{R}, s : \mathbb{R} \to \mathbb{R}$ is a possible history for the way supply flows in and out of the system and is stored in the system. The dissipation inequality is stated in the language of latent variable representations as follows.

Definition 4. Let $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathcal{B}_{\text{full}})$ be a latent variable dynamical system. The component $s : \mathbb{R} \to \mathbb{R}$ of a trajectory $(s, V) \in \mathcal{B}_{\text{full}}$ models the rate of supply absorbed by the system, while the component $V : \mathbb{R} \to \mathbb{R}$ models the supply stored. V is said to be a storage if $\forall (s, V) \in \mathcal{B}_{\text{full}}$ and $\forall t_0, t_1 \in \mathbb{R}, t_0 \leq t_1$, the dissipation inequality

$$V(t_1) - V(t_0) \leq \int_{t_0}^{t_1} s(t)\,\mathrm{d}t \qquad (\text{DissIneq}')$$

holds.

We now prove that dissipativity is equivalent to the existence of a non-negative storage.

Theorem 5. $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ is dissipative if and only if there exists a latent variable dynamical system $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathcal{B}_{\text{full}})$ with manifest behavior $\mathcal{B}$ such that the latent variable component of $(s, V) \in \mathcal{B}_{\text{full}}$ is a non-negative storage.

Proof. (if): Assume that $\Sigma_L = (\mathbb{R}, \mathbb{R}_+, \mathbb{R}, \mathcal{B}_{\text{full}})$ satisfies (DissIneq'), has manifest behavior $\mathcal{B}$, and $V \geq 0$. Let $s \in \mathcal{B}$. Then $\exists\, V : \mathbb{R} \to \mathbb{R}_+$ such that $(s, V) \in \mathcal{B}_{\text{full}}$, and hence

$$\forall t_0 \in \mathbb{R},$$
$$-\int_{t_0}^{T} s(t)\,\mathrm{d}t \leq V(t_0) - V(T) \leq V(t_0), \quad \text{for } T \geq t_0.$$

This shows that $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ is dissipative (take $K = V(t_0)$ in Definition 3).

(only if): Assume that $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ is dissipative. Define, for each trajectory $s \in \mathcal{B}$, an associated trajectory $V : \mathbb{R} \to \mathbb{R}$, as follows:

$$V(t) = \sup\left\{-\int_{t}^{T} s(t')\,\mathrm{d}t' \,\middle|\, T \geq t\right\}.$$

Obviously (take $T = t$ in the sup), $V \geq 0$. Since $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ is dissipative, $V(t) < \infty$ (in fact, $V(t_0) \leq K$, with K as in Definition 3). Hence, with the (s, V)'s so defined, we obtain a latent variable dynamical system $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathcal{B}_{\text{full}})$ with manifest behavior $\mathcal{B}$.

For $w \in \mathcal{B}$ and $t_0 \geq t_1$, there holds

$$\begin{aligned} V(t_0) &= \sup\left\{-\int_{t_0}^{T} s(t)\,\mathrm{d}t \,\middle|\, T \geq t_0\right\} \\ &\geq -\int_{t_0}^{t_1} s(t)\,\mathrm{d}t \\ &\quad + \sup\left\{-\int_{t_1}^{T} s(t)\,\mathrm{d}t \,\middle|\, T \geq t_1\right\} \\ &= -\int_{t_0}^{t_1} s(t)\,\mathrm{d}t + V(t_1). \end{aligned}$$

This proves the dissipation inequality.

□

The proof is based on the simple principle that a system is dissipative if and only if the 'free' supply is bounded. We use the term 'free' in the sense of 'free energy' as this is used in physics. Note that the construction of V in this proof leads to a non-negative $V \geq 0$. Moreover, if the system is time-invariant, i.e. if $\sigma^t \mathcal{B} = \mathcal{B}$ for all $t \in \mathbb{R}$, then the constructed full behavior of (s, V)'s is also time-invariant. We do not know a simple condition on a time-invariant system $\Sigma = (\mathbb{R}, \mathbb{R}, \mathcal{B})$ for the existence of *any* time-invariant latent variable representation $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathcal{B}_{\text{full}})$ with any storage (not necessarily non-negative, or what is equivalent, not necessarily bounded from below) such that the dissipation inequality holds.

9. QDF's as Supply Rates

Definition 3 gives a clean definition of dissipativity. It simply looks at the rate at which supply goes in and out of a system, and by considering all possible supply rate histories, comes up with a definition of dissipativity.

The main representations of dynamical systems studied in the literature depart either from behaviors defined as the set of solutions of differential equations, or, what basically is a special case, as transfer functions, or from state equations, or, more generally, from differential equations involving latent variables. However, there are also representations that start from image representations. In this section, we study such representations of supply rates.

One way to obtain a supply rate is by assuming that it is generated by a 'local' operator that acts on a *free* signal w, more precisely, a differential operator that acts on $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ to generate s. A very general situation of this type is obtained by a real polynomial in the variables $w_1, w_2, \cdots, w_w$ and their derivatives, and considering the supply rate histories that result from letting this polynomial act on an arbitrary $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. In this section, we examine the situation when the supply rate is generated by a homogeneous quadratic differential operator acting on a vector of free $\mathcal{C}^\infty$-functions and their derivatives. We call such differential operators quadratic differential forms.

Definition 6. A quadratic differential form (QDF) is a finite sum of quadratic expressions in the components of a vector-valued function $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ and its derivatives:

$$\Sigma_{r,k}\left(\frac{d^r}{dt^r}w\right)^\top \Phi_{r,k}\left(\frac{d^k}{dt^k}w\right),$$

with the $\Phi_{r,k} \in \mathbb{R}^{\bullet\times\bullet}$. Note that this defines a map from $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.

Denote by $\mathbb{R}[\zeta, \eta]$ the real polynomial matrices in the indeterminates ζ and η. Two-variable polynomial matrices lead to a compact notation and a convenient calculus for QDF's. Introduce the two-variable polynomial matrix Φ given by

$$\Phi(\zeta, \eta) = \Sigma_{r,k}\Phi_{r,k}\zeta^r\eta^k \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$$

and denote the expression in Definition 6 by $Q_\Phi(w)$. Hence

$$Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}),$$

$$w \mapsto Q_\Phi(w) := \Sigma_{r,k}\left(\frac{d^r}{dt^r}w\right)^\top \Phi_{r,k}\left(\frac{d^k}{dt^k}w\right).$$

Call $\Phi^\star$, defined by $\Phi^\star(\zeta, \eta) := \Phi^\top(\eta, \zeta)$, the *dual* of Φ; $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$ is called $[\![$symmetric$]\!] : \iff [\![\Phi = \Phi^\star]\!]$. Obviously, $Q_\Phi(w) = Q_{\Phi^\star}(w) = Q_{\frac{1}{2}(\Phi+\Phi^\star)}(w)$, which shows that in QDF's we can assume, without loss of generality, that Φ is symmetric. The QDF Q_Φ is said to be $[\![$*non−negative*$]\!]$ (denoted $Q_\Phi \geq 0$): $\iff [\![Q_\Phi(w)(0) \geq 0$ for all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)]\!]$. QDF's have been studied in depth in [12].

We now discuss supply rates defined by QDF's. Thus we consider the dynamical system $(\mathbb{R}, \mathbb{R}, \mathcal{B})$, with behavior $\mathcal{B}$ defined by a two-variable polynomial matrix $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$ as

$$\mathcal{B} = \{s : \mathbb{R} \to \mathbb{R} \,|\, \exists\, w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \text{ such that } s = Q_\Phi(w)\}.$$

Since this behavior is the image of the map Q_Φ, we denote it by im (Q_Φ).

The system $\Sigma_\Phi := (\mathbb{R}, \mathbb{R}\,\mathrm{im}(Q_\Phi))$ obtained this way is time-invariant but clearly nonlinear. At first sight it may appear that a supply rate that is a QDF deals with a rather special situation. But, to the contrary, it covers all controllable linear time-invariant differential systems and quadratic supply rates, as follows.

For elements of $\mathcal{L}^\bullet$, it can be shown that controllability is equivalent to the existence of an image representation. More precisely, $\mathcal{B} \in \mathcal{L}^\bullet$ is controllable if and only if there exists $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $w = M(\frac{d}{dt})\ell$ is a latent variable representation of $\mathcal{B}$, in other words, if and only if $\mathcal{B} = \mathrm{image}(M(\frac{d}{dt}))$, for some $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, with $M(\frac{d}{dt})$ viewed as a map $M(\frac{d}{dt}) : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{coldim}(M)}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{rowdim}(M)})$.

Now assume that we have an element of $\mathcal{B} \in \mathcal{L}^\bullet$, and that we wish to investigate its dissipativity with respect to a supply rate $s = Q_\Phi(w)$. Supply rates often contain derivatives of their own (e.g., in mechanical systems, the power equals $F^T \frac{d}{dt} q$, with F the force, and q the position). Then if $\mathcal{B}$ is controllable, we can use the image representation $w = M(\frac{d}{dt})\ell$ for $\mathcal{B} \in \mathcal{L}^\bullet$ and reduce the dissipativity question with $s = Q_\Phi(w), w \in \mathcal{B}$, to that of $s' = Q_{\Phi'}(\ell)$ with ℓ free, and $\Phi'(\zeta, \eta) = M^\top(\zeta)\Phi(\zeta, \eta)M(\eta)$. Hence this leads to a pure QDF, without constraints on the time-functions that the QDF is acting on. Basically, therefore, the constraints made by restricting attention to QDF's are only: linear, time-invariant, differential, controllable systems, and a QDF in the original system variables for the supply rate. These situations can be reduced to a supply rate behavior im (Q_Φ) for some $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$.

10. Dissipativity of QDF's

The question which we now deal with is to give conditions on the polynomial matrix Φ such that $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ is dissipative, or, more generally for the existence of a storage such that the dissipation inequality is satisfied. The paper [12] deals extensively with these questions. See also [10] for necessary and sufficient conditions for the existence for the existence of a non-negative storage function in the LQ case. The following proposition gives a necessary condition for dissipativity.

Proposition 7

$$[\![\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))\ \textit{dissipative}]\!]$$
$$\Rightarrow [\![\Phi(\lambda, \bar{\lambda}) + \Phi^\top(\bar{\lambda}, \lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \succcurlyeq 0]\!]$$
$$\Rightarrow [\![\Phi(i\omega, -i\omega) + \Phi^\top(-i\omega, i\omega) \succcurlyeq 0, \quad \forall \omega \in \mathbb{R}]\!]$$

In order to obtain necessary and sufficient conditions for dissipativity, we need to analyze the QDF further. Associate with $\Phi = \Phi^\star \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$, $\Phi(\zeta, \eta) = \Sigma_{r,k}\Phi_{r,k}\zeta^r\eta^k$, the matrix

$$\tilde{\Phi} = \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,k} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,k} & \cdots \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \\ \Phi_{r,0} & \Phi_{r,1} & \cdots & \Phi_{r,k} & \cdots \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots \end{bmatrix}.$$

$\tilde{\Phi}$ is symmetric, and, while infinite, it has only a finite number of non-zero entries. Consider the number of its positive and negative eigenvalues and its rank and, since they are uniquely determined by Φ, denote these by $\pi(\Phi)$, $\nu(\Phi)$, and $\mathrm{rank}(\Phi) = \pi(\Phi) + \nu(\Phi)$, respectively. $\tilde{\Phi}$ can be factored as $\tilde{\Phi} = \tilde{F}_+^\top \tilde{F}_+ - \tilde{F}_-^\top \tilde{F}_-$, with $\tilde{F}_+$ and $\tilde{F}_-$ matrices with an infinite number of columns but a finite number of rows. In fact, the number of rows of $\tilde{F}_+$ and $\tilde{F}_-$ can be taken to be equal to $\pi(\Phi)$ and $\nu(\Phi)$, respectively: $\mathrm{rowdim}(\tilde{F}_+) = \pi(\Phi)$ and $\mathrm{rowdim}(\tilde{F}_-) = \nu(\Phi)$ if and only if the rows of $\tilde{F} = \begin{bmatrix} \tilde{F}_+ \\ \tilde{F}_- \end{bmatrix}$ are linearly independent over $\mathbb{R}$. Define

$$F_+(\xi) = \tilde{F}_+\begin{bmatrix} I_w & I_w\xi & I_w\xi^2 & \cdots \end{bmatrix}^\top,$$
$$F_-(\xi) = \tilde{F}_-\begin{bmatrix} I_w & I_w\xi & I_w\xi^2 & \cdots \end{bmatrix}^\top.$$

This yields the factorization

$$\Phi(\zeta, \eta) = F_+^\top(\zeta)F_+(\eta) - F_-^\top(\zeta)F_-(\eta),$$

with $F_+ \in \mathbb{R}^{\bullet\times w}[\xi], F_- \in \mathbb{R}^{\bullet\times w}[\xi]$, yielding a decomposition of a QDF into a sum and difference of squares:

$$Q_\Phi(w) = |F_+\left(\frac{d}{dt}\right)w|^2 - |F_-\left(\frac{d}{dt}\right)w|^2.$$

The (controllable) linear time-invariant differential system with image representation

$$\begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \begin{bmatrix} F_+(\frac{d}{dt}) \\ F_-(\frac{d}{dt}) \end{bmatrix} w$$

plays an important role in the sequel. The above also holds, *mutatis mutandis*, for non-symmetric $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$, by replacing Φ by its symmetric part $\frac{1}{2}(\Phi + \Phi^*)$. We will use the notation $\pi(\Phi) = \pi(\frac{1}{2}(\Phi + \Phi^*))$, and $\nu(\Phi) = \nu(\frac{1}{2}(\Phi + \Phi^*))$ also in the non-symmetric case.

Hence every QDF can be factored as a sum and difference of squares:

$$Q_\Phi(w) = |F_+\left(\frac{d}{dt}\right)w|^2 - |F_-\left(\frac{d}{dt}\right)w|^2.$$

Define

$$F = \begin{bmatrix} F_+ \\ F_- \end{bmatrix} \in \mathbb{R}^{\bullet\times w}[\xi].$$

It is easy to see that for $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ with $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$, we can always assume that $\mathrm{rank}(F) = \dim(\Phi)$ to begin with, in the sense that for any $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$, there exists $\Phi' \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$ such that $\mathrm{im}(Q_{\Phi'}) = \mathrm{im}(Q_\Phi)$ and $\mathrm{rank}(F') = \dim(\Phi')$, leading to $F' = \begin{bmatrix} F'_+ \\ F'_- \end{bmatrix}$ corresponding to the factorization of $Q_{\Phi'}(w') = |F'_+(\frac{d}{dt})w'|^2 - |F'_-(\frac{d}{dt})w'|^2$ into a sum and difference of squares. Assume therefore that $\mathrm{rank}(\Phi) = \dim(\Phi)$. It can then be shown, using Proposition 7, that dissipativity implies that we can always assume that $\pi(\Phi) \geq \dim(\Phi)$. Of special interest is the situation in which there is a minimum number of positive squares: $\pi(\Phi) = \dim(\Phi)$. Then F_+ is square with $\det(F_+) \neq 0$. In this case, we can obtain a complete characterization of dissipativity of a QDF.

Recall the definition of the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ norms of $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$:

$$\|G\|_{\mathcal{L}_\infty} := \sup\{|G(i\omega)| \,|\, \omega \in \mathbb{R}\},$$
$$\|G\|_{\mathcal{H}_\infty} := \sup\{|G(s)| \,|\, s \in \mathbb{C}, \mathrm{Re}(s) \geq 0\},$$

where $|\cdot|$ denotes the matrix norm induced by the Euclidean norms. Note that $\|G\|_{\mathcal{L}_\infty} < \infty$ if and only if G is proper and has no poles on the imaginary axis, and that $\|G\|_{\mathcal{H}_\infty} < \infty$ if and only if G is proper and has no poles in the closed right half of the complex plane.

Theorem 8. Consider $\Phi = \Phi^\star \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}$, with $\pi(\Phi) = \mathrm{rank}(\Phi) = \dim(\Phi)$. The following are equivalent:

(i) $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ is dissipative,

(ii) there exists $\Psi \in \mathbb{R}[\zeta, \eta]^{\bullet\times\bullet}, Q_\Psi \geq 0$, such that
$$\frac{d}{dt}Q_\Psi(w) \leq Q_{\Phi(w)} \quad \forall\, w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w),$$

(iii) $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0 \quad \forall w \in \mathcal{D} \quad (\mathbb{R}, \mathbb{R}^{w})$,

(iv) $\Phi(\lambda, \bar{\lambda}) \geq 0 \quad \forall \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq 0$,

(v) $\|G\|_{\mathcal{H}_\infty} \leq 1$, with G defined as follows. Assume that Φ is given in terms of $F_+, F_- \in \mathbb{R}^{\bullet \times w}[\xi]$ by $F_+^\top(\zeta)F_+(\eta) - F_-^\top(\zeta)F_-(\eta)$, with $F_+ \in \mathbb{R}^{w \times w}[\xi]$, $F_- \in \mathbb{R}^{\bullet \times w}[\xi]$, and $\det(F_+) \neq 0$. Then $G := F_- F_+^{-1}$.

Theorem 8 applies to all situations in which the positive signature of Φ is equal to its dimension. The following theorem deals with another such situation, relevant to supply rates of the form $w_1^\top w_2$, as encountered in electrical circuits.

Theorem 9. Assume that $\Phi \in \mathbb{R}[\zeta, \eta]^{\bullet \times \bullet}$ is given by $\Phi(\zeta, \eta) = F_1^\top(\zeta)F_2(\eta)$, with $F_1, F_2 \in \mathbb{R}^{w \times w}[\xi]$, and $\det(F_1) \neq 0$. Define $G \in \mathbb{R}(\xi)^{w \times w}$ by $G = F_2 F_1^{-1}$. The following are equivalent:

(i) $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ is dissipative,

(ii) there exists $\Psi \in \mathbb{R}[\zeta, \eta]^{\bullet \times \bullet}, Q_\Psi \geq 0$, such that

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \quad \forall w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})^{w},$$

(iii) $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0 \quad \forall w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{w})$,

(iv) $\Phi(\lambda, \bar{\lambda}) + \Phi^\top(\bar{\lambda}, \lambda) \geq 0$

$\forall \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq 0$,

(v) G is positive real, i.e.

$$G(\lambda) + G^\top(\bar{\lambda}) \geq 0 \textit{ for } \mathrm{Re}(\lambda) > 0.$$

Important in the above theorem is the equivalence of dissipativity of a supply rate that is a QDF with the existence of a storage that is also a QDF:

$\exists \Psi \in \mathbb{R}[\zeta, \eta]^{\bullet \times \bullet}, Q_\Psi \geq 0$, such that

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \quad \forall w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w}).$$

It is easy to see (by writing this out in terms of the matrices associated with these QDF's) that this is an LMI in the space of two-variable polynomial matrices, with Φ given and Ψ an unknown.

We refer to [10] for more details on the material in this and the previous section.

It follows that the use of image representations greatly facilitates the analysis of dissipativity and the view that this question is an LMI. An obvious avenue of generalization is to deal with general polynomials in the vector $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$, and its derivatives, and analyze dissipativity by SOS methods.

11. The Storage as a State Function

In the classical definitions of the storage, it was assumed to be a state function. However, this is something one would like to prove rather than postulate. In fact, circumventing the explicit assumption that the storage is a state function is one of the main motivations that led to Definition 4. For supply rates that are QDF's, we can indeed prove that the storage is a state function. Assume that a behavior $\mathcal{B} \in \mathcal{L}^\bullet$ is given in terms of the latent variables x by

$$Bw + Ax + E\frac{d}{dt}x = 0,$$

with $A, B, E \in \mathbb{R}^{\bullet \times \bullet}$ constant matrices. The variables x are *state* variables. In fact, it can be shown that, for linear time-invariant differential systems, the state property is equivalent to the existence of such a representation by means of a differential equation that is first order in x and zero-th order in w.

The expansion of Q_Φ as $Q_\Phi(w) = |F_+(\frac{d}{dt})w|^2 - |F_-(\frac{d}{dt})w|^2$ leads to a state representation of a QDF, as follows. Let $Bf + Ax + E\frac{d}{dt}x = 0$ be a state representation of the (controllable) system in image representation

$$f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \begin{bmatrix} F_+(\frac{d}{dt}) \\ F_-(\frac{d}{dt}) \end{bmatrix} w.$$

Then

$$B\begin{bmatrix} f_+ \\ f_- \end{bmatrix} + Ax + E\frac{d}{dt}x = 0, s = |f_+|^2 - |f_-|^2$$

is a state representation of Q_Φ. In fact, by further partitioning the variables f_+ and f_- component-wise in inputs and outputs, we arrive at the following input/state/output representation of a QDF:

$$\frac{d}{dt}x = Ax + B\begin{bmatrix} u_+ \\ u_- \end{bmatrix},$$
$$\begin{bmatrix} y_+ \\ y_- \end{bmatrix} = Cx + D\begin{bmatrix} u_+ \\ u_- \end{bmatrix},$$
$$s = |u_+|^2 + |y_+|^2 - |u_-|^2 - |y_-|^2.$$

In [9] the notion of state is brought to bear on the storage. Assume that Q_Ψ satisfies the dissipation inequality

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \quad \forall w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w}).$$

Then it can be shown that Q_Ψ is actually a memoryless state function, i.e. there exists a matrix $K \in \mathbb{R}^{\bullet\times\bullet}$ such that

$$\left[\left(\begin{bmatrix} f_+ \\ f_- \end{bmatrix}, x\right) \text{ satisfies } B\begin{bmatrix} f_+ \\ f_- \end{bmatrix} + Ax + E\frac{\mathrm{d}}{\mathrm{d}t}x = 0 \right.$$
$$\left.\text{and } \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \begin{bmatrix} F_+(\frac{\mathrm{d}}{\mathrm{d}t}) \\ F_-(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} w\right] \Rightarrow \left[Q_\Psi(w) = x^\top K x\right].$$

If $Q_\Psi \geq 0$, then K can be taken to be symmetric and non-negative definite: $K = K^\top \geq 0$.

Summarizing, consider the following seven statements concerning the system $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ defined by a QDF.

- (i) Σ_Φ is dissipative,
- (ii) Σ_Φ admits a latent variable representation with a non-negative storage,
- (iii) Σ_Φ admits a latent variable representation with a non-negative QDF as storage,
- (iv) Σ_Φ admits a latent variable representation with a non-negative memoryless state function as storage,
- (v) Σ_Φ admits a latent variable representation with a non-negative memoryless quadratic state function as storage,
- (vi) $\int_{-\infty}^{0} Q_\Phi(w)\, dt \geq 0 \ \forall w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$,
- (vii) The frequency domain and Pick matrix condition of [9, condition 3 of Theorem 9.3] on Φ.

The following implications have been shown: (i) $\iff$ (ii) $\Leftarrow$ (iii) $\iff$ (iv) $\iff$ (v) $\iff$ (vi) '$\iff$' (vii) ('$\iff$') because there are additional assumptions in (vii)). This raises the question if (ii) $\Rightarrow$ (iii), i.e. if, assuming that the supply rate is a QDF, the existence of a non-negative storage is equivalent to the existence of a non-negative storage that is a QDF. We conjecture that this is the case.

If the signature condition $\pi(\Phi) = \dim(\Phi)$ of Theorem 8 holds, then we have proven that all these conditions are equivalent, in fact, with the frequency domain condition (vii) made more precise as an $\mathcal{H}_\infty$-norm condition.

It is useful to contrast this with the situation in which non-negativity of the storage is not required. Consider the following six statements concerning the system $\Sigma_\Phi = (\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi))$ defined by a QDF.

- (i) ' Σ_Φ admits a latent variable representation with a storage,
- (ii) ' Σ_Φ admits a latent variable representation with a QDF as storage,
- (iii) ' Σ_Φ admits a latent variable representation with a memoryless state function as storage,
- (iv) ' Σ_Φ admits a latent variable representation with a memoryless quadratic state function as storage,
- (v) ' $\int_{-\infty}^{+\infty} Q_\Phi(w)\, \mathrm{d}t \geq 0 \quad \forall w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$,
- (vi) ' $\Phi(i\omega, -i\omega) + \Phi^\top(-i\omega, i\omega) \geq 0 \ \forall \omega \in \mathbb{R}$.

The following implications have been shown: (ii)' $\Leftarrow$ (iii)' $\iff$ (iv)' $\iff$ (v)' $\iff$ (vi)' $\iff$ (vii)'. This raises the question if (ii)' $\Rightarrow$ (iii)', i.e. if assuming that the supply rate is a QDF, the existence of a storage is equivalent to the existence of a storage that is a QDF. We conjecture that also this is the case.

12. Synthesis of Passive Behaviors

We now pick up the discussion of a one-port circuit initiated in Section 2.1. Assume that we have a two-terminal circuit containing (a finite number of) passive elements: positive resistors, positive capacitors, positive inductors, transformers, and gyrators. It can be shown that the circuit acts as a port. Since all the laws (the constitutive laws of the elements and the interconnection laws) are linear, time-invariant, and differential, it follows (from what is called the 'elimination theorem') that the port behavior is also linear, time-invariant, and differential, implying that there exist a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet\times 2}$ such that $w = (V, I)$ belongs to the port behavior if and only if $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. From general consideration, only having to do with linear, time-invariant, differential, it follows that we can take R to have zero, one, or two rows. The fact that it is the port behavior of a passive circuit allows us to conclude that we can take $0 \neq R \in \mathbb{R}[\xi]^{1\times 2}$. More precisely, there exist $P, Q \in \mathbb{R}[\xi]$, not both zero, such that the the behavior is described by the kernel representation

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V = Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)I.$$

P and Q need not be co-prime. In fact, the specific circuit of Section 2.1, with $R_L = R_C = 1, L = 1, C = 1$ provides a not co-prime example. Co-primeness, in fact, means controllability of the port behavior. There are further conditions on (P, Q) that follow from the fact that the kernel representation describes the port behavior of a passive circuit. *But what are these conditions?* Clearly, the port behavior must be dissipative in the sense of Definition 3. But is this all? And what does this mean in terms of P, Q?

In the controllable case, these questions have been answered unequivocally. If the circuit is port controllable, then its kernel representation has P, Q co-prime. Assume that both P and Q are non-zero (of course, $P=1, Q=0$ or $P=0, Q=1$ are also possible, but we ignore these degenerate cases, these can be dealt with directly). Define $Z := P^{-1}Q \in \mathbb{R}(\xi)$, called the *driving point impedance* of the circuit. The port behavior allows an image representation,

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix} w,$$

with $D, N \in \mathbb{R}[\xi]$, and these may be chosen to be coprime. The conditions on D, N for passivity are precisely those of Theorem 9, with $F_1 \mapsto D, F_2 \mapsto N$. In particular, passivity requires that the driving point impedance is $Z = P^{-1}Q = ND^{-1}$ is positive real. That positive realness is necessary follows from Theorem 9. That it is also sufficient has an even more illustrious history: it requires synthesizing a positive real impedance using passive elements.

That a positive real impedance $Z(s)$ can be realized by means of a circuit containing positive resistors, positive capacitors, positive inductors, and transformers is a classical result (one of the highlights of electrical engineering) due to Brune [3]. Later, Bott and Duffin [2] proved that transformers are not needed. This synthesis theory is nicely explained in [14]. But the transfer function only captures the controllable part of the port behavior. These authors were not concerned with the question of obtaining a *controllable* port behavior with the correct impedance. In other words, the question whether the resulting polynomials P, Q in the kernel representation are also coprime, in addition to obeying $Z = P^{-1}Q$, was not considered. Actually, from the approach to circuit synthesis explained in [1], we can conclude that a positive real impedance can always be realized as the port behavior of a *port controllable* RLCT circuit in which the physical state, the inductor currents and the capacitor voltages, is observable from the port variables. Bott-Duffin's transformerless synthesis on the other hand usually leads to a non-controllable RLC circuit with the correct impedance, but not with the correct behavior if we define the behavior to be prespecified and controllable.

The question of which non-controllable port behaviors are realizable as passive *RLCT* circuits is an open problem. Necessary conditions are positive realness of the impedance $P^{-1}Q$ and stability of the common factor (i.e. only roots in the closed left part of the complex plane, and simple roots on the imaginary axis), but what additional conditions on the non-controllable part must hold for realizability is unknown. In particular, it is doubtful that for the one-port case, realizable RLCT behaviors are always realizable without transformers. In other words, Bott-Duffin did not resolve the transfomerless synthesis question in the sense of behaviors.

We emphasize that the external port behavior of a passive circuit may or may not be controllable (in the behavioral sense). For example, the specific circuit of Section 2.1 is not controllable if and only if $CR_C = L/R_L$ and $R_C = R_L$, in which case the circuit admits, in addition to (1), also the state representation

$$\begin{aligned} R_C C \frac{d}{dt}(R_L I_L - V_C) &= -(R_L I_L - V_C), \\ V &= (R_L I_L - V_C) + R_C I, \end{aligned} \tag{4}$$

which puts lack of controllability in evidence. Note that this circuit has impedance R_C, but that the behavior is different from the circuit with behavioral equation $V = R_C I$: non-controllable modes do matter in describing the port behavior of a physical circuit. Also observability is an issue. But what should we mean with observability (in the behavioral sense) of a circuit? Of course, we could mean this to refer to the possibility of deducing the 'physical' state, the inductor currents and the capacitor voltages, from the external port voltage and current. In this case the specific circuit of Section 2.1 is observable if and only if $CR_C \neq L/R_L$. If $CR_C = L/R_L$, then the natural storage, the stored energy, is not observable, it is not a function of the minimal state. For example, the state of the system (4) is observable, $Q(R_L I_L - V_C)^2$ with $Q > 0$ is an observable storage, but it is not equal to the physical stored energy. All this shows that assuming controllability and/or observability of the storage are far from evident assumptions.

It is instructive in this context to examine the equations, describing the circuit of Section 2.1 when $CR_C = L/R_L$. We started from an (unobservable) realization with two reactive elements. However, in the controllable case ($R_L \neq R_C$), there also exists an (observable) realization with only one reactive element. If, for example, $R_L > R_C$, then this port behavior can be realized as an RC-circuit consisting of a resistor in series with a parallel connection of a resistor and an RC-section consisting of a resistor in series with a capacitor. However, in the uncontrollable case, $CR_C = L/R_L, R_L = R_C$, it can be shown that the *RLCT* synthesis requires two reactive elements. Realizations with only one reactive elements require gyrators. So, reciprocal *RLCT* synthesis of non-controllable behaviors may require more than the minimal number of reactive elements, leading to a non-observable physical storage.

13. Distributed Systems

The results of Sections 9 and 10 can generalized to n-D systems, in particular to systems described by linear constant coefficient PDE's. Controllability, suitably defined, is again equivalent to the existence of an image representation. Maxwell's equations are a very important example of a linear shift-invariant differential system that is controllable. The linear quadratic case leads again to QDF's, now parametrized by real polynomials in 2n-variables, where n is the number of independent variables (often $n = 4$, reflecting time and space). The construction of storage function leads *linea recta* to Hilbert's 17-th problem on the factorization of a non-negative real polynomial in n variables as a sum of squares (the SOS problem is also relevant for obtaining conditions for dissipativity of nonlinear systems with polynomial right hand sides [5]).

An important consequence of the fact that this factorization can only be carried out over the rational functions is that the storage function is not observable, or, more precisely, that it is a function of a latent variable that is not observable from the manifest variables that enter in the PDE model and in the supply rate. This feature, it turns out, is already present in the internal energy for Maxwell's equations for electromagnetic fields. We refer to [8] for details. In the ECC presentation, the case of distributed dissipative systems will be presented.

Acknowledgments

I would like to thank Harry Trentelman who has worked with me on this topic since many years, and to Kiyotsugu Takaba for collaborating on [13].

This research is supported by the Research Council KUL project CoE EF/05/006 (OPTEC), Optimization in Engineering, and by the Belgian Federal Science Policy Office: IUAP P6/04 (Dynamical systems, Control and Optimization, 2007–2011).

References

1. Anderson BDO, Vongpanitlerd S. Network Analysis and Synthesis: A Modern Systems Theory Approach, by Brian D. O. Anderson, Prentice-Hall, 1973
2. Bott R, Duffin RJ. Impedance synthesis without transformers, J Appl Phys 1949; 20: 816
3. Brune O. Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency. J Math Phys 1931; 10: 191–236
4. Ebenbauer C, Allgöwer F. Advances in control theory and applications. In: Bonivento C, Isidori A, Marconi L, Rossi C. (eds). A dissipation inequality for the minimum phase property and performance limitations, vol 353, Lecture notes in control and information sciences series. Springer-Verlag, 2007
5. Ebenbauer C, Allgöwer F. Analysis and design of polynomial control systems using dissipation inequalities and sum of squares. Comp & Chem Eng 2006; 30: 1601–1614
6. Haddad WM, Chellaboina V, Nersenov S. Thermodynamics, A Dynamical Systems Approach, Princeton University Press, 2005
7. Polderman JW, Willems JC. Introduction to Mathematical Systems Theory: A Behavioral Approach, Springer-Verlag, 1998
8. Pillai H, Willems JC. Lossless and dissipative distributed systems. SIAM J Control Optim 2002; 40: 1406–1430
9. Trentelman HL, Willems JC. Every storage function is a state function. Syst Control Lett 1997; 32: 249–259
10. Trentelnan HL, Rapisarda P. Pick matrix conditions for sign-definite solutions of the algebraic Riccati equation. SIAM J Control Optim 2001; 40(3): 969–991
11. Willems JC. Dissipative dynamical systems – Part I: General theory, Part II: Linear systems with quadratic supply rates. Arch Ration Mech Anal 1972; 45: 321–351, 352–393
12. Willems JC, Trentelman HL. On quadratic differential forms. SIAM J Control Optim 1998; 36: 1703–1749
13. Willems JC, Takaba K. Dissipativity and stability of interconnections. Int J Robust Nonlinear Control, to appear
14. Willems JC. Review of the book thermodynamics, A Dynamical Systems Approach, by Wassim M. Haddad, VijaySekhar Chellaboina, and Sergey Nersesov, Princeton Series in Applied Mathematics, Princeton University Press, 2005, IEEE Trans Autom Control 2006; 51: 1217–1225