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Markov properties for systems described by PDEs and first-order representations

Paula Rocha[a,*], Jan C. Willems[b]

[a]Department of Mathematics, University of Aveiro, Campo de Santiago, 3810-193 Aveiro, Portugal
[b]K.U. Leuven, ESAT/SCD (SISTA), Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium

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Abstract

The relation between Markovianity and representability by means of first-order PDEs is investigated. We consider two versions of the Markovian property, weak and strong-Markovianity. The weak version has been introduced in [J.C. Willems, State and first-order representations, in: V.D. Blondel, A. Megretski (Eds.), Unsolved Problems in Mathematical Systems & Control Theory, Princeton University Press, Princeton, NJ, 2004, pp. 54–57] and conjectured to correspond to first-order representations. We provide a counterexample to this conjecture. For finite-dimensional behaviors, strong-Markovianity is proven to be indeed equivalent to the representability by means of first-order PDEs with a special structure.

Keywords: Behaviors; PDEs; Markov property; First-order representation

1. Introduction

Representing a dynamical system by means of first-order differential or difference equations not only guarantees easier recursive computations, but, in some cases, also allows to capture the system memory. Indeed, as shown in [2], the representability of a linear system with $\mathbb{R}$ or $\mathbb{Z}$ as time-axis by means of first-order linear equations is equivalent to the one-dimensional Markov property. A dynamical system with $\mathbb{R}$ or $\mathbb{Z}$ as time-axis is said to be *Markovian* whenever the concatenation of two system trajectories w_1, w_2 that coincide at one point (i.e., $w_1(t) = w_2(t)$, for some t) yields a function w (coinciding with w_1 on $(-\infty, t]$ and with w_2 on $[t, +\infty)$) which is still an admissible system trajectory [2]. This is a deterministic version of the stochastic Markovianity: independence of past and future given the present. The relation between first-order representations and the memory property is quite different for multidimensional systems: the existing results [3,4] deal mainly with discrete two-dimensional (2D) (meaning that the set of independent variables is $\mathbb{Z}^2$) systems, and show that a direct generalization of the Markov property for 1D systems (which in the sequel will be referred to as the weak-Markov property) does not correspond to the representability by means of first-order partial difference equations. However a stronger generalization has been introduced (the strong Markov property) which does correspond to the existence of first-order representations with, in fact, a special structure [5].

In this article, we consider systems described by linear constant coefficient PDEs, hence with a continuous set of independent variables equal to $\mathbb{R}^n$. Recently, a conjecture has been presented in [7], according to which these systems are thought to behave differently from the discrete ones, and the weak-Markov property is thought to be equivalent to the representability by means of a system of first-order linear PDEs. One of our purposes is to analyze this conjecture. After showing that it does not hold true, we prove that, for the particular case of *finite-dimensional* behaviors, it is a stronger version of the Markov property that indeed corresponds to representability by means of a system of first-order PDEs. This first-order representation is endowed with a special structure, since it exhibits a decoupling of the elementary partial differential operators. The

* Corresponding author. Tel.: +351 234370359; fax: +351 234382014.
E-mail addresses: procha@mat.ua.pt (P. Rocha), Jan.Willems@esat.kuleuven.ac.be (J.C. Willems).

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question whether the equivalence between strong-controllability and first-order representations also holds for general, not necessarily finite-dimensional, behaviors of PDEs, remains open.

2. nD Markovian properties

We consider multidimensional (nD) behavioral systems that can be represented as the solution set of a system of linear PDEs with constant coefficients. Formally, let $R \in \mathbb{R}^{\bullet\times w}[s_1, \ldots, s_n]$ (the real polynomial matrices in n variables with w columns). Associate with R the following system of PDEs

$$R\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) w = 0. \tag{1}$$

We define the behavior to be the set of solutions of this system of PDE's. There are many, more or less equivalent, ways to define this solution set: $\mathscr{C}^\infty$ solutions, distributions, etc. For the purposes of this paper it is convenient to consider the continuous solutions. Hence

$$\mathfrak{B} = \{w \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{R}^w) \mid \text{(1) holds in the distributional sense}\}.$$

As $\mathfrak{B}$ is the kernel of a partial differential operator, we refer to it as a *kernel behavior*, and denote it as $\ker(R(\partial/\partial x_1, \ldots, \partial/\partial x_n))$. The PDE (1) is called a *kernel representation* of $\mathfrak{B} = \ker(R(\partial/\partial x_1, \ldots, \partial/\partial x_n))$.

As mentioned in the introduction, the question which we investigate is the connection between the fact that a behavior $\mathfrak{B}$ is Markovian (in a sense to be made precise soon) and the possibility of representing it as the kernel of a system of *first-order* PDEs

$$R_0 w + R_1 \frac{\partial}{\partial x_1} w + \cdots + R_n \frac{\partial}{\partial x_n} w = 0. \tag{2}$$

We consider two versions of Markovianity. The first is the one used in [7]. We call it *weak-Markovianity*. Define Π to be the set of 3-way partitions (S_-, S_0, S_+) of $\mathbb{R}^n$ such that S_- and S_+ are open and S_0 is closed;given a partition $\pi = (S_-, S_0, S_+) \in \Pi$ and a pair of trajectories (w_-, w_+) that coincide on S_0, define the *concatenation* of (w_-, w_+) along π as the trajectory $w_- \wedge|_\pi w_+$ that coincides with w_- on $S_0 \cup S_-$ and with w_+ on$S_0 \cup S_+$.

Definition 1. A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^w)^{\mathbb{R}^n}$ is said to be *weak-Markovian* if for any partition $\pi \in \Pi$ and any pair of trajectories $w_-, w_+ \in \mathfrak{B}$ such that $w_{-|S_0} = w_{+|S_0}$, the trajectory $w_- \wedge|_\pi w_+$ is also an element of $\mathfrak{B}$.

The second version of Markovianity is called strong-Markovianity. It requires concatenability along partitions of linear subspaces of $\mathbb{R}^n$. Given a subspace $S \subseteq \mathbb{R}^n$, let Π_S be the set of 3-way partitions (S_-, S_0, S_+) of S such that S_- and S_+ are open (in S) and S_0 is closed (in S).

Definition 2. A multidimensional behavior $\mathfrak{B} \subseteq (\mathbb{R}^w)^{\mathbb{R}^n}$ is said to be *strong-Markovian* if for any subspace S, any partition $\pi_S \in \Pi_S$, and any pair of trajectories $w_-, w_+ \in \mathfrak{B}|_S$ such that $w_{-|S_0} = w_{+|S_0}$, the trajectory $w_- \wedge|_\pi w_+$ is an element of $\mathfrak{B}|_S$.

Obviously, strong-Markovianity implies weak-Markovianity. Note that strong-Markovianity coincides with weak-Markovianity for one-dimensional behaviors, and both can therefore be regarded as a generalization of the 1D Markov property.

Let $\mathfrak{B}$ be a behavior defined by a first-order PDE (1). It is easy to see that this implies weak-Markovianity. The question arises whether a behavior as (1) that is weak-Markovian admits an equivalent first-order representation (2) (equivalent in the sense that they have the same behavior). We provide a counterexample showing that, contrary as was put forward in [7], this converse does not hold true. The analogous questions arise for strong-Markovianity. Do first-order PDEs generate behaviors that are strong-Markovian? Do strongly Markovian behaviors of PDE's (1) admit equivalent first-order representations (2)? We will prove that for *finite-dimensional* behaviors, strong-Markovianity is equivalent to representability by means of a special type of first-order PDEs.

3. Weak-Markovianity and first-order representations

The next example shows that, similar to what happens in the discrete case, the direct generalization of the one-dimensional Markov property does not necessarily lead to the desired type of first-order representations, implying that the conjecture in [7] is false.

Consider the behavior $\mathfrak{B} \subseteq \mathscr{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$\mathfrak{B} = span\left\{\begin{bmatrix}1\\1\end{bmatrix}, e^x\begin{bmatrix}1\\0\end{bmatrix}, e^y\begin{bmatrix}0\\1\end{bmatrix}, e^{x+y}\begin{bmatrix}1\\-1\end{bmatrix}\right\}. \tag{3}$$

Obviously,

$$\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)$$

with

$$R(s_1, s_2) = \begin{bmatrix}(s_1-1)(s_2-1) & -(s_1-1)(s_2-1)\\ 0 & s_1(s_2-1)\\ s_2(s_1-1) & 0\\ s_1 s_2 & s_1 s_2\end{bmatrix}.$$

We will show that this behavior is weak-Markovian, but does not allow a first-order representation of form (2).

In order to check that $\mathfrak{B}$ is weak-Markovian, we show that if two trajectories w_1 and w_2 in $\mathfrak{B}$ coincide on two different points (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$, then they are the same trajectory. This obviously implies that any two trajectories coinciding on a set S_0 of a partition $\pi = (S_-, S_0, S_+) \in \Pi$ are concatenable in $\mathfrak{B}$. Assume that

$$w_1(x, y) = a_1\begin{bmatrix}1\\1\end{bmatrix} + b_1 e^x\begin{bmatrix}1\\0\end{bmatrix} + c_1 e^y\begin{bmatrix}0\\1\end{bmatrix} + d_1 e^{x+y}\begin{bmatrix}1\\-1\end{bmatrix},$$

and

$$w_2(x, y) = a_2\begin{bmatrix}1\\1\end{bmatrix} + b_2 e^x\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^y\begin{bmatrix}0\\1\end{bmatrix} + d_2 e^{x+y}\begin{bmatrix}1\\-1\end{bmatrix}$$

are two trajectories in $\mathfrak{B}$ such that $w_1(x_1, y_1) = w_2(x_1, y_1)$ and $w_1(x_2, y_2) = w_2(x_2, y_2)$, with $(x_1, y_1) \neq (x_2, y_2)$. This means that

$$\begin{aligned}(a_1-a_2)+(b_1-b_2)e^{x_1}+(d_1-d_2)e^{x_1+y_1}=0,\\ (a_1-a_2)+(c_1-c_2)e^{y_1}-(d_1-d_2)e^{x_1+y_1}=0,\\ (a_1-a_2)+(b_1-b_2)e^{x_2}+(d_1-d_2)e^{x_2+y_2}=0,\\ (a_1-a_2)+(c_1-c_2)e^{y_2}-(d_1-d_2)e^{x_2+y_2}=0.\end{aligned}$$

or, equivalently,

$$\underbrace{\begin{bmatrix}1 & e^{x_1} & 0 & e^{x_1+y_1}\\ 1 & 0 & e^{y_1} & -e^{x_1+y_1}\\ 1 & e^{x_2} & 0 & e^{x_2+y_2}\\ 1 & 0 & e^{y_2} & -e^{x_2+y_2}\end{bmatrix}}_{=:A}\begin{bmatrix}a_1-a_2\\ b_1-b_2\\ c_1-c_2\\ d_1-d_2\end{bmatrix}=0. \tag{4}$$

Since $\det(A)=e^{x_1+y_1+x_2+y_2}[e^{-(x_1-x_2)}(e^{x_1-x_2}-1)^2+e^{-(y_1-y_2)}(e^{y_1-y_2}-1)^2]$, which is clearly nonzero for $(x_1,y_1)\neq(x_2,y_2)$, we conclude that the only solution of (4) is the zero solution. In other words, we must have $a_1=a_2$, $b_1=b_2$, $c_1=c_2$, $d_1=d_2$, which means that $w_1=w_2$ as claimed.

We next show that $\mathfrak{B}$ does not allow a first-order representation. For that purpose we assume, to the contrary, that there exist real matrices R_0, R_1 and R_2, with two columns and the same number of rows, such that $\mathfrak{B}=\ker(R_0+R_1\partial/\partial x+R_2\partial/\partial y)$. Since the elements of the generating set in (3) are then obviously in $\ker(R_0+R_1\partial/\partial x+R_2\partial/\partial y)$, we have that

$$R_0\begin{bmatrix}1\\1\end{bmatrix}=0,$$

$$(R_0+R_1)\begin{bmatrix}1\\0\end{bmatrix}=0,\quad (R_0+R_2)\begin{bmatrix}0\\1\end{bmatrix}=0,$$

$$(R_0+R_1+R_2)\begin{bmatrix}1\\-1\end{bmatrix}=0.$$

Therefore, there exist column vectors X, Y such that

$$\begin{aligned}R_0+R_1s_1+R_2s_2&=[X(1-s_1)+Ys_2\ \ X(s_2-1)+Ys_1]\\ &=[X\ Y]Q(s_1,s_2),\end{aligned}$$

with

$$Q(s_1,s_2)=\begin{bmatrix}1-s_1 & s_2-1\\ s_2 & s_1\end{bmatrix}.$$

Consequently, $\ker(Q(\partial/\partial x,\partial/\partial y))\subseteq\ker(R_0+R_1\partial/\partial x+R_2\partial/\partial y)$. But this contradicts the fact that $\mathfrak{B}$ is finite-dimensional, since $\ker(Q(\partial/\partial x,\partial/\partial y))$ contains infinitely many linearly independent trajectories of the form $w(x,y)=e^{\alpha x+\beta y}w_0$, with (α,β) roots of $\det(Q(s_1,s_2))=-s_1^2-s_2^2+s_1+s_2$ and $0\neq w_0\in\mathbb{R}^2$ the associated solution of $Q(\alpha,\beta)w_0=0$. In this way we conclude that the given behavior cannot be represented by means of a set of first-order PDEs.

This example suggests that in order to guarantee first-order representability one should consider a stronger version of the Markov property. We will now examine if strong-Markovianity achieves this.

4. PDE's with a finite-dimensional behavior

In this section, we examine finite-dimensional behaviors. Of course, if the solution set of (1) is finite dimensional, all its elements are in $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}})$. Moreover, this set allows very special representations, as stated in the following result. In here we use the notion of a latent variable representation, a standard notion from the behavioral theory.

Proposition 1. *Let $\mathfrak{B}\subseteq\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}})$ be a finite-dimensional* n*D behavior that is the kernel of a PDE. Then it can be represented by a latent variable model of the form*

$$\begin{cases}\frac{\partial}{\partial x_1}z=A_1z,\\ \quad\vdots\\ \frac{\partial}{\partial x_{\mathtt{n}}}z=A_{\mathtt{n}}z,\\ \ w=Cz,\end{cases} \tag{5}$$

where $A_1,\dots,A_{\mathtt{n}}$ are square pairwise commuting matrices of size $\mathtt{N}=\dim(z)$, $z\in\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{N}})$ is the latent variable, and $w\in\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}})$ is the system variable. Note that $z(x_1,\dots,x_{\mathtt{n}})=Ce^{A_1x_1+\cdots+A_{\mathtt{n}}x_{\mathtt{n}}}z(0,\dots,0)$. Moreover, $(C;A_1,\dots,A_{\mathtt{n}})$ can be taken to be observable, in the sense that if $Ce^{A_1x_1+\cdots+A_{\mathtt{n}}x_{\mathtt{n}}}z(0,\dots,0)=0$ for all $x_i\in\mathbb{R}$, $i=1,\dots,\mathtt{n}$, then $z(0,\dots,0)=0$.

In order to prove this proposition, we make use of the following auxiliary lemma.

Lemma 1. *Let $\Phi_1,\dots,\Phi_{\mathtt{n}}$ be $\mathtt{N}\times\mathtt{N}$ commuting real matrices and $\Gamma\in\mathbb{R}^{\mathtt{p}\times\mathtt{N}}$. Then, there exists a nonsingular real matrix $T\in\mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ such that*

$$T\Phi_iT^{-1}=\begin{bmatrix}\Phi_i^{11} & 0\\ \Phi_i^{21} & \Phi_i^{22}\end{bmatrix},\quad i=1,\dots,\mathtt{n},\ \ \Gamma T^{-1}=[\Gamma_1\ 0],$$

with $(\Gamma_1;\Phi_1^{11},\dots,\Phi_{\mathtt{n}}^{11})$ observable.

This result is an immediate consequence of the fact that, similar to the 1D ($\mathtt{n}=1$) case, the unobservable subspace $\mathfrak{N}$ associated with $(\Gamma;\Phi_1,\dots,\Phi_{\mathtt{n}})$ is Φ_i-invariant and contains $\ker\Gamma$. Thus, in a basis of $\mathbb{R}^{\mathtt{N}}$ whose last elements constitute a basis for $\mathfrak{N}$ the matrices Φ_i and Γ have the desired form. A proof for the 2D ($\mathtt{n}=2$) case can be found in [6].

Proof of Proposition 1. We use the results of [8]. Assume that $\mathfrak{B}\subseteq\mathscr{U}$ is a finite-dimensional nD kernel behavior. Then it admits a kernel representation with $R(s_1,\dots,s_{\mathtt{n}})$ weakly zero prime, and hence there exist nD polynomial matrices $U_i(s_1,\dots,s_{\mathtt{n}})$ such that

$$U_i(s_1,\dots,s_{\mathtt{n}})R(s_1,\dots,s_{\mathtt{n}})=D_i(s_i),$$

where $D_i(s_i)=d_i(s_i)I_{\mathtt{w}\times\mathtt{w}}$ for $i=1,\dots,\mathtt{n}$. This implies that $\mathfrak{B}\subseteq\tilde{\mathfrak{B}}$, with $\tilde{\mathfrak{B}}$ described by

$$d_1\left(\frac{\partial}{\partial x_1}\right)w=0,\dots,d_{\mathtt{n}}\left(\frac{\partial}{\partial x_{\mathtt{n}}}\right)w=0.$$

Define a vector function $\tilde{z}$ whose components are the partial derivatives $(\partial^{\ell_1+\cdots+\ell_n}/\partial x_1^{\ell_1}\cdots\partial x_n^{\ell_n})w$ for $\ell_i=0,\ldots,deg(d_i)-1$. It is not difficult to check that this yields a latent variable representation for $\tilde{\mathfrak{B}}$ of the form

$$\begin{cases} \frac{\partial}{\partial x_1}\tilde{z} = F_1\tilde{z}, \\ \quad\vdots \\ \frac{\partial}{\partial x_n}\tilde{z} = F_n\tilde{z}, \\ w = H\tilde{z}, \end{cases} \tag{6}$$

with real commuting matrices $F_1,\ldots,F_n$. Therefore, $w\in\mathfrak{B}$ if and only if it satisfies (6) together with the equation $R(\partial/\partial x_1,\ldots,\partial/\partial x_n)w=0$. Let $R(\partial/\partial x_1,\ldots,\partial/\partial x_n)=\sum_{j_1,\ldots,j_n=0}^{J_1,\ldots,J_n}(\partial^{j_1+\cdots+j_n}/\partial x_1^{j_1}\cdots\partial x_n^{j_n})R_{(j_1,\ldots,j_n)}$.Taking (6) into account, the equation $R(\partial/\partial x_1,\ldots,\partial/\partial x_n)\ w=0$ becomes

$$\underbrace{\left(\sum_{j_1,\ldots,j_n=0}^{J_1,\ldots,J_n} R_{(j_1,\ldots,j_n)}HF_1^{j_1}\cdots F_n^{j_n}\right)}_{=:K}\tilde{z}=0.$$

In this way the following latent variable representation for $\mathfrak{B}$ is obtained

$$\begin{cases} \frac{\partial}{\partial x_1}\tilde{z} = F_1\tilde{z}, \\ \quad\vdots \\ \frac{\partial}{\partial x_n}\tilde{z} = F_n\tilde{z}, \\ K\tilde{z}=0, \\ w = H\tilde{z}. \end{cases}$$

It follows from Lemma 1 that there exists a nonsingular real matrix T such that

$$TF_iT^{-1}=\begin{bmatrix} F_i^{11} & 0 \\ F_i^{21} & F_i^{22} \end{bmatrix},\ i=1,\ldots,n,\ \ KT^{-1}=[K_1\ 0],$$

with $(K_1;F_1^{11},\ldots,F_n^{11})$ observable. Thus, partitioning $T\tilde{z}=col(\tilde{z}_1,\tilde{z}_2)$ accordingly, the equations for $\tilde{z}$ become

$$\begin{cases} \frac{\partial}{\partial x_i}\tilde{z}_1 = F_i^{11}\tilde{z}_1, \\ \frac{\partial}{\partial x_i}\tilde{z}_2 = F_i^{21}\tilde{z}_1+F_i^{22}\tilde{z}_2 \quad i=1,\ldots,n, \\ K_1\tilde{z}_1=0, \end{cases}$$

which, by observability, is equivalent to

$$\begin{cases} \tilde{z}_1=0, \\ \frac{\partial}{\partial x_i}\tilde{z}_2 = F_i^{22}\tilde{z}_2 \quad i=1,\ldots,n. \end{cases}$$

On the other hand, the equation $w=H\tilde{z}$ can be written as $w=H_2\tilde{z}_2$, where H_2 is such that $HT=[H_1\ H_2]$. Renaming $z=\tilde{z}_2$, $A_i=F_i^{22}$ and $C=H_2$, we obtain the following exact description for the dynamics of w:

$$\begin{cases} \frac{\partial}{\partial x_1}z = A_1z, \\ \quad\vdots \\ \frac{\partial}{\partial x_n}z = A_nz, \\ w = Cz, \end{cases} \tag{7}$$

where $A_1,\ldots,A_n$ are still pairwise commuting matrices.

The fact that $(C;A_1,\ldots,A_n)$ in (7) can be taken to be observable follows again from Lemma 1. This yields Proposition 1. □

5. Strong-Markovianity and first-order representations

It turns out that if, in addition to being finite-dimensional, $\mathfrak{B}$ has the strong-Markov property, then the matrix C in (5) can be shown to be injective.

Lemma 2. *Let $\mathfrak{B}\subseteq\mathscr{C}^\infty(\mathbb{R}^n,\mathbb{R}^w)$ be a finite-dimensional* n*D behavior that is the kernel of a PDE. If $\mathfrak{B}$ is strong-Markovian then it can be represented by a latent variable model of form* (5) *where the matrix C has full column rank.*

Proof. By Proposition 1, $\mathfrak{B}$ has a latent variable representation of form (5), with $(C;A_1,\ldots,A_n)$ observable. Note that in this case $\mathfrak{B}=\{w:\mathbb{R}^n\to\mathbb{R}^w|w(x_1,\ldots,x_n)=Ce^{A_1x_1+\cdots+A_nx_n}\bar{z},\ \bar{z}\in\mathbb{R}^N\}$.

We start by showing that if $\mathfrak{B}$ is strong-Markovian then, for $k=1,\ldots,n-1$, the behaviors $\mathfrak{B}_k:=\{w:\mathbb{R}^{n-k+1}\to\mathbb{R}^w|w(x_k,\ldots,x_n)=Ce^{A_kx_k+\cdots+A_nx_n}\bar{z},\ \bar{z}\in\mathbb{R}^N\}$ are also strong-Markovian with $(C;A_k,\ldots,A_n)$ observable. Strong-Markovianity of $\mathfrak{B}_k$ follows immediately from the definition. We now prove observability, by considering the case $k=2$, and proceeding by induction. Suppose that $z^*,z^{**}\in\mathbb{R}^N$ are such that

$$Ce^{A_2x_2+\cdots+A_nx_n}z^*=Ce^{A_2x_2+\cdots+A_nx_n}z^{**}$$
$$\text{for all } x_i\in\mathbb{R},\ i=2,\ldots,n.$$

Then the trajectories $w_*(x_1,x_2,\ldots,x_n)=Ce^{A_1x_1+A_2x_2+\cdots+A_nx_n}z^*$ and $w_{**}(x_1,x_2,\ldots,x_n)=Ce^{A_1x_1+A_2x_2+\cdots+A_nx_n}z^{**}$ of $\mathfrak{B}$ coincide on $S_0=\{(x_1,\ldots,x_n)\in\mathbb{R}^n|x_1=0\}$. If $\mathfrak{B}$ is strong-Markovian, this implies that $\hat{w}=w_*\wedge_{(S_-,S_0,S_+)}w_{**}$ (where $S_-=\{(x_1,\ldots,x_n)\in\mathbb{R}^n|x_1<0\}$ and $S_+=\{(x_1,\ldots,x_n)\in\mathbb{R}^n|x_1>0\}$) is a trajectory of $\mathfrak{B}$, i.e., there exists $\hat{z}\in\mathbb{R}^N$ such that $\hat{w}(x_1,\ldots,x_n)=Ce^{A_1x_1+A_2x_2+\cdots+A_nx_n}\hat{z}$. Since $\hat{w}$ coincides with w_* in S_- and with w_{**} in S_+, the observability of $(C;A_1,\ldots,A_n)$ implies that

$$z^*=\hat{z}=z^{**}$$

and hence that $(C;A_2,\ldots,A_n)$ is indeed observable.

We conclude in particular that the behavior of

$$\frac{\partial}{\partial x_n}z^n=A_nz^n \quad w^0(x_n)=Cz^n(x_n),$$

is strong-Markovian and observable. However by the results of the 1D case [2] this implies that C has full column rank. □

The previous lemma allows to state the main result of this paper.

Theorem 1. *Let $\mathfrak{B}\subseteq\mathscr{C}^\infty(\mathbb{R}^n,\mathbb{R}^w)$ be a finite-dimensional* n*D behavior that is the kernel of a PDE. Then it is strong-Markovian if and only if it can be represented by means of* *partial differential equations of the form*

$$\begin{bmatrix} \left(\frac{\partial}{\partial x_1} I_N - A_1\right) E \\ \left(\frac{\partial}{\partial x_2} I_N - A_2\right) E \\ \vdots \\ \left(\frac{\partial}{\partial x_n} I_N - A_n\right) E \\ F \end{bmatrix} w = 0, \tag{8}$$

where $A_1, A_2, \ldots, A_n$ *are square pairwise commuting matrices and the matrix* $V = \begin{bmatrix} E \\ F \end{bmatrix}$ *is invertible.*

Proof. Assume now that $\mathfrak{B}$ can be represented by a model of type (5) with C having full column rank. Let E be a left-inverse of C and F a suitable matrix such that $V = \begin{bmatrix} E \\ F \end{bmatrix}$ is invertible. Notice that Eqs. (5) yield (8).

Conversely, let $\mathfrak{B}$ have a representation as (8). In a suitable basis in $\mathbb{R}^{w}$, these equations look like

$$\begin{cases} \begin{bmatrix} \left(\frac{\partial}{\partial x_1} I_N - A_1\right) \\ \left(\frac{\partial}{\partial x_2} I_N - A_2\right) \\ \vdots \\ \left(\frac{\partial}{\partial x_n} I_N - A_n\right) \end{bmatrix} w_1 = 0 \\ w_2 = 0, \\ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \end{cases} \tag{9}$$

The corresponding w_1-behavior $\mathfrak{B}_1$ consists of all the trajectories of the form

$$w_1(x_1, \ldots, x_n) = e^{A_1 x_1 \cdots A_n x_n} z, \quad z \in \mathbb{R}^{w1}.$$

If suffices to prove that $\mathfrak{B}_1$ is strong-Markovian. But this is easy: any two trajectories which coincide on a subspace, have the same value at $x_1 = \cdots = x_n = 0$, and hence coincide, since $z = w_1(0, \ldots, 0)$. □

This theorem shows that, in the finite-dimensional case, strong-Markovianity is equivalent to the existence of a first-order representation with a special structure, where the elementary partial differential operators are decoupled. Note that the existence of such a representation may be difficult to check directly. However, a test for strong-Markovianity can be obtained as follows. The proof of Lemma 2 shows that if a finite-dimensional behavior $\mathfrak{B}$ is strong-Markovian then, in every corresponding observable $(C; A_1, \ldots, A_n)$ representation, the matrix C has full column rank. Moreover, it is easy to see that the converse also holds true. This allows to check whether $\mathfrak{B}$ is or not strong-Markovian by constructing an observable $(C; A_1, \ldots, A_n)$ representation (which can be done as in the proof of Proposition 1) and checking whether C has or not full column rank.

6. Conclusion

In this paper the conjecture of [7] on the correspondence between the nD weak-Markov property and first-order representability for PDE was proven to be false. In order to obtain equivalence with first-order representability, a strong-Markov property has been introduced, which can still be viewed as a generalization of 1D Markovianity to higher dimensions. For finite-dimensional behaviors this property was shown to be equivalent to the representability by means of a special type of first-order PDEs exhibiting a decoupling of the partial differentiation operators. This decoupling seems to be strictly connected with the finite-dimensionality of the associated behaviors. The obtained results suggest that strong-Markovianity constitutes a suitable extension of (1D) Markovianity to the nD case.

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