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ORIGINAL ARTICLE

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Conserved- and zero-mean quadratic quantities in oscillatory systems

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Abstract We study quadratic functionals of the variables of a linear oscillatory system and their derivatives. We show that such functionals are partitioned in conserved quantities and in trivially- and intrinsic zero-mean quantities. We also state an equipartition of energy principle for oscillatory systems.

Keywords Linear oscillatory systems · Two-variable polynomial matrices · Quadratic differential forms · Behavioral system theory · Equipartition of energy

1 Introduction

In this paper, we consider oscillatory systems, i.e. systems whose trajectories are linear combinations of sinusoidal functions

$$\sum_{k=1,\ldots,n} A_k \sin(\omega_k t + \phi_k)$$

with $\omega_k, A_k, \phi_k \in \mathbb{R}$ for all k. Among the many physical examples of systems of such type are mechanical systems consisting of connections of (frictionless) spring and masses, with external variables, the displacements or the velocities of the masses from the equilibrium positions; and electrical systems consisting of the interconnection of inductors and capacitors, with external variables, the voltages in the C components or the currents in the L components.

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In the context of oscillatory systems, we study quadratic functionals of the variables of the system and their derivatives, i.e. expressions of the form $Q_\Phi(w) = \sum_{i,j} (d^i w/dt^i)^{\mathrm{T}} \Phi_{ij} d^j w/dt^j$, where the indices i and j range over a finite set and $\Phi_{ij} = \Phi_{ji}^{\mathrm{T}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The first problem we set out to solve in this paper is the structure of the set of such quadratic functionals. We show that they are partitioned in *conserved quantities*, i.e. $Q_\Phi(w)$ is such that

$$\frac{d}{dt} Q_\Phi(w) = 0$$

for all trajectories w of the system; and in *zero-mean quantities*, i.e. $Q_\Phi(w)$ is such that its time average over the whole real axis is zero along the trajectories w of the system:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Q_\Phi(w)(t)dt = 0\,.$$

On physical considerations one can deduce the existence of at least one conserved quantity, namely the total energy of the system; however, in general there exist also other ones, which we characterize in this paper. As for zero-mean quantities, we formalize the intuitive notion that certain quadratic differential expressions are zero-mean quantities for all bounded (and consequently, also oscillatory) trajectories, since they are the derivative of some (necessarily bounded) function; consequently, we call them “trivial” zero-mean. Other zero-mean quantities, instead, are such only for the trajectories of the system at hand, and consequently will be called “intrinsic”, since they depend in an essential way on the dynamics.

Using this classification, we prove a decomposition theorem for quadratic differential functionals of the variables of an oscillatory system and their derivatives. Assuming that such functionals are “canonical” (this technical notion will be introduced in the course of the exposition), we show that they can be written in a unique way as the sum of three components: a conserved quantity, a trivially zero-mean quantity, and an intrinsic zero-mean quantity. We also state algorithms to compute bases for the spaces of (“canonical”) conserved quantities, and of trivial- and intrinsic zero-mean quantities.

Finally, we use the concept of conserved- and zero-mean quantity in order to state and prove an *equipartition of energy principle* for oscillatory systems: if such a system consists of symmetrically coupled identical subsystems, then the difference between the value of any quadratic functional of the variables of the one subsystem and their derivatives, and its value on the variables of the other and their derivatives is zero-mean. In particular, in the case of mechanical systems, the total energy (kinetic plus potential) of the different subsystems is the same. This result is inspired by and generalizes that of [BB1], in which classical state-space techniques are used in order to study the equipartition of energy of oscillators coupled in a lossless way.

The results reported here are obtained in the behavioral framework (see [PoW]), using the concept of quadratic differential form, introduced in [WT1]. In this framework, the properties of a system are defined and studied at the level of trajectories, independent of the actual representation of the system, be it state-space or transfer function as it is common in system- and control theory, or second-order in the positions as is the custom in classical mechanics.

Besides being conceptually simple, the choice of the behavioral framework entails some other relevant advantages. First, defining properties intrinsically leaves open the possibility of characterizing them in terms of any particular representation of the system which may be advantageous to use (be it transfer function, state-space, second-order) for conceptual or computational reasons. Another important advantage is that, by relying on the calculus developed in the behavioral framework (see Ch. 2 of [PoW] and the paper [WT1]), algorithms based on one- and two-variable polynomial algebra can be developed to determine the conserved quantities, the zero-mean quantities, etc. starting from a set of higher-order differential equations describing the system. This feature is of particular interest when considering the application of the results presented in this report to computer-assisted modeling and simulation.

The paper is organized as follows: in Sect. 2, we review some notions regarding linear differential systems, with special attention to oscillatory systems. In Sect. 3, we define bilinear and quadratic differential forms. In Sect. 4, we first give the definition of conserved quantity and of zero-mean quantity; we proceed to distinguish trivially zero-mean and intrinsic zero-mean functionals for a given behavior; and we give an algebraic characterization of them. Then we state a decomposition theorem for quadratic differential forms acting on oscillatory behaviors. In Sect. 5, we use these concepts in order to prove a general equipartition of energy principle, which we apply to the particular situation of identical oscillators symmetrically coupled. In Sect. 6, we discuss our results and outline some directions for future research.

The notation used in this paper is standard: the space of $\mathtt{n}$ dimensional real, respectively complex, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real, respectively complex, matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{C}^{\bullet\times\mathtt{n}}$ denotes the set of complex matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^{\bullet}$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font type!); similar considerations hold for matrices representing linear operators on such spaces. If $A_i \in \mathbb{R}^{\bullet\times\bullet}$, $i = 1, \ldots, r$ have the same number of columns, $\mathrm{col}(A_i)_{i=1,\ldots,r}$ denotes the matrix

$$\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$$

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. A polynomial p in the indeterminate ξ is called *even* if $p(\xi) = p(-\xi)$, i.e., if it is of the form $p(\xi^{2j})$. The space of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R^*(\xi) := R^{\mathrm{T}}(-\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$. If $R(\xi)$ has complex coefficients, then $R^*(\xi)$ denotes the matrix obtained from R by substituting $-\xi$ in place of ξ, transposing, and conjugating.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$, and with $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ consisting of compact support functions.

2 Oscillatory behaviors

In this section, we give the definition of oscillatory behavior and we study its properties. Since oscillatory behaviors are a particular case of linear differential behaviors, we introduce this notion first.

A *linear differential behavior* is a linear subspace $\mathfrak{B}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions w of a system of linear constant-coefficient differential equations:

$$R\left(\frac{d}{dt}\right) w = 0\,, \tag{1}$$

where $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, is called a *kernel representation* of the behavior

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w \text{ satisfies (1)}\}\,,$$

and w is called the *external variable* of $\mathfrak{B}$. The class of all such behaviors is denoted with $\mathfrak{L}^{\mathtt{w}}$. A given behavior $\mathfrak{B}$ can be described as the kernel of different polynomial differential operators; two kernel representations $R_1(d/dt)w = 0$ and $R_2(d/dt)w = 0$ with $R_1, R_2 \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ represent the same behavior if and only if there exist polynomial matrices F_1, F_2 with a suitable number of columns, such that $R_1 = F_1 R_2$ and $R_2 = F_2 R_1$; in particular if R_1 and R_2 are of full row rank, this means that there exists a unimodular matrix F such that $R_1 = F R_2$.

In this paper, we study linear differential autonomous systems. A behavior is *autonomous* if for all $w_1, w_2 \in \mathfrak{B}$

$$[w_1(t) = w_2(t) \quad \text{for} \quad t \leq 0] \Longrightarrow [w_1(t) = w_2(t) \quad \text{for all } t]\,.$$

Intuitively, a system is autonomous if the future of every trajectory in $\mathfrak{B}$ is uniquely determined by its past and by its present state. Note that in the behavioral framework "autonomous" means "closed", i.e. with no external influence. It can be shown that the behavior of an autonomous system is a *finite-dimensional* subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Equivalently, if the behavior admits kernel representations (1) in which the matrix R is square and nonsingular, it can be shown (see Theorem 3.6.4 in [PoW]) that a representation in which the matrix R is square and nonsingular has the minimal number of equations ($\mathtt{w}$, the number of variables of the system) needed in order to describe an autonomous behavior $\mathfrak{B}$, and is consequently called a *minimal* representation.

We now introduce a number of notions about the structure of autonomous behaviors which will be important in this paper.

The first one is that of the invariant polynomials of an autonomous behavior $\mathfrak{B}$. Since minimal kernel representations $R' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ of $\mathfrak{B}$ can all be obtained from a given one associated with $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ as $R' = UR$ with U unimodular, then all minimal representations have the same Smith form (for a definition, see for example Sect. 6.3.3 of [K]). The diagonal elements in such Smith forms are called the *invariant polynomials of* $\mathfrak{B}$; their product is denoted by $\chi_{\mathfrak{B}}$, and is called the *characteristic polynomial of* $\mathfrak{B}$. Observe that the nonzero invariant polynomials in the Smith form of any matrix $R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\mathfrak{B} = \ker R'(d/dt)$ also equal the invariant polynomials of $\mathfrak{B}$ (see Corollary 3.6.3 in [PoW]). In particular, $\chi_{\mathfrak{B}} = \det(\mathfrak{B})$ (the latter assumed monic).

We proceed by investigating the nature of the trajectories in an autonomous behavior. It can be proved (see Theorem 3.2.16 of [PoW]) that if $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, r$ are the distinct roots of the characteristic polynomial $\chi_{\mathfrak{B}}$, each with multiplicity n_i, then $w \in \mathfrak{B}$ if and only if

$$w(t) = \sum_{i=1}^{r} \sum_{j=0}^{n_j - 1} v_{ij} t^j e^{\lambda_i t} , \tag{2}$$

where the vectors $v_{ij} \in \mathbb{C}^{\mathtt{w}}$ satisfy $\sum_{j=k}^{n_i-1} \binom{j}{k} R^{(j-k)}(\lambda_i) v_{ij} = 0$, with $R^{(j-k)}$ denoting the $(j-k)$th derivative of the matrix polynomial R. In particular, every trajectory $w \in \mathfrak{B}$ is a linear combination of polynomial-exponential trajectories associated with the *characteristic frequencies* λ_i.

We now introduce the class of linear oscillatory behaviors.

Definition 1 $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *is an* oscillatory behavior *if*

$$[w \in \mathfrak{B}] \Longrightarrow [w \textit{ is bounded on } (-\infty, +\infty)] .$$

From the definition, it follows immediately that an oscillatory system is necessarily autonomous, since the presence of input variables in w implies that those components of w could be chosen to be unbounded. Physical examples of oscillatory behaviors are the evolution of the configuration variables in a mechanical system consisting of springs and masses, and the evolution of the voltages or current variables in any LC circuit.

The following is a characterization of oscillatory systems in terms of properties of its kernel representation.

Proposition 2 *Let* $\mathfrak{B} = \ker R(d/dt)$, *with* $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. *Then* $\mathfrak{B}$ *is oscillatory if and only if every nonzero invariant polynomial of* $\mathfrak{B}$ *has distinct and purely imaginary roots.*

Proof Without loss of generality, we can assume that the kernel representation induced by R is minimal, i.e. $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$. Let $R = U \Delta V$ be the Smith form of R, with U, V unimodular and Δ the diagonal matrix of the invariant polynomials of R. Observe that $R(d/dt)w = 0$ if and only if $\Delta(d/dt)V(d/dt)w = 0$; now define $\mathfrak{B}' := V(d/dt)\mathfrak{B}$, and observe that $\mathfrak{B}' = \ker \Delta(d/dt)$. Notice that since V is unimodular, it follows that $\mathfrak{B}'$ is oscillatory if and only if $\mathfrak{B}$ is.

Since $\mathfrak{B}'$ is described by the diagonal matrix $\Delta = \text{diag}(\psi_j)_{j=1,\ldots,\mathtt{w}}$, the claim of the Proposition is proved if we show that the scalar system $\mathfrak{B}'_j := \ker \psi_j(d/dt)$ is oscillatory if and only if $\psi_j \in \mathbb{R}[\xi]$ has distinct and purely imaginary roots.
(If) Observe that if the characteristic frequencies ω_{jk}, $k = 1, \ldots, \deg(\psi_j)$ of $\mathfrak{B}'_j$ lie on the imaginary axis and are distinct, then $w'_j \in \mathfrak{B}'_j$ if and only if

$$w'_j(t) = \sum_{k=1}^{\deg(\psi_j)} \alpha_{jk} e^{i\omega_{jk} t} \tag{3}$$

for $\alpha_{jk} \in \mathbb{C}$, $k = 1, \dots, \deg(\psi_j)$. Observe that the α_{jk}'s corresponding to conjugate characteristic frequencies $\pm i\omega_{jk}$ are also conjugate, since each entry of $\psi_j(\xi)$ has real coefficients. Conclude that (3) describes a linear combination of sinusoidal functions; thus, $\mathfrak{B}'_j$ is oscillatory.
(Only if) The proof is by contradiction. Assume that there is a characteristic frequency of $\mathfrak{B}'_j$ not lying on the imaginary axis; then it is easy to verify from (2) that this is in contradiction with the boundedness of the trajectories in $\mathfrak{B}'_j$ on the whole real axis. Now assume by contradiction that there is a characteristic frequency $i\omega_{jk}$, which is not simple. From (2), it follows that there exists one trajectory w'_j in $\mathfrak{B}'_j$ of the form $w'_j(t) = t\sin(\omega_{jk}t + \phi_{jk})$. Such w'_j is unbounded, and this is in contradiction with the oscillatory nature of $\mathfrak{B}'_j$. □

3 Quadratic differential forms

In modeling and control problems it is often necessary to study certain functionals of the system variables and their derivatives; when considering linear systems, such functionals are often quadratic. The parametrization of such functionals using two-variable polynomial matrices has been studied in detail in [WT1], resulting in the definition of bilinear- and quadratic differential form and in the development of a calculus for application in many areas. In this section we review the definitions and results which are used in this paper.

We first examine bilinear differential forms. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$; then

$$\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k,$$

where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ and N is a nonnegative integer. The two-variable polynomial matrix Φ induces the bilinear functional from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined as:

$$L_\Phi(w_1, w_2) = \sum_{h,k=0}^{N} \left(\frac{d^h w_1}{dt^h}\right)^{\mathrm{T}} \Phi_{h,k} \frac{d^k w_2}{dt^k}.$$

Such a functional is called a *bilinear differential form*, abbreviated as BDF. L_Φ is *symmetric*, meaning $L_\Phi(w_1, w_2) = L_\Phi(w_2, w_1)$ for all w_1, w_2, if and only if Φ is a *symmetric two-variable polynomial matrix*, i.e. if $\mathtt{w}_1 = \mathtt{w}_2$ and $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^{\mathrm{T}}$. The set of symmetric two-variable polynomial matrices of dimension $\mathtt{w} \times \mathtt{w}$ in the indeterminates ζ and η is denoted with $\mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$.

If the two-variable polynomial matrix Φ is symmetric, then it induces also a quadratic functional acting on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ as

$$Q_\Phi(w) := L_\Phi(w, w).$$

We will call Q_Φ the *quadratic differential form* (in the following abbreviated with QDF) associated with Φ.

With every $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ we associate its *coefficient matrix* $\tilde{\Phi}$, which is defined as the infinite matrix $\tilde{\Phi} := (\Phi_{i,j})_{i,j=0,\ldots}$. Indeed,

$$\Phi(\zeta, \eta) = \begin{bmatrix} I_{\mathtt{w}} & \zeta I_{\mathtt{w}} & \cdots & \cdots \end{bmatrix} \underbrace{\begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdots \\ \vdots & \ddots & \cdots & \ddots \end{bmatrix}}_{\tilde{\Phi}} \begin{bmatrix} I_{\mathtt{w}} \\ \eta I_{\mathtt{w}} \\ \vdots \\ \vdots \end{bmatrix}$$

Observe that although $\tilde{\Phi}$ is infinite, only a finite number of its entries are nonzero. Note that $\Phi \in R_S^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ if and only if its coefficient matrix is symmetric, $\tilde{\Phi}^{\mathrm{T}} = \tilde{\Phi}$.

We now introduce the concept of symmetric canonical factorization (see [WT1], p. 1709). Let $\Phi \in \mathbb{R}_S^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$; then its coefficient matrix $\tilde{\Phi}$ can be factored as $\tilde{\Phi} = \tilde{M}^{\mathrm{T}} \Sigma_\Phi \tilde{M}$, where $\tilde{M}$ is a full row rank infinite matrix with $\mathrm{rank}(\tilde{\Phi})$ rows and only a finite number of entries nonzero, and $\Sigma_\Phi \in \mathbb{R}^{\mathrm{rank}(\tilde{\Phi}) \times \mathrm{rank}(\tilde{\Phi})}$ is a signature matrix, i.e.

$$\Sigma_\Phi = \begin{bmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{bmatrix}$$

From such factorization, multiplying on the left by $\begin{pmatrix} I_{\mathtt{w}} & I_{\mathtt{w}}\zeta & I_{\mathtt{w}}\zeta^2 & \cdots \end{pmatrix}$ and on the right by $\mathrm{col}(\eta^k I_{\mathtt{w}})_{k=0,\ldots}$, we obtain the *symmetric canonical factorization* of Φ:

$$\Phi(\zeta, \eta) = M^{\mathrm{T}}(\zeta) \Sigma_\Phi M(\eta).$$

The association of two-variable polynomial matrices with BDF's and QDF's allows to develop a calculus that has applications in dissipativity theory and H_∞-control (see [PW, WT2, TW, WT3]). One important tool in such calculus is the map

$$\begin{aligned} \partial \quad &: \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta] \longrightarrow \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi] \\ \partial \Phi(\xi) &:= \Phi(-\xi, \xi) \end{aligned}$$

Observe that if $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is symmetric, then $\partial \Phi$ is *para-Hermitian*, i.e. $\partial \Phi = (\partial \Phi)^*$.

Another important role in the following is played by the notion of *derivative* of a QDF. Given a QDF Q_Φ, we define its *derivative* as the QDF $Q_{\overset{\bullet}{\Phi}}$ defined by

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. In terms of the two-variable polynomial matrices associated with the QDF's, the relationship between a QDF Q_Φ and $Q_{\overset{\bullet}{\Phi}}$ is

$$\overset{\bullet}{\Phi}(\zeta, \eta) = (\zeta + \eta) \Phi(\zeta, \eta) \,. \tag{4}$$

In the rest of this paper, we use integrals of BDFs/QDFs on closed finite intervals $[t_0, t_1] \subset \mathbb{R}$, defined as:

$$\int_{t_0}^{t_1} L_\Phi : \mathfrak{C}^\infty([t_0, t_1], \mathbb{R}^{\mathtt{v}}) \times \mathfrak{C}^\infty([t_0, t_1], \mathbb{R}^{\mathtt{w}}) \to \mathbb{R}$$

$$\int_{t_0}^{t_1} L_\Phi(v, w) := \int_{t_0}^{t_1} L_\Phi(v, w)(t)dt\,.$$

The notation for QDFs follows easily and will not be repeated here. We call $\int_{t_0}^{t_1} Q_\Phi(w)$ *independent of path* if for all intervals $[t_1, t_2]$, the value of the integral depends only on the value of w and (a finite number of) its derivatives at t_1 and at t_2, but not on the intermediate path used to connect these endpoints. The following algebraic characterization of path independence in terms of properties of two-variable polynomial matrices uses the notion of derivative of a QDF and the ∂ operator. Assume $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$; then $\int_{t_1}^{t_2} Q_\Phi$ is independent of path if and only if either of the following two equivalent conditions holds:

(a) There exists a $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$;
(b) $\partial\Phi(\xi) = \Phi(-\xi, \xi) = 0$.

(see Theorem 3.1 of [WT1]).

An essential role in this paper is played by QDFs evaluated along a linear differential behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Let $\Phi_1, \Phi_2 \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ and let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$; we say that Q_{Φ_1} is *equivalent to* Q_{Φ_2} *along* $\mathfrak{B}$, denoted

$$Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$$

if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ holds for all $w \in \mathfrak{B}$. It is a matter of straightforward verification to see that such relation is indeed an equivalence relation. This equivalence can be expressed in terms of a kernel representation (1) of $\mathfrak{B}$ as follows (see Proposition 3.2 of [WT1]): $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$ if and only if there exists $F \in \mathbb{R}^{\bullet\times\bullet}[\zeta, \eta]$ such that

$$\Phi_2(\zeta, \eta) = \Phi_1(\zeta, \eta) + R(\zeta)^{\mathrm{T}} F(\zeta, \eta) + F(\eta, \zeta)^{\mathrm{T}} R(\eta)\,. \tag{5}$$

If (5) holds, then we also say that Φ_1 and Φ_2 are *R-equivalent*, written $\Phi_1 \stackrel{R}{=} \Phi_2$.

If $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is autonomous, then each equivalence class of QDF's in the equivalence $\stackrel{\mathfrak{B}}{=}$ admits a canonical representative. In order to see this, choose a minimal kernel representation $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ of $\mathfrak{B}$; observe that since $\mathfrak{B}$ is autonomous, then $\det(R) \neq 0$. We call $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *R-canonical* if $(R(\zeta)^{\mathrm{T}})^{-1}\Phi(\zeta, \eta)(R(\eta))^{-1}$ is a matrix of strictly proper two-variable rational functions. It can be proved (see Proposition 4.9 p. 1716 of [WT1]) that if $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, then there exists exactly one QDF $\Phi' \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ which is R-canonical and such that $\Phi' \stackrel{R}{=} \Phi$; we call Φ' the *R-canonical representative* of Φ, denoted as $\Phi \bmod R$.

Example 3 As an illustration of the above definition, we consider the notion of R-equivalence for scalar systems. Assume that $\mathtt{w} = 1$, and let $\mathfrak{B} = \ker r(d/dt)$, with $r \in \mathbb{R}[\xi]$ having degree n. Observe that since

$$r_0 w + r_1 \frac{dw}{dt} + \ldots + r_n \frac{d^n w}{dt^n} = 0 \tag{6}$$

and $r_n \neq 0$, it follows that the derivatives of w of order higher than n can be rewritten as linear combinations of the derivatives of w of order less than or equal to $n-1$. Consequently, any quadratic differential form Q_Φ involving derivatives of w of order higher than or equal to n can be rewritten in an equivalent and unique way as a quadratic differential form $Q_{\Phi'}$ involving the derivatives of w up to the $(n-1)$ th one. Φ' is the r-canonical representative of Φ.

For example, observe that for the system described by (6), it holds that $Q_{\Phi_1}(w) = ((d^n/dt^n)w)^2$ and $Q_{\Phi_2}(w) = \left(-\frac{1}{r_n}\sum_{i=0}^{n-1} r_i d^i/dt^i w\right)^2$ are $\ker(r(d/dt))$-equivalent. Observe also that

$$\zeta^n \eta^n \stackrel{r}{=} \left(-\frac{1}{r_n}\sum_{i=0}^{n-1} r_i \zeta^i\right)\left(-\frac{1}{r_n}\sum_{i=0}^{n-1} r_i \eta^i\right),$$

which implies that the two-variable polynomial $\left(-\frac{1}{r_n}\sum_{i=0}^{n-1} r_i \zeta^i\right)\left(-\frac{1}{r_n}\sum_{i=0}^{n-1} r_i \eta^i\right)$ is the r-canonical representative of $\zeta^n \eta^n$.

We denote the set consisting of all $\mathtt{w}$-dimensional R-canonical symmetric two-variable polynomials with $\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. It is a matter of straightforward verification to prove that $\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is a vector space over $\mathbb{R}$. The following result establishes its dimension.

Proposition 4 *Let* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *be nonsingular, and let* $n := \deg(\det(R)) = \dim(\mathfrak{B})$. *The set of QDFs* Q_Φ *with* $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *taken modulo* $\mathfrak{B}$*, is a vector space over* $\mathbb{R}$ *of dimension* $\frac{n(n+1)}{2}$.

Proof It is easy to see that the set of QDFs modulo $\mathfrak{B}$ stands in one-to-one correspondence with the set

$$\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] = \{\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid \Phi \text{ is } R\text{-canonical}\}.$$

Now let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\xi]$, and let $\Phi(\zeta,\eta) = M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$ be a canonical factorization of Φ. Denote the rows of the matrix $M \in \mathbb{R}^{\mathrm{rank}(\tilde{\Phi})\times\mathtt{w}}[\xi]$ with $M_i \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$, $i = 1,\ldots,\mathrm{rank}(\tilde{\Phi})$.

It is easy to see that $\Phi(\zeta,\eta)$ is R-canonical if and only if $M_i(\xi)R(\xi)^{-1}$ is strictly proper for $i = 1,\ldots,\mathrm{rank}(\tilde{\Phi})$. Without loss of generality, we can assume that R is column-reduced, meaning that if the highest power of the indeterminate ξ in the ith column of R is k_i, then $\deg(\det(R)) = n = \sum_{i=1}^{\mathtt{w}} k_i$. It follows then from Lemma 6.3-11 of [K] that $v \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$ is such that vR^{-1} is strictly proper if and only if the degree of each of the entries of v is strictly less than the degree of the corresponding column of R. Conclude from this that the dimension of the vector space

$$\{v \in \mathbb{R}^{1\times\mathtt{w}}[\xi] | vR^{-1} \text{ is strictly proper}\}$$

over $\mathbb{R}$ equals $\sum_{i=1}^{\mathtt{w}} k_i = n$.

Let $v_i \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$ be a basis for this space. Such polynomial vectors induce the following basis for the space of two-variable symmetric R-canonical polynomial matrices:

$$v_i^{\mathrm{T}}(\zeta)v_j(\eta) + v_j^{\mathrm{T}}(\zeta)v_i(\eta)$$

for $1 \leq i \leq j \leq n$. Indeed, such $n(n+1)/2$ symmetric matrices are linearly independent since the v_i's are; moreover, it follows from the characterization of R-canonicity in terms of the factor M of a symmetrical canonical factorization that they span the set of R-canonical symmetric matrices. Conclude from this that the number of linearly independent symmetric R-canonical two-variable polynomial matrices is $n(n+1)/2$. □

Example 5 Consider a system with $\mathtt{w} = 1$ described by the differential equation $r(d/dt)w = 0$, with $r \in \mathbb{R}[\xi]$, $\deg(r) = n$. Consider that an r-canonical QDF Q_Φ is induced by a symmetric two-variable polynomial Φ in which only powers of ζ and η up to the $(n-1)$ th appear. It is evident that the space of such two-variable polynomials is in one-to-one correspondence with the space of symmetric matrices of dimension $n \times n$. This observation yields a simpler proof of the statement of Proposition 4 for the scalar case.

In the rest of this paper, we also need the notion of nonnegativity and positivity of a QDF. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$; we call it nonnegative, denoted $\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. We call Φ positive, denoted $Q_\Phi > 0$, if $\Phi \geq 0$ and $(Q_\Phi(w) = 0) \Longrightarrow (w = 0)$. Using the two-variable matrix representation of Q_Φ and the concept of symmetric canonical factorization, it can be verified that

$$\begin{aligned}
&[Q_\Phi \geq 0] \Longleftrightarrow [\exists\, D \in \mathbb{R}^{\bullet\times\mathtt{w}} \text{ such that } \Phi(\zeta,\eta) = D^{\mathrm{T}}(\zeta)D(\eta)]\,,\\
&[Q_\Phi > 0] \Longleftrightarrow [\exists\, D \in \mathbb{R}^{\bullet\times\mathtt{w}} \text{ such that } \Phi(\zeta,\eta) = D^{\mathrm{T}}(\zeta)D(\eta)\,,\\
&\qquad\qquad \text{and } \operatorname{rank}(D(\lambda)) = \mathtt{w} \text{ for all } \lambda \in \mathbb{C}]\,.
\end{aligned}$$

Often, in the following, we study whether a given QDF is zero-, nonnegative-, or positive along a behavior $\mathfrak{B}$. We call Q_Φ *zero along* $\mathfrak{B}$, denoted with

$$Q_\Phi \stackrel{\mathfrak{B}}{=} 0 \text{ or } \Phi \stackrel{\mathfrak{B}}{=} 0$$

if $Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}$; we call Q_Φ *nonnegative along* $\mathfrak{B}$, denoted

$$Q_\Phi \stackrel{\mathfrak{B}}{\geq} 0$$

or $\Phi \stackrel{\mathfrak{B}}{\geq} 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$. The notion of *positivity along a behavior* is analogous and will not be repeated here. These concepts translate in terms of properties of the one- and two-variable polynomial matrices representing $\mathfrak{B}$ and the QDFs as follows. From the notion of $\mathfrak{B}$-equivalence and from its characterization (5) we can conclude that

$$\begin{aligned}
[Q_\Phi \stackrel{\mathfrak{B}}{=} 0] \Longleftrightarrow\ & [\exists\, F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that}\\
& \Phi(\zeta,\eta) = R(\zeta)^{\mathrm{T}}F(\zeta,\eta) + F^{\mathrm{T}}(\eta,\zeta)R(\eta)]\,.
\end{aligned} \tag{7}$$

Also, $\Phi \stackrel{\mathfrak{B}}{\geq} 0$ if and only if there exists Φ' such that $\Phi' \stackrel{\mathfrak{B}}{=} \Phi$ and $\Phi' \geq 0$; equivalently,

$$\begin{aligned}
[\Phi \stackrel{\mathfrak{B}}{\geq} 0] \Longleftrightarrow\ & [\exists\, D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] \text{ and } F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that}\\
& \Phi(\zeta,\eta) = D(\zeta)^{\mathrm{T}}D(\eta) + R(\zeta)^{\mathrm{T}}F(\zeta,\eta) + F^{\mathrm{T}}(\eta,\zeta)R(\eta)]\,.
\end{aligned}$$

4 A decomposition theorem for QDFs

We begin this section with the definition of conserved and zero-mean quantities; among the latter, we distinguish between trivially- and intrinsic zero-mean quantities. We proceed to parametrize these in terms of properties of the two-variable polynomial matrices representing the QDFs. Finally, we give the main result of this section, a decomposition theorem for QDFs, and we illustrate this result with an example.

The definition of conserved quantity is as follows.

Definition 6 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *be an oscillatory system, and let* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then a QDF* Q_Φ *is a* conserved quantity *for* $\mathfrak{B}$ *if*

$$[w \in \mathfrak{B}] \Longrightarrow [\frac{d}{dt} Q_\Phi(w) = 0] \,.$$

The definition of zero-mean quantity is as follows.

Definition 7 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *be an oscillatory system, and let* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then QDF* Q_Φ *is a* zero-mean quantity *for* $\mathfrak{B}$ *if*

$$[w \in \mathfrak{B}] \Longrightarrow [\lim_{T\to\infty} \frac{1}{T} \int_0^{\mathrm{T}} Q_\Phi(w)(t)dt = 0] \,.$$

We illustrate these definitions with an example, in which we also point out some aspects of conserved- and zero-mean quantities which will be treated in detail in the following.

Example 8 Assume that two equal masses m connected to "walls" by springs of equal stiffness k, are coupled together with a spring of stiffness k'.

We interpret this situation as the *symmetric interconnection*, through the spring with elastic constant k', of two identical *oscillators*, each consisting of a mass m and a spring with elastic constant k. Take as external variables the displacements w_1 and w_2 of the masses from their equilibrium positions; in such case two equations describing the system are

$$\begin{aligned} m\tfrac{d^2 w_1}{dt^2} &= k'(w_2 - w_1) - k w_1 \,, \\ m\tfrac{d^2 w_2}{dt^2} &= k'(w_1 - w_2) - k w_2 \,. \end{aligned} \tag{8}$$

Assume that this system has $m = 13$ kg, $k = 7$ (N/m), and $k' = 5$ (N/m), and that it is excited by some arbitrary nonzero initial conditions, for example $w_1 = 1$, $dw_1/dt = 0$, $w_2 = 0$, $dw_2/dt = 0$. Define the *energy of the* i*th oscillator* as $E_i(t) := \frac{1}{2}k w_i^2 + \frac{1}{2}m(dw_i/dt)^2$, $i = 1, 2$.

The energy of the first oscillator is depicted in Fig. 1, together with its time-average

$$\bar{E}_1(t) := \frac{1}{t} \int_0^t E_1(\tau)d\tau$$

at time t.

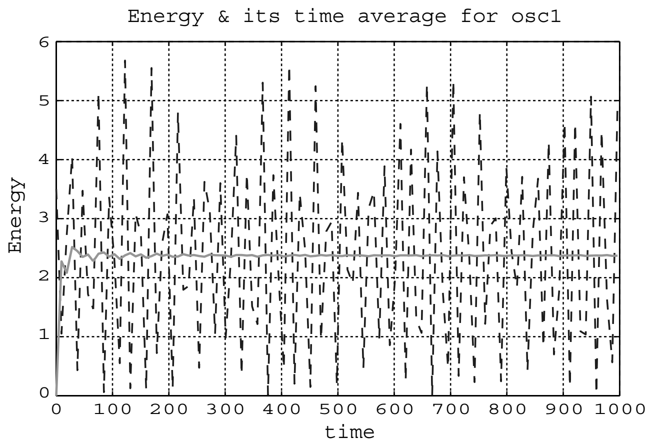

Fig. 1 Energy (*dashed line*) and its time-average (*solid line*) for oscillator 1

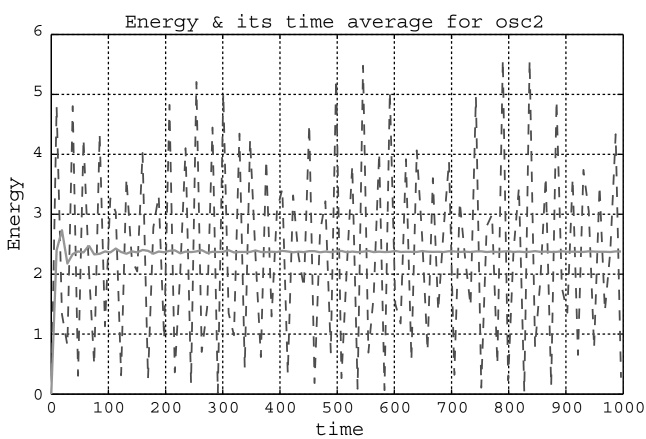

Fig. 2 Energy (*dashed line*) and its time-average (*solid line*) for oscillator 2

The energy of the second oscillator and its time average are depicted in Fig. 2. It follows from Fig. 3 that the difference $E_1(t) - E_2(t)$ of the energies of the oscillators is *zero-mean*, meaning that

$$\lim_{t \to \infty} \bar{E}_1(t) - \bar{E}_2(t) = 0\,.$$

It is not difficult to see that the quadratic expression $w_1(dw_1/dt)$ also has zero-mean. Indeed,

$$\int_0^t w_1(\tau)\frac{dw_1}{dt}(\tau)\mathrm{d}\tau = \frac{1}{2}(w_1(t)^2 - w_1(0)^2)\,.$$

Given the oscillatory nature of the system, w_1 is bounded, and consequently

$$\lim_{t \to \infty} \frac{1}{t}\int_0^t w_1(\tau)\frac{dw_1}{dt}(\tau)d\tau = 0\,.$$

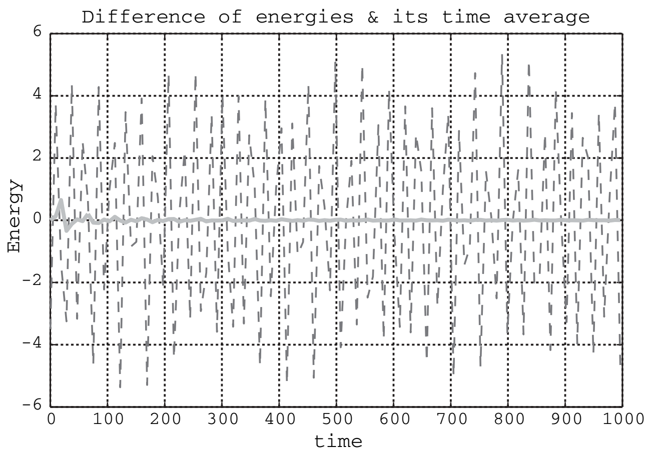

Fig. 3 Graph of $E_2 - E_1$ (*dashed line*) and $\bar{E}_2 - \bar{E}_1$ (*solid line*)

Observe the qualitative difference between the zero-mean quantities $w_1(dw_1/dt)$ and $E_1(t) - E_2(t)$; the first one is zero-mean *for all* bounded differentiable arguments w, while the second one is zero-mean *only* for the trajectories satisfying the differential equations (8).

The system under study also has conserved quantities. Because of physical considerations, namely the absence of dissipative elements, we can conclude that one of them is the *total energy* of the system at time t, given by

$$E(t) = E_1(t) + E_2(t) .$$

For the trajectories (w_1, w_2) corresponding to the given initial conditions, $E(t)$ is constant, equal to 20 J (Joules). The system also admits another conserved quantity, linearly independent of $E(\cdot)$. One possible choice for such conserved quantity is the functional

$$C(t) = -\frac{k'}{2} w_1(t)^2 - \frac{k'}{2} w_2(t)^2 + (k + k')w_1(t)w_2(t) + m\frac{dw_1}{dt}(t)\frac{dw_2}{dt}(t)$$

whose dimension is that of an energy. For the trajectories (w_1, w_2) at hand, the constant value of such a functional is 11.5 J.

In Example 8 it has been pointed out that certain zero-mean quantities are such for *every* oscillatory system: their zero-mean nature has nothing to do with the dynamics of the particular system at hand, but follows instead from the fact that such quadratic differential forms are derivatives of some other QDF. The following definition addresses this issue.

Definition 9 *Let* $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then a QDF* Q_Φ *is a* trivially zero-mean quantity *if*

$$[w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}),\ w \text{ bounded }] \Longrightarrow [\lim_{T\to\infty} \frac{1}{T} \int_0^{\mathrm{T}} Q_\Phi(w)(t)dt = 0] .$$

It is easy to see that a QDF Q_Φ is trivially zero-mean if and only if there exists Q_Ψ such that $(d/dt)Q_\Psi = Q_\Phi$, or equivalently $\partial\Phi = 0$.

It is a matter of straightforward verification to see that given $\mathfrak{B} = \ker R(d/dt)$, the sets of conserved-, zero-mean, and trivially zero-mean quantities for $\mathfrak{B}$ are linear subspaces of the vector space of R-canonical two-variable polynomials. We denote such subspaces respectively with $\mathfrak{C}$, $\mathfrak{Z}$ and $\mathfrak{T}$; thus

$$\begin{aligned}
\mathfrak{C} &:= \{\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is conserved }\}\,,\\
\mathfrak{Z} &:= \{\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is zero-mean }\}\,,\\
\mathfrak{T} &:= \{\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is trivially zero-mean }\}\,.
\end{aligned}$$

It is a matter of straightforward verification to prove that the sets of R-canonical conserved-, zero-mean, and trivially zero-mean quantities are linear subspaces of $\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, the set of R-canonical quadratic differential forms. We denote such subspaces respectively with $\mathfrak{C}_R$, $\mathfrak{Z}_R$ and $\mathfrak{T}_R$.

We now give parametrizations of the elements of $\mathfrak{C}_R$, $\mathfrak{Z}_R$ and $\mathfrak{T}_R$, beginning with the conserved quantities.

Proposition 10 *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be oscillatory, and let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be such that $\mathfrak{B} = \ker R(d/dt)$. Then $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is a conserved quantity if and only if there exists $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that*

$$(\zeta+\eta)\Phi(\zeta,\eta) = R(\zeta)^{\mathrm{T}}Y(\zeta,\eta) + Y(\eta,\zeta)^{\mathrm{T}}R(\eta)\,. \tag{9}$$

Proof Q_Φ is a conserved quantity if and only if $(d/dt)Q_\Phi \overset{\mathfrak{B}}{=} 0$. Given $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, the derivative of Q_Φ is represented by $(\zeta+\eta)\Phi(\zeta,\eta)$, and the fact that $(d/dt)Q_\Phi$ is zero along $\mathfrak{B}$ is expressed as in (9) (see equation (7)). □

From Proposition 10, we conclude that in order to compute a conserved quantity, the following algorithm can be used: first solve the one-variable *polynomial Lyapunov equation* (*PLE*)

$$R^{\mathrm{T}}(-\xi)X(\xi) + X^{\mathrm{T}}(-\xi)R(\xi) = 0 \tag{10}$$

in the unknown matrix $X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. A conserved quantity $\Phi(\zeta,\eta)$ is then obtained taking the R-canonical representative of

$$\Phi(\zeta,\eta) = \frac{R^{\mathrm{T}}(\zeta)X(\eta) + X^{\mathrm{T}}(\zeta)R(\eta)}{\zeta+\eta}\,.$$

If the solution X of (10) is taken to be R-canonical, then the corresponding Φ is also R-canonical; moreover, every conserved quantity is obtained in this way (see Proposition 4.1 of [PR]).

We now establish the dimension of the subspace of R-canonical conserved quantities.

Proposition 11 *Assume that* $\ker R(d/dt)$ *is oscillatory, without characteristic frequencies in zero. Let $\pm i\omega_i$, $i = 1,\ldots,r$, be the distinct roots of* $\det(R)$*, with algebraic multiplicity μ_i, $i = 1,\ldots,r$. Then*

$$\dim \mathfrak{C}_R = \sum_{i=1}^{r} \mu_i^2\,.$$

Proof In order to prove the claim, we use several concepts developed in [PR], and proceed as follows. We first introduce a linear map $\mathcal{L}$ on the space of R-canonical matrices, which associates to $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ the R-canonical representative of $(\zeta+\eta)\Phi(\zeta,\eta)$. Given the characterization of Proposition 10, the kernel of $\mathcal{L}$ coincides with the space of R-canonical conserved quantities. Consequently, in order to compute the dimension of the space of R-canonical conserved quantities, we need to determine the dimension of the eigenspace of $\mathcal{L}$ associated with the eigenvalue zero.

The map $\mathcal{L}$ is defined as

$$\mathcal{L}:\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]\rightarrow\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$$
$$\mathcal{L}(\Phi(\zeta,\eta)):=(\zeta+\eta)\Phi(\zeta,\eta)\bmod R$$

where $(\zeta+\eta)\Phi(\zeta,\eta)\bmod R$ denotes the R-canonical representative of $(\zeta+\eta)\Phi(\zeta,\eta)$. It is easy to see that $\mathcal{L}$ is well defined, since $\Phi_1\overset{\mathfrak{B}}{=}\Phi_2$ implies that $\mathcal{L}(\Phi_1)=\mathcal{L}(\Phi_2)$. Observe also that $\mathcal{L}$ is linear. Moreover, the set $\mathfrak{C}_R$ of conserved quantities coincides with the kernel of $\mathcal{L}$. In order to find its dimension, we study the dimension of the eigenspace of $\mathcal{L}$ associated with the eigenvalue zero. In order to do this, we will have to consider one- and two-variable polynomial matrices with complex coefficients; observe that the notion of R-canonicity is valid also in such cases.

Consider the equivalence relation in $\mathbb{C}^{1\times\mathtt{w}}[\xi]$ defined by $v_1\overset{R}{=}v_2$ if and only if $v_1-v_2=fR$ for some $f\in\mathbb{C}^{1\times\mathtt{w}}[\xi]$. We denote with $v\bmod R$ the canonical representative of the equivalence class of $v\in\mathbb{C}^{1\times\mathtt{w}}[\xi]$, defined as the only vector in the equivalence class

$$[v]:=\{v'\mid \text{ exists } f\in\mathbb{C}^{1\times\mathtt{w}}[\xi] \text{ such that } v-v'=fR\}$$

such that $v'R^{-1}$ is strictly proper.

The set of canonical representatives (equivalently, of the equivalence classes) is the $\deg(\det(R))$-dimensional vector space over $\mathbb{C}$

$$\mathbb{C}_R^{1\times\mathtt{w}}[\xi]:=\{v\in\mathbb{C}^{1\times\mathtt{w}}[\xi]\mid vR^{-1}\text{ is strictly proper }\}$$

Now consider the map

$$\mathcal{S}:\mathbb{C}_R^{1\times\mathtt{w}}[\xi]\rightarrow\mathbb{C}_R^{1\times\mathtt{w}}[\xi]$$
$$\mathcal{S}(p(\xi)):=\xi p(\xi)\bmod R$$

It is easy to see that $\mathcal{S}$ is linear. We now prove that its eigenvalues coincide with the roots of $\det(R)$. Indeed, assume that $\lambda\in\mathbb{C}$ is a root of $\det(R)$ with associated left singular vector $v\in\mathbb{C}^{1\times\mathtt{w}}$; then $vR(\lambda)=0$, and therefore $vR(\xi)=v(R(\xi)-R(\lambda))$. Observe that the polynomial matrix $R(\xi)-R(\lambda)$ is zero for $\xi=\lambda$; consequently, all of its entries must have $\xi-\lambda$ as a factor. Consequently $v'(\xi):=v(R(\xi)-R(\lambda))/(\xi-\lambda)=vR(\xi)/(\xi-\lambda)$ is a vector polynomial. Moreover, $v'(\xi)R(\xi)^{-1}$ is strictly proper. Now observe that $\xi v'(\xi)=\lambda v'(\xi)+vR(\xi)$; this implies that λ is an eigenvalue of $\mathcal{S}$ with associated eigenvector $v'\in\mathbb{C}^{1\times\mathtt{w}}[\xi]$.

Conversely, assume that $\lambda\in\mathbb{C}$ is an eigenvector of $\mathcal{S}$ with associated eigenvector $v\in\mathbb{C}^{1\times\mathtt{w}}[\xi]$; then $\xi v(\xi)\bmod R=\lambda v(\xi)$. Now for $v=\operatorname{col}(v_i)_{i=1,\dots,\mathtt{w}}\in\mathbb{C}_R^{1\times\mathtt{w}}[\xi]$, define $\deg(v)=\max_{i=1,\dots,\mathtt{w}}\{\deg(v_i)\}$. Consider that $\deg(\xi v(\xi))=\deg(v(\xi))+1$, and consequently $\xi v(\xi)R(\xi)^{-1}$ must have a nonzero constant part, which we denote with $v_c \in \mathbb{C}^{1\times \mathtt{w}}$ in the following. Observe that $\xi v(\xi) \bmod R = \lambda v(\xi) = \xi v(\xi) - v_c R(\xi)$. The last equality implies that $v_c R(\xi) = (\xi - \lambda) v(\xi)$ from which, letting $\xi = \lambda$, we conclude that v_c is a left singular vector of R associated with the root λ of $\det(R)$.

We now return to the proof of the claim of the proposition. Apply Proposition 2 in order to conclude that since $\mathfrak{B}$ is oscillatory and without characteristic frequencies in zero, $\mathbb{C}_R^{1\times \mathtt{w}}[\xi]$ admits a basis $\{b_i\}_{i=1,\dots,\deg(\det(R))}$ consisting of eigenvectors of $\mathcal{S}$. Consequently,

$$\{\bar{b}_i^{\mathrm{T}}(\zeta) b_j(\eta) + \bar{b}_j^{\mathrm{T}}(\zeta) b_i(\eta)\}_{1\le i\le j\le \deg(\det(R))}$$

is a basis for $\mathbb{C}_R^{\mathtt{w}\times \mathtt{w}}[\zeta, \eta]$ consisting of eigenvectors of $\mathcal{L}$ respectively associated with $\bar{\lambda}_i + \lambda_j$, where $\bar{b}_i(\xi)$ is obtained from $b_i(\xi)$ by conjugating the coefficients (see Proposition 3.4 of [PR]). Conclude from this that the characteristic polynomial of $\mathcal{L}$ is $\Pi_{1\le i\le j\le \deg(\det(R))}(\xi - (\bar{\lambda}_i + \lambda_j)$. In order to complete the proof, observe that for each $i\omega_i$, there exist exactly μ_i roots of $\det(R)$ equal to $-i\omega_i$. Conclude that $i\omega_i$ contributes $\mu_i \cdot \mu_i = \mu_i^2$ zero eigenvalues of $\mathcal{L}$. This concludes the proof of the claim. □

Corollary 12 *Assume that* ker $R(d/dt)$ *is oscillatory, without characteristic frequencies in zero; assume that the roots of* $\det(R)$ *are all simple. Then*

$$\dim \mathfrak{C}_R = \frac{\deg(\det(R))}{2} .$$

Remark 13 The parametrization (9) of conserved quantities can be further refined in the case $\mathtt{w} = 1$. Then R is an even polynomial with distinct roots, and it is easy to see that a polynomial $X \in \mathbb{R}[\xi]$ solves the PLE (10) if and only if it is odd. It follows that a basis for the set $\mathfrak{C}_R$ of R-canonical conserved quantities is the family

$$C_j(\zeta, \eta) := \frac{\zeta^{2j-1} R(\eta) + R(\zeta)\eta^{2j-1}}{\zeta + \eta} , \tag{11}$$

$j = 1, \dots, \frac{\deg(\det(R))}{2}$.

Using such characterization, it is a matter of straightforward verification to prove that each conserved quantity $\Phi(\zeta, \eta)$ can then be expressed as

$$\Phi(\zeta, \eta) = \Phi'(\zeta, \eta) + \zeta\eta\Phi''(\zeta, \eta)$$

where Φ' and Φ'' contain only *even* powers of ζ and η, that is $\Phi'(\zeta, \eta) = \sum_{i,j} \Phi'_{i,j} \zeta^{2i}\eta^{2j}$ and $\Phi''(\zeta, \eta) = \sum_{i,j} \Phi''_{i,j}\zeta^{2i}\eta^{2j}$. This means that $Q_{\Phi'}(w)$ is a quadratic functional of the even derivatives of w, while the QDF induced by $\zeta\eta\Phi''(\zeta, \eta)$ is a quadratic functional of the odd derivatives of w. This result generalizes to higher-order systems the decomposition of the total energy of a mechanical system as the sum of the potential energy (which in the case of a mechanical system is a quadratic functional of the positions, that is of even derivatives of the configuration variables) and of the kinetic energy (which in the case of a mechanical system is a quadratic functional involving the velocities, that is odd derivatives of the configuration variables).

Observe that for the case of one oscillator governed by the equation $md^2w/dt^2 + kw = 0$, the characterization (11) yields

$$\Phi(\zeta,\eta) = \frac{\zeta(m\eta^2+k)+\eta(m\zeta^2+k)}{\zeta+\eta} = m\zeta\eta + k\,,$$

which is the two-variable polynomial corresponding to the total energy (kinetic+ potential) $Q_\Phi(w) = m((d/dt)w)^2 + kw^2$ of the oscillator. Observe also that since $\deg(\det(R)) = 2$ in this case, the space $\mathfrak{C}_R$ has dimension 1; consequently, in the case of one oscillator the total energy is the *only* conserved quantity for such a system.

The following example illustrates the computation of conserved quantities for the system considered in Example 8.

Example 14 The system is described by the two second-order differential equations (8), and consequently a matrix $R \in \mathbb{R}^{2\times 2}[\xi]$ such that $\mathfrak{B} = \ker R(d/dt)$ is

$$R(\xi) = \begin{bmatrix} m\xi^2+k+k' & -k' \\ -k' & m\xi^2+k+k' \end{bmatrix} \tag{12}$$

Observe that this matrix is column proper, and consequently the R-canonical matrices $X \in \mathbb{R}[\xi]$ have column degree less than or equal to one. Since $\deg(\det(R)) = 4$, it follows from Proposition 4 that $\dim \mathbb{R}_R^{2\times 2}[\zeta,\eta] = 10$, and from Proposition 11 that there are two linearly independent R-canonical conserved quantities. We now proceed to construct a basis for $\mathfrak{C}_R$ using the characterization (9).

We first solve the PLE $R(-\xi)^\mathrm{T} X(\xi) + X(-\xi)^\mathrm{T} R(\xi) = 0$ in the R-canonical matrix $X \in \mathbb{R}^{2\times 2}[\xi]$. It follows from simple computations that two independent solutions are

$$X_1(\xi) := \frac{1}{2}\begin{bmatrix} \xi & 0 \\ 0 & \xi \end{bmatrix} \text{ and } X_2(\xi) := \frac{1}{2}\begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix}$$

corresponding to the two linearly independent conserved quantities

$$\Phi_{\mathfrak{C}_R,1}(\zeta,\eta) := \frac{1}{2}\begin{bmatrix} k+k'+m\zeta\eta & -k' \\ -k' & k+k'+m\zeta\eta \end{bmatrix}$$
$$\Phi_{\mathfrak{C}_R,2}(\zeta,\eta) := \frac{1}{2}\begin{bmatrix} -k' & k+k'+m\zeta\eta \\ k+k'+m\zeta\eta & -k' \end{bmatrix}$$

Observe that $\Phi_{\mathfrak{C}_R,1}$ induces the total energy of the system, and that $\Phi_{\mathfrak{C}_R,2}$ induces the conserved quantity C of Example 8.

We now give a parametrization of zero-mean quantities.

Proposition 15 *Let $\mathfrak{B} \in \mathcal{L}^\mathtt{w}$ be oscillatory, and let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be such that $\mathfrak{B} = \ker R(d/dt)$. Then $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is a zero-mean quantity if and only if there exist $\Psi, X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that*

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + R(\zeta)^\mathrm{T} X(\zeta,\eta) + X(\eta,\zeta)^\mathrm{T} R(\eta)\,. \tag{13}$$

Proof (If) Observe that if (13) holds, then for every $w \in \mathfrak{B}$ it holds

$$\frac{1}{T}\int_0^{\mathrm{T}} Q_\Phi(w)(t)dt = \frac{1}{T}\int_0^{\mathrm{T}} \frac{d}{dt} Q_\Psi(w)(t)dt = \frac{1}{T}\left(Q_\Psi(w)(T) - Q_\Psi(w)(0)\right) .$$

Now consider that $w \in \mathfrak{B}$ implies w and its derivatives are bounded on $(-\infty, \infty)$. Conclude that

$$\lim_{t\to\infty} \frac{1}{T}\int_0^{\mathrm{T}} Q_\Phi(w)(t)dt = \lim_{\mathrm{t}\to\infty} \frac{1}{T}\left(Q_\Psi(w)(T) - Q_\Psi(w)(0)\right) = 0 .$$

(Only if) Assume by contradiction that (13) does not hold; then there exists a $\Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ that is not the derivative of some QDF, is nonzero along $\mathfrak{B}$, and such that

$$\Phi(\zeta, \eta) = \Phi'(\zeta, \eta) + (\zeta + \eta)\Psi(\zeta, \eta) + R(\zeta)^{\mathrm{T}} X(\zeta, \eta) + X(\eta, \zeta)^{\mathrm{T}} R(\eta) . \quad (14)$$

Given the assumptions on Φ', there exists a root $i\omega$ of $\det(R)$, with associated right singular vector v, i.e. $R(i\omega)v = 0$, such that $v^{\mathrm{T}}\Phi'(-i\omega, i\omega)v \neq 0$. Now substitute ζ with $-i\omega$ and η with $i\omega$ in (14), multiply both sides of the equation by $\bar{v}^{\mathrm{T}}$ on the left and by v on the right, and conclude that $\bar{v}^{\mathrm{T}}\Phi(-i\omega, i\omega)v$ is nonzero. This means that $Q_\Phi(ve^{i\omega t})$ contains a nonzero constant term, and therefore that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{\infty} Q_\Phi(ve^{i\omega t} + \bar{v}e^{-i\omega t})dt \neq 0,$$

is a contradiction. □

From Proposition 15, we conclude that in order to find a zero-mean quantity, the following algorithm can be used. Let $X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be R-canonical, and define $Z(\xi)$ as

$$R^{\mathrm{T}}(-\xi)X(\xi) + X^{\mathrm{T}}(-\xi)R(\xi) =: Z(\xi) .$$

Now find a two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $\partial\Phi = Z$; for example, Φ can be chosen as the R-canonical representative of $\frac{1}{2}Z^{\mathrm{T}}(\zeta) + \frac{1}{2}Z(\eta)$. Then Φ is a zero-mean quantity; indeed, equation (13) holds, with

$$\Psi(\zeta, \eta) := \frac{\Phi(\zeta, \eta) - R^{\mathrm{T}}(\zeta)X(\eta) - X^{\mathrm{T}}(\zeta)R(\eta)}{\zeta + \eta} .$$

Remark 16 In the scalar case, if $\mathfrak{B}$ is oscillatory and without characteristic frequencies in zero, then the polynomial $R \in \mathbb{R}[\xi]$ such that $\mathfrak{B} = \ker R(d/dt)$ is even, that is $R^* = R$. It then follows from equation (13) and Theorem 3.1 of [WT1] that Q_Φ is zero-mean if and only if there exists $Y \in \mathbb{R}[\xi]$, YR^{-1} strictly proper, such that $\partial\Phi = R^*Y + Y^*R = R(Y + Y^*)$. This leads to the following result.

Proposition 17 *Let $\mathfrak{B} \in \mathcal{L}^1$ be oscillatory and without characteristic frequencies in zero. The QDF induced by $\Phi \in \mathbb{R}[\zeta, \eta]_R$ is zero-mean if and only if $\partial\Phi$ has R as a factor.*

Example 18 Consider the single oscillator described by the differential equation $md^2w/dt^2 + kw = 0$. The result of Proposition 15 allows us to conclude that the following are zero-mean quantities for $\ker(md^2/dt^2 + k)$:

$$m\zeta\eta - k = \underbrace{(\zeta+\eta)\frac{1}{2}(\zeta+\eta)m}_{\Psi_1} - (m\zeta^2 + k)\cdot\frac{1}{2} - \frac{1}{2}\cdot(m\eta^2 + k)\,,$$

$$(\zeta+\eta)k = \underbrace{-(\zeta+\eta)m\zeta\eta}_{\Psi_2} + (m\zeta^2+k)\cdot\eta + (m\eta^2+k)\cdot\zeta\,.$$

Observe that the first of these zero-mean quantities is none other than the Lagrangian of the system, while the second one is evidently a trivially zero-mean quantity, being $(d/dt)kw^2$.

Observe also that these two zero-mean quantities are linearly independent, and also linearly independent from the total energy of the system, represented by the two-variable polynomial $m\zeta\eta + k$, which is a conserved quantity. Conclude that there exists a basis of the space of R-canonical QDFs, which in this case is three-dimensional, consisting of the direct sum of the zero-mean, trivially zero-mean and conserved quantities subspaces. As will be shown later in this section, this is no coincidence: for an oscillatory system, any R-canonical QDFs is the sum of a conserved quantity and a zero-mean quantity.

We now establish the dimension of the subspace $\mathfrak{Z}_R$ of R-canonical zero-mean QDFs.

Proposition 19 *Assume that* $\ker R(d/dt)$ *is oscillatory, without characteristic frequencies in zero. Let* $\pm i\omega_i$, $i = 1, \ldots, r$, *be the distinct roots of* $\det(R)$, *with algebraic multiplicity* μ_i, $i = 1, \ldots, r$. *Then*

$$\dim \mathfrak{Z}_R = \frac{\deg(\det(R))(\deg(\det(R)) + 1)}{2} - \sum_{i=1}^{r} \mu_i^2$$

Proof Consider the map $\mathcal{L}$ introduced in the proof of Proposition 11. From equation (13), it follows that $\mathfrak{Z}_R$ is the image of $\mathcal{L}$, and that $\mathfrak{C}_R = \ker \mathcal{L}$. Recall from Proposition 4 that $\dim(R_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]) = \deg(\det(R))(\deg(\det(R)) + 1)/2$ and from Proposition 11 that $\dim(\mathfrak{C}_R) = \sum_{i=1}^{r} \mu_i^2$. The proof is complete. □

Corollary 20 *Assume that* $\ker R(d/dt)$ *is oscillatory, without characteristic frequencies in zero, and assume that the roots of* $\det(R)$ *are all simple. Then*

$$\dim \mathfrak{Z}_R = \frac{(\deg(\det(R)))^2}{2}\,.$$

Example 21 We consider again the system illustrated in Example 8. A basis for $\mathfrak{C}_R$ has been computed in Example 14. In order to compute a basis for $\mathfrak{Z}_R$, we use the characterization (13). Choosing linearly independent $X \in \mathbb{R}^{2\times 2}[\xi]$ and generating the corresponding $\partial\Phi$ according to the polynomial Lyapunov equation

$$\partial\Phi(\xi) = R^{\mathrm{T}}(-\xi)X(\xi) + X^{\mathrm{T}}(-\xi)R(\xi)$$

we obtain the following basis for $\mathfrak{Z}_R$:

$$\Phi_{\mathfrak{Z},1}(\zeta,\eta) := \begin{pmatrix} 2(k+k') - 2m\zeta\eta & -k' \\ -k' & 0 \end{pmatrix}, \; \Phi_{\mathfrak{Z},2}(\zeta,\eta) := \begin{pmatrix} 0 & k'\eta \\ k'\zeta & 0 \end{pmatrix},$$

$$\Phi_{\mathfrak{Z},3}(\zeta,\eta) := \begin{pmatrix} 0 & k+k'-m\zeta\eta \\ k+k'-m\zeta\eta & -2k' \end{pmatrix},$$

$$\Phi_{\mathfrak{Z},4}(\zeta,\eta) := \begin{pmatrix} -2k' & k+k'-m\zeta\eta \\ k+k'-m\zeta\eta & 0 \end{pmatrix},$$

$$\Phi_{\mathfrak{Z},5}(\zeta,\eta) := \begin{pmatrix} 2k'\zeta\eta & 0 \\ 0 & -2k'\zeta\eta \end{pmatrix},$$

$$\Phi_{\mathfrak{Z},6}(\zeta,\eta) := \begin{pmatrix} 0 & -k' \\ -k' & 2(k+k') - 2m\zeta\eta \end{pmatrix}, \; \Phi_{\mathfrak{Z},7}(\zeta,\eta) := \begin{pmatrix} 0 & k'\zeta \\ k'\eta & 0 \end{pmatrix},$$

$$\Phi_{\mathfrak{Z},8}(\zeta,\eta) := (\zeta+\eta)\begin{pmatrix} -k' & k+k'+m\zeta\eta \\ k+k'+m\zeta\eta & -k' \end{pmatrix}.$$

We now analyze trivially zero-mean quantities, giving first their characterization in terms of two-variable polynomial matrices, and then determining the dimension of the subspace $\mathfrak{T}_R$. Since it has been proved before that Φ is trivially zero-mean if and only if there exists $\Psi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $(d/dt)Q_\Psi = Q_\Phi$, we only need to prove that if Φ is R-canonical, then so is Ψ.

Proposition 22 *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be oscillatory. Then $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is a trivially zero-mean quantity if and only if there exists $\Psi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that*

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta). \tag{15}$$

Proof Condition (15) is evidently sufficient. In order to prove its necessity, we proceed as follows. Using equation (13), observe that if Q_Φ is zero-mean, then $Q_\Phi - (d/dt)Q_\Psi$ is zero along $\mathfrak{B}$; observe also that $Q_\Phi - (d/dt)Q_\Psi$ is induced by some $\Phi' \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ of the form $\Phi'(\zeta,\eta) = R^{\mathrm{T}}(\zeta)X(\zeta,\eta) + X^{\mathrm{T}}(\eta,\zeta)R(\eta)$ for some $X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. It follows from Proposition 4.1 of [PR] that X can be chosen univariate and such that XR^{-1} is strictly proper; consequently, Φ' is R-canonical. □

We now proceed to establish the dimension of $\mathfrak{T}_R$, the subspace of R-canonical trivially zero-mean quantities.

Proposition 23 *Assume that* ker $R(d/dt)$ *is oscillatory, without zero characteristic frequencies. Then*

$$\dim \mathfrak{T}_R = \frac{\deg(\det(R))(\deg(\det(R))+1)}{2} - 2\sum_{i=1}^{r}\mu_i^2.$$

Proof Introduce the following equivalence relation on $\mathfrak{Z}_R$:

$$[\Phi_1 \sim \Phi_2] \Longleftrightarrow [\Phi_1 - \Phi_2 \in \mathfrak{T}_R].$$

Observe that the set consisting of all equivalence classes of $\sim$ is in one-to-one correspondence with $\mathfrak{T}_R$. We proceed to determine its dimension. In order to do so, we determine first the dimension of the zero equivalence class considered as a subspace of $\mathfrak{Z}_R$.

It follows from equation (13) that the equivalence class of zero is in one-to-one correspondence with the set $\mathfrak{C}_R$ of R-canonical conservation laws, since

$$0 = (\zeta + \eta)\Psi(\zeta, \eta) + R(\zeta)^{\mathrm{T}} X(\zeta, \eta) + X(\eta, \zeta)^{\mathrm{T}} R(\eta)$$

if and only if Q_Ψ is a conservation law.

According to Proposition 11 $\mathfrak{C}_R$ has dimension equal to $\sum_{i=1}^{r} \mu_i^2$. Conclude that the set of equivalence classes of $\sim$ has dimension $\dim \mathfrak{Z}_R - \sum_{i=1}^{r} \mu_i^2 = \frac{\deg(\det(R))(\deg(\det(R))+1)}{2} - 2\sum_{i=1}^{r} \mu_i^2$ as claimed. □

Corollary 24 *Assume that* $\ker R(d/dt)$ *is oscillatory, without zero characteristic frequencies, and assume that all the roots of* $\det(R)$ *are simple. Then*

$$\dim \mathfrak{T}_R = \left(\frac{\deg(\det(R))}{2}\right)^2 - \frac{\deg(\det(R))}{2}.$$

The set $\mathfrak{T}_R$ of trivially zero-mean quantities is a subspace of $\mathfrak{Z}_R$, the space of zero-mean quantities. Let $\mathfrak{I}_R$ be a complement of $\mathfrak{T}_R$ in $\mathfrak{Z}_R$; then $\mathfrak{I}_R$ consists of those zero-mean quantities which are not trivial ones. We call the elements of $\mathfrak{I}_R$ the *intrinsically zero-mean quantities*, in order to emphasize that their zero-mean nature depends in an essential way on the dynamics of the system. Observe that the dimension of $\mathfrak{I}_R$ is $\sum_{i=1}^{r} \mu_i^2$.

Remark 25 Using the characterization of zero-mean quantities for the case $\mathtt{w} = 1$ given in Remark 16, it is not difficult to see that the following two-variable polynomials form a basis for a choice of the space of intrinsically zero-mean quantities:

$$\Phi_i(\zeta, \eta) := \frac{R(\zeta)\eta^{2i-1} - \zeta^{2i-1} R(\eta)}{\zeta - \eta}$$

where $i = 1, \ldots, \frac{\deg(\det(R))}{2}$. It is a matter of straightforward verification to see that each $\Phi_i(\zeta, \eta)$ can be written as

$$\Phi_i(\zeta, \eta) = \Phi_i'(\zeta, \eta) - \zeta\eta\Phi_i''(\zeta, \eta)$$

where $\Phi_i'(\zeta, \eta)$ and $\Phi_i''(\zeta, \eta)$ contain only even powers of ζ and η, $1 \le i \le \frac{\deg(\det(R))}{2}$. Following the line of thought illustrated in Remark 13, one can think of the basis Φ_i, $i = 1, \ldots, \frac{\deg(\det(R))}{2}$ as consisting of "generalized Lagrangians". Indeed, in the case of one oscillator described by the equation $md^2w/dt^2 + kw = 0$, the only element of the basis of $\mathfrak{I}_R$ constructed in this way is $m\zeta\eta - k$, the Lagrangian of the system.

Remark 26 In Remark 25, we have shown how to construct a basis for a choice of the space of intrinsically zero-mean quantities for the scalar case. An alternative basis can be constructed as follows.

Observe that if $\mathfrak{B} = \ker R(d/dt)$ is oscillatory and it has no characteristic frequency in zero, then $R \in \mathbb{R}[\xi]$ is an even polynomial: $R(\xi) = \sum_{i=0}^{\frac{\deg R}{2}} R_i \xi^{2i}$. Then it easily follows from Remark 16 that the following two-variable polynomials:

$$\Phi_i(\zeta, \eta) := R(-\zeta\eta)(-\zeta\eta)^{2i},$$

$i = 0, \ldots, \frac{\deg R}{2} - 1$, also form a basis for $\mathfrak{I}_R$.

Remark 27 In Proposition 11, we have assumed for simplicity of exposition that the behavior $\mathfrak{B}$ under study had no characteristic frequencies in zero. If this assumption does not hold, it can be proved that the following result holds true (see also statement (i) of Theorem 6.1 in [BB2]):

Proposition 28 *Assume that* $\ker R(d/dt)$ *is oscillatory. Let* $\pm i\omega_i$, $i = 1, \ldots, r$, *be the distinct roots of* $\det(R)$, *with algebraic multiplicity* μ_i, $i = 1, \ldots, r$, *and let* 0 *be a root of* $\det(R)$ *with multiplicity* μ_0. *Then*

$$\dim \mathfrak{C}_R = \frac{\mu_0(\mu_0+1)}{2} + \sum_{i=1}^{r} \mu_i^2 .$$

We can now state the main result of this section, a decomposition theorem for R-canonical QDFs.

Theorem 29 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *be oscillatory, and let* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *be such that* $\mathfrak{B} = \ker R(d/dt)$. *Assume that* $\mathfrak{B}$ *has no characteristic frequencies in zero. Denote with* $\mathfrak{C}_R$, $\mathfrak{Z}_R$, *and* $\mathfrak{T}_R$, *respectively, the space of* R*-canonical conserved, zero-mean, and trivially zero-mean quantities. Let* $\mathfrak{I}_R$ *be a complementary subspace of* $\mathfrak{T}_R$ *in* $\mathfrak{Z}_R$. *Then every* $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *admits a unique decomposition as*

$$\Phi = \Phi_{\mathfrak{C}_R} + \Phi_{\mathfrak{T}_R} + \Phi_{\mathfrak{I}_R} ,$$

where $\Phi_{\mathfrak{C}_R} \in \mathfrak{C}_R$, $\Phi_{\mathfrak{T}_R} \in \mathfrak{T}_R$, $\Phi_{\mathfrak{I}_R} \in \mathfrak{I}_R$.

Proof Observe first that $\mathfrak{C}_R \cap \mathfrak{Z}_R = \{0\}$. Recall respectively from Proposition 4, from Proposition 11, and from Proposition 15 that

$$\begin{aligned}
\dim \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] &= \frac{\deg(\det(R))(\deg(\det(R))+1)}{2} , \\
\dim \mathfrak{C}_R &= \sum_{i=1}^{r} \mu_i^2 , \\
\dim \mathfrak{Z}_R &= \frac{\deg(\det(R))(\deg(\det(R))+1)}{2} - \sum_{i=1}^{r} \mu_i^2 .
\end{aligned}$$

Conclude that

$$\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] = \mathfrak{C}_R \oplus \mathfrak{Z}_R .$$

Use the definition of $\mathfrak{T}_R$ and of $\mathfrak{I}_R$ to conclude that since $\mathfrak{Z}_R = \mathfrak{T}_R \oplus \mathfrak{I}_R$, it follows

$$\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] = \mathfrak{C}_R \oplus \mathfrak{T}_R \oplus \mathfrak{I}_R .$$

This concludes the proof of the claim. □

5 An equipartition of energy principle

In Example 8, we examined a system consisting of identical parts (the two oscillators) interconnected in a “symmetrical” way and we simulated the behavior of the system, finding that the average total energy of each oscillator is the same. The purpose of this section is to state and prove a general result valid for oscillatory systems consisting of identical subsystems connected in a symmetrical way, which following [BB1] we call the *deterministic equipartition of energy principle*.

In order to do so, we first need to formalize the notion of symmetry. As usual in the behavioral framework, we define such property at an intrinsic level, that of the trajectories of the system (see [FW] for a thorough investigation of symmetries and the related representational issues).

Definition 30 *Let* $\mathfrak{B}$ *be a linear differential behavior with* $\mathtt{w}$ *external variables, and let* $\Pi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *be a linear involution, i.e.* $\Pi^2 = I_{\mathtt{w}}$. $\mathfrak{B}$ *is called* Π-symmetric *if* $\Pi\mathfrak{B} = \mathfrak{B}$.

This definition is an operational one, as is common in physics: a behavior is symmetric if it can be subjected to a certain operation (the transformation Π of the external variables) without altering it.

In the following, we use the symmetry induced by the permutation matrix

$$\Pi = \begin{pmatrix} 0 & I_{\mathtt{m}} \\ I_{\mathtt{m}} & 0 \end{pmatrix} \tag{16}$$

or equivalently, we consider systems with $2\mathtt{m}$ external variables

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ w_{m+1} \\ \vdots \\ w_{2m} \end{pmatrix}$$

for which

$$[w \in \mathfrak{B}] \Longleftrightarrow \left[\begin{pmatrix} w_{m+1} \\ \vdots \\ w_{2m} \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \Pi w \in \mathfrak{B} \right]. \tag{17}$$

In order to state the main result of this section, we need to introduce the notion of observability. Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$, with its external variable w partitioned as $w = (w_1, w_2)$; then w_2 is *observable* from w_1 if for all $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $w_2 = w_2'$. Thus, the variable w_2 is observable from w_1 if w_1 and the dynamics of the system uniquely determine w_2; in other words, the variable w_1 contains all the information about the trajectory $w = (w_1, w_2)$. For linear differential systems, observability of w_2 from w_1 is equivalent with the following property of the kernel representation of $\mathfrak{B}$. Let $R_1(d/dt)w_1 = R_2(d/dt)w_2$ be a kernel representation of $\mathfrak{B}$, with $R_1 \in \mathbb{R}^{\bullet\times \mathtt{w}_1}[\xi]$, $R_2 \in \mathbb{R}^{\bullet\times \mathtt{w}_2}[\xi]$. Then w_2 is observable from w_1 if and only if rank $R_2(\lambda) = \mathtt{w}_2$ for all $\lambda \in \mathbb{C}$ (see [PoW], Th. 5.3.3). It is easy to see that this is equivalent with the existence of a polynomial matrix $F \in \mathbb{R}^{\mathtt{w}_2\times \mathtt{w}_1}[\xi]$ such that $w_2 = F(d/dt)w_1$ for all $w_1, w_2 \in \mathfrak{B}$.

Theorem 31 *Let $\mathfrak{B}$ be an oscillatory behavior with $\mathtt{w} = 2m$ external variables. Assume that $\mathfrak{B}$ is Π-symmetric, with Π given by (16), i.e. (17) holds. Moreover, assume that*

(*a*) $w_2, \dots, w_{\mathtt{m}}, w_{\mathtt{m}+1}$ *observable from* w_1*; and*
(*b*) $w_{\mathtt{m}+2}, \dots, w_{2\mathtt{m}}$ *observable from* $w_{\mathtt{m}+1}$.

Let $\Psi \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$, and consider the QDF Q_Φ induced by the $2m \times 2m$ two-variable matrix

$$\Phi(\zeta,\eta) := \begin{pmatrix} \Psi(\zeta,\eta) & 0 \\ 0 & -\Psi(\zeta,\eta) \end{pmatrix}$$

on $\mathfrak{B}$. Then Q_Φ is a zero-mean quantity for $\mathfrak{B}$.

Proof In order to prove the claim, we reduce ourselves to the case of two external variables as follows. Since $\mathfrak{B}$ is Π-symmetric and since $w_2, \dots, w_m$ is observable from w_1, and $w_{m+2}, \dots, w_{2m}$ is observable from w_{m+1}, we can write

$$\begin{pmatrix} w_2 \\ \vdots \\ w_{\mathtt{m}} \end{pmatrix} = F\left(\frac{d}{dt}\right) w_1 \text{ and } \begin{pmatrix} w_{\mathtt{m}+2} \\ \vdots \\ w_{2\mathtt{m}} \end{pmatrix} = F\left(\frac{d}{dt}\right) w_{\mathtt{m}+1}$$

for some $F \in \mathbb{R}^{(\mathtt{m}-1)\times 1}[\xi]$. Consequently, we can write

$$Q_\Phi(w) = Q_\Psi \begin{pmatrix} w_1 \\ F(\frac{d}{dt})w_1 \end{pmatrix} - Q_\Psi \begin{pmatrix} w_{\mathtt{m}+1} \\ F(\frac{d}{dt})w_{\mathtt{m}+1} \end{pmatrix} = Q_{\Psi'}(w_1) - Q_{\Psi'}(w_{\mathtt{m}+1})$$

where the symmetric two-variable polynomial $\Psi'(\zeta,\eta)$ is defined as

$$\Psi'(\zeta,\eta) = \begin{pmatrix} 1 & F^{\mathrm{T}}(\zeta) \end{pmatrix} \Psi(\zeta,\eta) \begin{pmatrix} 1 \\ F(\eta) \end{pmatrix}.$$

We now prove that the QDF induced by

$$\begin{pmatrix} \Psi'(\zeta,\eta) & 0 \\ 0 & -\Psi'(\zeta,\eta) \end{pmatrix} \in \mathbb{R}^{2\times 2}[\xi]$$

and acting on the projection of $w \in \mathfrak{B}$ on the components w_1 and $w_{\mathtt{m}+1}$ is zero-mean, using the characterization (13) of zero-mean quantities. In order to do so, observe first that $\mathfrak{B}$ being oscillatory implies also that its projection on the w_1 and $w_{\mathtt{m}+1}$ variable is such. Moreover, Π-symmetry of $\mathfrak{B}$ implies that its projection on the w_1 and $w_{\mathtt{m}+1}$ variable is J-symmetric, with

$$J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In order to complete the claim of the Theorem, therefore, it suffices to show that given any J-symmetric oscillatory behavior $\mathfrak{B}$ with two external variables one of which is observable from the other, and given any $\Gamma \in \mathbb{R}_S[\zeta, \eta]$, the QDF $Q_\Gamma(w_1) - Q_\Gamma(w_2)$ is zero-mean.

We first prove that this system admits a kernel representation of special structure using the results of [FW]. Argue analogously to the proof of Theorem 4 p. 9 of [FW] in order to conclude that $\mathfrak{B}$ admits a kernel representation of the form

$$\begin{pmatrix} r_1 & \varepsilon_1 r_1 \\ r_2 & \varepsilon_2 r_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}[\xi]$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$. Observe that the determinant of this matrix equals $r_1 r_2(\varepsilon_2 - \varepsilon_1)$, and that it is nonzero if and only if $\varepsilon_1 \neq \varepsilon_2$, that is, if and only if $\varepsilon_1 = -\varepsilon_2$. Conclude from this that $\mathfrak{B}$ can also be described in kernel form by

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} r_1 & \varepsilon_1 r_1 \\ r_2 & \varepsilon_2 r_2 \end{pmatrix} &= \begin{pmatrix} r_1 + r_2 & \varepsilon_1 r_1 + \varepsilon_2 r_2 \\ \varepsilon_1 r_1 + \varepsilon_2 r_2 & r_1 + r_2 \end{pmatrix} \\ &=: R' = \begin{pmatrix} r_1' & r_2' \\ r_2' & r_1' \end{pmatrix} \end{aligned}$$

Observe that $\det(R') = r_1'^2 - r_2'^2$ is an even polynomial, since $\mathfrak{B}$ is oscillatory. Conclude from $r_1'^2 - r_2'^2 = (r_1' + r_2')(r_1' - r_2')$, that r_1' and r_2' are even polynomials. Moreover, since the second external variable is observable from the first one, then col $(r_2'(\lambda), r_1'(\lambda))$ has rank 1 for all $\lambda \in \mathbb{C}$, in other words, $GCD(r_1', r_2') = 1$. This implies that there exist $a, b \in \mathbb{R}[\xi]$ such that

$$a r_1' + b r_2' = 1\,.$$

Observe that since r_1' and r_2' are even, a and b can also be taken to be even polynomials.

Now let $\Gamma \in \mathbb{R}_S[\zeta, \eta]$, and define

$$X(\xi) := \partial\Gamma(\xi) \begin{pmatrix} a(\xi) & -b(\xi) \\ b(\xi) & -a(\xi) \end{pmatrix}\,.$$

It is a matter of straightforward manipulations to see that

$$R'^{\mathrm{T}}(-\xi) X(\xi) + X^{\mathrm{T}}(-\xi) R'(\xi) = \begin{pmatrix} \partial\Gamma(\xi) & 0 \\ 0 & -\partial\Gamma(\xi) \end{pmatrix}\,.$$

We conclude from the characterization of zero-mean quantities given in Proposition 15 that the QDF $Q_\Gamma(w_1) - Q_\Gamma(w_2)$ is zero-mean. This concludes the proof of the theorem. □

Example 32 Consider the system described in Example 8. It is easy to see that this system satisfies the assumptions of Theorem 31. It has already been remarked that the difference between the energies of the two oscillators, associated with the two-variable polynomial matrix

$$\Gamma(\zeta, \eta) := \begin{pmatrix} m\zeta\eta + k & 0 \\ 0 & -(m\zeta\eta + k) \end{pmatrix}$$

is zero-mean. In fact, it can be easily verified that with R defined as in (12), equation (13) has the solution

$$X(\xi) = \begin{pmatrix} -\frac{1}{2} & -\frac{2k+k'}{2k'} \\ -\frac{2k+k'}{2k'} & \frac{1}{2} \end{pmatrix} .$$

From the result of Theorem 31, we can conclude that the difference between the *kinetic energies* of the two oscillators, represented by the two-variable polynomial matrix

$$\Gamma(\zeta,\eta) := \begin{pmatrix} m\zeta\eta & 0 \\ 0 & -m\zeta\eta \end{pmatrix}$$

is also zero-mean. In fact, in this case equation (13) has the solution

$$X(\xi) = \begin{pmatrix} -\frac{1}{2} & \frac{k+k'}{2k'} \\ -\frac{k+k'}{2k'} & \frac{1}{2} \end{pmatrix} .$$

Of course, the difference between the *potential energies* of the two oscillators, represented by the two-variable polynomial matrix

$$\Gamma(\zeta,\eta) := \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$$

is also zero-mean. In fact, in this case equation (13) has the solution

$$X(\xi) = \begin{pmatrix} 0 & \frac{k}{2k'} \\ -\frac{k}{2k'} & 0 \end{pmatrix} .$$

Example 33 Consider the following oscillatory system. Two oscillators with mass m are attached to walls by means of springs with stiffness constant k. The oscillators are coupled symmetrically by means of an oscillatory system consisting of a mass m' attached on either side to the oscillators by means of springs of equal stiffness constant k'. We consider as external variables of this system the displacements from the equilibrium positions of the two masses m (labeled w_1 and w_3), and as latent variable the displacement from the equilibrium position of the third mass m' (labeled w_2). A kernel description of such behavior $\mathfrak{B}$ is given by the matrix

$$R(\xi) = \begin{pmatrix} m\xi^2 + k + k' & -k' & 0 \\ -k' & m'\xi^2 + 2k' & -k' \\ 0 & -k' & m\xi^2 + k + k' \end{pmatrix} .$$

Consider for example the case $k = 7\frac{N}{m}$, $k' = 2\frac{N}{m}$, $m = 13\,\text{kg}$, $m' = 10\,\text{kg}$. Eliminating the w_2 variable from the equations (see Sect. 6.2 of [PoW]) yields a kernel representation of the projection of $\mathfrak{B}$ onto (w_1, w_3) as

$$R'(\xi) = \begin{pmatrix} -9 - 13\xi^2 & 9 + 13\xi^2 \\ 32 + 142\xi^2 + 130\xi^4 & -4 \end{pmatrix} .$$

It is easy to verify that all assumptions of Theorem 31 are satisfied. It follows that the difference of the potential energies of the two oscillators is zero-mean. This can also be verified via (13), since the equation $R'^{\mathrm{T}}(-\xi)X(\xi) + X^{\mathrm{T}}(-\xi)R(\xi) = \mathrm{diag}(k, -k)$ has the solution

$$X(\xi) = \begin{pmatrix} -\frac{7}{2} - \frac{35}{4}\xi^2 & 0 \\ -\frac{7}{4} & \frac{7}{4} \end{pmatrix}.$$

The same conclusion holds for the difference of the kinetic energies, associated with the two-variable polynomial matrix $\mathrm{diag}(m\zeta\eta, -m\zeta\eta)$. Indeed, the equation $R'^{\mathrm{T}}(-\xi)X(\xi) + X^{\mathrm{T}}(-\xi)R(\xi) = \mathrm{diag}(-m\xi^2, m\xi^2)$ has the solution

$$X(\xi) = \begin{pmatrix} -4 - \frac{45}{4}\xi^2 & \frac{1}{2} \\ -\frac{9}{4} & \frac{9}{4} \end{pmatrix}.$$

Of course, this implies that the difference of the actual energies of the two oscillators is also zero-mean.

6 Conclusions

The main results of this paper are the decomposition presented in Theorem 29 and the equipartition principle stated in Theorem 31, which are proved using the framework of quadratic differential forms. The computation of the conserved- and zero-mean quantities for a given system is reduced to the solution of polynomial matrix equations such as (9) and (13). As such, these results can be applied to systems described by higher-order equations, and they can be implemented easily using standard polynomial computations.

Research efforts are now being pursued in incorporating Lagrangian and symplectic methods in our framework, with the ultimate goal of automatizing the work- and energy methods used in mechanics and engineering through their implementation with standard polynomial computations. The use of quadratic differential forms in energy flow modeling and control of complex interconnected structures (see [KB1, KB2, KBH]) is also under investigation.

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