

THE BEHAVIORAL APPROACH AS A PARADIGM FOR MODELLING INTERCONNECTED SYSTEMS

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Abstract

This article contains the outline of a 3-hour course that will be given by the authors at the ECC 2003. The course consists of 4 parts: 1. The basic concepts, 2. Linear differential systems, 3. Control in a behavioral setting, and 4. The synthesis of dissipative systems.

Keywords: Behavior, differential systems, controllability, observability, control, implementability, quadratic differential forms, dissipative systems, storage functions.

Introduction

The purpose of this short course at the ECC 2003 is to outline the basics of a mathematical language for the modeling, analysis, and the synthesis of systems. The framework that we will present considers the *behavior* of a system as the main object of study. It treats *a priori* all the system variables on equal footing. This differs in an essential way from the input/output paradigm which has dominated the development of the field of systems and control in the 20-th century. This paradigm-shift calls for a reconsideration of many of the basic concepts, of the model classes, of the problem formulations, and of the algorithms in the field.

It is impossible to do justice to all these aspects in the span of a three-hour course. We will therefore concentrate on the following main themes:

1. The basic concepts and motivation of the mathematical framework used.
2. A discussion of linear differential systems.
3. The formulation of control as interconnection.
4. Quadratic differential forms and the synthesis of dissipative systems.

It is not the purpose to develop mathematical ideas for their own sake. To the contrary, we will downplay mathematical issues of a technical nature. The main aim is to convince that the behavioral framework is a cogent systems-theoretic setting that

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properly deals with physical systems and that uses modeling as the essential motivation for choosing appropriate mathematical concepts.

The present article contains a brief outline of the course. More details can be found in the references given, and on the website

<http://www.esat.kuleuven.ac.be/~jwillems>.

Lecture 1: The behavior, manifest and latent variables

The framework that we will use for discussing mathematical models views a model as follows. Assume that we have a phenomenon that we wish to model. Nature (that is, the reality that governs this phenomenon) can produce certain events (we will also call them outcomes). The totality of these possible events (*before* we have modelled the phenomenon) forms a set \mathbb{W} , called the *universum*. A *mathematical model* of this phenomenon restricts the outcomes that are declared possible to a subset \mathfrak{B} of \mathbb{W} ; \mathfrak{B} is called the *behavior* of the model. We refer to $(\mathbb{W}, \mathfrak{B})$ (or to the behavior \mathfrak{B} itself, since \mathbb{W} is usually obvious from the context) as a mathematical model.

In the study of (dynamical) systems we are, more specifically, interested in situations where the events are signals, trajectories, i.e., maps from a set of *independent variables* (time, or space, or time and space) to a set of *dependent variables* (the values taken on by the signals). In this case the universum is the collection of all maps from the set of independent variables to the set of dependent variables. It is convenient to distinguish these sets explicitly in the notation for a mathematical model: \mathbb{T} for the set of independent variables, and \mathbb{W} for the set of dependent variables. \mathbb{T} suggests 'time', but in distributed parameter systems \mathbb{T} is often time and space - we have incorporated distributed systems because of their importance in chemical engineering models. Whence we define a *system* as a triple

$$\Sigma = (\mathbb{W}, \mathbb{T}, \mathfrak{B})$$

with \mathfrak{B} , *the behavior*, a subset of $\mathbb{W}^{\mathbb{T}}$ ($\mathbb{W}^{\mathbb{T}}$ is the standard mathematical notation for the set of all maps from \mathbb{T} to \mathbb{W}). The behavior is the central object in this definition. It formalizes which signals $w : \mathbb{T} \rightarrow \mathbb{W}$ can, according to the model, occur: those in \mathfrak{B} , and which cannot occur: those not in \mathfrak{B} .

With these definitions, we aim principally at 'open' systems, that is, systems that interact with their environment. It has been customary to deal with such systems by viewing them as input/output systems, and by assuming that the input is imposed by the environment. The input/output setting imposes an unnecessary - and unphysical - signal flow structure on systems in interaction with our environment. The input/output point of view has many virtues as a vehicle of studying physical systems, but as a starting point, it is simply inappropriate. First principles laws in physics always state that some events can happen (those satisfying the model equations) while others cannot happen (those violating the model equations). This is a far distance from specifying a system as being driven from the outside by free inputs which together with an initial state specifies the other variables, the outputs. The behavioral framework treats a model for what it is: an exclusion law.

In the basic equations describing systems, very often other variables appear in addition to those whose behavior the model aims at describing. The origin of these auxiliary variables varies from case to case. They may be state variables (as in automata and input/state/output systems); they may be potentials (as in the well-known expressions for

the solutions of Maxwell's equations); most frequently, they are interconnection variables. It is important to incorporate these variables in our basic modeling language *ab initio*, and to distinguish clearly between the variables whose behavior the model aims at, and the auxiliary variables introduced in the modeling process. We call the former *manifest* variables, and the latter *latent* variables.

A *mathematical model with latent variables* is defined as a triple $(\mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ with \mathbb{W} the universum of manifest variables, \mathbb{L} the universum of latent variables, and $\mathfrak{B}_{\text{full}} \subseteq \mathbb{W} \times \mathbb{L}$ the *full behavior*. It induces the *manifest model* $(\mathbb{W}, \mathfrak{B})$, with $\mathfrak{B} = \{w \in \mathbb{W} \mid \text{there exists } \ell \in \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$. A *system with latent variables* is defined completely analogously as

$$\boxed{(\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})}$$

with $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$. The notion of a system with latent variables is the natural end-point of a modeling process and hence a very natural starting point for the analysis and synthesis of systems. We shall see that latent variables enter also very forcefully in representation questions.

Situations in which models use latent variables either for mathematical reasons or in order to express the behavioral constraints abound: internal voltages and currents in electrical circuits, momenta in mechanics, chemical potentials, entropy and internal energy in thermodynamics, prices in economics, state variables, the wave function in quantum mechanics in order to explain observables, the basic probability space Ω in probability, etc.

In the first lecture, these concepts will be illustrated by means of concrete examples. Also general system properties, as linearity and time-invariance will be introduced.

The behavioral approach is discussed, including the mathematical technicalities, in the recent textbook [12]. The three part paper [19] provides the first detailed exposition of the behavior framework, although the roots go back earlier [18]. It has been further elaborated in [20] and in [21]. This latter reference contains a comprehensive overview. We also mention [15] for generalizations to n-D systems, and [11] where PDE's are viewed in this setting. Informal expositions of the behavioral approach can be found in [22, 24].

As already mentioned, state variables emerge as latent variables *par excellence*. State and state construction problems are discussed in [13, 14]. A setting where the behavioral framework and latent variables coming from the interconnection constraints occur very naturally is modelling of interconnected systems by *tearing and zooming*. This application is discussed in [4, 5].

Lecture 2: Linear differential systems

The 'ideology' that underlies the behavioral approach is the belief that in a model of a dynamical (physical) phenomenon, it is the behavior \mathfrak{B} , i.e., a set of possible trajectories $w: \mathbb{T} \rightarrow \mathbb{W}$, that is the central object of study. However, as we have seen, in first principles modeling, also latent variables enter *ab initio*. But, the sets \mathfrak{B} or $\mathfrak{B}_{\text{full}}$ of trajectories must be specified somehow, and it is here that differential equations (and difference equations in discrete-time systems) enter the scene. Of particular interest (in control, signal processing, circuit theory, econometrics, etc.) are systems with a signal space that is a finite-dimensional vector space and behavior described by linear constant coefficient differential (or difference) equations. Such systems occur not only

when the dynamics are linear, but also after linearization around an equilibrium point, when studying the 'small signal behavior'.

A *linear time-invariant differential system* is a dynamical system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{W} = \mathbb{R}^w$ a finite-dimensional (real) vector space, whose behavior consists of the solutions of a system of differential equations of the form

$$R_0 w + R_1 \frac{d}{dt} w + \dots + R_n \frac{d^n}{dt^n} w = 0,$$

with R_0, R_1, \dots, R_n matrices of appropriate size that specify the system parameters, and $w = (w_1, w_2, \dots, w_w)$ the vector of (real-valued) system variables. These systems call for polynomial matrix notation. It is convenient to denote the above system of differential equations as

$$\boxed{R\left(\frac{d}{dt}\right)w = 0},$$

with $R \in \mathbb{R}^{\bullet \times w}[\xi]$ a real polynomial matrix with w columns. The behavior of this system is defined as

$$\mathfrak{B} = \{w : \mathbb{R} \rightarrow \mathbb{R}^w \mid R\left(\frac{d}{dt}\right)w = 0\}.$$

The precise definition of what we consider a solution of $R\left(\frac{d}{dt}\right)w = 0$ is an issue that we will slide over, but for the results that follow, it is convenient to consider solutions in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. Since \mathfrak{B} is the kernel of the differential operator $R\left(\frac{d}{dt}\right)$, we often write $\mathfrak{B} = \ker(R\left(\frac{d}{dt}\right))$, and call $R\left(\frac{d}{dt}\right)w = 0$ a *kernel representation* of the associated linear time-invariant differential system. We denote the set of differential systems or their behaviors by \mathfrak{L}^\bullet , or by \mathfrak{L}^w when the number of variables is w .

Of course, the number of columns of the polynomial matrix R equals the dimension of \mathbb{W} . The number of rows of R , which represents the number of scalar differential equations, is arbitrary. In fact, when the row dimension of R is less than its column dimension, as is usually the case, $R\left(\frac{d}{dt}\right)w = 0$ is an under-determined system of differential equations which is typical for models in which the influence of the environment is taken into account.

In the linear time-invariant case with latent variables, this becomes

$$\boxed{R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell},$$

with R and M polynomial matrices of appropriate sizes. Define the *full* behavior of this system as

$$\{(w, \ell) : \mathbb{R} \rightarrow \mathbb{R}^{w \times \ell} \mid R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell\}.$$

Hence the *manifest* behavior of this system is

$$\{w : \mathbb{R} \rightarrow \mathbb{R}^w \mid \text{there exists } \ell : \mathbb{R} \rightarrow \mathbb{R}^\ell \text{ such that } R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell\}.$$

We call the $R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$ a *latent variable* representation of the manifest behavior \mathfrak{B} .

There is a very extensive theory about these linear differential systems. It is a natural starting point for a theory of dynamical systems. Besides being a natural outcome of modeling (perhaps after linearization), it incorporates high order differential equations, the ubiquitous first order state systems and transfer function models, implicit (descriptor) systems, etc., as special cases. The study of these systems is therefore intimately connected with the study of polynomial matrices. In the second lecture, we will deal with some of the fundamental facts which emerges. For instance,

1. *Mathematical structure:* there is a $1 \leftrightarrow 1$ correspondence between \mathcal{L}^w and the sub-modules of $\mathbb{R}^w[\xi]$ viewed as a module over $\mathbb{R}[\xi]$. This $1 \leftrightarrow 1$ correspondence associates with each $\mathfrak{B} \in \mathcal{L}^w$ simply its *annihilators* $\mathfrak{N}_{\mathfrak{B}}$, i.e., the linear differential relations (sometimes called *consequences*) that annihilate \mathfrak{B} . The set $\mathfrak{N}_{\mathfrak{B}} \subseteq \mathbb{R}^w[\xi]$ consists of the polynomial vectors $n \in \mathbb{R}^w[\xi]$ such that $n^\top (\frac{d}{dt})\mathfrak{B} = 0$, and is easily seen to be a sub-module of $\mathbb{R}^w[\xi]$. This $1 \leftrightarrow 1$ relationship brings many issues in the study of \mathcal{L}^\bullet back to the structure of these sub-modules.
2. *Inclusion:* the behavior of $R_1(\frac{d}{dt})w = 0$ is contained in that of $R_2(\frac{d}{dt})w = 0$ if and only if $R_2 = FR_1$ with F a polynomial matrix of suitable dimensions.
3. *Minimality of kernel representations:* every $\mathfrak{B} \in \mathcal{L}^w$ has a kernel representation $R(\frac{d}{dt})w = 0$ with R a polynomial matrix of full row rank. Such kernel representations are called *minimal*, because of all kernel representations of a given $\mathfrak{B} \in \mathcal{L}^w$, the number of rows of R (i.e., the number of scalar differential equations defining \mathfrak{B}) is as small as possible.
4. *Uniqueness of minimal kernel representations:* two minimal kernel representations $R_1(\frac{d}{dt})w = 0$ and $R_2(\frac{d}{dt})w = 0$ represent the same system if and only if $R_1 = UR_2$ with U a unimodular polynomial matrix.
5. *Elimination:* the question occurs whether the manifest behavior \mathfrak{B} of a latent variable representation belongs to \mathcal{L}^w , i.e., whether it can also be described by a linear constant coefficient differential equation. We call this the *elimination problem*. The following *elimination theorem* holds: For any real polynomial matrices (R, M) with $\text{rowdim}(R) = \text{rowdim}(M)$, there exists a real polynomial matrix R' such that the manifest behavior of $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ has the kernel representation $R'(\frac{d}{dt})w = 0$.

It is easy to prove that a linear differential system admits an *input/output representation*. This means that for every $\mathfrak{B} \in \mathcal{L}^w$, there exists a permutation matrix Π and a partition $\Pi w = \text{col}(u, y)$ such that for any $u^* \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^u)$, there exist a $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^y)$ such that $(u^*, y) \in \Pi\mathfrak{B}$. Moreover, the y 's that such $(u^*, y) \in \Pi\mathfrak{B}$ form a linear finite dimensional variety, implying that such a y is uniquely determined by its derivatives at $t = 0$.

Thus in linear differential systems, the variables can always be partitioned into two groups. The first group act a free inputs, the second group a bound outputs: they are completely determined by the inputs and their initial conditions. In traditional systems theory, it is customary to view system interconnection as identifying inputs of one system with outputs of another system. Unfortunately, this is often precisely the opposite of what happens physically. For example, in fluidics, interconnection calls for identifying two pressures (often both inputs) and two flows (often both outputs), in mechanics, interconnection equates two positions (often both outputs), and puts the sum of two forces (often both inputs) equal to zero. If the field of systems and control wants to take modeling seriously, it should retrace the *faux pas* of taking input/output thinking as the basic framework, and cast models in the language of behaviors. It is only when considering the more detailed signal flow graph structure of a system that input/output thinking becomes useful. Signal flow graphs are useful building blocks for interpreting information processing systems, but physical systems simply need a different, a more flexible framework.

An important property in the analysis and synthesis of systems is controllability. Controllability refers to the ability of transferring a system from one mode of operation to another. By viewing the first mode of operation as undesired and the second one as desirable, the relevance to control and other areas of applications becomes clear. The concept of controllability has originally been introduced in the context of state space systems. A disadvantage of the classical notion of controllability as formulated above is that it refers to a particular representation of a system, notably to a state space representation. Thus a system may be uncontrollable either for the intrinsic reason that the control has insufficient influence on the system variables, or because the state has been chosen in an inefficient way. It is clearly not desirable to confuse these reasons. In the context of behavioral systems, a definition of controllability has been put forward that involves the manifest system variables directly.

Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ be a dynamical system with $\mathbb{T} = \mathbb{R}$ or \mathbb{Z} , and assume that it is time-invariant, that is $\sigma^t \mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{T}$, where σ^t denotes the t -shift (defined by $(\sigma^t f)(t') := f(t' + t)$); Σ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists $T \in \mathbb{T}$, $T \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - T)$ for $t \geq T$. Thus controllability refers to the ability to switch from any one trajectory in the behavior to any other one, allowing some time-delay.

Two questions that occur are the following: What conditions on the parameters of a system representation imply controllability? Do controllable systems admit a particular representation in which controllability becomes apparent? For linear time-invariant differential systems, these questions are answered in the following theorem. Let $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ be a linear time-invariant differential system. The following are equivalent:

1. $\mathfrak{B} \in \mathcal{L}^w$ is controllable.
2. The polynomial matrix R in a kernel representation $R(\frac{d}{dt})w = 0$ of $\mathfrak{B} \in \mathcal{L}^w$ satisfies $\text{rank}(R(\lambda)) = \text{rank}(R)$ for all $\lambda \in \mathbb{C}$.
3. The behavior $\mathfrak{B} \in \mathcal{L}^w$ is the image of a linear constant-coefficient differential operator, that is, there exists a polynomial matrix $M \in \mathbb{R}^{w \times \bullet}[\xi]$ such that $\mathfrak{B} = \{w \mid w = M(\frac{d}{dt})\ell \text{ for some } \ell\}$.
4. The compact support elements of $\mathfrak{B} \in \mathcal{L}^w$ are dense in \mathfrak{B} .
5. The $\mathbb{R}[\xi]$ -module $\mathbb{R}^w[\xi]/\mathfrak{N}_{\mathfrak{B}}$, where $\mathfrak{N}_{\mathfrak{B}}$ denotes the annihilators of $\mathfrak{B} \in \mathcal{L}^w$, is torsion free.

There exist various algorithms for verifying controllability of a system starting from the coefficients of the polynomial matrix R in a kernel (or a latent variable) representation of Σ . The basic idea is to compute *syzygies* associated with R . These are standard elements in computer algebra packages. We will however not enter into these algorithmic aspects.

A point of the above theorem that is worth emphasizing is that, as stated in the above theorem, controllable systems admit a representation as the manifest behavior of the latent variable system of the special form

$$w = M\left(\frac{d}{dt}\right)\ell.$$

We call this an *image* representation of the system with behavior

$$\mathfrak{B} = \{w \mid \text{there exists } \ell \text{ such that } w = M\left(\frac{d}{dt}\right)\ell\}.$$

It follows from the elimination theorem that every system in image representation can be brought in kernel representation. But not every system in kernel representation can be brought in image representation: it are precisely the controllable ones for which this is possible.

The controllability question has been pursued for many other classes of systems. In particular (more difficult to prove) generalizations have been derived for differential-delay [16, 8], for nonlinear systems, and for PDE's [11].

The notion of observability is always introduced hand in hand with controllability. In the context of the input/state/output system $\frac{d}{dt}x = f(x, u), y = h(x, u)$, it refers to the possibility of deducing, using the laws of the system, the state from observation of the input and the output. The definition that is used in the behavioral context is more general in that the variables that are observed and the variables that need to be deduced are kept general.

In observability, we ask the question: *Can the trajectory w_1 be deduced from the trajectory w_2 ?* Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ be a dynamical system, and assume that \mathbb{W} is a product space: $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2$. Then w_1 is said to be *observable* from w_2 in Σ if $(w_1, w_2') \in \mathfrak{B}$ and $(w_1, w_2'') \in \mathfrak{B}$ imply $w_2' = w_2''$. Observability thus refers to the possibility of deducing the w_1 from observation of w_2 and from the laws of the system (\mathfrak{B} is assumed to be known).

The theory of observability runs parallel to that of controllability. We mention only the result that for linear time-invariant differential systems, w_1 is observable from w_2 if and only if there exists a set of differential equations satisfied by the behavior of the system (i.e., a set of consequences) of the following form, that puts observability into evidence: $w_1 = R_2'(\frac{d}{dt})w_2$. This condition is again readily turned into a standard problem in computer algebra.

We call a latent variable system *observable* if the latent variables are observable from the manifest ones. It can be shown that a controllable linear differential system always admits an observable image representation. This is only true for 1-D systems. For n-D linear differential systems, controllability still implies the existence of an image representation, but an observable one may not exist (for example in Maxwell's equations) [11]. Systems with an observable image representation have received much attention for nonlinear differential-algebraic systems, where they are referred to as *flat* systems [7].

In the second lecture of the short course, we will also review this behavioral theory of controllability and observability.

Lecture 3: Control as interconnection

The behavioral point of view has received broad acceptance as an approach for modeling dynamical systems. It is generally agreed upon that when modeling a dynamic component it makes no sense to prejudice oneself (as one would be forced to do in a transfer function setting) as to which variables should be viewed as inputs and which variables should be viewed as outputs. This is not to say, by any means, that there are no situations where the input/output structure is natural. Quite to the contrary. In fact, whenever logic devices are involved, in information processing, the input/output structure is often a must.

The behavioral approach has, until now, met with much less acceptance in the context of control. We can offer a number of explanations for this fact. Firstly, there is something very natural in viewing control variables as inputs and measured variables

as outputs. Control then becomes decision making on the basis of observations. When subsequently a controller is regarded as a feedback processor, one ends up with the feeling that the input/output structure is in fact an essential feature of control. Secondly, as mentioned, it is possible to prove that every linear time-invariant system admits a component-wise input/output representation. This leaves the mistaken impression that the input/output framework can be adopted without second thoughts, that nothing is lost by taking it as a universal starting point.

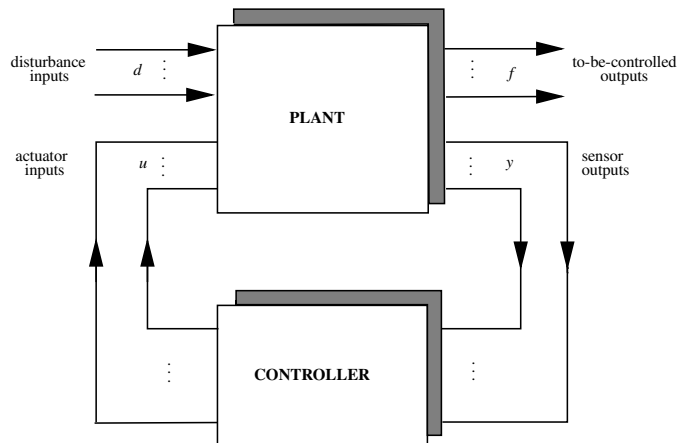


Figure 1: Intelligent control

Present-day control theory centers around the signal flow graph shown in figure 1. The plant has four terminals (supporting variables which will typically be vector-valued). There are two input terminals, one for the control, one for the other exogenous variables (disturbances, set-points, reference signals, tracking inputs, etc.) and there are two output terminals, one for the measurements, and one for the to-be-controlled variables. By using feed-through terms in the plant equations this configuration accommodates, by incorporating these variables in the measurements, for the possible availability to the controller of set-point settings, reference signals, or disturbance measurements for feed-forward control, and, by incorporating the control input in the to-be-controlled outputs, for penalizing excessive control action. The control inputs are generated by means of actuators and the measurements are made available through sensors. Usually, the dynamics of the actuators and of the sensors are considered to be part of the plant. We call this structure *intelligent control*. In intelligent control, the controller is thought of as a micro-processor type device which is driven by the sensor outputs and which produces the actuator inputs through an algorithm involving a combination of feedback, identification, and adaptation. Also, often loops expressing model uncertainty are incorporated in the above as well. Of course, many variations, refinements, and special cases of this structure are of interest, but the basic idea is that of supervisor reacting in an intelligent way to observed events and measured signals.

The paradigm embodied in figure 1 has been universally prevalent ever since the beginning of the subject, from the usual interpretation of the Watt regulator, Black's feedback amplifier, and Wiener's cybernetics, to the ideas underlying modern adaptive and robust control. It is indeed a deep and very appealing paradigm, which will undoubtedly gain in relevance and impact as logic devices become ever more prevalent,

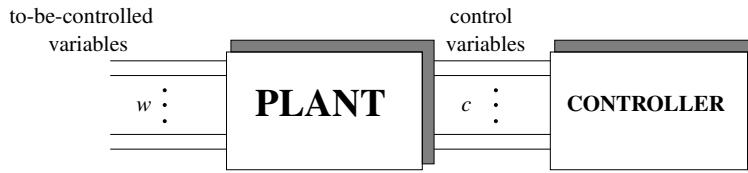


Figure 2: Control as interconnection

reliable, and inexpensive.

However, as one can conclude from analyzing simple passive controllers (as shock absorbers, heat fins, pressure control valves, etc.), notwithstanding all its merits, the intelligent control paradigm of figure 1 gives an unnecessarily restrictive view of control theory. In many practical control problems, the signal-flow-graph interpretation of figure 1 is untenable. Many, if not most, practical controllers do not act as sensor-output-to-actuator-input devices. The solution which we propose to this dilemma is the following. We will keep the distinction between plant and controller with the understanding that this distinction is justified only from an *evolutionary* point of view, in the sense that it becomes evident only *after* we comprehend the genesis of the controlled system, after we understand the way in which the closed loop system has come into existence as a purposeful system. However, we will abandon the intelligent control signal flow graph as a paradigm for control. We will abandon the distinction between control inputs and measured outputs. Instead, we will view *interconnection of a controller to a plant* as the central paradigm in control theory. However, we by no means claim that the intelligent control paradigm is without merits. To the contrary, it is extremely useful and important. Claiming that the input/output framework is *not always* the suitable framework to approach a problem does not mean that one claims that it is *never* the suitable framework. However, a good universal framework for control should be able to deal both with interconnection, with designing subsystems, and with intelligent control. The behavioral framework does, the intelligent control framework does not.

In order to illustrate the nature of control that we would like to transmit, consider the system configuration depicted in figure 2. In the top part of the figure, there are two systems, shown as black-boxes with terminals. It is through their terminals that systems interact with their environment. The black-box imposes relations on the variables that ‘live’ on its terminals. These relations are formalized by the behavior of the system in the black-box. The system to the left in figure 2 is called the *plant*, the one to the right the *controller*. The terminals of the plant consist of *to-be-controlled variables* w , and *control variables* c . The controller has only terminals with the control variables c . In the bottom part of the figure, the control terminals of the plant and of the controller are connected. Before interconnection, the variables w and c of the plant have to satisfy the laws imposed by the plant behavior. But, after interconnection, the variables c also have to satisfy the laws imposed by the controller. Thus, after interconnection, the restrictions imposed on the variables c by the controller will be transmitted to the variables w . Choosing the black-box to the right so that the variables w have a desirable behavior in the interconnected black-box is, in our view, the basic problem of control.

This leads to the following mathematical formulation. The *plant* and the *controller* are both dynamical systems, given by $\Sigma_{plant} = (\mathbb{T}, \mathbb{W} \times \mathbb{C}, \mathfrak{B}_{plant})$ and $\Sigma_{controller} = (\mathbb{T}, \mathbb{C}, \mathfrak{B}_{controller})$ where \mathbb{W} denotes the signal space of the to-be-controlled variables, \mathbb{C} denotes the signal space of control variables (hence, here \mathbb{C} does not mean the complex plane), and both systems are assumed to have the same time axis \mathbb{T} . The intercon-

nection of Σ_{plant} and $\Sigma_{controller}$ leads to the system $\Sigma_{full} = (\mathbb{T}, \mathbb{W}, \mathbb{C}, \mathfrak{B}_{full})$ which is a system with latent variables (\mathbb{C}) and full behavior defined by

$$\mathfrak{B}_{full} = \{(w, c) : \mathbb{T} \rightarrow \mathbb{W} \times \mathbb{C} \mid (w, c) \in \mathfrak{B}_{plant} \text{ and } c \in \mathfrak{B}_{controller}\}$$

The manifest system obtained by Σ_{full} is the *controlled system* and is hence defined as $\Sigma_{controlled} = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_{controlled})$ with

$$\mathfrak{B}_{controlled} = \{w : \mathbb{T} \rightarrow \mathbb{W} \mid \text{there exists } c : \mathbb{T} \rightarrow \mathbb{C} \\ \text{such that } (w, c) \in \mathfrak{B}_{plant} \text{ and } c \in \mathfrak{B}_{controller}\}$$

The notion of a controller put forward by the above view considers interconnection as the basic idea of control. The special controllers that consist of sensor-outputs to actuator-inputs signal processors emerge as (very important) special cases. We think of these as controllers as *feedback*, or *intelligent*, controllers. However, our view of control as the design of suitable subsystems greatly enhances the applicability of control, since it views control as integrated subsystem design.

A question that arises in this context is the following. Assume that Σ_{plant} is given. What systems $\Sigma_{controlled}$ can be obtained by suitably choosing $\Sigma_{controller}$? This question can be answered very explicitly, at least for linear time-invariant differential systems. Assume that the plant is given by $\Sigma_{plant} = (\mathbb{R}, \mathbb{R}^w \times \mathbb{R}^c, \mathfrak{B}_{plant}) \in \mathbb{L}^{w+c}$ with \mathfrak{B}_{plant} described by a system of linear constant differential equations $R(\frac{d}{dt})w = R(\frac{d}{dt})c$. Let $\Sigma_{controller} = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}_{controller}) \in \mathbb{L}^c$, and assume that $\mathfrak{B}_{controller}$ is similarly described by a system of linear constant coefficient differential equations $C(\frac{d}{dt})c = 0$. Then, by elimination theorem $\Sigma_{controlled} = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B}_{controlled})$ has also a behavior that is described by a system of linear constant coefficient differential equations. It turns out that the behaviors $\mathfrak{B}_{controlled}$ that can be obtained this way can be characterized very nicely.

Define therefore two behaviors, \mathfrak{P} and \mathfrak{N} ; \mathfrak{P} is called the *realizable (plant) behavior* and \mathfrak{N} the *hidden behavior*. They are defined as follows: \mathfrak{P} is the manifest behavior of the system, i.e.,

$$\mathfrak{P} = \{w : \mathbb{R} \rightarrow \mathbb{R}^w \mid \text{there exists } c : \mathbb{R} \rightarrow \mathbb{R}^c \text{ such that } (w, c) \in \mathfrak{B}_{full}\},$$

and \mathfrak{N} is defined as

$$\mathfrak{N} = \{w : \mathbb{R} \rightarrow \mathbb{R}^w \mid (w, 0) \in \mathfrak{B}_{plant}\}.$$

Hence \mathfrak{N} is the behavior of the plant variables that are compatible with the control variables equal to zero. We say that $\mathfrak{B}_{controller}$ *implements* $\mathfrak{B}_{controlled}$ if there exists a controller such that the controlled behavior after interconnecting the controller with behavior $\mathfrak{B}_{controller}$ to the plant, yields $\mathfrak{B}_{controlled}$ as the controlled behavior.

The controller implementability problem asks what behaviors $\mathfrak{B}_{controlled}$ can be obtained this way. For linear time-invariant systems it is possible to prove that $\mathfrak{B}_{controlled}$ is implementable if and only if

$$\mathfrak{N} \subseteq \mathfrak{B}_{controlled} \subseteq \mathfrak{P}.$$

This result shows that the behaviors that are implementable by means of a controller are precisely those that are wedged in between the hidden behavior \mathfrak{N} and the realizable plant \mathfrak{P} .

The problem of control can therefore be reduced to finding such a behavior. Of course, the issue of how to construct $\Sigma_{controller}$ (for example, as a signal processor from the sensor outputs to the actuator inputs) must be addressed as well, but this can be done. This leads to the important issue of *regular* controllers, which will not be discussed here. The behavioral theory of control has been introduced in [23] and is further elaborated in [1, 2], and, for filtering, in [17]. In the fourth lecture this approach is used for the design of \mathcal{H}_∞ -controllers, where we also discuss the implementability of behavioral controllers as feedback controllers.

We believe that the point of view of control that emerges from this, as designing a subsystem (with feedback control as a special case) greatly enhances the scope and applicability of control as a discipline. In this setting, control comes down to subsystem design. The third lecture of the short course will highlight this view of control.

Lecture 4: Synthesis of dissipative systems

The subject of the fourth lecture is shaping the behavior of a linear system by attaching a controller to it. Conditions are derived that make it possible to render the system dissipative, for example, contractive, or passive. This problem is basically what is usually called the \mathcal{H}_∞ -problem. We show that it can be reformulated in an elegant way as that of finding a behavior that is wedged in between two given behaviors and makes a quadratic differential form non-negative. As discussed in the third lecture, the ‘upper bound’ results from the fact that the controlled behavior must be physically realizable, and hence included in the (unconstrained) plant behavior. The ‘lower bound’ expresses in a subtle way the restriction that the controlled behavior must be implementable by a controller that acts through the control variables only.

The conditions for solvability use the theory of dissipative systems and their associated storage functions. The surprising aspect of the main result is a coupling condition among certain storage functions, more than reminiscent of the clever coupling condition between the solutions of algebraic Riccati equations that first appeared in the classic paper [6]. Our solvability conditions also require the dissipativeness of the hidden behavior and of a suitable orthogonal complement of the plant behavior. These conditions feature prominently also in [9, 10].

We will cast the development completely in the language of behaviors and the associated quadratic differential forms. This not only allows a clean problem statement, but it results in a formulation that is representation-free and flexible in the algorithms that can be used for verifying the existence and the specification of the controller.

We use the abbreviation QDF for ‘quadratic differential form’. Quadratic forms play an important role in linear system theory: as performance criteria, as Lyapunov functions, etc. In the context of behavioral differential systems, quadratic functionals are most naturally formulated as QDF’s. These notions are key elements in the behavioral approach to control. They are now briefly introduced. We only consider the elements that are needed in the formulation and solution of the problem that we discuss in this lecture.

The quadratic form on \mathbb{R}^\bullet induced by the matrix $S = S^T \in \mathbb{R}^{\bullet \times \bullet}$ is denoted by $q_S(x) = |x|_S^2 = x^T S x$. When $S = I$, the subscript in $|x|_S^2$ is usually deleted. Denote the signature of S by $\text{signature}(S) = (\sigma_-(S), \sigma_+(S))$, with $\sigma_-(S)$ and $\sigma_+(S)$ the number of negative and positive eigenvalues of S respectively. Bilinear and quadratic forms in the setting of differential systems are parametrized very effectively by two-variable polynomial matrices. Let $\Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$, written out in terms of its co-

efficient matrices as the (finite) sum $\Phi(\zeta, \eta) = \sum_{k, \ell \in \mathbb{Z}_+} \Phi_{k, \ell} \zeta^k \eta^\ell$. It induces the map $L_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by

$$L_\Phi(w_1, w_2) = \sum_{k, \ell \in \mathbb{Z}_+} \left(\frac{d^k}{dt^k} w_1 \right)^T \Phi_{k, \ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right).$$

This map is called the *bilinear differential form* induced by Φ . When $w_1 = w_2 = w$, L_Φ induces the map $Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mapsto \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by $Q_\Phi(w) = L_\Phi(w, w)$, i.e.,

$$Q_\Phi(w) = \sum_{k, \ell \in \mathbb{Z}_+} \left(\frac{d^k}{dt^k} w \right)^T \Phi_{k, \ell} \left(\frac{d^\ell}{dt^\ell} w \right).$$

This map is called the *quadratic differential form (QDF)* induced by Φ . Denote $\Phi^T(\eta, \zeta)$ as $\Phi^*(\zeta, \eta)$. Note that when considering QDF's, we may as well assume that Φ defining Q_Φ is symmetric, that is $\Phi = \Phi^*$, i.e. $\Phi_{k\ell} = \Phi_{\ell k}^T$ for all $k, \ell \in \mathbb{Z}_+$. Indeed, $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$ entails symmetry without loss of generality.

Let $\Phi = \Phi^*$ and $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$. The system \mathfrak{B} is said to be *dissipative with respect to* Q_Φ , (briefly, Φ -dissipative) if $\int_{-\infty}^{+\infty} Q_\Phi(w) dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$ (\mathfrak{D} denotes the \mathcal{C}^∞ functions with compact support). It is said to be *dissipative on \mathbb{R}_- with respect to Q_Φ* , (briefly, Φ -dissipative on \mathbb{R}_-) if $\int_{-\infty}^0 Q_\Phi(w) dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Dissipativity on \mathbb{R}_+ is analogously defined. Obviously, dissipativity on \mathbb{R}_- or \mathbb{R}_+ implies dissipativity. Dissipativity on \mathbb{R}_- combines dissipativity on the whole of \mathbb{R} with stability.

For an intuitive interpretation, identify $Q_\Phi(w)(t)$ with the power, the rate of energy, delivered to the system at time t , and $\int_{-\infty}^{+\infty} Q_\Phi(w) dt$ with the total net energy delivered to the system by taking it through the history w . Dissipativity states that the system absorbs energy during any history in \mathfrak{B} that starts and ends with the system at rest. Dissipativity on \mathbb{R}_- states that *at any time* the net flow of energy *up to that time* has been *into* the system. Note also that we have limited our definition of dissipativeness to controllable systems.

Let, for $\mathfrak{B} \in \mathcal{L}^\bullet$, $m(\mathfrak{B})$ denote the *input cardinality* of \mathfrak{B} , i.e., the number of free variables, of inputs in \mathfrak{B} .

The QDF Q_Σ with $\Sigma = \Sigma^T \in \mathbb{R}^{v \times v}$ defines the *weighting functional* that enters in the control performance. Denote by $\text{signature}(\Sigma) = (\sigma_-(\Sigma), \sigma_+(\Sigma))$ its signature. The problem that we solve in this lecture may succinctly be formulated as follows.

Let $\mathfrak{N}, \mathcal{P} \in \mathcal{L}_{\text{cont}}^v$, and $\Sigma = \Sigma^T \in \mathbb{R}^{v \times v}$ non-singular; \mathcal{P} is called the *plant behavior*, \mathfrak{N} the *hidden behavior*, and Q_Σ the *weighting functional*. The problem is to find $\mathcal{K} \in \mathcal{L}_{\text{cont}}^v$ (called the *controlled behavior*) such that:

1. $\mathfrak{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ (*implementability*),
2. \mathcal{K} is Σ -dissipative on \mathbb{R}_- (*dissipativity*),
3. $m(\mathcal{K}) = \sigma_+(Q_\Sigma)$ (*liveness*).

We now explain informally the interpretation of these conditions. The first condition has been explained in lecture 3. The inclusion $\mathcal{K} \subseteq \mathcal{P}$ signifies that the controlled behavior is physically possible: the controller merely restricts the plant behavior. We

view this as *realizability*. The inclusion $\mathcal{K} \supset \mathfrak{N}$ is more subtle. It means that the controlled behavior is implementable by a controller that acts through the control variables only.

That the controlled behavior \mathcal{K} must be Σ -dissipative is the basic control design specification. As is well-known, by suitably choosing Σ , it implies disturbance attenuation, or passivation. The fact that Σ -dissipativity is required to hold on \mathbb{R}_- , and not just on \mathbb{R} , implies stability of the controlled behavior.

The liveness requirement states that $\sigma_+(\Sigma)$ components of v must remain free in the controlled behavior. It expresses that the controlled system must still be able to accept free exogenous inputs: the controlled behavior is not allowed to restrict the exogenous inputs directly, it only serves to shape the influence of the exogenous inputs on the endogenous outputs. It is easy to prove that the liveness condition is equivalent to the requirement that in the controlled behavior there are as many free variables as possible. The problem statement can thus be rephrased as: *When does there exist there a controlled behavior that is Σ -dissipative on \mathbb{R}_- and of maximal input cardinality?*

In order to state the solution to the problem formulated above, we need a couple of more preliminaries: the notion of a storage function, and orthogonality of behaviors.

Let $\mathfrak{B} \in \mathcal{L}^w$, $\Phi = \Phi^*$, and $\Psi = \Psi^* \in \mathbb{R}^{w \times w}[\zeta, \eta]$. Then Q_Ψ is said to be a *storage function for \mathfrak{B} with respect to the supply rate Q_Φ* if the *dissipation inequality* $\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w)$ holds for all $w \in \mathfrak{B}$. For $f: A \rightarrow \mathbb{R}$, $f \geq 0$ means $f(t) \geq 0 \forall t \in A$. There is an immediate relation between dissipativity and the existence of a storage function, with its sign related to half-line dissipativity. Indeed, let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$ and $\Phi = \Phi^*$. Then \mathfrak{B} is Φ -dissipative if and only if there exists $\Psi = \Psi^*$ such that Q_Ψ is a storage function for \mathfrak{B} with respect to the supply rate Q_Φ . Furthermore, \mathfrak{B} is Φ -dissipative on \mathbb{R}_- if and only if Q_Ψ can be taken to be non-negative on \mathfrak{B} , i.e., $Q_\Psi(w) \geq 0$ for all $w \in \mathfrak{B}$, and Φ -dissipative on \mathbb{R}_+ if and only if Q_Ψ can be taken to be non-positive on \mathfrak{B} .

This theme is a recurrent one: it identifies a ‘global’ statement (dissipativity: an inequality involving an integral over \mathbb{R}) with a ‘local’ statement (the dissipation inequality: an inequality that is *point-wise* on \mathbb{R}). Intuitively, therefore, a system globally dissipates supply along any trajectory on the whole of \mathbb{R} if and only if this dissipation can be brought into evidence through a storage function whose rate of increase does not exceed, point-wise in time, the rate of supply delivered to the system.

We also need the orthogonal complement of a controllable behavior, with orthogonality viewed with respect to a bilinear differential form induced by a constant matrix. Let $\Phi \in \mathbb{R}^{w \times w}$, and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^w$; \mathfrak{B}_1 and \mathfrak{B}_2 are said to be *orthogonal with respect to L_Φ* (briefly, *Φ -orthogonal*) if $\int_{-\infty}^{+\infty} L_\Phi(w_1, w_2) dt = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathcal{D}$ and $w_2 \in \mathfrak{B}_2 \cap \mathcal{D}$. We denote this as $\mathfrak{B}_1 \perp_\Phi \mathfrak{B}_2$. Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^w$, and define the Φ -orthogonal complement $\mathfrak{B}^{\perp_\Phi}$ of \mathfrak{B} as

$$\mathfrak{B}^{\perp_\Phi} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \int_{-\infty}^{+\infty} L_\Phi(w, w') dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathcal{D}\}.$$

It is easy to see that $\mathfrak{B}^{\perp_\Phi} \in \mathcal{L}_{\text{cont}}^w$. When $\Phi = I_w$, we denote \perp_Φ simply as \perp . Note that $\mathfrak{B}^{\perp_\Phi} = (\Phi \mathfrak{B})^\perp = (\Phi^T)^{-1} \mathfrak{B}^\perp$ ($^{-1}$ denotes the set-theoretic inverse), and, if Φ is non-singular, then $\mathfrak{B} = (\mathfrak{B}^{\perp_\Phi})^{\perp(\Phi^T)^{-1}}$.

In order to state our main result, we need the following result. Let $\Phi \in \mathbb{R}^{w \times w}$, and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^w$. Then there exists a $\Psi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ such that $\frac{d}{dt}L_\Psi(w_1, w_2) = L_\Phi(w_1, w_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$, if and only if $\mathfrak{B}_1 \perp_\Phi \mathfrak{B}_2$. Moreover, Ψ is essentially unique, in the sense that if $\Psi_1, \Psi_2 \in \mathbb{R}^{w \times w}[\zeta, \eta]$ both satisfy this equality,

then $L_{\Psi_1}(w_1, w_2) = L_{\Psi_2}(w_1, w_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$. The idea is again the equivalence of a ‘local’ and a ‘global’ property, this time for orthogonality. We call the bilinear differential form L_Ψ (or simply Ψ) of this proposition, $[(\mathfrak{B}_1, \mathfrak{B}_2); \Phi]$ -*adapted*.

Equipped with these additional preliminaries, the notion of a storage function and existence of an adapted bilinear differential form, we are able to state the solution of the problem formulated above. This problem allows an explicit and representation-free solution, involving the storage functions associated with dissipative systems in a subtle way.

The controlled behavior $\mathcal{K} \in \mathfrak{L}_{\text{cont}}^v$ described in the problem formulation exists if and only if the following conditions are satisfied:

1. \mathfrak{N} is Σ -dissipative,
2. $\mathcal{P}^{\perp\Sigma}$ is $(-\Sigma)$ -dissipative,
3. there exist $\Psi_{\mathfrak{N}}, \Psi_{\mathcal{P}^{\perp\Sigma}}, \Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})} \in \mathbb{R}^{v \times v}[\zeta, \eta]$, defining
 - a storage function $Q_{\Psi_{\mathfrak{N}}}$ for \mathfrak{N} as a Σ -dissipative system, i.e.,
 $\frac{d}{dt} Q_{\Psi_{\mathfrak{N}}}(v_1) \leq Q_{\Sigma}(v_1)$ for $v_1 \in \mathfrak{N}$,
 - a storage function $Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}$ for $\mathcal{P}^{\perp\Sigma}$ as a $(-\Sigma)$ -dissipative system, i.e.,
 $\frac{d}{dt} Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}(v_2) \leq -Q_{\Sigma}(v_2)$ for $v_2 \in \mathcal{P}^{\perp\Sigma}$,
 - and the $[(\mathfrak{N}, \mathcal{P}^{\perp\Sigma}); \Sigma]$ -adapted bilinear differential form $L_{\Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})}}$, i.e.,
 $\frac{d}{dt} L_{\Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})}}(v_1, v_2) = L_{\Sigma}(v_1, v_2)$, for $v_1 \in \mathfrak{N}, v_2 \in \mathcal{P}^{\perp\Sigma}$,

such that the QDF

$$Q_{\Psi_{\mathfrak{N}}}(v_1) - Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}(v_2) + 2L_{\Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})}}(v_1, v_2)$$

is non-negative for all $v_1 \in \mathfrak{N}$ and $v_2 \in \mathcal{P}^{\perp\Sigma}$.

Note that these storage functions are well-defined by the assumed dissipativeness of \mathfrak{N} and $\mathcal{P}^{\perp\Sigma}$, and that $\Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})}$ is well-defined, since $\mathfrak{N} \subseteq \mathcal{P}$.

The surprising condition in the above result is the required non-negativity. We refer to this condition as the *coupling condition*. It implies in particular that $Q_{\Psi_{\mathfrak{N}}}$ is non-negative on \mathfrak{N} , which shows that \mathfrak{N} is Σ -dissipative on \mathbb{R}_- , clearly (since $\mathfrak{N} \subseteq \mathcal{K}$) a necessary condition for the existence of \mathcal{K} . It also implies that $Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}$ is non-positive on $\mathcal{P}^{\perp\Sigma}$, which in turn shows that $\mathcal{P}^{\perp\Sigma}$ is $(-\Sigma)$ -dissipative on \mathbb{R}_+ . It is not difficult to see that this is also a necessary condition for the existence of \mathcal{K} . In fact, it can be shown that Σ -dissipativity of \mathcal{K} on \mathbb{R}_- combined with $\mathfrak{m}(\mathcal{K}) = \sigma_+(Q_{\Sigma})$, implies that $\mathcal{K}^{\perp\Sigma}$ is $(-\Sigma)$ -dissipative on \mathbb{R}_+ . In addition, $\mathcal{K} \subseteq \mathcal{P}$ implies $\mathcal{P}^{\perp\Sigma} \subseteq \mathcal{K}^{\perp\Sigma}$. Therefore $(-\Sigma)$ -dissipativity of $\mathcal{P}^{\perp\Sigma}$ on \mathbb{R}_+ is also a necessary condition for the existence of \mathcal{K} . As already mentioned, both these elements of the solution are present in [9, 10]. What makes the result surprising is the coupling of $Q_{\Psi_{\mathfrak{N}}}$ and $Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}$ through $L_{\Psi_{(\mathfrak{N}, \mathcal{P}^{\perp\Sigma})}}$, thus strengthening the required non-negativity of the storage functions $Q_{\Psi_{\mathfrak{N}}}$ and $-Q_{\Psi_{\mathcal{P}^{\perp\Sigma}}}$,

and coupling the dissipativeness of \mathfrak{R} and $\mathcal{P}^{\perp\Sigma}$. This condition is analogous to (but a representation-free generalization of) the remarkable condition coupling solutions of algebraic Riccati equations that first appeared in the instant-classic paper [6].

In the fourth lecture of the course, we will present this result. Details may be found in [25, 3].

Acknowledgment: This research is supported by the Belgian Federal Government under the DWTC program Interuniversity Attraction Poles, Phase V, 2002 - 2006, Dynamical Systems and Control: Computation, Identification and Modelling.

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