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Time-autonomy versus time-controllability

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Abstract

We study systems whose dynamics are described by systems of linear constant coefficient partial differential equations in a behavioural framework. Questions about the notion of autonomy and controllability are addressed with special importance given to the time-evolution. We investigate consequences of time-autonomy for time-controllability.

Keywords: Behaviours; Time-controllability; Time-autonomy; Partial differential equations

1. Introduction

The purpose of this paper is to study autonomy and controllability of dynamical systems described by linear constant coefficient partial differential equations in the behavioural theory of Willems (see [11]). Traditionally, behaviours arising from systems of partial differential equations are studied in a general setting in which the time-axis does not play a distinguished role in the formulation of the definitions pertinent to control theory (see [5–7,9] etc.) However, it is reasonable to suggest that in the study of systems with "dynamics" arising from (engineering) applications, it is useful to give special importance to the time variable in defining system theoretic concepts. This also highlights the similarities with the definitions in the case of 1D dynamical systems. (For an excellent elementary introduction to the behavioural theory in the 1D case, we refer to Polderman and Willems [8].) In this paper, we study time-autonomy and time-controllability properties (see also [1]).

The paper is organized as follows: In Section 2, we give the mathematical preliminaries that we need and recall the notion of time-controllability of the behaviours. We establish two main theorems about time-autonomy of behaviours in Section 3. Finally in Section 4, we study the relationship between time-autonomy and time-controllability.

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2. Preliminaries

In this section, we recall some facts about distributions and dynamical systems. We shall tailor our choice of definitions and theorems to the needs of the remainder of this paper.

This paper concerns *dynamical systems* $\Sigma = (\mathbb{R}^{m+1}, \mathbb{C}^{w}, \mathfrak{B})$, where $\mathbb{C}^{w}$ is called the *signal space* and $\mathfrak{B} \subset \mathscr{D}'(\mathbb{R}^{m+1})$ is called the *behaviour* of the system Σ (see for example [7]).

Let us denote the polynomial ring $\mathbb{C}[\xi, \eta_1, \ldots, \eta_m]$ by $\mathscr{A}$. Consider the polynomial matrix

$$R = \begin{bmatrix} r_{11} & \ldots & r_{1w} \\ \vdots & & \vdots \\ r_{g1} & \ldots & r_{gw} \end{bmatrix} \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_m]^{g \times w}$$

with each entry in $\mathscr{A}$. Consider each row of R as an element of the free module $\mathscr{A}^{w}$. Let $\langle R \rangle$ denote the $\mathscr{A}$-submodule of $\mathscr{A}^{w}$ generated by the rows of the polynomial matrix R. The polynomial matrix R gives rise to a map $D_R : \mathscr{W}^{w} \to \mathscr{W}^{g}$, where $\mathscr{W} \subset \mathscr{D}'(\mathbb{R}^{m+1})$, which acts as follows:

$$D_R \begin{bmatrix} w_1 \\ \vdots \\ w_w \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{w} r_{1k}(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}) w_k \\ \vdots \\ \sum_{k=1}^{w} r_{gk}(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}) w_k \end{bmatrix}.$$

Such maps will be called *differential maps* in the sequel.

Throughout this paper, it is assumed that behaviours of dynamical systems are actually ones which are distributional kernels of differential maps D_R, where R is a polynomial matrix. So in particular, the behaviours are time-invariant subspaces.

The $\mathscr{W}$-behaviour corresponding to $R \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_m]^{g \times w}$, is defined to be $\mathfrak{B}_{R,\mathscr{W}} = \{ w \in \mathscr{W}^{w} \mid D_R(w) = 0 \}$. Given a $\mathscr{W}$-behaviour, say $\mathfrak{B}$, define

$$\langle R \rangle_{\mathfrak{B}} = \{ r = [r_1 \ \ldots \ r_w] \in \mathscr{A}^{w} \mid D_r(w) = 0 \text{ for all } w \in \mathfrak{B} \}.$$

It was shown in [5] that given any $R \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_m]^{g \times w}$, $\langle R \rangle_{\mathfrak{B}_{R,\mathscr{W}}} = \langle R \rangle$, if $\mathscr{W}$ is $\mathscr{C}^{\infty}(\mathbb{R}^{m+1}, \mathbb{C})$ or $\mathscr{D}'(\mathbb{R}^{m+1})$.

Let $\mathscr{W} \subset \mathscr{D}'(\mathbb{R}^{m+1})$. The behaviour $\mathfrak{B}$ of a dynamical system is said to be *time-autonomous with respect to* $\mathscr{W}$ if

$$\begin{bmatrix} w \in \mathfrak{B} \cap \mathscr{W}^{w} \text{ and} \\ \langle w, \varphi \rangle = 0 \text{ for all } \varphi \in \mathscr{D}(\mathbb{R}^{m+1}) \text{ with } \operatorname{supp}(\varphi) \subset (-\infty, 0) \times \mathbb{R}^{m} \end{bmatrix} \Rightarrow [w = 0].$$

The behaviour $\mathfrak{B}$ of a dynamical system is said to be *time-controllable with respect to* $\mathscr{W}$ if for any w_1 and w_2 in $\mathfrak{B} \cap \mathscr{W}^{w}$, there exists a $w \in \mathfrak{B} \cap \mathscr{W}^{w}$ and a $\tau \geqslant 0$ such that

$$\langle w, \varphi \rangle = \begin{cases} \langle w_1, \varphi \rangle & \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^{m+1}) \text{ with } \operatorname{supp}(\varphi) \subset (-\infty, 0) \times \mathbb{R}^{m}, \\ \langle \sigma_\tau w_2, \varphi \rangle & \text{for all } \varphi \in \mathscr{D}(\mathbb{R}^{m+1}) \text{ with } \operatorname{supp}(\varphi) \subset (\tau, \infty) \times \mathbb{R}^{m}. \end{cases}$$

w is then said to *concatenate* w_1 and w_2. Roughly speaking, a time-invariant dynamical system is time-controllable if it is controllable with respect to the open sets $(-\infty,0)\times\mathbb{R}^{\mathtt{m}}$ and $(\tau,\infty)\times\mathbb{R}^{\mathtt{m}}$ (see [7] for a definition of controllability).

We will denote the interval $(0,\infty)$ by $\mathbb{R}_+$ in the sequel. Given $w\in\mathscr{D}'(\mathbb{R}^{\mathtt{m}+1})$, define $w|_{t>0}\in\mathscr{D}'(\mathbb{R}_+\times\mathbb{R}^{\mathtt{m}})$ as follows:

$$\langle w|_{t>0},\varphi\rangle=\langle w,\varphi\rangle \quad \text{for all } \varphi\in\mathscr{D}(\mathbb{R}^{\mathtt{m}+1}) \quad \text{with } \operatorname{supp}(\varphi)\subset\mathbb{R}_+\times\mathbb{R}^{\mathtt{m}}.$$

$\mathscr{W}|_{t>0}$ is the set $\{w|_{t>0}\mid w\in\mathscr{W}\}$. Suppose that $\mathscr{W}|_{t>0}$ is a topological vector space. The behaviour $\mathfrak{B}$ of a dynamical system is said to be *approximately time-controllable with respect to* $\mathscr{W}$ if for any w_1 and w_2 in $\mathfrak{B}\cap\mathscr{W}^{\mathtt{w}}$, and for any neighbourhood of $w_2|_{t>0}$, there exists a $w\in\mathfrak{B}\cap\mathscr{W}^{\mathtt{w}}$ and a $\tau\geqslant 0$ such that

$$\langle w,\varphi\rangle=\langle w_1,\varphi\rangle \quad \text{for all } \varphi\in\mathscr{D}(\mathbb{R}^{\mathtt{m}+1}) \quad \text{with } \operatorname{supp}(\varphi)\subset(-\infty,0)\times\mathbb{R}^{\mathtt{m}} \text{ and } (\sigma_{-\tau}w)|_{t>0}\in N.$$

It is easy to see that if a behaviour is time-controllable with respect to $\mathscr{W}$, then it is approximately time-controllable with respect to $\mathscr{W}$. If $\mathscr{W}=\mathscr{D}'(\mathbb{R}^{\mathtt{m}+1})$, then one simply speaks of time-autonomy, time-controllability or approximate time-controllability.

If $T\in\mathscr{D}'(\mathbb{R}^{m+1})$, then one can associate a continuous linear map $\iota T:\mathscr{D}(\mathbb{R})\to\mathscr{D}'(\mathbb{R}^{\mathtt{m}})$ as follows: $\langle(\iota T)(\varphi),\psi\rangle=\langle T,\varphi\otimes\psi\rangle$ for all $\varphi\in\mathscr{D}(\mathbb{R})$ and $\psi\in\mathscr{D}'(\mathbb{R}^{\mathtt{m}})$. We recall below the Schwartz kernel theorem (see for instance [4, p. 128, Theorem 5.2.1] or [10]).

Lemma 2.1 (The Schwartz kernel theorem). *The map* $T\mapsto\iota T$ *is an isomorphism from* $\mathscr{D}'(\mathbb{R}^{m+1})$ *onto* $\mathscr{L}(\mathscr{D}(\mathbb{R}),\mathscr{D}'(\mathbb{R}^{\mathtt{m}}))$.

We abbreviate $\mathfrak{B}\cap\mathbb{C}^\infty(\mathbb{R}^{\mathtt{m}+1},\mathbb{C})^{\mathtt{w}}$ by $\mathfrak{B}_{\mathscr{C}^\infty}$. The *smooth null behaviour* of a dynamical system is defined as the set

$$\mathfrak{B}^0_{\mathscr{C}^\infty}=\{w\in\mathfrak{B}_{\mathscr{C}^\infty}\mid w|_{t\leqslant 0}=0\},$$

where $w|_{t\leqslant 0}$ is the restriction of the function w to the closed left half-space $\{(t,\underline{x})\in\mathbb{R}^{\mathtt{m}+1}\mid t\leqslant 0\}$. By $\mathfrak{B}_{\mathscr{C}^\infty}|_{t>0}$ we mean the set

$$\{w\in\mathscr{C}^\infty(\mathbb{R}_+\times\mathbb{R}^{\mathtt{m}},\mathbb{C})^{\mathtt{w}}\mid\exists w_0\in\mathfrak{B}_{\mathscr{C}^\infty} \text{ such that } w_0|_{t>0}=w\},$$

where $w_0|_{t>0}$ is the restriction of the function w_0 to the open right half-space $\{(t,\underline{x})\in\mathbb{R}^{\mathtt{m}+1}\mid t>0\}$.

Lemma 2.2. *The behaviour* $\mathfrak{B}$ *of a dynamical system is approximately time-controllable with respect to* $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1},\mathbb{C})$ *if*

$$\overline{\mathfrak{B}^0_{\mathscr{C}^\infty}}=\mathfrak{B}_{\mathscr{C}^\infty}|_{t>0},$$

that is, the closure of the smooth null behaviour in the topology of $(\mathscr{E}(\mathbb{R}_+\times\mathbb{R}^{\mathtt{m}}))^{\mathtt{w}}$ *is the restriction of the smooth behaviour to the open right half-space* $t>0$.

Proof. Let w_1 and w_2 belong to $\mathfrak{B}_{\mathscr{C}^\infty}$, $\tau\geqslant 0$, and N be a neighbourhood of $w_2|_{t>0}$. Then $-w_1(\bullet+\tau)+w_2(\bullet)\in\mathfrak{B}_{\mathscr{C}^\infty}$, and $-w_1(\bullet+\tau)+N$ is a neighbourhood of $(-w_1(\bullet+\tau)+w_2(\bullet))|_{t>0}$. From the hypothesis, it follows that there exists a null solution $\tilde{w}$ approximating $(-w_1(\bullet+\tau)+w_2(\bullet))|_{t>0}$ in $(\mathscr{E}(\mathbb{R}_+\times\mathbb{R}^{\mathtt{m}}))^{\mathtt{w}}$, that is $\tilde{w}(t)=0$ for all $t\leqslant 0$, and $\tilde{w}|_{t>0}\in-w_1(\bullet+\tau)+N$. Defining $w=\tilde{w}+w_1$, we have that $w\in\mathfrak{B}_{\mathscr{C}^\infty}$, and moreover, $w(t)=w_1(t)$ for all $t<0$, and $w(\bullet+\tau)|_{t>0}\in N$, hence establishing the approximate time-controllability of the behaviour with respect to $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1},\mathbb{C})$. $\square$

3. Time-autonomy

In this section, we study the notion of time-autonomy behaviours, and we will establish two main results about the time-autonomy of a partial differential equation with constant coefficients.

3.1. All time-autonomous scalar behaviours with respect to $\iota^{-1}\mathscr{L}(\mathscr{D}(\mathbb{R}),\mathscr{E}'(\mathbb{R}^{m}))$[1]

The space of distributions with compact support is denoted by $\mathscr{E}'(\mathbb{R}^{m})$. In the sequel, we denote the Fourier transform from $\mathscr{E}'(\mathbb{R}^{m})$ to $\mathscr{E}(\mathbb{R}^{m})$ by the symbol $\mathscr{F}$.

Theorem 3.1. *The behaviour corresponding to $p\in\mathscr{A}$ is time-autonomous with respect to $\iota^{-1}\mathscr{L}(\mathscr{D}(\mathbb{R}),\mathscr{E}'(\mathbb{R}^{m}))$ iff $p\neq 0$.*

Proof. Let $p=a_0(\underline{\eta})+a_1(\underline{\eta})\xi+\cdots+a_N(\underline{\eta})\xi^N$, where $a_0,\dots,a_N\in\mathbb{C}[\underline{\eta}]$. Let $w\in\iota^{-1}\mathscr{L}(\mathscr{D}(\mathbb{R}),\mathscr{E}'(\mathbb{R}^{m}))\subset\mathscr{D}'(\mathbb{R}^{m+1})$ be a solution of

$$0=p\left(\frac{\partial}{\partial t},\frac{\partial}{\partial \underline{x}}\right)w=\sum\nolimits_{k=0}^{N}a_k\left(\frac{\partial}{\partial\underline{x}}\right)\frac{\partial^k}{\partial t^k}w=0 \tag{1}$$

with $\langle\iota w,\varphi\rangle=0$ if $\varphi\in\mathscr{D}(\mathbb{R})$ and $\mathrm{supp}(\varphi)\subset(-\infty,0)$. For $\varphi\in\mathscr{D}(\mathbb{R})$ and $y\in\mathbb{C}^{m}$, define

$$(\mathfrak{F}w)(\varphi,y):=(\mathscr{F}\langle\iota w,\varphi\rangle)(y)=\langle\langle\iota w,\varphi\rangle,\mathrm{e}^{-2\pi\mathrm{i}\langle\bullet,y\rangle}\rangle=\langle w,\varphi\otimes\mathrm{e}^{-2\pi\mathrm{i}\langle\bullet,y\rangle}\rangle. \tag{2}$$

Then $(\mathfrak{F}w)(\varphi,\bullet)$ is analytic in y and $(\mathfrak{F}w)(\bullet,y)\in\mathscr{D}'(\mathbb{R})$. From (2), we infer

$$\left[\mathfrak{F}\left(\frac{\partial}{\partial t}w\right)\right](\varphi,y)=\left\langle w,-\frac{\partial\varphi}{\partial t}\otimes\mathrm{e}^{-2\pi\mathrm{i}\langle\bullet,y\rangle}\right\rangle=\left(\mathscr{F}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\iota w,\varphi\right\rangle\right)(y),$$

$$\left[\mathfrak{F}\left(\frac{\partial}{\partial x_k}w\right)\right](\varphi,y)=\langle w,-\varphi\otimes(-2\pi y_k)\mathrm{e}^{-2\pi\mathrm{i}\langle\bullet,y\rangle}\rangle=2\pi y_k\left(\mathscr{F}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\iota w,\varphi\right\rangle\right)(y).$$

The differential equation (1) thus implies

$$\sum\nolimits_{k=0}^{N}a_k(2\pi\mathrm{i}y)\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathfrak{F}w)(\bullet,y)=0\quad\text{for all } y\in\mathbb{C}^{m}.$$

If $a_N(2\pi y)\neq 0$, then the distributional solution $(\mathfrak{F}w)(\bullet,y)\in\mathscr{D}'(\mathbb{R})$ of this nonzero ordinary differential equation is indeed analytic and therefore zero since it is zero on $(-\infty,0)$ by assumption. Hence

$$(\mathfrak{F}w)(\varphi,y)a_N(2\pi\mathrm{i}y)=0\quad\text{for all }\varphi\text{ and }y.$$

For fixed φ this is a product of analytic functions. Since $a_N(2\pi\mathrm{i}\bullet)\neq 0$ and since the rings of local or global analytic functions have no zero-divisors (power series rings are integral domains) this implies

$$(\mathfrak{F}w)(\varphi,y)=0,\ \ \mathscr{F}(\langle\iota w,\varphi\rangle)=0,\ \ \langle\iota w,\ \ \varphi\rangle=0\quad\text{and}\quad w=0.\qquad\square$$

3.2. All time-autonomous scalar behaviours

Our main result in this section is Theorem 3.4. In order to establish this result, we recall a few notions from the theory of linear partial differential equations.

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[1] See Theorem 2.1.

Let $p \in \mathbb{C}[\zeta_1, \dots, \zeta_n]$ be of the form

$$p = \sum_{|(\alpha_1,\dots,\alpha_n)| \leqslant N} a_{(\alpha_1,\dots,\alpha_n)} \zeta_1^{\alpha_1} \dots \zeta_n^{\alpha_n}$$

with $a_{(\alpha_1,\dots,\alpha_n)} \neq 0$ for some $(\alpha_1, \dots, \alpha_n)$ with $|(\alpha_1, \dots, \alpha_n)| = N$. The *degree of* p, denoted by $\deg(p)$, is N. The *principal part of* p (denoted by p_N) is defined by

$$p_N = \sum_{|(\alpha_1,\dots,\alpha_n)| = N} a_{(\alpha_1,\dots,\alpha_n)} \zeta_1^{\alpha_1} \dots \zeta_n^{\alpha_n}.$$

The hyperplane with normal $\hat{n} \in \mathbb{R}^n$, that is, $\{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle = 0\}$ is said to be *characteristic with respect to* p if $p_N(\hat{n}) = 0$.

We quote the following from Hörmander [4] (Theorem 8.6.8, p. 312):

Lemma 3.2. *Let X_1 and X_2 be open convex sets in $\mathbb{R}^n$ such that $X_1 \subset X_2$, and let $p \in \mathbb{C}[\zeta_1, \dots, \zeta_n]$. Then the following conditions are equivalent*:

1. *Every $T \in \mathscr{D}'(X_2)$ satisfying the equation $D_p T = 0$ in X_2 and vanishing in X_1 must also vanish in X_2.*
2. *Every hyperplane which is characteristic with respect to p and intersects X_2 also intersects X_1.*

Let us denote by $\jmath$ the homomorphism

$$p(\xi, \eta_1, \dots, \eta_m) \mapsto p(\xi, 0, \dots, 0) : \mathbb{C}[\xi, \eta_1, \dots, \eta_m] \to \mathbb{C}[\xi].$$

Lemma 3.3. *The hyperplane with the normal $\hat{n}$ mbox:$= (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$ is characteristic with respect to $p \in \mathbb{C}\ [\xi, \eta_1, \dots, \eta_m]$ iff* $\deg(p) \neq \deg(\jmath(p))$.

Proof. If

$$p = \sum_{\alpha \in \mathbb{N}, \underline{\beta} \in \mathbb{N}^m} a_{\alpha, \underline{\beta}} \xi^{\alpha} \underline{\eta}^{\underline{\beta}}, \text{ then } p(\xi, \underline{0}) = \sum_{\alpha \leqslant N} a_{\alpha \underline{0}} \xi^{\alpha}.$$

Furthermore,

$$p_N = \sum_{\alpha, \underline{\beta}, \alpha + |\underline{\beta}| = N} a_{\alpha, \underline{\beta}} \xi^{\alpha} \underline{\eta}^{\underline{\beta}} \quad \text{and} \quad p_N(1, \underline{0}) = a_{N\underline{0}}.$$

Consequently, $p_{N\underline{0}} = 0$ iff $a_{N\underline{0}} = 0$, that is, iff $\deg(p(\xi, \underline{0})) < N = \deg(p)$. □

With $X_2 = \mathbb{R}^{m+1}$ and X_1 as the half-space $\{(t, \underline{x}) \mid t < 0\}$, we obtain the following theorem:

Theorem 3.4. *The behaviour $\mathfrak{B}$ corresponding to $0 \neq p \in \mathbb{C}[\xi, \eta_1, \dots, \eta_m]$ is time-autonomous iff* $\deg(p) = \deg(\jmath(p))$.

Proof. Let $X_2 = \mathbb{R}^{m+1}$ and X_1 be the half-space $\{(t, x_1, \dots, x_m) \mid t < 0\}$ in Lemma 3.2. Then item 1 in Lemma 3.2 is exactly our definition of time-autonomy of the behaviour $\mathfrak{B}$ corresponding to p. Every hyperplane intersects the whole space $X_2 = \mathbb{R}^{m+1}$. Moreover, every hyperplane, with the exception of the one with the normal $\hat{n} = (1, 0, \dots, 0)$, intersects the half-space X_1. Consequently, time-autonomy of $\mathfrak{B}$ is equivalent to the hyperplane with the normal $(1, 0, \dots, 0)$ being not characteristic with respect to p. Thus in light of the Lemma 3.3 above, the desired assertion is proved. □

Example 3.5. The following table gives some simple examples.

Time-autonomous	Not time-autonomous
$\mathbb{C}[\xi]\backslash\{0\}$	$\mathbb{C}[\eta_1,\ldots,\eta_m]$
$\xi+\eta_1+\cdots+\eta_m$	$\xi(\eta_1+\cdots+\eta_m)$
$\xi^2-(\eta_1^2+\cdots+\eta_m^2)$	$(1+\eta_1+\cdots+\eta_m)+(1+\eta_1+\cdots+\eta_m)\xi$

This table shows that $\imath^{-1}\mathscr{L}(\mathscr{D}(\mathbb{R}),\mathscr{E}'(\mathbb{R}^m))$-time-autonomy is different from $\mathscr{D}'(\mathbb{R}^{m+1})$-time-autonomy.

Next, we discuss the following refractory example, which shows that the underlying space is crucial in discussing time-autonomy.

Example 3.6 (The diffusion equation). The behaviour corresponding to the polynomial

$$p=\xi-(\eta_1^2+\cdots+\eta_m^2)$$

is not time-autonomous. Indeed, $\deg(p)=2\neq 1=\deg(\jmath(p))$.

1. We give the construction of a smooth nonzero trajectory with zero past. The following example of a $\mathscr{C}^\infty(\mathbb{R}^2,\mathbb{C})$ solution to the diffusion equation with one 1D space is based on A.N. Tychonov's example, which can be found for instance in [2] (Example 2, pp. 50–51):

$$w(x,t)=\sum_{k=0}^{\infty} f^{(k)}(t)\frac{x^{2k}}{(2k)!},\quad -\infty<x,\ t<\infty$$

with

$$f(t)=\begin{cases} e^{-1/t^2} & \text{for } t>0,\\ 0 & \text{for } t\leqslant 0.\end{cases}$$

2. However, with an "appropriate" choice of the space of solutions, for instance $\imath^{-1}\mathscr{C}^\infty(\mathbb{R}_+,L_1(\mathbb{R},\mathbb{C}))$, the Cauchy initial value problem for the diffusion equation is well-posed:

$$\left[\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right]w=0 \quad\text{and}\quad w(0,x)=w_0(x)$$

has the unique solution in $\imath^{-1}\mathscr{C}^\infty(\mathbb{R}_+,L_1(\mathbb{R},\mathbb{C}))$ given by the Weierstrass formula

$$w(t,x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(((x-\xi)^2)/4t)}w_0(\xi)\,d\xi \quad \text{for all } t>0$$

if $w_0\in L_1(\mathbb{R},\mathbb{C})$, guaranteeing time-autonomy with respect to $\mathscr{C}^\infty(\mathbb{R}_+,L_1(\mathbb{R},\mathbb{C}))$.
In this case, if $w_0>0$, then one can prove that $||w(t,\bullet)||_1=||w_0(\bullet)||_1$ for all $t\geqslant 0$, which conforms with the physical intuition that in a diffusion process, matter is conserved. We remark that the anomaly of nonuniqueness is a consequence of the fact that the plane $t=0$, carrier of the initial data, is a characteristic of the diffusion equation.

3.3. A necessary condition for time-autonomy with respect to $\mathscr{C}^\infty(\mathbb{R}^{m+1},\mathbb{C})$ of a scalar behaviour

We quote the following from Hörmander [4] (Theorem 8.6.7, p. 310):

Lemma 3.7. *Let the plane hyperplane with normal $\hat{n}$ be characteristic with respect to $p \in \mathbb{C}[\zeta_1, \ldots, \zeta_n]$. Then there exists a $w \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C})$ such that $D_p w = 0$ and supp$(w) = \{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle \geqslant 0\}$.*

An easy application of this lemma yields the following corollary:

Corollary 3.8. *If the behaviour corresponding to p is time-autonomous with respect to $\mathscr{C}^\infty(\mathbb{R}^{m+1}, \mathbb{C})$, then $p \neq 0$ and* $\deg(p) = \deg(\jmath(p))$.

4. Time-autonomy versus time-controllability

4.1. Time-autonomy versus exact time-controllability

Lemma 4.1. *If the behaviour $\mathfrak{B}$ of a dynamical system is nontrivial in $\mathscr{W}^w$ and time-autonomous with respect to $\mathscr{W}$, then $\mathfrak{B}$ is not time-controllable with respect to $\mathscr{W}$.*

Proof. The proof is immediate from the definitions. □

Example 4.2. The behaviour $\mathfrak{B}$ of the dynamical system corresponding to $p = \xi + \eta \in \mathbb{C}[\xi, \eta]$ is not time-controllable: indeed, this is now easy to see from the above theorem, since from Theorem 3.4 of the previous section, we know that $\mathfrak{B}$ is time-autonomous. Moreover, the behaviour is nontrivial since it contains for instance the nonzero trajectory e^{x-t}.

Theorem 4.3 (A necessary condition for time-controllability). *The behaviour $\mathfrak{B}$ of a dynamical system corresponding to $R \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_m]^{g \times w}$ is time-controllable only if*

$$\neg[\exists \chi \in \mathscr{A}^w \setminus \langle R \rangle \text{ and } \exists (0 \neq) p \in \mathscr{A} \text{ such that } p \cdot \chi \in \langle R \rangle, \text{ and } \deg(p) = \deg(\jmath(p))]. \tag{3}$$

Proof. Suppose that (3) does not hold and $\mathfrak{B}$ is time-controllable. Then there exists an element $\chi \in \mathscr{A}^w \setminus \langle R \rangle$ and a $p \in \mathscr{A}$ such that $p \cdot \chi \in \langle R \rangle$, and $\deg(p) = \deg(\jmath(p))$.

Clearly the trivial zero trajectory $w_1 := 0$ belongs to $\mathfrak{B}$. Since χ is not in $\langle R \rangle$, it follows that it does not kill every element in $\mathfrak{B}$. Let $w_2 \in \mathscr{D}'(\mathbb{R}^{m+1})^w$ be a trajectory such that $D_\chi w_2 \neq 0$.

Let w be a trajectory in $\mathfrak{B}$ which concatenates w_1 and w_2. Define $u = D_\chi w$. Since $p \cdot \chi \in \langle R \rangle$, and $w \in \mathfrak{B}$, it follows that $D_p u = 0$. But since $\langle w, \varphi \rangle = 0$ for all φ with support in $(-\infty, 0) \times \mathbb{R}^m$, it follows from the definition of u that $\langle u, \varphi \rangle = 0$ for all such φ. Hence it follows from Corollary 3.4 that u must be zero. But this a contradiction since u matches the nonzero future of $D_\chi w_2$ (w_2 might have to be shifted in order to achieve this). This completes the proof. □

Remarks. 1. The above theorem says the following: If there exists a row vector χ in $\mathscr{A}^w$ which is not in $\langle R \rangle$ and there exists a nonzero polynomial p satisfying $\deg(p) = \deg(\jmath(p))$ such that $p \cdot \chi \in \langle R \rangle$, then the behaviour corresponding to R is not time-controllable.

2. We observe that (3) implies that the $\mathbb{C}[\xi]$-module $\mathscr{A}^w / \langle R \rangle$ is torsion free, and hence we recover the necessary condition for time-controllability, Theorem 3.9 in [1]. Note that $\langle R \rangle$ is a $\mathscr{A}$-submodule of the $\mathscr{A}$-module $\mathscr{A}^w$. Hence $\mathscr{A}^w / \langle R \rangle$ makes sense as an $\mathscr{A}$-module. But since $\mathbb{C}[\xi]$ is a subring of $\mathscr{A}$, $\mathscr{A}^w / \langle R \rangle$ is also a $\mathbb{C}[\xi]$-module: indeed, we simply restrict scalar multiplication to elements belonging to $\mathbb{C}[\xi]$ instead of the full ring $\mathscr{A}$.

Example 4.4. Let $\mathfrak{B}$ be the behaviour of the dynamical corresponding to $p = \xi\eta \in \mathbb{C}[\xi, \eta]$. Since $\deg(p) = 2 \neq 1 = \deg(\jmath(p))$, it follows from Theorem 3.4 that $\mathfrak{B}$ is not time-autonomous. $\mathfrak{B}$ is not time-controllable, since the $\mathbb{C}[\xi]$-module $\mathscr{A}/\langle \xi\eta \rangle$ is not torsion free. For instance, $\bar{\eta}$ is a nonzero torsion element.

Example 4.5. The following table gives some simple examples.

Time-autonomous and $\neg$ time-controllable	$\neg$ time-autonomous and time-controllable	$\neg$ time-autonomous and $\neg$ time-controllable
$\mathbb{C}[\xi] \setminus \{0\}$, $\xi + \eta_1 + \cdots + \eta_{\mathtt{m}}$, $\xi^2 - (\eta_1^2 + \cdots + \eta_{\mathtt{m}}^2)$	$\mathbb{C}[\eta_1, \ldots, \eta_{\mathtt{m}}]$	$\xi(\eta_1 + \cdots + \eta_{\mathtt{m}})$

4.2. Time-autonomy versus approximate time-controllability

Theorem 4.6 (A necessary condition for approximate time-controllability). *The behaviour $\mathfrak{B}$ of a dynamical system corresponding to $R \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_{\mathtt{m}}]^{\mathtt{g}\times\mathtt{w}}$ is approximately time-controllable with respect to $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C})$ only if* (3) *holds.*

Proof. Suppose that (3) does not hold and the behaviour is approximately time-controllable with respect to $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C})$. Then there exists an element $\chi \in \mathscr{A}^{\mathtt{w}} \setminus \langle R \rangle$ and a $p \in \mathscr{A}$ such that $p \cdot \chi \in \langle R \rangle$, and $\deg(p) = \deg(\jmath(p))$.

As χ is not in $\langle R \rangle$, it follows that, it does not kill every element in $\mathfrak{B}_{\mathscr{C}^\infty}$. Let $w_0 \in \mathfrak{B}_{\mathscr{C}^\infty}$ be a trajectory such that $D_\chi w_0 \neq 0$. Without loss of generality, we may assume that $(D_\chi w_0)|_{t>0} \neq 0$ (otherwise w_0 can be shifted to achieve this). Since the topology of $\mathscr{E}(\Omega)$ is Hausdorff, it follows that there exists a neighbourhood N in $(\mathscr{E}(\mathbb{R}_+ \times \mathbb{R}^{\mathtt{m}}))^{\mathtt{w}}$ of $(D_\chi w_0)|_{t>0}$ that does not contain 0.

Since the map $D_\chi : (\mathscr{E}(\mathbb{R}_+ \times \mathbb{R}^{\mathtt{m}}))^{\mathtt{w}} \to \mathscr{E}(\mathbb{R}_+ \times \mathbb{R}^{\mathtt{m}})$ is continuous, it follows that there exists a neighbourhood N_1 in $(\mathscr{E}(\mathbb{R}_+ \times \mathbb{R}^{\mathtt{m}}))^{\mathtt{w}}$ of $w_0|_{t>0}$ such that $w_1 \in N_1$ implies that $D_\chi w_1 \in N$. Consequently $(D_\chi w_1)|_{t>0} \neq 0$.

Let $\tau \geqslant 0$ and let $w \in \mathfrak{B}_{\mathscr{C}^\infty}$ be such that $w(t) = 0$ for all $t \leqslant 0$, and $w(\bullet + \tau) \in N_1$. Defining $u = D_\chi w$, we have $u|_{t>0} \neq 0$.

Since $p \cdot \chi \in \langle R \rangle$, and $w \in \mathfrak{B}_{\mathscr{C}^\infty}$, it follows that $D_p u = 0$. But since $w(t, \bullet) = 0$ for all $t \in (0, \infty)$, it follows from the definition of u that $u(t, \bullet) = 0$ for all $t \in (0, \infty)$. Hence it follows from Theorem 3.4 that u must be zero. But this a contradiction since $u \neq 0$. This completes the proof. □

In the case of the scalar behaviour of a single partial differential equation, the necessary condition for approximate time-controllability also turns out to be sufficient. This is a corollary of Theorem 1, p. 255 from Hörmander [3].

Theorem 4.7. *If $p \in \mathbb{C}[\xi, \eta_1, \ldots, \eta_{\mathtt{m}}]$ is such that each irreducible factor $p_{\mathtt{i}}$ of p satisfies* $\deg(p_{\mathtt{i}}) \neq \deg(\jmath(p_{\mathtt{i}}))$, *then the behaviour corresponding to p is approximately time-controllable with respect to $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C})$.*

Proof. From Theorem 1 [3, p. 255], it follows that $\overline{\mathfrak{B}^0_{\mathscr{C}^\infty}} = \mathfrak{B}_{\mathscr{C}^\infty}|_{t>0}$. Consequently, from Lemma 2.2 we have that the behaviour corresponding to p is approximately time-controllable with respect to $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{m}+1}, \mathbb{C})$. □

Example 4.8 (The diffusion equation). The diffusion equation is approximately time-controllable: indeed $\xi-\eta^2$ is the only irreducible factor, and $\deg(\xi - \eta^2) = 2 \neq 1 = \deg(\xi)$.

Concluding remark. The main feature of a differential equation as a mathematical model of a physical phenomenon is the "local" nature of the law. In the case of an ordinary differential equation, "local" usually refers to "local in time", while for a partial differential equation, "local" refers to "local in time and space". The question arises if for PDEs this point of view is adequate. As we have seen, the diffusion equation $(\partial/\partial t)w = (\partial^2/\partial x^2)w$, with solutions in $\mathscr{C}^\infty(\mathbb{R}^2, \mathbb{C})$, shows odd behaviour. The variable w is not free and hence there are no inputs, but nevertheless the system is approximately controllable! However, if a global condition (for example the requirement that $w(\bullet, t)$ is bounded for all t) is added, this odd behaviour disappears: the only solution compatible with $w(\bullet, t) = 0$ for $t \leqslant 0$ is the zero solution. It is clear, however, that what global condition is reasonable to impose will depend on the specific physical system that is being modelled and on the specific question one wishes to analyse. Is there a general way to proceed?

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