Synthesis of Dissipative Systems Using Quadratic Differential Forms: Part II

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***Abstract*—In this second part of this paper, we discuss several important special cases of the problem solved in Part I. These are: disturbance attenuation and passivation, the full information case, the filtering problem, and the case that the to-be-controlled plant is given in input–state–output representation. An interesting aspect is the notion of full information, which we define in terms of the observability of the to-be-controlled variables from the control variables. When the system is given in state space form, we obtain conditions for the existence of a controller that renders a system dissipative in terms of two coupled algebraic Riccati inequalities. The controller turns out to be a feedback system with a transfer function that is proper, but, in general, not strictly proper. Another issue that we study in this paper is feedback implementability. We find conditions under which, in the context of synthesis of dissipative systems, a controlled behavior can implemented by a feedback controller.**

***Index Terms*—Disturbance attenuation, feedback implementability, filtering, full information, $\mathcal{H}_\infty$-control, passivation, Riccati equations, state space systems.**

I. Introduction

In THE SECOND part of this paper, we continue studying the control configuration shown schematically in Fig. 1. We will use the same notation and terminology as in [23]. For a review of the background material on linear differential systems and quadratic differential forms needed in this paper, we refer to [23, Sec. II, IV.1, and VI.1].

The basic ingredients for the problems discussed in this two-part paper are

- the *full plant behavior* $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\text{v}+\text{c}}$, consisting of all trajectories (v, c) that satisfy the equations of the to-be-controlled plant;
- the *plant behavior* $\mathcal{P} \in \mathfrak{L}^{\text{v}}$, consisting of the to-be-controlled trajectories v that the full plant behavior allows, before control is applied;
- the *hidden behavior* $\mathcal{N} \in \mathfrak{L}^{\text{v}}$, consisting of the trajectories v in the full plant behavior that are compatible with the control trajectories $c = 0$;
- the *weighting functional*, given by a nonsingular matrix $\Sigma = \Sigma^* \in \mathbb{R}^{\text{v}\times\text{v}}$ that defines, through the integral $\int_{-\infty}^{+\infty} Q_\Sigma(v)\, dt$ of the quadratic differential form Q_Σ, the control performance functional that needs to be made nonnegative.

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In [23], we obtained a precise characterization of the controlled behaviors that can be obtained by interconnecting the full plant behavior with a controller through the control variables c. These implementable behaviors consist exactly of the behaviors $\mathcal{K} \in \mathfrak{L}^{\text{v}}$ that are wedged in between $\mathcal{N}$ and $\mathcal{P}$. Hence, $\mathcal{K}$ is an implementable behavior if and only if $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$. In [23], we obtained necessary and sufficient conditions for the existence of an implementable behavior $\mathcal{K}$ that meets the control specifications. Such an implementable behavior exists if and only if $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$ are dissipative and a certain quadratic differential form, made up by coupling storage functions for $\mathcal{P}, \mathcal{N}$, is nonnegative.

In the present part, we discuss some special cases that are of much independent interest. They serve to illustrate the problem statement and its solution, and the generality and unifying features of our approach and of the results.

The first two special cases that we consider have to do with the weighting functional Q_Σ. Taking for Q_Σ the difference of two norms, leads to disturbance attenuation. Taking for Q_Σ an ordinary inner product, leads to passivation. The specifications on $\mathcal{K}$ can then be expressed as conditions on the transfer function of the controlled system.

The third and fourth special case are $\mathcal{N} = 0$ and $\mathcal{P} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{v}})$, respectively. Both have very nice interpretations. The first case corresponds to what we call "full information" control. In the control literature, full information control is usually taken to mean that the sensor outputs consist of the full state of the plant. Unfortunately, such a definition is not particularly intrinsic. First, because the state itself is not intrinsic, due to the possible nonminimality of the state, and second, because in optimal $\mathcal{H}_2$- or $\mathcal{H}_\infty$-control problems, the state also incorporates variables that originate from the cost-functional, whose measurability in terms of sensor outputs is not particularly meaningful. In a behavioral context, however, we obtain a very crisp definition of full information control. We say that we have full information control if the to-be-controlled variables are observable from the control variables, equivalently, $\mathcal{N} = 0$. The fourth special case, $\mathcal{P} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{v}})$, corresponds to filtering. The fact that both full information control and filtering both appear as simple special cases of our general control problem, is a matter of interest.

The final special case that we consider is the state space case, i.e., when the full plant behavior is given in input-state-output representation, and the weighting functional is a constant two-variable polynomial matrix. In this case, our general results of [23] lead to solvability conditions in terms of algebraic Riccati equations and inequalities. This problem has been studied very intensively in the 1990s, for example, in [1], [4], [7], [12]–[15], [3], [18], [16], [17], [5], [6], [9], and [10].

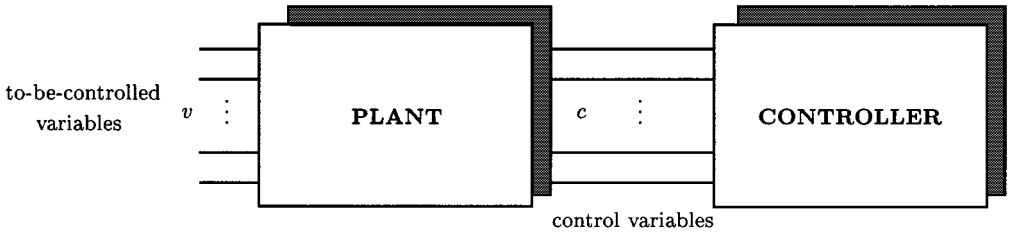

Fig. 1. Plant and controller configuration.

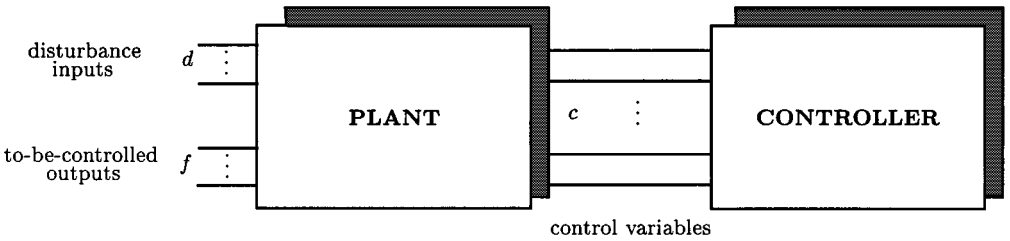

Fig. 2. Disturbance attenuation control.

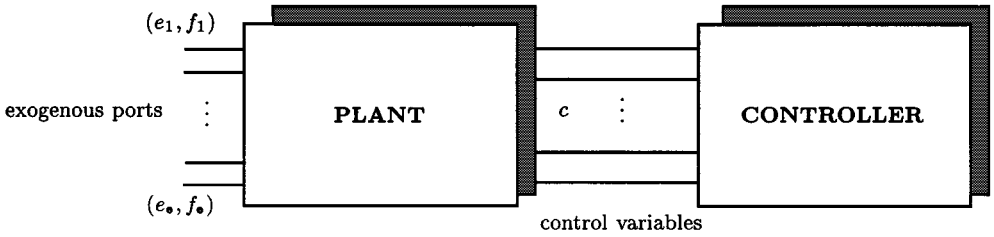

Fig. 3. Passivation control.

In Section VI of this paper, we address the important issue of feedback implementability. We find conditions under which a controller that renders a system dissipative, can be implemented as a feedback controller.

The proofs of the results can be found in Section VII.

II. Disturbance Attenuation and Passivation

We now discuss the first two special cases of the main problem solved in [23]. In particular, we show how to convert the dissipativity requirement into more classical statements about the input-output behavior and the transfer function of the controlled system. It is quite apparent, of course, that $\mathcal{H}_\infty$-disturbance attenuation and passivation are special cases of our general problem formulation. However, the translation to the abstract setting of [23] needs some extra work.

A. Disturbance Attenuation

In the important case of $\mathcal{H}_\infty$-disturbance attenuation (see Fig. 2) we have $v = (d, f)$ with d exogenous disturbance variables, f endogenous to-be-controlled variables, and $Q_\Sigma(v) = |d|^2 - |f|^2$. Whence, $\Sigma = \mathrm{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$. The following proposition reformulates the problem in this case into a more transparent one.

Proposition 1: Assume that $v = (d, f)$ and $Q_\Sigma(v) = |d|^2 - |f|^2$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$. Then the following conditions are equivalent:

i) $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)\,(=\mathtt{d})$;
ii) $\|f\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})} \leq \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})}$ for all $(d, f) \in \mathcal{K} \cap \mathcal{L}_2$, the exogenous disturbances d are free in $\mathcal{K}$ [i.e., for all $d \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$, there exists $f \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{f}})$ such that $(d, f) \in \mathcal{K}$], and $(0, f) \in \mathcal{K}$ implies that $f(t)$ goes to zero as $t \to \infty$;
iii) in $\mathcal{K}$, d is input and f is output, and the transfer function $G_{d \mapsto f}$ from d to f satisfies $\|G_{d \mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

B. Passivation

A similar story holds, with disturbance attenuation replaced by passivity, when $v = (e, f)$ (e for "effort," f for "flow") and $Q_\Sigma(e, f) = e^T f$, with $e_k f_k$ the "power" flowing into the plant through the kth exogenous port or terminal (see Fig. 3), whence $\Sigma = 1/2\left[\begin{smallmatrix} 0 & I_{\mathtt{e}} \\ I_{\mathtt{e}} & 0 \end{smallmatrix}\right]$. Here, $\mathtt{e} = \mathtt{f} = \dim(e) = \dim(f)$. The following proposition in turn reformulates the problem in this case into a more transparent one.

Proposition 2: Assume that $v = (e, f)$ and $Q_\Sigma(v) = e^T f$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$. Then the following conditions are equivalent:

i) $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)\,(=\mathtt{e} = \mathtt{f})$;
ii) $\int_{-\infty}^{+\infty} e^T f\, dt \geq 0$ for all $(e, f) \in \mathcal{K} \cap \mathcal{L}_2$, there is a component-wise input/output partition (u, y) of $v = (e, f)$ such that for all $1 \leq i \leq \mathtt{e} = \mathtt{f}$, either e_i or f_i, is input, and the other is output, and $(e, f) \in \mathcal{K}$ and $u = 0$ imply that $(e, f)|_{\mathbb{R}_+}$ is bounded [$(e, f)|_{\mathbb{R}_+}$ denotes the restriction of (e, f) to $\mathbb{R}_+$];
iii) there is a component-wise input/output partition (u, y) of $v = (e, f)$ such that for all $1 \leq i \leq \mathtt{e} = \mathtt{f}$, either e_i, or f_i, is input, and the other is output, and the transfer matrix $G_{u \mapsto y}$ from u to y in $\mathcal{K}$ is positive real, i.e., $G_{u \mapsto y}(\lambda) + G^T_{u \mapsto y}(\overline{\lambda}) \geq 0$ for all $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) > 0$.

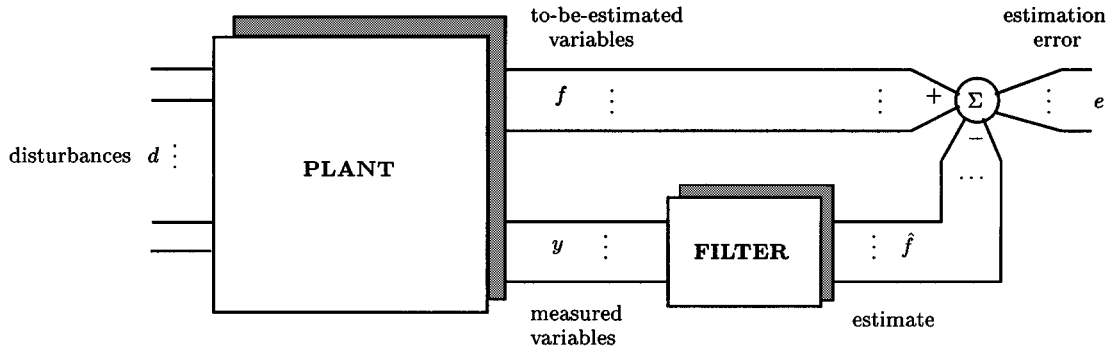

Fig. 4. Plant and filter configuration.

III. Full Information Control

Consider the plant shown in Fig. 1, and recall its description as discussed in [23, Sec. II], with the formal definitions of the full plant behavior $\mathcal{P}_{\text{full}}$, the manifest plant behavior $\mathcal{P}$, and the hidden behavior $\mathcal{N}$. The controller is assumed to act through the control variables c.

In keeping with the behavioral definition of observability (see [23, Sec. VI.1]), we call the to-be-controlled variables v *observable* from the control variables c in $\mathcal{P}_{\text{full}}$ if $(v_1, c), (v_2, c) \in \mathcal{P}_{\text{full}}$ implies $v_1 = v_2$, equivalently if there exists a polynomial matrix $F \in \mathbb{R}^{\mathtt{v}\times\mathtt{c}}[\xi]$ such that $(v, c) \in \mathcal{P}_{\text{full}}$ implies $v = F(d/dt)c$. We call this situation *full information control*, since in this case knowledge of the control variables allows to reconstruct completely the to-be-controlled variables. In particular, the controller then has complete information of the exogenous disturbances. It is as if the constraint that the controller is restricted to act through the control variables is inoperative. In particular, if we consider the control variables as being measured, these measurements allow to deduce all relevant to-be-controlled signals acting on the plant. The following proposition relates full information control to the hidden behavior.

Proposition 3: Full information control holds if and only if $\mathcal{N} = 0$.

In the case of full information control, our main result ([23, Th. 5]) immediately specializes to the following theorem.

Theorem 4 (Full Information Control): Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ and $\Sigma = \Sigma^* \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be nonsingular. Then there exists $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ such that

1) $\mathcal{K} \subset \mathcal{P}$ (*implementability*);
2) $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ (*dissipativity*);
3) $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ (*liveness*);

if and only if $\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative on $\mathbb{R}_+$, equivalently, if and only if there exists a two-variable polynomial matrix $\Psi_{\mathcal{P}^{\perp_\Sigma}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta, \eta]$, such that $Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v) \geq 0$ and $(d/dt)Q_{\Psi_{\mathcal{P}^{\perp_\Sigma}}}(v) \geq Q_\Sigma(v)$ for $v \in \mathcal{P}^{\perp_\Sigma}$.

This theorem is an immediate consequence of [23, Th. 5]. It is actually the main result of [19], where, however, a much more involved proof was given.

IV. Filtering

The problem discussed in [23] solves some interesting system theoretic problems that are, properly speaking, not control problems. It turns out, in fact, that the "dual" of full information control is filtering.

Consider the signal processing problem depicted in Fig. 4. In this set-up, the *plant* relates three types of variables: *disturbances* d, *to-be-estimated variables* f, and *measured variables* y. The problem is to design a *filter* that relates the measured variables y to the *estimate* $\hat{f}$, such that the *estimation error* $e = f - \hat{f}$ is small in an appropriate sense. Denote the number of components of d by $\mathtt{d}$, of f (and hence $\hat{f}$ and e) by $\mathtt{f}$, and of y by $\mathtt{y}$. The variables of interest whose relationship we are trying to shape by means of a filter are (d, e).

Define the *full plant behavior*, $\mathcal{P}_{\text{full}}$, to be the signals (d, f, y) that the plant allows, the *manifest plant behavior*, $\mathcal{P}$, to be the signals (d, f) that the plant allows, hence with the measured variables y eliminated, and the *hidden behavior*, $\mathcal{N}$, to be those signals (d, f) that are compatible with the plant equations *and* with the measured variables equal to zero. Define further $\mathcal{D}$, the *disturbance behavior*, to be the signals d that are possible, whence with f and y eliminated from $\mathcal{P}_{\text{full}}$. The formal definition of these behaviors is hence

$$\begin{aligned}
\mathcal{P}_{\text{full}} &= \big\{(d, f, y) \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}+\mathtt{y}}\big) \,|\, (d, f, y) \\
&\qquad \text{satisfies the plant equations}\big\} \\
\mathcal{P} &= \big\{(d, f) \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}\big) \,|\, \exists\, y \\
&\qquad \text{such that } (d, f, y) \in \mathcal{P}_{\text{full}}\big\} \\
\mathcal{N} &= \big\{(d, f) \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}\big) \,|\, (d, f, 0) \in \mathcal{P}_{\text{full}}\big\} \\
\mathcal{D} &= \big\{d \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{d}}\big) \,|\, \exists\, (f, y) \\
&\qquad \text{such that } (d, f, y) \in \mathcal{P}_{\text{full}}\big\}.
\end{aligned}$$

We assume throughout this section that in $\mathcal{P}_{\text{full}}$ d is free, i.e., that $\mathcal{D} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$.

A *filter* is a dynamical system that relates the measured variables y to the estimate $\hat{f}$ of f. The filter imposes a relation on the variables $(y, \hat{f})$. We take this to mean $(y, \hat{f}) \in \mathcal{F}$, with $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ the behavior of the filter. Before the filter acts, the variables d, f, y, $\hat{f}$ and e are constrained to satisfy $(d, f, y) \in \mathcal{P}_{\text{full}}$ and $e = f - \hat{f}$. However, with the filter in action, they have to obey also $(y, \hat{f}) \in \mathcal{F}$. This yields the manifest behavior $\mathcal{E}$ of the variables (d, e) in the interconnected system shown in Fig. 4, formally defined as

$$\begin{aligned}
\mathcal{E} = \Big\{(d, e) \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}\big) \,|\, \exists (f, y, \hat{f}) \in \mathfrak{C}^\infty\big(\mathbb{R}, \mathbb{R}^{\mathtt{f}+\mathtt{y}+\mathtt{f}}\big) \\
\text{such that } (d, f, y) \in \mathcal{P}_{\text{full}},\ (y, \hat{f}) \in \mathcal{F},\ e = f - \hat{f}\Big\}.
\end{aligned}$$

The behavior $\mathcal{E}$ is called *the estimation error behavior*. Obviously, by the elimination theorem, $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$. If, for a given element $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$, there exists $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ such that the above relation holds, then we say that the filter $\mathcal{F}$ *implements* $\mathcal{E}$. The question what $\mathcal{E}$'s are implementable is answered in the following theorem.

Theorem 5 (Filter Implementability Theorem): The behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ is implementable by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ if and only if $\mathcal{N} \subset \mathcal{E}$. Moreover, if $\mathcal{E}$ is implementable, and if d is input and e is output in $\mathcal{E}$, then it can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}_{\mathrm{cont}}$ such that in $\mathcal{F}$, y is input and $\hat{f}$ output.

The problem that we consider is to find a filter that renders the estimation error behavior dissipative with respect to a QDF in the variables (d, e). We consider only the case of $\mathcal{H}_\infty$-filtering. The following theorem shows when such a filter exists.

Theorem 6 ($\mathcal{H}_\infty$**-Filtering**): Assume that $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$. Then there exists $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$ such that

1. $\mathcal{N} \subset \mathcal{E}$ $(implementability)$
2. the disturbances d are free in $\mathcal{E}$ $(liveness)$
3. $(d, e) \in \mathcal{E} \cap \mathcal{L}_2$ implies $\|e\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})} \leq \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})}$ $(disturbance\ attenuation)$
4. $(d, e) \in \mathcal{E}$ and $d(t) = 0$ for $t \geq 0$ implies $e(t) \to \infty$ $(stability)$

if and only if $\mathcal{N}$ is Σ-dissipative on $\mathbb{R}_-$, with $\Sigma = \mathrm{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$, equivalently, if and only if there exists a two-variable polynomial matrix $\Psi_{\mathcal{N}} \in \mathbb{R}^{(\mathtt{d}+\mathtt{f})\times(\mathtt{d}+\mathtt{f})}[\zeta, \eta]$, such that $Q_{\Psi_{\mathcal{N}}}(d, e) \geq 0$ and $(d/dt)Q_{\Psi_{\mathcal{N}}}(d, e) \leq |d|^2 - |e|^2$ for $(d, e) \in \mathcal{N}$.

We now explain the meaning of these conditions. The idea is that before the filter acts, the variables (d, e) are free: for e, this is trivially so, and for d, it holds by assumption. With the filter put into place, as shown in Fig. 4, the variables (d, e) are constrained to belong to $\mathcal{E}$. The first condition is thus merely the implementability condition of Theorem 5. The second condition states that the filter is not allowed to restrict the free exogenous disturbances d: the interconnected systems should still be allowed to accept arbitrary ds. By [23, Prop. 2], and with also the third condition, this is equivalent to $\mathtt{m}(\mathcal{E}) = \mathtt{d}$. The third condition expresses disturbance attenuation: for all $(d, e) \in \mathcal{E} \cap \mathcal{L}_2$ there should hold $\int_{-\infty}^{+\infty} |e|^2\, dt \leq \int_{-\infty}^{+\infty} |d|^2\, dt$. The fourth condition states that without the disturbances acting, the estimation error must go to zero. Actually conditions 3 and 4 combined are equivalent to Σ-dissipativity of $\mathcal{E}$ on $\mathbb{R}_-$: for all $(d, e) \in \mathcal{E} \cap \mathfrak{D}$ there should hold $\int_{-\infty}^{0} |e|^2\, dt \leq \int_{-\infty}^{0} |d|^2\, dt$. The theorem therefore states that dissipativity of $\mathcal{N}$ on $\mathbb{R}_-$, an obvious necessary condition (since $\mathcal{N} \subset \mathcal{E}$), is also sufficient.

Theorem 6 shows that, in a sense, $\mathcal{H}_\infty$-filtering is a rather easy problem. It is possible to obtain disturbance attenuation from d to e if and only if disturbance attenuation already holds from d to e when $y = 0$. The content of Theorem 6 is to show that this obvious necessary condition is also sufficient. We note that our result does not involve any particular representation of the full plant behavior, nor an *a priori* given input/output partition. For systems in state space representation, a similar result is found in [7] and [8]. We also refer to [9] and [11].

We remark, without going into the details, that theorem 6 is readily extended to the case of weighted $\mathcal{H}_\infty$-norms. It is a simple matter to include frequency weighting in the formulation of the filtering problem, by letting d and e be related to the "physical" exogenous input disturbance d' and endogenous estimation error $e' = f' - \hat{f}'$ by $Q(d/dt)d' = P(d/dt)d$, $N(d/dt)e = D(d/dt)e'$, with P and D square, nonsingular, and Hurwitz.

We emphasize that the problem with which we deal here is filtering, in contrast to *smoothing*. The smoothing problem corresponds to the case that $\mathcal{E}$ is dissipative w.r.t. $|d|^2 - |e|^2$ on $\mathbb{R}$ (instead of $\mathbb{R}_-$). Equivalently, the problem solved in theorem 6, but without condition 4, the stability requirement. When $\mathcal{E}$ is dissipative w.r.t. $|d|^2 - |e|^2$ on $\mathbb{R}$, then there exists for each $d \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$, a $e \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})$, such that $(d, e) \in \mathcal{E}$. However, in order to obtain this e, the initial conditions of the corresponding $\mathcal{F}$ should be chosen well, and this choice involves both the past and the future of d.

Assume that the dissipativity condition on the hidden behavior $\mathcal{N}$ of theorem 6 is satisfied. Then there exists an implementable $\mathcal{E}$ such that conditions 2, 3, and 4 are satisfied. By [23, Prop. 2], condition 3 implies that $\mathtt{m}(\mathcal{E}) \leq \mathtt{d}$. Together with condition 2, this yields that in $\mathcal{E}$, in fact, d is input and e is output. Thus, by applying Theorem 5 we find that there exists a filter which implements $\mathcal{E}$, and has y as input and $\hat{f}$ as output. In other words, there exists a filter that acts as a *signal processor* that accepts any input signal $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{y}})$ and produces as output the estimate $\hat{f}$ of f. There is no *a priori* reason, of course, for the transfer function of this signal processor from y to $\hat{f}$ to be proper, since singular filtering is very much part of our setup. However, properness may be obtained by imposing some additional structure on the plant. We will return to this issue in Section VI.

V. The State Space $\mathcal{H}_\infty$-Problem

In this section we apply the results of [23] to the special case that the plant is given in input/state/output representation. We shall see that our results and proofs concerning the general problem set-up lead to a solution for the state space case, analogous to those on the standard $\mathcal{H}_\infty$ problem obtained in [1]. This double Riccati equation solution and its variations have been the subject of very intensive research, see, e.g., [7], [18] and generalizations in [12], [13], [16], [17], [9], [10], [5], [6], and [3].

Whereas most of the existing literature deals with the problem of finding an internally stabilizing controller such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function is *strictly* less than 1, we deal with the problem of making the $\mathcal{H}_\infty$ norm of this transfer function *less than or equal* to 1, see [4], [14], [15]. Also this problem turns out to admit a solution in terms of two Riccati equations, together with a coupling condition. A difference with the strict suboptimal problem is, that the dimension of the state space of the controller depends on the solutions of the Riccati

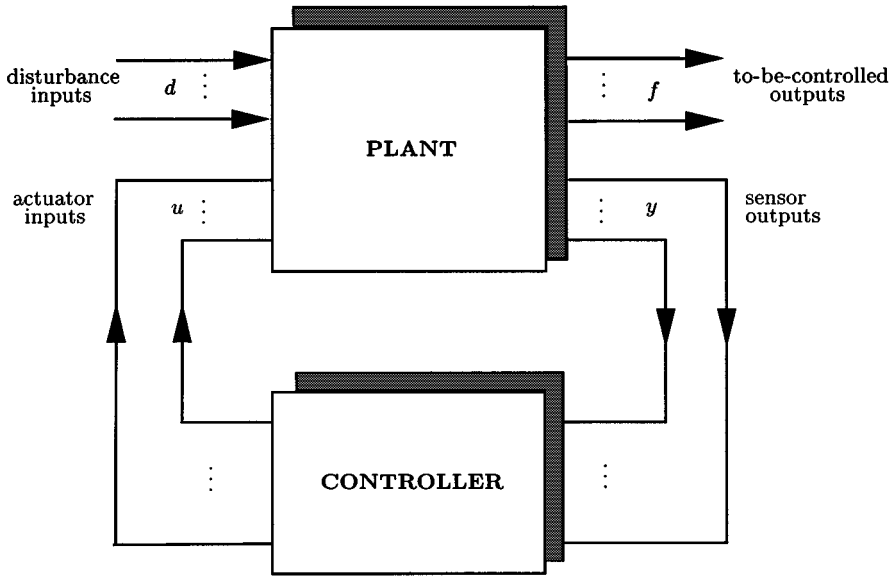

Fig. 5. Feedback control.

equation, and may be smaller than the dimension of the state space of the plant.

The structure of this section is as follows. In Section V-A we state the problem, and in Sections V-B–V-H we derive conditions for the existence of suitable controllers, and convenient representations for them. Finally, in Section V-I we summarize these results in a theorem. Connections with the double Riccati equation solution from [1] are discussed in Section V-J.

A. Plant Description

Consider the plant $\mathcal{P}_{\text{full}}$ given in input-state-output representation by

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu + Gd, \\ y &= Cx + \quad + Dd \cdot \\ f &= Hx + Ju. \end{aligned} \tag{1}$$

Assume that the following three regularity conditions hold:

- A.1) D is surjective and J is injective;
- A.2) $(A - GD^T(DD^T)^{-1}C,\ G(I_{\mathtt{d}} - D^T(DD^T)^{-1}D))$ is a controllable pair of matrices;
- A.3) $(A - B(J^TJ)^{-1}J^TH,\ (I_{\mathtt{f}} - J(J^TJ)^{-1}J^T)H)$ is an observable pair of matrices.

In terms of the usual feedback diagram (see Fig. 5) u are the inputs to the actuators, y are the outputs of the sensors, d the exogenous disturbances, and f the endogenous to-be-controlled outputs. The problem is to find a controller acting on the control variables (u, y) such that the controlled system meets certain specifications. We are looking for a controller that is also in state representation, more exactly, in input–state–output representation, with y the input, u the output, and with the controller state denoted as x_c:

$$\begin{aligned} \frac{d}{dt}x_c &= A_cx_c + B_cy \\ u &= C_cx_c + D_cy. \end{aligned} \tag{2}$$

Our aim is to derive conditions for the existence and algorithms for the computation of the controller parameter matrices (A_c, B_c, C_c, D_c) such that the controlled system has the following properties:

1) *disturbance attenuation* with gain factor normalized to 1, i.e., for all $(d, f) \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$ for which there exist (u, y, x, x_c) satisfying both the plant equations (1) and the controller equations (2), there should hold $\|f\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})} \leq \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})}$;
2) *internal stability,* meaning that in the controlled system $d = 0$ should imply that the signals (x, x_c, u, f) all go to zero as $t \to \infty$.

Note that in the controlled system $\mathcal{K}_{\text{full}}$ given by the combined equations (1) and (2), d is free. This implies that d is also free in $\mathcal{K}$. Hence, from Proposition 1, it follows that conditions 1 and 2 above are equivalent to requiring that the controlled system is internally stable and has transfer function $G_{d\mapsto f}$ satisfying $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

In terms of the notation used in [23], we have $v = (d, f)$ as the to-be-controlled variables, $c = (u, y)$ as the control variables, and $\Sigma = \text{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$ as the weighting matrix. In this section, hence, $\Sigma = \text{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$.

Observe that internal stability is a slightly stronger stability notion than the one used in the disturbance attenuation problem treated in Section II-A. There it is only required that $d = 0$ implies $f(t) \to 0$ as $t \to \infty$. We refer to this latter property as *external stability*.

B. Calculation of Subbehaviors

In order to apply [23, Th. 5], we first derive the various behaviors that are involved. In particular, for the full plant behavior $\mathcal{P}_{\text{full}}$ represented by (1), we will derive specific representations for the manifest plant behavior $\mathcal{P}$ and its Σ-orthogonal complement $\mathcal{P}^{\perp_\Sigma}$, and the hidden behavior $\mathcal{N}$ and its Σ-orthogonal complement. Subsequently, we will derive conditions under which $\mathcal{P}$ and $\mathcal{N}$ satisfy the conditions of [23, Th. 5].

Eliminating (u, y) from (1), yields the following driving variable representation for the plant behavior $\mathcal{P}$:

$$\frac{d}{dt}x_{\mathcal{P}} = Ax_{\mathcal{P}} + \begin{bmatrix} B & G \end{bmatrix} d_{\mathcal{P}},$$
$$v_{\mathcal{P}} = \begin{bmatrix} 0 \\ H \end{bmatrix} x_{\mathcal{P}} + \begin{bmatrix} 0 & I \\ J & 0 \end{bmatrix} d_{\mathcal{P}}.$$

Putting $u = y = 0$ yields the following output nulling representation for the hidden behavior $\mathcal{N}$:

$$\frac{d}{dt}x_{\mathcal{N}} = Ax_{\mathcal{N}} + \begin{bmatrix} G & 0 \end{bmatrix} v_{\mathcal{N}},$$
$$0 = \begin{bmatrix} C \\ H \end{bmatrix} x_{\mathcal{N}} + \begin{bmatrix} D & 0 \\ 0 & -I \end{bmatrix} v_{\mathcal{N}}.$$

Assumptions A.2) and A.3) made in the beginning of this section ensure that the behaviors $\mathcal{N}$ and $\mathcal{P}$ are controllable. Moreover, their state space representations obtained above are controllable and observable.

From these equations and the relations between an output nulling representation and driving variable representation of a behavior and its orthogonal complement (see [23, Sec. IV.1]), we obtain the following output nulling representation for $\mathcal{P}^{\perp_\Sigma}$:

$$\frac{d}{dt}z_{\mathcal{P}} = -A^T z_{\mathcal{P}} + \begin{bmatrix} 0 & -H^T \end{bmatrix} v_{\mathcal{P}^{\perp_\Sigma}},$$
$$0 = \begin{bmatrix} B^T \\ G^T \end{bmatrix} z_{\mathcal{P}} + \begin{bmatrix} 0 & J^T \\ -I & 0 \end{bmatrix} v_{\mathcal{P}^{\perp_\Sigma}}$$

and driving variable representation for $\mathcal{N}^{\perp_\Sigma}$

$$\frac{d}{dt}z_{\mathcal{N}} = -A^T z_{\mathcal{N}} + \begin{bmatrix} C^T & H^T \end{bmatrix} d_{\mathcal{N}^{\perp_\Sigma}},$$
$$v_{\mathcal{N}^{\perp_\Sigma}} = \begin{bmatrix} G^T \\ 0 \end{bmatrix} z_{\mathcal{N}} + \begin{bmatrix} -D^T & \\ 0 & -I \end{bmatrix} d_{\mathcal{N}^{\perp_\Sigma}}.$$

These equations immediately yield the following output nulling representation for $\mathcal{N} + \mathcal{P}^{\perp_\Sigma}$:

$$\frac{d}{dt}\begin{bmatrix} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{bmatrix} = \begin{bmatrix} A & -GG^T \\ H^T H & -A^T \end{bmatrix}\begin{bmatrix} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & -H^T \end{bmatrix} v_{\mathcal{N}+\mathcal{P}^{\perp_\Sigma}},$$
$$0 = \begin{bmatrix} C & -DG^T \\ -J^T H & B^T \end{bmatrix}\begin{bmatrix} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & J^T \end{bmatrix} v_{\mathcal{N}+\mathcal{P}^{\perp_\Sigma}}$$

and driving variable representation for $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$

$$\frac{d}{dt}\begin{bmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{bmatrix} = \begin{bmatrix} -A^T & -H^T H \\ GG^T & A \end{bmatrix}\begin{bmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} C^T & -H^T J \\ -GD^T & B \end{bmatrix} d_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}},$$
$$v_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}} = \begin{bmatrix} G^T & 0 \\ 0 & H \end{bmatrix}\begin{bmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} -D^T & 0 \\ 0 & J \end{bmatrix} d_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}.$$

We frequently use the obvious partitions $v_{\mathcal{P}} = (v'_{\mathcal{P}}, v''_{\mathcal{P}})$, etc., for the vs, and the analogs for the ds.

C. Verification of Dissipativity

The next step, after cataloging these behaviors, is to verify Σ-dissipativity of $\mathcal{N}$ and $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$. The following lemma comes in useful in these verifications.

Lemma 7: Let $M = M^T$, N, P, Q be real matrices of appropriate dimensions, and assume that Q is surjective. Then the quadratic form $x_1^T M x_1 + 2x_1^T N x_2 + x_2^T x_2$ is nonnegative on the subspace defined by $Px_1 + Qx_2 = 0$ if and only if

$$0 \le M - NN^T + (P - QN^T)^T (QQ^T)^{-1}(P - QN^T) = L^T L$$

in which case the quadratic form on the subspace equals

$$|Lx_1|^2 + |x_2 + N^T x_1|^2_{(I - Q^T(QQ^T)^{-1}Q)}.$$

Using [23, Prop. 3], we know that Σ-dissipativity of $\mathcal{N}$ is equivalent to the existence of a storage function. It is implied in [23, Prop. 10] that a storage function is a state function. Hence $\mathcal{N}$ is Σ-dissipative if and only if there exists a matrix $K_{\mathcal{N}} = K_{\mathcal{N}}^T \in \mathbb{R}^{n\times n}$ [with $n = \dim(x)$] such that

$$\frac{d}{dt}|x_{\mathcal{N}}|^2_{K_{\mathcal{N}}} \le |v'_{\mathcal{N}}|^2 - |v''_{\mathcal{N}}|^2$$

for all $(v'_{\mathcal{N}}, v''_{\mathcal{N}}, x_{\mathcal{N}})$ satisfying the equations for $\mathcal{N}$, equivalently

$$|v'_{\mathcal{N}}|^2 - 2(v'_{\mathcal{N}})^T G^T K_{\mathcal{N}} x_{\mathcal{N}} - |Hx_{\mathcal{N}}|^2 - x_{\mathcal{N}}^T (A^T K_{\mathcal{N}} + K_{\mathcal{N}} A) x_{\mathcal{N}} \ge 0$$

for all $(v'_{\mathcal{N}}, x_{\mathcal{N}})$ satisfying $Cx_{\mathcal{N}} + Dv'_{\mathcal{N}} = 0$. We now express this as an inequality in terms of the system matrices. Using Lemma 7, it follows that Σ-dissipativity of $\mathcal{N}$ is equivalent to the existence of a matrix $K_{\mathcal{N}} = K_{\mathcal{N}}^T \in \mathbb{R}^{n\times n}$ such that the algebraic Riccati inequality

$$\boxed{\begin{aligned} 0 \le\ & -A^T K_{\mathcal{N}} - K_{\mathcal{N}} A - H^T H - K_{\mathcal{N}} G G^T K_{\mathcal{N}} \\ & + (C^T + K_{\mathcal{N}} G D^T)(DD^T)^{-1}(C + DG^T K_{\mathcal{N}}) \\ & = L_{\mathcal{N}}^T L_{\mathcal{N}} \end{aligned}} \quad (3)$$

holds, in which case

$$|v'_{\mathcal{N}}|^2 - |v''_{\mathcal{N}}|^2 - \frac{d}{dt}|x_{\mathcal{N}}|^2_{K_{\mathcal{N}}} = |v'_{\mathcal{N}} - G^T K_{\mathcal{N}} x_{\mathcal{N}}|^2_{(I - D^T(DD^T)^{-1}D)} + |L_{\mathcal{N}} x_{\mathcal{N}}|^2.$$

Similarly, $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ is equivalent to the existence of $K_{\mathcal{P}} = K_{\mathcal{P}}^T \in \mathbb{R}^{n\times n}$ such that

$$\frac{d}{dt}|z_{\mathcal{P}}|^2_{K_{\mathcal{P}}} \le -|v'_{\mathcal{P}^{\perp_\Sigma}}|^2 + |v''_{\mathcal{P}^{\perp_\Sigma}}|^2$$

for all $(v'_{\mathcal{P}^{\perp_\Sigma}}, v''_{\mathcal{P}^{\perp_\Sigma}}, z_{\mathcal{P}})$ satisfying the equations for $\mathcal{P}^{\perp_\Sigma}$. Using lemma 7 again, it readily follows that $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ is equivalent to the existence of $K_{\mathcal{P}} = K_{\mathcal{P}}^T \in \mathbb{R}^{n\times n}$ such that the algebraic Riccati inequality

$$\boxed{\begin{aligned} 0 \le\ & AK_{\mathcal{P}} + K_{\mathcal{P}} A^T - GG^T - K_{\mathcal{P}} H^T H K_{\mathcal{P}} \\ & + (B - K_{\mathcal{P}} H^T J)(J^T J)^{-1}(B^T - J^T H K_{\mathcal{P}}) \\ & = L_{\mathcal{P}}^T L_{\mathcal{P}} \end{aligned}} \quad (4)$$

holds, in which case

$$-|v'_{\mathcal{P}^{\perp_\Sigma}}|^2 + |v''_{\mathcal{P}^{\perp_\Sigma}}|^2 - \frac{d}{dt}|z_{\mathcal{P}}|^2_{K_{\mathcal{P}}} = |v''_{\mathcal{P}^{\perp_\Sigma}} + HK_{\mathcal{P}} z_{\mathcal{P}}|^2_{(I - J(J^T J)^{-1}J^T)} + |L_{\mathcal{P}} z_{\mathcal{P}}|^2.$$

Now apply [23, Th. 5], using the interpretation of $x_{\mathcal{N}}^T K_{\mathcal{N}} x_{\mathcal{N}}$ and $z_{\mathcal{P}}^T K_{\mathcal{P}} z_{\mathcal{P}}$ as storage functions for $\mathcal{N}$ and $\mathcal{P}^{\perp\Sigma}$ respectively, and the fact that the minimal states $(x_{\mathcal{N}}, z_{\mathcal{P}})$ appearing in the equations for $\mathcal{N}$ and $\mathcal{P}^{\perp\Sigma}$ satisfy $(d/dt)(x_{\mathcal{N}}^T z_{\mathcal{P}}) = v_{\mathcal{N}}^T \Sigma v_{\mathcal{P}}$. It follows that a necessary and sufficient condition for the existence of a controlled behavior $\mathcal{K}$ satisfying $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, Σ-dissipativity, external stability, and d free in $\mathcal{K}$, is that there exist solutions $K_{\mathcal{N}}$ and $K_{\mathcal{P}}$ to the algebraic Riccati inequalities (3), (4) such that

$$\boxed{K = \begin{bmatrix} K_{\mathcal{N}} & I \\ I & -K_{\mathcal{P}} \end{bmatrix} \geq 0.}$$

This nonnegativity is easily seen to be equivalent to the combined inequalities

1) $K_{\mathcal{N}} > 0$;
2) $K_{\mathcal{P}} < 0$;
3) $K_{\mathcal{N}} \geq (-K_{\mathcal{P}})^{-1}$, i.e., $\rho(K_{\mathcal{N}} K_{\mathcal{P}}) \geq 1$, where ρ denotes the spectral radius.

The theory of the algebraic Riccati equation and its relation with the algebraic Riccati inequalities allows to analyze the situation further. The final conclusion becomes that a necessary and sufficient condition for the existence of a required controlled behavior $\mathcal{K}$ is that the two algebraic Riccati equations

$$\begin{aligned} &-A^T K_{\mathcal{N}} - K_{\mathcal{N}} A - H^T H - K_{\mathcal{N}} G G^T K_{\mathcal{N}} \\ &\quad + (C^T + K_{\mathcal{N}} G D^T)(DD^T)^{-1}(C + DG^T K_{\mathcal{N}}) = 0 \quad (5) \\ &A K_{\mathcal{P}} + K_{\mathcal{P}} A^T - G G^T - K_{\mathcal{P}} H^T H K_{\mathcal{P}} \\ &\quad + (B - K_{\mathcal{P}} H^T J)(J^T J)^{-1}(B^T - J^T H K_{\mathcal{P}}) = 0 \quad (6) \end{aligned}$$

have symmetric solutions $K_{\mathcal{N}} = K_{\mathcal{N}}^T$ and $K_{\mathcal{P}} = K_{\mathcal{P}}^T$, and that the maximal real symmetric solution $K_{\mathcal{N}}^+$ of (5) combined with the minimal real symmetric solution $K_{\mathcal{P}}^-$ of (6) should satisfy

$$\begin{bmatrix} K_{\mathcal{N}}^+ & I \\ I & -K_{\mathcal{P}}^- \end{bmatrix} \geq 0.$$

There are actually four K matrices which may be obtained from combining the storage functions for $\mathcal{N}$ and $\mathcal{P}^{\perp\Sigma}$ that are all four relevant in our development. One is any K derived from storage functions obtained from algebraic Riccati inequalities. The second is the one obtained by substituting both the extreme storage functions obtained from algebraic Riccati equations. The third and fourth are obtained by substituting one extreme storage function. We denote these four cases as

$$\begin{aligned} K &= \begin{bmatrix} K_{\mathcal{N}} & I \\ I & -K_{\mathcal{P}} \end{bmatrix} \quad K_{-}^{+} = \begin{bmatrix} K_{\mathcal{N}}^+ & I \\ I & -K_{\mathcal{P}}^- \end{bmatrix} \\ K^+ &= \begin{bmatrix} K_{\mathcal{N}}^+ & I \\ I & -K_{\mathcal{P}} \end{bmatrix} \quad K_{-} = \begin{bmatrix} K_{\mathcal{N}} & I \\ I & -K_{\mathcal{P}}^- \end{bmatrix}. \end{aligned}$$

Obviously, $K \leq K^+ \leq K_{-}^{+}$, $K \leq K_{-} \leq K_{-}^{+}$. The condition $K \geq 0$ is necessary and sufficient for the existence of a controlled behavior $\mathcal{K}$. However, in order to construct $\mathcal{K}$, we need K^+ (or, in the dual case, K_{-}). The standard solutions of the state space $\mathcal{H}_\infty$-problem work with K_{-}^{+}.

D. Specification of the Controlled Behavior

Next, we derive an inequality that makes it evident in the state space case how to specify the controlled behavior, using ideas from the proof of [23, Th. 5]. Introduce the variables θ, $\chi \in \mathbb{R}^{\mathbf{n}}$ and impose the constraint $\left[\begin{smallmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$. Note that this equation merely expresses that $\left[\begin{smallmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{smallmatrix}\right]$ must belong to $\mathrm{im}(K)$. Combined with the equations for $\mathcal{P} \cap \mathcal{N}^{\perp\Sigma}$, this constraint actually defines the behavior $\mathcal{M}$ introduced in the proof of [23, Th. 5]. In the regular case [which corresponds to $K > 0$, since for the case at hand it is readily shown that $\mathcal{N} + \mathcal{N}^{\perp\Sigma} = \mathcal{P} + \mathcal{P}^{\perp\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{d}+\mathbf{f}})$] the existence of (θ, χ) obviously imposes no conditions on $(z_{\mathcal{N}}, x_{\mathcal{P}})$, but in the singular case, it does. A straightforward calculation using the algebraic Riccati inequalities yields

$$\begin{aligned} &|v'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 - |v''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 - \frac{d}{dt}\begin{bmatrix} \theta \\ \chi \end{bmatrix}^T K \begin{bmatrix} \theta \\ \chi \end{bmatrix} \\ &\quad = |d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} - (DD^T)^{-1}(C + DG^T K_{\mathcal{N}})\theta|^2_{DD^T} - |L_{\mathcal{N}}\theta|^2 \\ &\qquad - |d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} + (J^T J)^{-1}(B^T - J^T H K_{\mathcal{P}})\chi|^2_{J^T J} \\ &\qquad + |L_{\mathcal{P}}\chi|^2 \end{aligned}$$

for all $(v'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}, v''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}, z_{\mathcal{N}}, x_{\mathcal{P}})$ satisfying the equations for $\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}$, and with $\left[\begin{smallmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$. Combining this equality with the analogous ones derived in Section V-C for $\mathcal{N}$ and $\mathcal{P}^{\perp\Sigma}$ yields that for $v_{\mathcal{N}} \in \mathcal{N}$, $v_{\mathcal{P}^{\perp\Sigma}} \in \mathcal{P}^{\perp\Sigma}$, and $v_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} \in \mathcal{P} \cap \mathcal{N}^{\perp\Sigma}$, with $x_{\mathcal{N}}, z_{\mathcal{P}}, z_{\mathcal{N}}, x_{\mathcal{P}}$, the variables introduced in the state representations of these behaviors, and $\left[\begin{smallmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$, there holds

$$\begin{aligned} &|v'_{\mathcal{N}} + v'_{\mathcal{P}^{\perp\Sigma}} + v'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 - |v''_{\mathcal{N}} + v''_{\mathcal{P}^{\perp\Sigma}} + v''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 \\ &\qquad - \frac{d}{dt}\left|\begin{bmatrix} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} \theta \\ \chi \end{bmatrix}\right|^2_K \\ &\quad = |v'_{\mathcal{N}} - G^T K_{\mathcal{N}} x_{\mathcal{N}}|^2_{(I - D^T(DD^T)^{-1}D)} + |L_{\mathcal{N}} x_{\mathcal{N}}|^2 \\ &\qquad - |v''_{\mathcal{P}^{\perp\Sigma}} + H K_{\mathcal{P}} z_{\mathcal{P}}|^2_{(I - J(J^T J)^{-1}J^T)} - |L_{\mathcal{P}} z_{\mathcal{P}}|^2 \\ &\qquad + |d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} - (DD^T)^{-1}(C + DG^T K_{\mathcal{N}})\theta|^2_{DD^T} \\ &\qquad - |L_{\mathcal{N}}\theta|^2 - |d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} + (J^T J)^{-1} \\ &\qquad \cdot (B^T - J^T H K_{\mathcal{P}})\chi|^2_{J^T J} + |L_{\mathcal{P}}\chi|^2. \end{aligned} \quad (7)$$

In the case under consideration, we have $\mathcal{N} + \mathcal{P}^{\perp\Sigma} + \mathcal{P}\cap\mathcal{N}^{\perp\Sigma} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{d}+\mathbf{f}})$. The above equation shows how the right hand side can be made nonnegative, thereby achieving a Σ-dissipative sub-behavior $\mathcal{K}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathbf{d}+\mathbf{f}})$. Indeed, we should make sure that

1) $v_{\mathcal{P}^{\perp\Sigma}} = 0$, and $z_{\mathcal{P}} = 0$; this ensures that only $\mathcal{N}$, the Σ-dissipative part of $\mathcal{N} + \mathcal{P}^{\perp\Sigma}$, is incorporated in $\mathcal{K}$;
2) $L_{\mathcal{N}} = 0$; this is achieved by taking $K_{\mathcal{N}} = K_{\mathcal{N}}^+$, yielding $K = K^+$;
3) $d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} = -(J^T J)^{-1}(B^T - J^T H K_{\mathcal{P}})\chi$; this ensures that only a Σ-dissipative part of $\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}$ is incorporated in $\mathcal{K}$.

It goes without saying that there is also a dual construction. Note that also here, as was the case in the proof of the main result in [23], one of the storage functions needs to be an extreme one, while the other is arbitrary. In the case at hand, this corresponds to taking a solution of an algebraic Riccati equation combined with a solution of an algebraic Riccati inequality.

The resulting controlled behavior $\mathcal{K}$ is obtained by combining these relations with the equations for $\mathcal{N}+\mathcal{P}^{\perp_\Sigma}$ and $\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}$. This yields

$$\begin{aligned}
\frac{d}{dt}x_{\mathcal{N}} &= Ax_{\mathcal{N}} + Gv'_{\mathcal{N}},\\
0 &= Cx_{\mathcal{N}} + Dv'_{\mathcal{N}},\\
\frac{d}{dt}K^+\begin{bmatrix}\theta\\ \chi\end{bmatrix} &= \begin{bmatrix}-A^T & -H^TH\\ GG^T & A\end{bmatrix}K^+\begin{bmatrix}\theta\\ \chi\end{bmatrix}\\
&\quad + \begin{bmatrix}C^T & -H^TJ\\ -GD^T & B\end{bmatrix}\begin{bmatrix}d'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\\ d''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\end{bmatrix},\\
d''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}} &= -(J^TJ)^{-1}(B^T - J^THK_{\mathcal{P}})\chi,\\
v_{\mathcal{K}} &= (G^T(K^+_{\mathcal{N}}\theta+\chi) + v'_{\mathcal{N}} - D^Td'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}},\\
&\quad Hx_{\mathcal{N}} + H(\theta - K_{\mathcal{P}}\chi) + Jd''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}})
\end{aligned}$$

as a state-space representation of $\mathcal{K}$. Specializing (7) to $\mathcal{K}$ leads to

$$\begin{aligned}
&|v_{\mathcal{K}}|^2_\Sigma - \frac{d}{dt}\left|\begin{bmatrix}x_{\mathcal{N}}+\theta\\ \chi\end{bmatrix}\right|^2_{K^+}\\
&\quad = |v'_{\mathcal{N}} - G^TK_{\mathcal{N}}x_{\mathcal{N}}|^2_{(I-D^T(DD^T)^{-1}D)}\\
&\qquad + |d'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}} - (DD^T)^{-1}(C+DG^TK_{\mathcal{N}})\theta|^2_{DD^T}\\
&\qquad + |L_{\mathcal{P}}\chi|^2. \qquad (8)
\end{aligned}$$

The fact that $\mathcal{K}$ as defined above satisfies the controller specifications follows from the proof of [23, Th. 5], since the construction of $\mathcal{K}$ given here for the state-space case is exactly the one used in the proof of [23, Th. 5] for the general case. However, for the case at hand, it is instructive to verify the satisfaction of the specifications directly. That $\mathcal{K}$ satisfies $\mathcal{N}\subset\mathcal{K}\subset\mathcal{P}$ follows immediately from the construction of $\mathcal{K}$. That $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ follows from the equality (8) combined with the nonnegative definiteness of K^+. The inequality $\int_{-\infty}^0(|d|^2-|f|^2)\,dt\geq 0$ for all $(d,\,f)\in\mathcal{K}$ of compact support, also yields external stability. Proving that $\mathfrak{m}(\mathcal{K})=\dim(d)$ is, as it is for [23, Th. 5] the difficult part, especially in the case that $K^+\geq 0$ is singular. We will give a proof in the next sections, together with the specification of the controller and a proof of internal stability.

E. Specification of the Controller

It is implied in [23, Th. 1], since $\mathcal{N}\subset\mathcal{K}\subset\mathcal{P}$, that $\mathcal{K}$ can be implemented by a controller acting on the variables $(u,\,y)$. However, the equations for $\mathcal{K}$ that we just derived fail to make this apparent. In these equations, the controlled behavior is given as $\mathcal{N}$ added to a suitable sub-behavior of $\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}$. What we need to do, is rewrite $\mathcal{K}$ as the manifest behavior of the variables $(d,\,f)$ of $\mathcal{P}_{\text{full}}$, interconnected with a control law acting on the variables $(u,\,y)$. Note that $\mathcal{K}$ is given by

$$\begin{aligned}
\frac{d}{dt}x_{\mathcal{N}} &= Ax_{\mathcal{N}} + Gv'_{\mathcal{N}},\\
0 &= Cx_{\mathcal{N}} + Dv'_{\mathcal{N}},\\
\frac{d}{dt}\begin{bmatrix}z_{\mathcal{N}}\\ x_{\mathcal{P}}\end{bmatrix} &= \begin{bmatrix}-A^T & -H^TH\\ GG^T & A\end{bmatrix}\begin{bmatrix}z_{\mathcal{N}}\\ x_{\mathcal{P}}\end{bmatrix}\\
&\quad + \begin{bmatrix}C^T & -H^TJ\\ -GD^T & B\end{bmatrix}\begin{bmatrix}d'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\\ d''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}\end{bmatrix},\\
\begin{bmatrix}z_{\mathcal{N}}\\ x_{\mathcal{P}}\end{bmatrix} &= K\begin{bmatrix}\theta\\ \chi\end{bmatrix}\\
d''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}} &= -(J^TJ)^{-1}(B^T - J^THK_{\mathcal{P}})\chi,\\
v_{\mathcal{K}} &= (G^Tz_{\mathcal{N}} + v'_{\mathcal{N}} - D^Td'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}},\\
&\quad Hx_{\mathcal{N}} + Hx_{\mathcal{P}} + Jd''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}})
\end{aligned}$$

Now introduce new state variables $x = x_{\mathcal{N}}+x_{\mathcal{P}}$, $z = z_{\mathcal{N}} - K^+_{\mathcal{N}}x_{\mathcal{P}}$, and the new variables $u = d''_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}$, $d = G^Tz_{\mathcal{N}} + v'_{\mathcal{N}} - D^Td'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}$, $y = Cx+Dd$. Rewritten in terms of these new variables, the equations for $\mathcal{K}$ become

$$\begin{aligned}
\frac{d}{dt}x &= Ax + Bu + Gd, \quad v_{\mathcal{K}} = (d,\,Hx+Ju),\\
y &= Cx + Dd,\\
\frac{d}{dt}z &= -(A^T + K^+_{\mathcal{N}}GG^T)z - (C^T + K^+_{\mathcal{N}}GD^T)\\
&\quad \cdot (DD^T)^{-1}(y - DG^Tz) - (H^TJ + K^+_{\mathcal{N}}B)u,\\
z &= (I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}})\chi,\\
u &= -(J^TJ)^{-1}(B^T - J^THK_{\mathcal{P}})\chi,\\
\frac{d}{dt}x_{\mathcal{P}} &= GG^Tz + (A + GG^TK^+_{\mathcal{N}})x_{\mathcal{P}}\\
&\quad + Bu + GD^T(DD^T)^{-1}\\
&\quad \cdot ((y - DG^Tz) - (C + DG^TK^+_{\mathcal{N}})x_{\mathcal{P}}),\\
x_{\mathcal{P}} &= \theta - K_{\mathcal{P}}\chi,\\
d'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}} &= -(DD^T)^{-1}(Dd - DG^T(z + K^+_{\mathcal{N}}x_{\mathcal{P}})\\
&\quad + C(x - x_{\mathcal{P}})).
\end{aligned}$$

The last three equations merely serve to define $x_{\mathcal{P}}$, θ, and $d'_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}$ and do not contribute to $v_{\mathcal{K}}$. Hence, the controlled behavior is given by the plant equations

$$\begin{aligned}
\frac{d}{dt}x &= Ax + Bu + Gd,\\
v &= (d,\,Hx+Ju),\\
y &= Cx + Dd
\end{aligned}$$

combined with the control law

$$\begin{aligned}
\frac{d}{dt}(I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}})\chi &= -(A^T + K^+_{\mathcal{N}}GG^T)(I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}})\chi\\
&\quad - (C^T + K^+_{\mathcal{N}}GD^T)(DD^T)^{-1}\\
&\quad \cdot (y - DG^T(I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}})\chi)\\
&\quad - (H^TJ + K^+_{\mathcal{N}}B)u,\\
u &= -(J^TJ)^{-1}(B^T - J^THK_{\mathcal{P}})\chi.
\end{aligned}$$

Note that when $I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}}$ is singular, this is a singular-state system. The manifest behavior of the equations in $(u,\,y,\,\chi)$ can be viewed as a control law restricting $(u,\,y)$ that, when acting on the plant, achieves the control specifications.

These equations, while rather simple, still have two drawbacks. First, when $I_{\text{n}} + K^+_{\mathcal{N}}K_{\mathcal{P}}$ is singular, they fail to make apparent that the transfer function from y to u of the controller exists, or is proper. They do not even make apparent that y is free in the controller. Second, the equations do not display the cherished observer-error-feedback structure commonly found in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-controllers.

F. Error Feedback

In order to overcome these drawbacks, we rewrite the controller equations using the following relation that is readily de-

duced from the algebraic Riccati equation for $K_{\mathcal{N}}^{+}$ and the algebraic Riccati inequality for $K_{\mathcal{P}}$:

$$\begin{aligned}-(A^T+K_{\mathcal{N}}^{+}GG^T)(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})&\\ =(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})(A-GG^TK_{\mathcal{P}}^{-1})&\\ +(K_{\mathcal{P}}^{-1}B-H^TJ)(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)&\\ -(C^T+K_{\mathcal{N}}^{+}GD^T)(DD^T)^{-1}(C+DG^TK_{\mathcal{N}}^{+})&\\ -K_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}.&\end{aligned}\tag{9}$$

Substituting this relation into the controller equations, and defining $\hat{x}=-K_{\mathcal{P}}\chi$, yields

$$\begin{aligned}\frac{d}{dt}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})\hat{x}=&(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})(A\hat{x}+Bu+G\hat{d})\\ &+(C^T+K_{\mathcal{N}}^{+}GD^T)(DD^T)^{-1}(y-\hat{y})\\ &-K_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}\hat{x}\\ \hat{y}=&C\hat{x}+D\hat{d}\\ \hat{d}=&-G^TK_{\mathcal{P}}^{-1}\hat{x}\\ u=&(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)\hat{x}.\end{aligned}$$

We can also write these equations as

$$\begin{aligned}\frac{d}{dt}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})\hat{x}=&(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})(A\hat{x}+Bu\\ &+G\hat{d}+GD^T(DD^T)^{-1}(y-\hat{y}))\\ &+(C^T-K_{\mathcal{P}}^{-1}GD^T)(DD^T)^{-1}\\ &\cdot(y-\hat{y})-K_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}\hat{x},\\ \hat{y}=&C\hat{x}+D\hat{d},\\ \hat{d}=&-G^TK_{\mathcal{P}}^{-1}\hat{x},\\ u=&(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)\hat{x}.\end{aligned}\tag{10}$$

Note that this very attractive expression displays the controller both as an input/output system driven by the sensor outputs that returns the actuator inputs, and as an observer driven by error-feedback, with $\hat{y}$ the estimate of the sensor output, and $\hat{d}$ the estimate of the worst disturbance.

G. Properness of the Transfer Function of the Controller

We now address the issue of the properness of the transfer function of the controller. In order to do this, we decompose the singular state system that specifies the controller in its regular and singular parts. Let $R\in\mathbb{R}^{n\times\dim(\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$ and $N\in\mathbb{R}^{n\times\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$ be matrices whose columns span $\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$ and $\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})=(\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))^{\perp}$, respectively. Define $\hat{x}_1$, $\hat{x}_2$ by $\hat{x}=[R\quad N]\left[\begin{smallmatrix}\hat{x}_1\\ \hat{x}_2\end{smallmatrix}\right]$. Whence if the columns of R and N are orthonormal, $R\hat{x}_1$ and $N\hat{x}_2$ are the orthogonal projections of $\hat{x}$ onto $\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$ and $\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$. Pre-multiplying equation (10) for the controller by N^T yields the following relation between y, $\hat{x}_1$, $\hat{x}_2$

$$\begin{aligned}&N^T(C^T-K_{\mathcal{P}}^{-1}GD^T)(DD^T)^{-1}\\ &\cdot(y-(C-DG^TK_{\mathcal{P}}^{-1})(R\hat{x}_1+N\hat{x}_2))\\ &-N^TK_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}(R\hat{x}_1+N\hat{x}_2)=0.\end{aligned}$$

This equation shows that $\hat{x}_2$ may be solved in terms of y and $\hat{x}_1$. In order to express the solution conveniently, denote

$$\begin{aligned}P=&(C^T-K_{\mathcal{P}}^{-1}GD^T)(DD^T)^{-1}(C-DG^TK_{\mathcal{P}}^{-1})\\ Q=&K_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}.\end{aligned}$$

Define $L_{\mathrm{y}}\in\mathbb{R}^{\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))\times\mathrm{y}}$, $F_{\mathrm{y}}\in\mathbb{R}^{n\times\mathrm{y}}$, $L_1\in\mathbb{R}^{n\times\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$, and $F_1\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ by

$$N^T(P+Q)NL_{\mathrm{y}}=N^T(C^T-K_{\mathcal{P}}^{-1}GD^T)(DD^T)^{-1}\tag{11}$$

$$F_{\mathrm{y}}=NL_{\mathrm{y}}\tag{12}$$

$$N^T(P+Q)NL_1=-N^TQ,\quad F_1=NL_1.\tag{13}$$

Since $\mathrm{im}(N^T(P+Q)N)$ contains both $\mathrm{im}(N^T(C^T-K_{\mathcal{P}}^{-1}GD^T)(DD^T)^{-1}$ and $\mathrm{im}(N^TQ)$, these equations may be solved for L_{y}, F_{y}, L_1. Hence, $N\hat{x}_2$ is given in terms of y and $R\hat{x}_1$ by

$$N\hat{x}_2=F_{\mathrm{y}}(y-(C-DG^TK_{\mathcal{P}}^{-1})R\hat{x}_1)+F_1R\hat{x}_1+Na$$

with a some signal taking values in $\ker(N^T(P+Q)N)$.

We now prove, using (9), that the variable a does not appear in the equation for the controller, by showing that it is annihilated both by the controller gain and by the differential equation that governs the evolution of $\hat{x}_1$. To show that $(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)Na=0$, pre- and postmultiply (9) by N^T and N. Subsequently, use this equation again to show that $(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})(A-GG^TK_{\mathcal{P}}^{-1})Na=0$.

It follows that $\mathcal{K}$ is also represented by (1) combined with the controller

$$\begin{aligned}&\frac{d}{dt}R^T(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})R\hat{x}_1\\ &\quad=R^T(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})(A(R\hat{x}_1+N\hat{x}_2)+Bu+G\hat{d})\\ &\qquad+R^T(C^T+K_{\mathcal{N}}^{+}GD^T)(DD^T)^{-1}(y-\hat{y})\\ &\qquad-R^TK_{\mathcal{P}}^{-1}L_{\mathcal{P}}^TL_{\mathcal{P}}K_{\mathcal{P}}^{-1}(R\hat{x}_1+N\hat{x}_2),\\ &N\hat{x}_2=F_{\mathrm{y}}(y-(C-DG^TK_{\mathcal{P}}^{-1})R\hat{x}_1)+F_1R\hat{x}_1\\ &\quad\hat{y}=C(R\hat{x}_1+N\hat{x}_2)+D\hat{d},\\ &\quad\hat{d}=-G^TK_{\mathcal{P}}^{-1}(R\hat{x}_1+N\hat{x}_2),\\ &\quad u=(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)(R\hat{x}_1+N\hat{x}_2)\end{aligned}\tag{14}$$

with $R\in\mathbb{R}^{n\times\dim(\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$ and $N\in\mathbb{R}^{n\times\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$ matrices whose columns span $\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$ and $\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})=(\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))^{\perp}$ respectively, and $F_{\mathrm{y}}\in\mathbb{R}^{n\times\mathrm{y}}$ and $F_1\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ as defined above.

These equations show that the transfer function of the controller is indeed proper. The feed-through term is given by $(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1}-J^TH)NF_{\mathrm{y}}$, while the strictly proper part is given by the differential equation part of the above expression. This differential equation is a regular one, since

$$R^T(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})R\in\mathbb{R}^{\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))\times\dim(\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1}))}$$

is a nonsingular matrix.

The above expression for the controller also makes it apparent that d is free in $\mathcal{K}$.

H. Internal Stability

In this section, we show that the controlled system obtained by interconnecting the plant (1) with the controller (14) is internally stable, i.e., we prove that any $(x, \hat{x}_1)$ satisfying both (1) and (14), satisfies $(x(t), \hat{x}_1(t)) \to 0$ as $t \to \infty$, when $d = 0$. For any $\hat{x}_1$ and y satisfying (14), $\hat{x} = R\hat{x}_1 + N\hat{x}_2$, with $N\hat{x}_2 = F_y(y - (C - DG^T K_{\mathcal{P}}^{-1})R\hat{x}_1) + F_1 R\hat{x}_1$, and y are related by (10). Obviously, $R^T(K_{\mathcal{N}} + K_{\mathcal{P}}^{-1})R\hat{x} = R^T(K_{\mathcal{N}}^{+} + K_{\mathcal{P}}^{-1})R\hat{x}_1$, so, to prove that $(x(t), \hat{x}_1(t)) \to 0$ as $t \to \infty$, it suffices to prove that $(x(t), \hat{x}(t)) \to 0$ as $t \to \infty$.

Note that the controlled system obtained by interconnecting (1) and (10) (with $d = 0$) is a singular state system with state $(x, \hat{x})$. The following Lyapunov function argument that applies to singular systems (see [22, Th. 4.3]), is the basis of the proof of internal stability.

Lemma 8: Consider the singular state system $(d/dt)Ez = Fz$, where $E, F \in \mathbb{R}^{n\times n}$. Let $P = P^T \in \mathbb{R}^{n\times n}$ satisfy $P \geq 0$, and define the Lyapunov function $V(z) = |z|_P^2$. Assume that $Q = Q^T \in \mathbb{R}^{n\times n}$, $Q \geq 0$, is such that for all z satisfying $(d/dt)Ez = Fz$, we have

i) $(d/dt)V(z) = -|z|_Q^2$;
ii) $(Qz = 0) \Rightarrow (z = 0)$.

Then the system is asymptotically stable, i.e., all solutions z satisfy $z(t) \to 0$ as $t \to \infty$.

We now apply this lemma to the controlled system. Consider the Lyapunov function

$$V(x, \hat{x}) := -|x|_{K_{\mathcal{P}}^{-1}}^2 + |x - \hat{x}|_{K_{\mathcal{N}}^{+} + K_{\mathcal{P}}^{-1}}^2.$$

Clearly $V(x, \hat{x}) \geq 0$ for all $(x, \hat{x})$. A straightforward computation shows that for all $(x, \hat{x})$ satisfying the singular state equations (1), (10) we have

$$\begin{aligned}\frac{d}{dt}V(x,\hat{x}) &= -|Hx + J(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1} - J^TH)\hat{x}|^2 \\ &\quad - |G^TK_{\mathcal{P}}^{-1}\hat{x} - (G^TK_{\mathcal{N}} - D^T(DD^T)^{-1}(C + DG^TK_{\mathcal{N}})) \\ &\qquad \cdot (x - \hat{x})|_{(DD^T)^{-1}}^2.\end{aligned}$$

Hence, along solutions $(x, \hat{x})$ of the controlled system, the derivative of $V(x, \hat{x})$ is a negative–semidefinite quadratic form. This yields condition i) of Lemma 8.

Now, turn to condition ii). Clearly $(d/dt)V(x, \hat{x}) = 0$, along solutions of the system described by the combined equations (1), (10) if and only if $(x, \hat{x})$ satisfies the following two additional equations:

$$\begin{aligned}&Hx + J(J^TJ)^{-1}(B^TK_{\mathcal{P}}^{-1} - J^TH)\hat{x} = 0 \\ &G^TK_{\mathcal{P}}^{-1}\hat{x} - (G^TK_{\mathcal{N}} - D^T(DD^T)^{-1}(C + DG^TK_{\mathcal{N}})) \\ &\quad \cdot (x - \hat{x}) = 0.\end{aligned}$$

Premultiplying the first equation by J^T yields $J^THx = -(B^TK_{\mathcal{P}}^{-1} - J^TH)\hat{x}$, so, using (10), $u = -(J^TJ)^{-1}J^THx$. Using (1), and $d = 0$, yields

$$\frac{d}{dt}x = (A - B(J^TJ)^{-1}J^TH)x,\ (I - J(J^TJ)^{-1}J^T)Hx = 0.$$

By regularity condition A.3), i.e., observability of the pair $(A - B(J^TJ)^{-1}J^TH, (I - J(J^TJ)^{-1}J^T)H)$, this implies $x = 0$. The second equation yields

$$G^T(K_{\mathcal{N}} + K_{\mathcal{P}}^{-1})\hat{x} - D^T(DD^T)^{-1}(C + DG^TK_{\mathcal{N}})\hat{x} = 0.$$

After multiplication with D, it yields $(DG^TK_{\mathcal{P}}^{-1} - C)\hat{x} = 0$. Now define $x_c = (K_{\mathcal{N}} + K_{\mathcal{P}}^{-1})\hat{x}$. Combining the previous two equations, we obtain $(I - D^T(DD^T)^{-1}D)G^Tx_c = 0$. By combining the controller equations (10) with (9), and using these relations derived from $(d/dt)V(x, \hat{x}) = 0$, we obtain, after some calculations, that x_c satisfies the differential equation

$$\begin{aligned}\frac{d}{dt}x_c = (&-A^T - C^T(DD^T)^{-1}DG^T \\ &- K_{\mathcal{N}}G(I - D^T(DD^T)^{-1}D)G^T)x_c.\end{aligned}$$

Consequently, x_c satisfies

$$\begin{aligned}\frac{d}{dt}x_c &= (-A^T - C^T(DD^T)^{-1}DG^T)x_c \\ 0 &= (I - D^T(DD^T)^{-1}D)G^Tx_c.\end{aligned}$$

By the regularity condition (A.2), which is equivalent to the observability of the pair $(A^T - C^T(DD^T)^{-1}DG^T, (I - D^T(DD^T)^{-1}D)G^T)$, we hence obtain $x_c = 0$.

We conclude that if $(x, \hat{x})$ satisfy both the plant and controller equations, and if $(d/dt)V(x, \hat{x}) = 0$, then $x = 0$, and $x_c = 0$. In order to complete the proof, we show that $x = 0$, $x_c = 0$ implies $x = 0$, $\hat{x} = 0$. Recall that $R^T(K_{\mathcal{N}} + K_{\mathcal{P}}^{-1})R\hat{x}_1 = R^Tx_c$. This shows that $x_c = 0$ implies $\hat{x}_1 = 0$. Also, for $d = 0$, $N\hat{x}_2 = F_y(Cx - (C - DG^TK_{\mathcal{P}}^{-1})R\hat{x}_1) + F_1R\hat{x}_1$. Thus $x = 0$, $\hat{x}_1 = 0$ implies $N\hat{x}_2 = 0$ and hence $\hat{x} = 0$. Now use lemma 8 to conclude that the controlled system is indeed internally stable.

I. Statement of the Results for State Representations

We collect our results on the state-space $\mathcal{H}_\infty$-control problem in the following theorem.

Theorem 9: Consider the plant (1). Assume that the regularity conditions A.1), A.2), and A.3) are satisfied. Then, the following statements are equivalent.

i) There exists a feedback controller (2) such that the controlled system is internally stable, and the closed loop transfer function $G_{d\mapsto f}$ satisfies $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.
ii) There exist real symmetric solutions $K_{\mathcal{N}}$ and $K_{\mathcal{P}}$ of the algebraic Riccati inequalities (3) and (4), satisfying the conditions $K_{\mathcal{N}} > 0$, $K_{\mathcal{P}} < 0$, and $K_{\mathcal{N}} \geq (-K_{\mathcal{P}})^{-1}$.
iii) There exist real symmetric solutions of the algebraic Riccati equations (5) and (6), and the supremal real symmetric solution $K_{\mathcal{N}}^{+}$ of (5) and the infimal real symmetric solution $K_{\mathcal{P}}^{-}$ of (6) satisfy $K_{\mathcal{N}}^{+} > 0$, $K_{\mathcal{P}}^{-} < 0$, and $K_{\mathcal{N}}^{+} \geq (-K_{\mathcal{P}}^{-})^{-1}$.

Assume that any of these conditions hold. Let $K_{\mathcal{N}}^{+}$ be the largest real symmetric solution of the algebraic Riccati equation (5), and let $K_{\mathcal{P}}$ be any real symmetric solution of the algebraic Riccati inequality (4). Then a suitable feedback controller that satisfies i) is given by the singular state space representation (10).

Alternatively, a suitable feedback controller is given by the regular state space representation (14). In these equations, R is an injective matrix whose columns spans $\mathrm{im}(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$, N is an injective matrix whose columns spans $\ker(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})$, and $F_{\mathrm{y}} \in \mathbb{R}^{n\times \mathrm{y}}$ and $F_{1} \in \mathbb{R}^{\mathrm{n}\times \mathrm{n}}$ are given by (11)–(13). The dimension of the state space of this controller is equal to $\mathrm{rank}(K_{\mathcal{N}}^{+}+(K_{\mathcal{P}})^{-1})$.

J. Remarks

1) We first recapitulate the procedure followed in obtaining the controller. Application of [23] leads to a representation of a controlled behavior that meets the design specifications. By suitably rewriting the expression of the controlled behavior, it achieves the structure of the plant interconnected with a controller that has the sensor output y as its input, and the actuator input u as its output. This controller is a singular state space system, and it is not evident that it has a proper transfer function. However, by rewriting the controller, it may be de-singularized by introducing a feed-through term in order to obtain an expression that puts the properness of the controller transfer function into evidence. In the case that $(I_{\mathrm{n}}+K_{\mathcal{N}}^{+}K_{\mathcal{P}}^{-})$ is nonsingular, this transfer is, in fact, strictly proper. The controller obtained also has the structure of an observer driven by error feedback.
2) It is worth emphasizing that here we solve the $\mathcal{H}_{\infty}$-problem with nonstrict inequality specifications: $\|G_{d\mapsto f}\|_{\mathcal{H}_{\infty}} \leq 1$. It is this nonstrict inequality that leads to the feed-through term in the controller.
3) We now demonstrate that our conditions for the existence of a controlled behavior $\mathcal{K}$ that meets the specifications, as given in Theorem 9, and the construction of the controller, as given in the proof, specialize in the state space case to conditions involving a double algebraic Riccati equation as in [1]. By pre- and postmultiplying (5) and (6) by $K_{\mathcal{N}}^{-1}$ and $K_{\mathcal{P}}^{-1}$, we obtain the following necessary and sufficient conditions for the existence of a suitable $\mathcal{H}_{\infty}$-controller: there exist real symmetric solutions P and Q of the "mixed sign" algebraic Riccati equations

$$\begin{aligned} &A^{T}P+PA+PGG^{T}P-(PB+H^{T}J)(J^{T}J)^{-1}\\ &\quad\cdot(B^{T}P+J^{T}H)+H^{T}H=0,\\ &AQ+QA^{T}+QH^{T}HQ-(QC^{T}+GD^{T})(DD^{T})^{-1}\\ &\quad\cdot(CQ+DG^{T})+GG^{T}=0. \end{aligned}$$

satisfying $P>0$, $Q>0$, and $Q^{-1}\geq P$.

Since we only require the closed loop transfer function to satisfy the nonstrict inequality $\|G_{d\mapsto f}\|_{\mathcal{H}_{\infty}}\leq 1$, our conditions are somewhat weaker than those obtained in [1], where the strict inequality $\|G_{d\mapsto f}\|_{\mathcal{H}_{\infty}}<1$ is required. Concretely, the strict inequality $Q^{-1}>P$ of [1] is weakened to $Q^{-1}\geq P$, and the condition of [1] that the matrices $A-B(J^{T}J)^{-1}(B^{T}P+J^{T}H)+GG^{T}P$ and $A-(QC^{T}+GD^{T})(DD^{T})^{-1}C+QH^{T}H$ should have their eigenvalues in the open left half plane does not show up in our result.
4) In our formulas for the controller, we need the supremal real symmetric solution of the algebraic Riccati equation (5), and *any* real symmetric solution of the algebraic Riccati inequality (4). This may be "dualized" into a controller that uses any real symmetric solution of the algebraic Riccati inequality (3), and the infimal solution of the algebraic Riccati equation (6). Note also that the formulas for the controller (10) simplify somewhat if we use both the supremal real symmetric solution of (5) and the infimal real symmetric solution of (6). In that case the last term of the right-hand side of the formula for $(d/dt)(K_{\mathcal{N}}^{+}+K_{\mathcal{P}}^{-1})\hat{x}$ in (10) is absent. The combined use here of both the algebraic Riccati equation and the algebraic Riccati inequality is one of the two main differences with our earlier paper [20]. The other difference is that in [20], we treat only the so-called "standard" state-space system structure.

VI. Implementability by Feedback Controllers

In [23, Th. 5], we obtain the controlled behavior directly. By [23, Th. 1], we are guaranteed that the controlled behavior is implementable by a controller that acts on the control variables. However, in the classical view of control, a controller is always regarded as a feedback signal processor that accepts the sensor outputs of the plant as its own inputs, and that produces the actuator inputs to the plant as its own outputs. It is important to be able to conclude *a priori* when a controlled behavior is implementable by a feedback processor. Some, less definitive, results on this problem can be found in [21]. In this section, we show how, by imposing certain restrictions on the input/output transfer functions of the plant and of the controlled behavior, we can obtain such feedback implementability results.

We refer to [23, Sec. IV.1 and VI.1] for a review of what is meant in a behavioral context by inputs, outputs, input/output representations, and the associated transfer function. Assume that the plant variables consist of disturbance and actuator inputs, and of to-be-controlled and sensor outputs (see Fig. 5). More precisely, assume that the to-be-controlled plant variables are partitioned as $v=(d,\,f)$, with d (exogenous) disturbance inputs, and f (endogenous) to-be-controlled outputs, and that the control variables are partitioned as $c=(u,\,y)$, with u the actuator inputs, and y the sensor outputs. Let the full plant behavior $\mathcal{P}_{\mathrm{full}}$ be represented by

$$P\left(\frac{d}{dt}\right)\begin{bmatrix} f\\ y\end{bmatrix}=Q\left(\frac{d}{dt}\right)\begin{bmatrix} d\\ u\end{bmatrix}$$

and assume that P is square with $\det(P)\neq 0$. This assumption on P implies that the variables $(d,\,u)$ are indeed inputs, and that the variables $(f,\,y)$ are outputs. The matrix of rational functions $G_{(d,\,u)\mapsto(f,\,y)}=P^{-1}Q$ is the transfer function associated with $\mathcal{P}_{\mathrm{full}}$. In the obvious partition suggested by the notation, we have

$$G_{(d,\,u)\mapsto(f,\,y)}=\begin{bmatrix} G_{d\mapsto f} & G_{u\mapsto f}\\ G_{d\mapsto y} & G_{u\mapsto y}\end{bmatrix}.$$

In both theorems of this section, $G_{(d,\,u)\mapsto(f,\,y)}$ is a matrix of proper rational functions. Define $G^{\infty}_{(d,\,u)\mapsto(f,\,y)}=\lim_{s\in\mathbb{C},\,s\to\infty}G_{(d,\,u)\mapsto(f,\,y)}(s)$. The matrix $G^{\infty}_{(d,\,u)\mapsto(f,\,y)}\in$

$\mathbb{R}^{(\mathtt{f}+\mathtt{y})\times(\mathtt{d}+\mathtt{u})}$ is called the *feed-through term* of the plant. In the obvious partition again, we have

$$G^{\infty}_{(d,\,u)\mapsto(f,\,y)} = \begin{bmatrix} G^{\infty}_{d\mapsto f} & G^{\infty}_{u\mapsto f} \\ G^{\infty}_{d\mapsto y} & G^{\infty}_{u\mapsto y} \end{bmatrix}.$$

The theorem that follows shows that if the transfer functions of the plant and the controlled behavior are both proper, and if certain surjectivity and injectivity conditions on the feed-through term of the plant are met, then the following holds. For any implementable behavior, and any controller that implements that behavior we have: the controller is a feedback controller with a proper transfer function if and only if the implementable behavior itself already has a proper transfer function.

Theorem 10: Assume that the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{u}+\mathtt{y}}$ has the input/output structure described in the pre-amble, with input $(d,\,u)$ and output $(f,\,y)$. Assume also that the transfer function $G_{(d,\,u)\mapsto(f,\,y)}$ associated with $\mathcal{P}_{\text{full}}$ has the following properties:

i) $G_{(d,\,u)\mapsto(f,\,y)}$ is proper;
ii) $G^{\infty}_{u\mapsto f}$ is injective;
iii) $G^{\infty}_{d\mapsto y}$ is surjective;
iv) $G^{\infty}_{u\mapsto y} = 0$.

Let $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ be the hidden behavior, and $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ be the plant behavior associated with $\mathcal{P}_{\text{full}}$. Assume that the behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ satisfies $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, i.e., $\mathcal{K}$ is an implementable behavior. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ be a controller that implements $\mathcal{K}$. Then, the following statements are equivalent:

1) in $\mathcal{K}$, d is input and f is output, and the transfer function $K_{d\mapsto f}$ from d to f in $\mathcal{K}$ is proper.
2) in $\mathcal{C}$, y is input and u is output, and the transfer function $C_{y\mapsto u}$ from y to u in $\mathcal{C}$ is proper.

The above theorem gives natural conditions under which implementability by means of feedback control is a straightforward consequence of certain properness assumptions of the transfer functions of the plant and of the controlled system. These assumptions are often implicit in many controller design questions. It shows that the behavioral idea of control, while being formulated as a general interconnection, captures feedback by simply imposing natural conditions on the plant and the to-be-controlled system.

A. Feedback Implementability and Dissipativity

In the case of the synthesis of dissipative systems, the problem discussed in [23], we readily obtain the following useful corollary of Theorem 10.

Corollary 11: Consider the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{u}+\mathtt{y}}$. Assume it has the input/output structure described in the pre-amble of theorem 10, with input variables $(d,\,u)$ and output variables $(f,\,y)$, and assume that conditions i) to iv) of Theorem 10 are satisfied. Let $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ and $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ be the hidden behavior and the plant behavior, respectively, associated with the plant $\mathcal{P}_{\text{full}}$ and $\Sigma = \operatorname{diag}(I_{\mathtt{d}},\,-I_{\mathtt{f}})$. If there exists $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$ such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathcal{K}) = \mathtt{d}$, then every controller $\mathcal{C}$ that implements K has the property that in $\mathcal{C}$, y is input and u is output, and the transfer function $C_{y\mapsto u}$ from y to u in $\mathcal{C}$ is proper.

We remark, without going into details, that under the conditions of Theorem 10 and Corollary 11, in both cases the transfer function of the controller is uniquely specified by the controlled behavior, equivalently, there is a *unique* controllable controller that implements $\mathcal{K}$. This follows from the fact that under the assumptions of the theorem and the corollary, for a given $K_{d\mapsto f}$ the equation (20) in the proof of theorem 10 is uniquely solvable for $C_{y\mapsto u}$.

Our second implementability result gives conditions on the feed-through term of the plant such that if there exists *any* controller that achieves a certain $\mathcal{H}_\infty$-gain, then there exists a *strictly proper* one that achieves an $\mathcal{H}_\infty$-gain that is arbitrarily close to the original one. Basically, the result is a behavioral restatement of a result that dates back to the earliest work on $\mathcal{H}_\infty$ control, see, e.g., [24], [2]. We include it here for the sake of completeness, but we omit the proof.

Theorem 12: Assume that the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}+\mathtt{u}+\mathtt{y}}$ has the input/output structure described in the pre-amble of theorem 10, with input variables $(d,\,u)$ and output variables $(f,\,y)$. Assume also that the transfer function $G_{(d,\,u)\mapsto(f,\,y)}$ associated with $\mathcal{P}_{\text{full}}$ has the following properties:

i) $G_{(d,\,u)\mapsto(f,\,y)}$ is proper;
ii) $G^{\infty}_{d\mapsto f} = 0$;
iii)′ either $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} < \infty$, or
iii)″ the behavior $\{(u,\,f,\,y) \in \mathfrak{C}^{\infty}(\mathbb{R},\,\mathbb{R}^{\mathtt{u}+\mathtt{f}+\mathtt{y}}) \mid (u,\,0,\,f,\,y) \in \mathcal{P}_{\text{full}}\} \in \mathfrak{L}^{\mathtt{f}+\mathtt{u}+\mathtt{y}}$ is controllable, and f is observable from $(d,\,u,\,y)$ in $\mathcal{P}_{\text{full}}$.

Let $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ be the hidden behavior, and $\mathcal{P} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ the plant behavior associated with $\mathcal{P}_{\text{full}}$. Assume that there exists a behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ such that

iv) $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, i.e., $\mathcal{K}$ is an implementable controlled behavior;
v) in $\mathcal{K}$, d is input and f is output;
vi) the transfer function $K_{d\mapsto f}$ from d to f in $\mathcal{K}$ is proper and has finite $\mathcal{H}_\infty$-norm.

Then for all $\gamma > \|K_{d\mapsto f}\|_{\mathcal{H}_\infty}$, there is a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ such that

1) in $\mathcal{C}$, y is input, u is output, and the transfer function $C_{y\mapsto u}$ from y to u in $\mathcal{C}$ is strictly proper;
2) in $\mathcal{K}'$, the controlled behavior implemented by $\mathcal{C}$, d is input and f is output;
3) the transfer function $K'_{d\mapsto f}$ from d to f in $\mathcal{K}'$ has $\mathcal{H}_\infty$-norm less than γ: $\|K'_{d\mapsto f}\|_{\mathcal{H}_\infty} < \gamma$.

The combination of Theorems 10 and 12 with [23, Th. 5] give methods for constructing a feedback controller with input y, output u, and (strictly) proper transfer function, such that the closed loop transfer function from d to f has $\mathcal{H}_\infty$-norm less than or equal to 1 (or, at least, less than $1+\varepsilon$ with $\varepsilon > 0$ sufficiently small). First verify that the appropriate conditions of [23, Th. 5] are met in the case of disturbance attenuation, i.e., with respect to the quadratic differential form $\|d\|^2 - \|f\|^2$. If these conditions are indeed satisfied, then there exists a controlled behavior that achieves $\mathcal{H}_\infty$-norm less than or equal to 1. This controlled behavior $\mathcal{K}$ may not be implementable by a controller that has the desired input/output structure of a feedback controller. However, as shown in Corollary 11, if on the one hand conditions i) to

iv) of Theorem 10 are satisfied, then theorem 10 guarantees that any controller that implements $\mathcal{K}$ is in fact a feedback controller with y as input and u as output, and proper transfer function. If, on the other hand, conditions i) to iii)$''$ of Theorem 12 are satisfied, then, for any $\varepsilon > 0$, there exists a feedback controller with y as input and u as output, and strictly proper transfer function that achieves $\mathcal{H}_\infty$-norm less than $1+\varepsilon$. An alternative to Theorem 12 is to *regularize* the system first, such that conditions i) to iv) of Theorem 10 are satisfied. This can be achieved, for example, by replacing $f \mapsto (f, \varepsilon' u)$, $d \mapsto (d, d')$, $y \mapsto y + \varepsilon' d'$, with ε' sufficiently small. Next, one can compute, using theorem 10, a feedback controller with proper transfer function that achieves $\mathcal{H}_\infty$-norm less than $1+\varepsilon$.

B. Implementability by Filters With a Proper Transfer Matrix

In Section IV we explained, in the context of the filtering problem, that if for the full plant behavior $\mathcal{P}_{\text{full}}$ there exists a filter that implements a given estimation error behavior $\mathcal{E}$, then there also exists a filter which implements $\mathcal{E}$, and has y as input and $\hat{f}$ as output. Hence, such filter acts as a signal processor from y to $\hat{f}$. We noted that there is no *a priori* reason for the transfer function from y to $\hat{f}$ of this signal processor to be proper. Now, we give conditions on $\mathcal{P}_{\text{full}}$ which guarantee that this transfer matrix is proper.

Assume that in the full plant $\mathcal{P}_{\text{full}}$, d is input and y and f are output. Denote the transfer functions from d to (f, y) in $\mathcal{P}_{\text{full}}$ by $G_{d\mapsto f}$ and $G_{d\mapsto y}$, respectively.

Theorem 13: Assume that in the plant d is input and y and f output, with the transfer functions $G_{d\mapsto f}$ and $G_{d\mapsto y}$ proper. Assume further that $G^{\infty}_{d\mapsto y} := \lim_{s\to\infty} G_{d\mapsto y}(s)$, the feedthrough term of $G_{d\mapsto y}$, is surjective. Then the estimation error behavior $\mathcal{E}$ of Theorem 6 (assuming it exists) can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}_{\text{cont}}$ such that in $\mathcal{F}$, y is input and $\hat{f}$ output, and $G_{y\mapsto\hat{f}}$, the transfer function of the $\mathcal{H}_\infty$-filter, is also proper.

VII. Proofs

Proof of Proposition 1: i) $\Rightarrow$ iii): We first prove that i) implies that the variables d must be free in $\mathcal{K}$. Since $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma) = \dim(d)$, there are $\dim(d)$ free variables in $\mathcal{K}$. Therefore, if d is not free, the behavior $\mathfrak{F} = \{f \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{f}}) | (0, f) \in \mathcal{K}\}$ contains free variables, whence $\mathfrak{F} \cap \mathfrak{D}$ contains nonzero elements. However, since $\mathcal{K}$ is Σ-dissipative, $f \in \mathfrak{F} \cap \mathfrak{D}$ implies $\|f\|^2_{\mathcal{L}_2} \leq 0$, a contradiction. Note that, as a consequence, in $\mathcal{K}$ d is input and f is output.

Let $P(d/dt)f = Q(d/dt)d$ be a full row rank kernel representation of $\mathcal{K}$. Then, since d is input and f is output, P is square and $\det(P) \neq 0$,and $G_{d\mapsto f}$, the transfer function from d to f, equals $P^{-1}Q$. We now prove that P is Hurwitz. Let $\hat{f}$ be any element of the autonomous behavior $P(d/dt)f = 0$. Then, since $\mathcal{K}$ is controllable, there exists $(d, f) \in \mathcal{K}$ and $t' < 0$ such that $(d, f)(t) = 0$ for $t \leq t'$, and $(d, f)(t) = (0, \hat{f}(t))$ for $t \geq 0$. Now, consider for this (d, f), the integral $\int_{-\infty}^{T} (|d|^2 - |f|^2)\, dt$. Since $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, this integral is nonnegative for all $T \in \mathbb{R}$. This implies that $\int_0^{+\infty} |\hat{f}|^2\, dt < \infty$. Consequently, all solutions of $P(d/dt)f = 0$ satisfy $\int_0^{+\infty} |f|^2\, dt < \infty$. Therefore, P is Hurwitz.

We will now prove that $\|G_{d\mapsto f}\|_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} |G_{d\mapsto f}(i\omega)| \leq 1$. For this, let

$$\begin{bmatrix} d \\ f \end{bmatrix} = \begin{bmatrix} D\left(\frac{d}{dt}\right) \\ N\left(\frac{d}{dt}\right) \end{bmatrix} \ell$$

be an observable image representation of $\mathcal{K}$. Then we have $G_{d\mapsto f} = ND^{-1}$. Also, for all latent variable trajectories $\ell \in \mathfrak{D}$ we have $\int_{-\infty}^{+\infty} |N(d/dt)\ell|^2\, dt - |D(d/dt)\ell|^2\, dt \leq 0$. Taking Fourier transforms in the integrand, it follows from Parseval's theorem that $N^T(-i\omega)N(i\omega) - D^T(-i\omega)D(i\omega) \leq 0$ for all ω, whence $G^T_{d\mapsto f}(-i\omega)G_{d\mapsto f}(i\omega) \leq I$ for all ω. Together with the fact that P is Hurwitz, this yields $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

iii) $\Rightarrow$ i): In $\mathcal{K}$, d is input and f is output, hence $\mathtt{m}(\mathcal{K}) = \dim(d) = \sigma_+(\Sigma)$. To prove that $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$ and $(d, f) \in \mathcal{K} \cap \mathfrak{D}$ imply $\int_{-\infty}^0 |d|^2\, dt \geq \int_{-\infty}^0 |f|^2\, dt$, write $\mathcal{K}$ in minimal i/o representation as $(d/dt)x = Ax + Bd$, $f = Cx + Dd$. Now use [22, Th. 6.4] to prove that there exists $K = K^T > 0$ such that $(d/dt)x^T Kx \leq |d|^2 - |f|^2$. Integrating yields $\int_{-\infty}^0 |d|^2\, dt \geq \int_{-\infty}^0 |f|^2\, dt$.

iii) $\Rightarrow$ ii): Let $P(d/dt)f = Q(d/dt)d$ be a full row rank kernel representation of $\mathcal{K}$. Then $P^{-1}Q = G_{d\mapsto f} \in \mathcal{H}_\infty$. Since $\mathcal{K}$ is controllable, this implies that P is Hurwitz. From this it follows that $(0, f) \in \mathcal{K}$ implies that $f(t)$ goes to zero as $t \to \infty$. The inequality $\|f\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})} \leq \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})}$ for $(d, f) \in \mathcal{K} \cap \mathcal{L}_2$ follows from $\sup_{\omega\in\mathbb{R}} |G_{d\mapsto f} + (i\omega)| \leq 1$ and Parseval's theorem.

ii) $\Rightarrow$ iii): From the fact that d free and that $(0, f) \in \mathcal{K}$ implies that $f(t)$ goes to zero as $t \to \infty$, it follows that d is input, and f output. Let $P(d/dt)f = Q(d/dt)d$ be a full-row rank kernel representation for $\mathcal{K}$. Hence, P is square, $\det(P) \neq 0$, and P is Hurwitz. Finally, $\|f\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{f}})} \leq \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})}$ for all $(d, f) \in \mathcal{K} \cap \mathcal{L}_2$ and Parseval's theorem again yield $\sup_{\omega\in\mathbb{R}} |G_{d\mapsto f}(i\omega)| \leq 1$. Combined with P Hurwitz, this yields $\|G_{d\mapsto f}\|_{\mathcal{H}_\infty} \leq 1$.

Proof of Proposition 2: We first prove that i) implies the existence of an input/output representation of the type claimed in ii) and iii). Obviously, i) implies that $\mathcal{K}$ is dissipative on $\mathbb{R}_-$ with respect to $Q_\Sigma(v) = |e+f|^2 - |e-f|^2$, and that $\mathtt{m}(\mathcal{K}) = \mathtt{e} = \mathtt{f}$. It hence follows from proposition 1 that $\mathcal{K}$ admits a minimal input/state/output representation of the form $(d/dt)x = Ax + B(e+f)$, $e - f = Cx + D(e+f)$, with $D^T D \leq 1$. We need the following matrix lemma, stated here without proof.

Lemma 14: Let $M \in \mathbb{R}^{n\times 2n}$ and assume that $\dim(\ker(M)) = n$. Let $P(d/dt)f = Q(d/dt)d$ $J = \left[\begin{smallmatrix} 0_{n\times n} & I_n \\ I_n & 0_{n\times n} \end{smallmatrix}\right]$. Assume that $x \in \ker(M)$ implies $x^T Jx \geq 0$. Let M_i denote the ith column of M. Then there is a selection of n linearly independent columns $\{c_1, c_2, \ldots, c_n\}$ of M such that for all $1 \leq i \leq n$, either M_i, or M_{i+n}, but not both, belongs to $\{c_1, c_2, \ldots, c_n\}$.

Now apply this lemma to the matrix $[I - D \quad I + D]$. The equation $(I-D)e = Cx + (I+D)f$ may hence be rewritten as $y = C'x + D'u$, with for all $1 \leq i \leq \mathtt{e} = \mathtt{f}$, either e_i, or f_i, but not both, components of u, and the others the components of y.

It follows that without loss of generality, we may hence assume that $e = u$ and $y = f$, for otherwise, reverse the roles of e_i and f_i appropriately. We will make this assumption in the

remainder of the proof. The state equations for $\mathcal{K}$ may then be written as $(d/dt)x = A'x + B'e$, $f = C'x + D'e$.

i) $\Rightarrow$ iii): The existence of the required input/output partition has already been proven. Let $R(d/dt)v = 0$, i.e., $R_e(d/dt)e = R_f(d/dt)f$ be a full row rank kernel representation of $\mathcal{K}$. Assume, as in the pre-amble, that $e = u$ and $y = f$. Using the input/state/output representation of $\mathcal{K}$ derived in the pre-amble shows that R_f is square and $\det(R_f) \neq 0$ (and also that $R_f^{-1}R_e$ is proper, but we do not need this). To prove positive realness, note that the transfer functions $G_{e\mapsto f}$ and $G_{(e+f)\mapsto(e-f)}$ are related by $G_{(e+f)\mapsto(e-f)} = (I - G_{e\mapsto f})(I + G_{e\mapsto f})^{-1}$. From here it follows that $G_{e\mapsto f}$ is positive real if and only if $\|G_{(e+f)\mapsto(e-f)}\|_{\mathcal{H}_\infty} \leq 1$. The latter is a consequence of dissipativity of $\mathcal{K}$ on $\mathbb{R}_-$ with respect to $Q_\Sigma(v) = |e+f|^2 - |e-f|^2$, and proposition 1.

iii) $\Rightarrow$ i): Use again the fact that $G_{e\mapsto f}$ is positive real if and only if $\|G_{(e+f)\mapsto(e-f)}\|_{\mathcal{H}_\infty} \leq 1$. Subsequently, use the implication iii) $\Rightarrow$ i) of proposition 1.

i) $\Rightarrow$ ii): The input/output partition has already been proven. To show $\int_{-\infty}^{+\infty} e^T f\, dt \geq 0$ for all $(e, f) \in \mathcal{K} \cap \mathcal{L}_2$, use that i) implies that $\mathcal{K}$ is dissipative on $\mathbb{R}_-$ with respect to $Q_\Sigma(v) = |e+f|^2 - |e-f|^2$, and proposition 1. Finally, note that $\mathcal{K}$ may hence be written as $(d/dt)x = A'x + B'e$, $f = C'x + D'e$, and that, by [22, Th. 6.4], there exists $K = K^T > 0$ such that $(d/dt)x^TKx \leq e^Tf$. Now take $e = 0$, and deduce that x, and, hence, f, are bounded on $\mathbb{R}_+$.

ii) $\Rightarrow$ i): Follows from [22, theorem 6.4 (in particular 3′)], interpreting that theorem as a statement about transfer functions.

Proof of Proposition 3: We have to prove that $\mathcal{N} = 0$ if and only if v is observable from c. Assume $\mathcal{N} = 0$ and let $(v_1, c), (v_2, c) \in \mathcal{P}_{\text{full}}$. Then $(v_1 - v_2, 0) = (v_1, c) - (v_2, c) \in \mathcal{P}_{\text{full}}$, so $v_1 - v_2 \in \mathcal{N}$. This implies $v_1 = v_2$, hence observability. Conversely, assume v observable from c. Let $v \in \mathcal{N}$. Then $(v, 0) \in \mathcal{P}_{\text{full}}$. Since also $(0, 0) \in \mathcal{P}_{\text{full}}$, we conclude, by observability, $v = 0$.

Proof of Theorem 4: Apply [23, Th. 3] with $\mathcal{N} = 0$.

Proof of Theorem 5: We give two proofs of the first part of the theorem. The second proof also covers the second statement of the theorem.

First Proof: This theorem is actually an immediate consequence of [23, Th. 1]. Simply make the following identifications (the variables on the left are those appearing in [23, Th. 1], those on the right are those appearing in Theorem 5):

$$\begin{aligned} v &\leftarrow \text{col}(d, e)\\ c &\leftarrow \text{col}(y, \hat{f})\\ \mathcal{P}_{\text{full}} &\leftarrow \{\text{col}(d, e, y, \hat{f}) | \text{col}(d, e+\hat{f}, y) \in \mathcal{P}_{\text{full}}\}\\ \mathcal{P} &\leftarrow \mathfrak{C}^\infty(\mathbb{R}\mathbb{R}^{\mathfrak{d}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathfrak{f}})\\ \mathcal{N} &\leftarrow \mathcal{N}\\ \mathcal{C} &\leftarrow \mathcal{F}\\ \mathcal{K} &\leftarrow \mathcal{E}. \end{aligned}$$

Second Proof: We also give a direct proof. That the conditions of are necessary for implementability is obvious. The proof of sufficiency uses kernel representations. Let

$$R_d\left(\frac{d}{dt}\right)d + R_f\left(\frac{d}{dt}\right)f + R_y\left(\frac{d}{dt}\right)y = 0 \qquad (15)$$

be a kernel representation of $\mathcal{P}_{\text{full}}$. Then

$$R_d\left(\frac{d}{dt}\right)d + R_f\left(\frac{d}{dt}\right)f = 0 \qquad (16)$$

is obviously a kernel representation of $\mathcal{N}$. Since $\mathcal{N} \subset \mathcal{E}$, there exists, by [23, Lemma 14], a $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that

$$F\left(\frac{d}{dt}\right)R_d\left(\frac{d}{dt}\right)d + F\left(\frac{d}{dt}\right)R_f\left(\frac{d}{dt}\right)e = 0 \qquad (17)$$

is a kernel representation of $\mathcal{E}$. We claim that

$$F\left(\frac{d}{dt}\right)R_f\left(\frac{d}{dt}\right)\hat{f} + F\left(\frac{d}{dt}\right)R_y\left(\frac{d}{dt}\right)y = 0 \qquad (18)$$

is a kernel representation of a filter $\mathcal{F}$ that implements $\mathcal{E}$. To see this, eliminate f, $\hat{f}$, and y from (15), (18), and $e = f - \hat{f}$, and show that (17) is the resulting equation governing (d, e). Pre-multiply (15) by $F(d/dt)$ and subtract (18) from it, to obtain equivalent equations (15), (17), and $e = f - \hat{f}$. Now, observe that since d is free in (15), for any d, e satisfying (17), there exist f, y such that (15) holds. Hence, (17) is the resulting equation governing (d, e).

Now assume $\mathcal{E}$ implementable, and, in $\mathcal{E}$, d is input and e is output. Repeat the above argument, but now choose F such that (17) is a minimal kernel representation of $\mathcal{E}$, i.e., $[FR_d \;\; FR_f]$ has full row rank. Then (18) represents a filter that implements $\mathcal{E}$. However, since in $\mathcal{E}$, d is input and e is output, FR_f is square and nonsingular. Consequently, in $\mathcal{F}$, $\hat{f}$ is output and y is input.

Proof of Theorem 6: From [23, Prop. 3] and the main result, [23, Th. 5], we deduce that Σ-dissipativity on $\mathbb{R}_-$ of $\mathcal{N}$ is equivalent to the existence of $\mathcal{E} \in \mathfrak{L}_{\text{cont}}^{\mathfrak{d}+\mathfrak{f}}$ such that i) $\mathcal{N} \subset \mathcal{E}$, ii) $\mathfrak{m}(\mathcal{E}) = \mathfrak{d}$, and iii) $\mathcal{E}$ is Σ-dissipative on $\mathbb{R}_-$. In turn, by Proposition 1, ii) and iii) are equivalent to the conditions 2, 3, and 4 in the Theorem statement.

Proof of Lemma 7: Denote $I - Q^T(QQ^T)^{-1}Q$ by S. If $Px_1 + Qx_2 = 0$, then

$$\begin{bmatrix} x_1\\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\\ -Q^T(QQ^T)^{-1}Px_1 \end{bmatrix} + \begin{bmatrix} 0\\ Sx_2 \end{bmatrix}.$$

A simple calculation yields that for all such x_1, x_2 we have

$$\begin{aligned} &x_1^TMx_1 + 2x_1^TNx_2 + x_2^Tx_2\\ &\quad = x_1^T(M - NN^T + (P - QN^T)^T(QQ^T)^{-1}(P - QN^T))x_1\\ &\qquad + |x_2 + N^Tx_1|_S^2. \end{aligned} \qquad (19)$$

Assume now that for all x_1 and x_2 such that $Px_1 + Qx_2 = 0$, we have $x_1^TMx_1 + 2x_1^TNx_2 + x_2^Tx_2 \geq 0$. Take an arbitrary x_1 and let x_2 be such that $Px_1 + Qx_2 = 0$ and $S(x_2 + N^Tx_1) = 0$, equivalently

$$\begin{bmatrix} Q\\ S \end{bmatrix} x_2 = -\begin{bmatrix} P\\ SN^T \end{bmatrix} x_1.$$

This equation is solvable for any given x_1, since the image of $\text{col}(P, SN^T)$ is contained in the image of $\text{col}(Q, S)$. Applying (19) to this x_1 and x_2 yields

$$x_1^T(M - NN^T + (P - QN^T)^T(QQ^T)^{-1}(P - QN^T))x_1 \geq 0.$$

Since x_1 is arbitrary, this proves the required nonnegativity of $M - NN^T + (P - QN^T)^T(QQ^T)^{-1}(P - QN^T)$. To prove the converse, note that $S \geq 0$. Hence the right-hand side of

(19) is nonnegative for all x_1 and x_2. As a consequence, the left-hand side of (19) is nonnegative for all x_1 and x_2 satisfying $Px_1 + Qx_2 = 0$. The remaining statement of the theorem is immediate.

Proof of Theorem 10: Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ be an implementable behavior, and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ be a controller that implements $\mathcal{K}$.

(1. $\Rightarrow$ 2.) Assume in $\mathcal{K}$, d is input and f is output, and the transfer function $\mathcal{K}_{d\mapsto f}$ from d to f in $\mathcal{K}$ is proper. This part of the proof is divided into a number of steps.

1) Analyze the input/output structure of $\mathcal{P}_{\text{full}}$, and observe that surjectivity of $G^{\infty}_{d\mapsto y}$, and hence of $G_{d\mapsto y}$, implies that (u, y) is free in $\mathcal{P}_{\text{full}}$.
2) Define $\mathcal{K}_{\text{full}} = \{(d, f, u, y) \in \mathcal{P}_{\text{full}} \,|\, (d, f) \in \mathcal{K}\}$. We now analyze the input/output structure of $\mathcal{K}_{\text{full}}$, and prove that in $\mathcal{K}_{\text{full}}$, d is input and (f, u, y) output. Since (d, u) is input and (f, y) output in $\mathcal{P}_{\text{full}}$, it suffices to prove that in $\mathcal{K}_{\text{full}}$, d is free, and that there are no additional free components in u. Since d is in put in $\mathcal{K}$, it is also free in $\mathcal{K}_{\text{full}}$. By assumption ii), $G^{\infty}_{u\mapsto f}$ and, hence, $G_{u\mapsto f}$ are injective. This implies that, when $d = 0$, there are indeed no free components in u in $\mathcal{K}_{\text{full}}$, since injectivity of $G_{u\mapsto f}$ implies that when $d = 0$, a free component in u is carried over to f.
3) Prove that $\mathtt{p}(\mathcal{C}) = \mathtt{u}$. In order to see this, we revert to the notation $v = (d, f)$, $c = (u, y)$. There obviously exists a minimal kernel representation for $\mathcal{P}_{\text{full}}$ of the form $R(d/dt)v = M(d/dt)c$, $0 = M_2(d/dt)c$, with R and M_2 polynomial matrices of full-row rank. Since, as shown in step 1, c is free, the term $M_2(d/dt)c = 0$ is absent. Let $C(d/dt)c = 0$ be a minimal kernel representation of $\mathcal{C}$. This yields $R(d/dt)v = M(d/dt)c$ and $C(d/dt)c = 0$. as minimal (full row rank) kernel representations of the plant and the controller, respectively. These equations combined form a minimal kernel representation of $\mathcal{K}_{\text{full}}$. Hence row $\dim([R - M]) = \mathtt{f} + \mathtt{y}$, and row $\dim([R-M])+$row $\dim(C) = \mathtt{f}+\mathtt{u}+\mathtt{y}$. It follows that row $\dim(C) = \mathtt{p}(\mathcal{C}) = \mathtt{u}$. Write the representation $C(d/dt)c = 0$ in terms of (u, y) as $N(d/dt)y = D(d/dt)u$. From $\mathtt{p}(\mathcal{C}) = \mathtt{u}$ it follows that D is square.
4) We want to prove that D is in fact nonsingular. For this, we first prove that y is free in $\mathcal{K}_{\text{full}}$. In order to see this, we first prove that the transfer function $K_{d\mapsto u}$ in $\mathcal{K}_{\text{full}}$ is proper. Indeed, $K_{d\mapsto f} = G_{d\mapsto f}+G_{u\mapsto f}K_{d\mapsto u}$. Whence, since $K_{d\mapsto f}$, $G_{d\mapsto f}$ and $G_{u\mapsto f}$ are proper and $G^{\infty}_{u\mapsto f}$ is injective, $K_{d\mapsto u}$ is proper. The transfer function from d to y in $\mathcal{K}_{\text{full}}$, $K_{d\mapsto y}$, equals $G_{d\mapsto y} + G_{u\mapsto y}K_{d\mapsto u}$. This transfer function is proper, and its feed-through term equals $G^{\infty}_{d\mapsto y}$. Since $G^{\infty}_{d\mapsto y}$ is surjective, so is $K_{d\mapsto y}$. This implies that y is free in $\mathcal{K}_{\text{full}}$. Next, we prove that D is nonsingular. Indeed, if not, there exists a polynomial row vector $h \neq 0$ such that $hD = 0$. Since $[D\ N]$ has full row rank, $hN \neq 0$. This implies that all ys appearing in $\mathcal{K}_{\text{full}}$ satisfy the differential equation $hN(d/dt)y = 0$, which contradicts the fact that y is free in $\mathcal{K}_{\text{full}}$.
5) We have now shown that in $\mathcal{C}$, y is input and u is output. The transfer function $C_{y\mapsto u}$ of $\mathcal{C}$ from y to u is equal to $D^{-1}N$. It remains to be proven that $C_{y\mapsto u}$ is *proper*. In order to do this, we first prove that $I - G_{u\mapsto y}C_{y\mapsto u}$ is nonsingular. Let

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u + S\left(\frac{d}{dt}\right)d$$

be a minimal kernel representation of the behavior

$$\{(d, u, y) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{u}+\mathtt{y}}) | \exists\, f \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{f}}) \text{ such that } (d, f, u, y) \in \mathcal{P}_{\text{full}}\}.$$

Obviously, since in this behavior (d, u) is input, and y output, P is square and invertible, and $P^{-1}Q = G_{u\mapsto y}$. Then

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u + S\left(\frac{d}{dt}\right)d$$
$$N\left(\frac{d}{dt}\right)y = D\left(\frac{d}{dt}\right)u$$

is a kernel representation of

$$\{(d, u, y) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{u}+\mathtt{y}}) | \exists\, f \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{f}}) \text{ such that } (d, f, u, y) \in \mathcal{K}_{\text{full}}\}.$$

Since, by step 2, d is input and (u, y) output in this behavior, it follows that the polynomial matrix $\left[\begin{smallmatrix} P & Q \\ N & D \end{smallmatrix}\right]$ is square and nonsingular. This is equivalent to the nonsingularity of $I - P^{-1}QD^{-1}N$, which equals $I - G_{u\mapsto y}C_{y\mapsto u}$.
6) Observe that the transfer function $K_{d\mapsto f}$ in $\mathcal{K}$ is given by

$$K_{d\mapsto f} = G_{d\mapsto f} + G_{u\mapsto f}C_{y\mapsto u}(I - G_{u\mapsto y}C_{y\mapsto u})^{-1}G_{d\mapsto y}. \tag{20}$$

Now examine the feed-through terms and use surjectivity of $G^{\infty}_{d\mapsto y}$ and injectivity of $G^{\infty}_{u\mapsto f}$ to conclude that $X := C_{y\mapsto u}(I - G_{u\mapsto y}C_{y\mapsto u})^{-1}$ is proper. Finally, since $C_{y\mapsto u} = (I + XG_{u\mapsto y})^{-1}X$ and since $G_{u\mapsto y}$ is strictly proper, this implies that $C_{y\mapsto u}$ is indeed proper.

(2. $\Rightarrow$ 1.) The converse implication is immediate. This concludes the proof of Theorem 10.

Proof of Corollary 11: Since $\mathcal{K}$ satisfies $\mathtt{m}(\mathcal{K}) = \mathtt{d}$ and $\|f\|^2 \leq \|d\|^2$ for all $(d, f) \in \mathcal{K} \cap \mathfrak{D}$, d is input and f output in $\mathcal{K}$, and the $\mathcal{L}_{\infty}$-norm of the transfer function $K_{d\mapsto f}$ from d to f is finite. In particular, $K_{d\mapsto f}$ is proper. Now apply Theorem 10.

Proof of Theorem 13: Examine the transfer function $G_{d\mapsto e}$ from d to e. Note that it equals $G_{d\mapsto f} - G_{y\mapsto \hat{f}}G_{d\mapsto y}$. The $\mathcal{H}_{\infty}$-norm of $G_{d\mapsto e}$ is bounded (actually, by 1). Since $G_{d\mapsto f}$ is proper, this implies that $G_{y\mapsto \hat{f}}G_{d\mapsto y}$ is proper. Since $G^{\infty}_{d\mapsto y}$ is surjective, this implies that $G_{y\mapsto \hat{f}}$ is also proper, as claimed.

VIII. Conclusion

In this second part, we have discussed several special cases of the results obtained in the first part of this paper.

The first two special cases disturbance attenuation and passivation. We have shown that the control synthesis question then reduces, as expected, to an $\mathcal{H}_{\infty}$-norm restriction or a positive

realness condition on the transfer function of the controlled behavior.

We have also discussed two extreme situations: when the hidden behavior is zero, and when the plant behavior contains all trajectories. The first of these corresponds to what we call full information control. It implies that the to-be-controlled variables are observable from the control variables. In the context of the synthesis of dissipative systems, this leads to the result that a controlled behavior that meets the specifications exists if and only the orthogonal complement of the plant behavior is dissipative. The second of these situations leads to $\mathcal{H}_\infty$-filtering. The resulting existence condition for a filter that meets the specifications is intuitively very appealing and easy to comprehend: a filter exists if and only if the dissipativity required for the estimation error behavior is already satisfied for the hidden behavior.

We have also given ample attention to the state-space case, the system representation that has been commonly used in the synthesis of $\mathcal{H}_\infty$-controllers. We obtained conditions for the existence of a controlled behavior that meets the specifications in terms of two coupled algebraic Riccati inequalities. The synthesis of the controller, however, requires the solution of one algebraic Riccati inequality, and one algebraic Riccati equation. The controlled behavior can in this case always be implemented by a feedback controller. However, because we treat the $\mathcal{H}_\infty$-problem with nonstrict inequality, the resulting controller is proper, but in general not strictly proper.

We have also derived two general results on the implementability of a controlled behavior by means of a sensor output to actuator input feedback processor. These results require a suitable input-output structure of the plant and the controlled behavior, and properness conditions of their transfer functions. These results recover many of the classical problem formulations in control. However, in our case, the feedback structure is imported as a consequence of the properties of the plant and the control problem formulation.

A few words about the (many) research directions that emanate from this 2-part paper. The most pressing direction is the development of specific algorithms. The main result of [23] gives necessary and sufficient conditions for the existence of an implementable dissipative controlled behavior in terms of the full plant behavior. However, in applications, the plant behavior will be given in a parameterized form, for example in terms of polynomial matrices, or (what is actually not all that different) by matrices of rational functions, or by matrices (as in the state-space case discussed in Part II, or as descriptor systems). The most flexible parametrization, of course, is in terms of latent variables, since this model class has most other parametrization as special cases. The algorithmic question that emerges is to find algorithms for the construction of the storage functions and verification of the existence conditions in terms of these parametrization. This leads to polynomial-matrix versions of semidefinite programming. Some results in this direction were included in an earlier draft of these papers, but had to be removed because of length restrictions. A subsidiary problem is to develop algorithms that aim directly at the specification of the controller, instead of via the controlled behavior. A fruitful avenue of generalization is to systems described by PDEs, where the framework of models in terms of polynomial matrices (in many indeterminates) and QDFs makes perfect sense. And, of course, there is also the question of how to cast robust control in the framework QDFs, the generalization to nonlinear systems, $\mathcal{H}_2$-versions, treating the case when the weighting functional is a general two-variable polynomial matrix, etc.

The combination of the two parts of this paper gives a rather complete and self-contained theory of the synthesis a dissipative systems. We emphasize that our treatment was completely representation independent. This was achieved by systematically using the behavioral approach, not only for the system models, but also for the specification of the desired controlled behavior.

REFERENCES

[1] J. C. Doyle, K. Glover, P. Khargonekar, and B. A. Francis, "State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, Aug. 1989.

[2] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory*: Springer, 1987.

[3] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to $\mathcal{H}_\infty$-control," *Int. J. Robust Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[4] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, "A characterization of all solutions to the four block general distance problem," *SIAM J. Control Optim.*, vol. 29, pp. 283–324, 1991.

[5] M. Green, "H_∞ controller synthesis by J-lossless coprime factorization," *SIAM J. Control Optim.*, vol. 30, pp. 522–547, 1992.

[6] M. Green, K. Glover, D. Limebeer, and J. C. Doyle, "A J-spectral factorization approach to H_∞ control," *SIAM J. Control Optim.*, vol. 28, pp. 1350–1371, 1990.

[7] P. P. Khargonekar, "State-space $\mathcal{H}_\infty$ control theory," in *Mathematical System Theory: The Influence of R.E. Kalman*, A. C. Antoulas, Ed. New York: Springer Verlag, 1991, pp. 159–176.

[8] P. P. Khargonekar and K. M. Nagpal, "Filtering and smoothing in an $\mathcal{H}_\infty$ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 151–166, Feb. 1991.

[9] G. Meinsma, "Frequency domain methods in H_∞ Control," Ph.D. dissertation, University of Twente, The Netherlands, 1993.

[10] ——, "Polynomial solutions to H_∞ problems," *Int. J. Robust Nonlinear Control*, vol. 4, pp. 323–351, 1994.

[11] G. Meinsma and M. Green, "On strict passivity and its application to interpolation and $\mathcal{H}_\infty$ control," in *Proc. 31st Conf. Decision Control*, Tucson, AZ, 1992, pp. 2802–2807.

[12] C. Scherer, "$\mathcal{H}_\infty$-control by state-feedback for plants with zeros on the imaginary axis," *SIAM J. Control Optim.*, vol. 30, pp. 123–142, 1992.

[13] ——, "$\mathcal{H}_\infty$-optimization without assumptions on finite or infinite zeros," *SIAM J. Control Optim.*, vol. 30, pp. 143–166, 1992.

[14] ——, "$\mathcal{H}_\infty$-control by state feedback and fast algorithms for the computation of optimal $\mathcal{H}_\infty$ norms," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1090–1099, Oct. 1990.

[15] C. W. Scherer, "The state-feedback $\mathcal{H}_\infty$-problem at optimality," *Automatica*, vol. 30, pp. 293–305, 1994.

[16] A. A. Stoorvogel and H. L. Trentelman, "The quadratic matrix inequality in singular $\mathcal{H}_\infty$ control with state feedback," *SIAM J. Control Optim.*, vol. 28, pp. 1190–1208, 1990.

[17] A. A. Stoorvogel, "The singular $\mathcal{H}_\infty$ control problem with dynamic measurement feedback," *SIAM J. Control Optim.*, vol. 29, pp. 160–184, 1991.

[18] G. Tadmor, "Worst-case design in the time domain: The maximum principle and the standard $\mathcal{H}_\infty$ problem," *Math. Control Signals Syst.*, vol. 3, pp. 301–324, 1990.

[19] H. L. Trentelman and J. C. Willems, "$\mathcal{H}_\infty$-control in a behavioral context: The full information case," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 521–536, Mar. 1999.

[20] ——, "Dissipative differential systems and the state space $\mathcal{H}_\infty$ control problem," *Int. J. Robust Nonlinear Control*, vol. 10, pp. 1039–1057, 2000.

[21] J. C. Willems, "On interconnections, control, and feedback," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 326–339, Mar. 1997.

[22] J. C. Willems and H. L. Trentelman, "On quadratic differential forms," *SIAM J. Control Optim.*, vol. 36, pp. 1703–1749, 1998.

[23] ——, "Synthesis of dissipative systems using quadratic differential forms, Part I," *IEEE Trans. Automat. Contr.*, vol. 47, pp. XXX–XXX, Jan. 2002.

[24] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 301–320, 1981.

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