runs on the `Nserver`, the Nomad simulator. These test runs, where the robot tracks a circle from different initial positions and configurations, clearly indicate that our proposed second control algorithm works globally in a stable and robust way. It should be emphasized that if we were to use the first, local algorithm on these test runs, it would fail since the initial positions and orientations would make $\Delta xp'(s)+\Delta yq'(s) = 0$, and thus $\dot{s}$ would not be defined anymore if the first algorithm were to be used.

V. Conclusion

In this note, two intuitive, model independent path following control strategies are proposed, and the stability analysis is done with respect to two different platforms. What is new here is that by combining the conventional trajectory tracking approach and the more recent geometric path following approach, we can design a virtual vehicle that moves on the reference path and is regulated in a closed-loop fashion by exploiting the position error. In the first algorithm, the velocity is kept constant, while the other, global method depends on the possibility of fine velocity control.

Implementing these ideas on actual robots gives us some experimental data that show that our controllers work in practice as well as in theory, which is what we were aiming for, since our main design strategy was to keep the control algorithms model independent and as simple as possible.

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A Polynomial Approach to Nonlinear System Controllability

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***Abstract*—This note uses a polynomial approach to present a necessary and sufficient condition for local controllability of single-input–single-output (SISO) nonlinear systems. The condition is presented in terms of common factors of a noncommutative polynomial expression. This result exposes controllability properties of a nonlinear system in the input–output framework, and gives a computable procedure for examining nonlinear system controllability using computer algebra.**

***Index Terms*—Common factor, controllability, differential fields, noncommutative ring, nonlinear systems.**

I. Introduction

Controllability is one of the central notions of modern control theory. The results on controllability of linear systems have been seminal in the development of the field, and the literature on controllability of nonlinear systems is vast. See, for example, [16], [1], [7], [8], [11], [15], and [19]. Traditionally, controllability is defined for linear state space systems and refers to the possibility of transferring a system from any initial to any terminal state. For nonlinear state-space systems the notion of controllability or strong accessibility refers to the case where the control can act on the system state, but may be insufficient to transfer it to a specified terminal state. Often, nonlinear system controllability is defined in terms of system state equation and tested by means of Lie distributions or their dual form.

The notion of controllability is recently extended to systems in more general framework. For linear systems, controllability is viewed in [18] in terms of system trajectories which may not necessarily be the system state. A system is defined to be controllable if one can switch from any feasible past trajectory in the system behavior to any feasible future trajectory, after some time delay. It is observed that the lack of behavioral controllability implies the existence of an autonomous system 'output', which is a nontrivial function of the system variables. It turns out that a linear time-invariant input–output system is controllable if and only if it does not have autonomous variables in its behavior and if and only if the polynomial matrices that specify the system behavior are left coprime.

The notion of autonomous variables is also used to describe controllability of nonlinear systems [1], [5], [7], [19]. In [7], local controllability of nonlinear state space systems is described in terms of the absence of local first integrals which are autonomous variables of the system state. In [1] and [19], controllability of nonlinear state-space systems is described by the absence of autonomous variables in terms of differential one-forms. Moreover, the need for a controllability concept for nonlinear input/output systems is discussed in [13], where a

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notion of constrained observable is proposed for describing controllability of nonlinear input–output systems and nonlinear systems in partial differential equations. The concept of constrained observable is equivalent to that of autonomous variable.

The purpose of the present paper is to further explore controllability of nonlinear input–output systems and develop a new approach to testing nonlinear system controllability. We follow the early work [18], [7], [1], [19], [12], [13] to define controllability of nonlinear systems in terms of the nonexistence of autonomous variables.

A polynomial expression of nonlinear systems is developed in this paper to examine the controllability of nonlinear input/output systems. It is shown that a nonlinear system is controllable if and only if there are no common factors in the system polynomial expression. This leads to a novel algebraic approach to nonlinear system controllability beyond the conventional approach using Lie distributions or their dual form. Two distinctive features of our polynomial approach are

- the nonlinear system controllability in terms of common factors remarkably coincides with the corresponding linear systems result. This serves to provide deeper insights into controllability of dynamical systems;
- the factorization of the nonlinear system polynomial expression for examining common factors and controllability can be readily programmed and carried out by computer algebra. Thus, it gives the first computable result for nonlinear system controllability using computer algebra.

The rest of this paper is organized as follows. Section 2 defines controllability of nonlinear systems and presents fundamentals of the differential field and differential vector space of nonlinear systems. Section 3 presents a polynomial expression of nonlinear systems. Using the polynomial expression, Section 4 presents a necessary and sufficient condition for controllability of nonlinear systems using the polynomial. In Section 5, a computational procedure for testing controllability of nonlinear system is developed and an illustrative example is given.

II. Problem Statement and Preliminaries

Consider a scalar nonlinear system defined by

$$y^{(n)} = f\left(y, y^{(1)}, \ldots, y^{(n-1)}, u, u^{(1)}, \ldots, u^{(m)}\right) \tag{1}$$

where u and y are the input and output variables, $y^{(i)}$, $1 \le i \le n$, is the ith derivative of $y, u^{(j)}$, $1 \le j \le m$, is the jth derivative of u and $f : \mathcal{R}^n \times \mathcal{R}^{m+1} \to \mathcal{R}$ is a meromorphic function. Assume $(\partial f/\partial u^{(m)}) \not\equiv 0$. We confine our attention in this paper to local controllability of nonlinear system (1) over an open subset $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ with the input u and output y of nonlinear system (1) satisfying

$$\left(y, y^{(1)}, \ldots, y^{(n-1)}\right)^T \in Y, \quad \left(u, u^{(1)}, \ldots, u^{(m)}\right)^T \in U. \tag{2}$$

We use the notion of autonomous variable to define local controllability of the nonlinear system (1) as follows.

Definition 1: The nonlinear system (1) is locally controllable over an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ if under the local condition (2) there exist no meromorphic functions $h : \mathcal{R}^{l+1} \to \mathcal{R}$ and $z : \mathcal{R}^{\hat{n}+1} \times \mathcal{R}^{\hat{m}+1} \to \mathcal{R}$ with $l \ge 1$, $\hat{n} + l = n$ and $\hat{m} + l = m$ such that

$$\begin{cases} h\left(z, z^{(1)}, \ldots, z^{(l)}\right) = 0 \\ z = z(y, y^{(1)}, \ldots, y^{(\hat{n})}, u, u^{(1)}, \ldots, u^{(\hat{m})}). \end{cases} \tag{3}$$

Remark: The nonlinear system (1) and its controllability are defined in terms of meromorphic functions. Meromorphic functions are elements of the quotient field of the ring of analytic functions [5]. Thus the functions which define the nonlinear system and its controllability are analytic over an open dense set of $\mathcal{R}^n \times \mathcal{R}^{m+1}$. This allows us to define the local controllability of (1) over the open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$. The use of meromorphic functions is essential for carrying out arithmetic operations, particularly division, over the meromorphic function field in computation of the polynomial equations and common factors in this note.

Remark: An abuse of notation is involved in Definition 1, where z denotes a variable as well as a function of y and u and their derivatives. While z is a function of $y^{(i)}$ and $u^{(j)}$, $0 \le i \le n-l$, $0 \le j \le m-l$, it is governed by the homogeneous differential equation $h(z, z^{(1)}, \ldots, z^{(l)}) = 0$. For any initial condition, the solution of z is uniquely determined by this homogeneous differential equation and is consequently independent of the external input u. In this sense, z is an autonomous variable which represents the lack of controllability of the nonlinear system. It follows that the nonlinear system (1) is controllable if and only if it contains no autonomous variables.

Remark: The term *local controllability* was also used for nonlinear state space systems which is akin to *strong accessibility* under some conditions [16]. Throughout our paper, this term is used for nonlinear input–output systems following from Definition 1.

We now recall the following two basic definitions.

A differential field $\mathcal{F}$ is a field equipped with a derivative operation $\dot{(\ )} : \mathcal{F} \to \mathcal{F}$.

A differential vector space $\mathcal{V}^*$ over a differential field $\mathcal{F}$ is a vector space equipped with a derivative operation $\dot{(\ )} : \mathcal{V}^* \to \mathcal{V}^*$.

A differential field is closed under addition, multiplication and derivative operations and a differential vector space is closed under addition, scalar multiplication and derivative operations over the differential field. We further define a polynomial ring as follows.

Let ξ be an indeterminate over a differential field $\mathcal{F}$, then the polynomial ring $\mathcal{F}[\xi]$ is defined by the following multiplication:

$$\xi f = f\xi + \dot{f} \tag{4}$$

for any $f \in \mathcal{F}$.

The polynomial ring $\mathcal{F}[\xi]$ with multiplication rule (4) is noncommutative and is an example of an *Ore ring* ([14]). In the remainder of this note, we let the indeterminate ξ be the derivative operation d/dt.

Let $\mathcal{K}$ be the field of all meromorphic functions of $y^{(i)}$, $0 \le i \le n-1$, and $u^{(j)}$, $j \ge 0$. Thus, each meromorphic function $\psi \in \mathcal{K}$ is such that $\psi : \mathcal{R}^n \times \mathcal{R}^r \to \mathcal{R}$ for some $r \ge 0$, and may be written in the form $\psi(y, y^{(1)}, \ldots, y^{(n-1)}, u, u^{(1)}, \ldots, u^{(r-1)})$. It is straightforward to verify, by the quotient rule of calculus, that

$$\begin{aligned} &\frac{\partial \psi}{\partial y^{(i)}} \in \mathcal{K}, \quad \text{for } 0 \le i \le n-1 \\ &\frac{\partial \psi}{\partial u^{(j)}} \in \mathcal{K}, \quad \text{for } 0 \le j \le r-1. \end{aligned} \tag{5}$$

For any $\psi \in \mathcal{K}$, we define the *derivative operation* $d/dt : \mathcal{K} \to \mathcal{K}$ on $\mathcal{K}$ as follows:

$$\frac{d\psi}{dt} = \sum_{i=0}^{n-1} \frac{\partial \psi}{\partial y^{(i)}} y^{(i+1)} + \sum_{j=0}^{r-1} \frac{\partial \psi}{\partial u^{(j)}} u^{(j+1)} \tag{6}$$

We will use $\psi^{(i)}$ to denote the ith derivative of ψ. To see that $\mathcal{K}$ is closed under the derivative operation defined by (6), with (1) we note that

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial y^{(n-1)}} f + \sum_{i=0}^{n-2} \frac{\partial \psi}{\partial y^{(i)}} y^{(i+1)} + \sum_{j=0}^{r-1} \frac{\partial \psi}{\partial u^{(j)}} u^{(j+1)}$$

where each term in the above summation belongs to $\mathcal{K}$. $\mathcal{K}$ also satisfies the rules for the differentiation of sums and products and is therefore a differential field with the derivative operation d/dt. It is the differential field uniquely defined by nonlinear system (1).

We denote by $\mathcal{D}^*$ the vector space spanned over $\mathcal{K}$ by $dy^{(i)}, du^{(j)}$, $0 \le i \le n-1,\ j \ge 0$, namely $\mathcal{D}^* := \mathrm{span}_{\mathcal{K}}\{dy^{(i)}, du^{(j)} \,|\, 0 \le i \le n-1, j \ge 0\}$. Thus each element $\omega \in \mathcal{D}^*$ is in the form

$$\omega = \sum_{i=0}^{n-1} \alpha_i dy^{(i)} + \sum_{j=0}^{r} \beta_j du^{(j)}$$

for some $r \ge 0$, with $\alpha_i, \beta_j \in \mathcal{K}$. We can now define an operator $d:\ \mathcal{K} \to \mathcal{D}^*$, called the *differential operation* as follows:

$$d\colon \psi \mapsto d\psi = \sum_{i=0}^{n-1} \frac{\partial \psi}{\partial y^{(i)}} dy^{(i)} + \sum_{j=0}^{r-1} \frac{\partial \psi}{\partial u^{(j)}} du^{(j)} \in \mathcal{D}^*, \quad \forall \psi \in \mathcal{K} \tag{7}$$

and define the *derivative operation* $(d/dt)\ :\ \mathcal{D}^* \to \mathcal{D}^*$ on $\mathcal{D}^*$ as follows:

$$\frac{d\omega}{dt} = \sum_{i=0}^{n-1} \left(\alpha_i^{(1)} dy^{(i)} + \alpha_i dy^{(i+1)}\right) + \sum_{j=0}^{r} \left(\beta_j^{(1)} du^{(j)} + \beta_j du^{(j+1)}\right). \tag{8}$$

We will use $\omega^{(i)}$ to denote the ith derivative of ω. To see that the vector space $\mathcal{D}^*$ is closed under the derivative operation d/dt, we note that $dy^{(n)} = \sum_{i=0}^{n-1} (\partial f/\partial y^{(i)}) dy^{(i)} + \sum_{j=0}^{m} (\partial f/\partial u^{(j)}) du^{(j)} \in \mathcal{D}^*$. Thus each term in the summation (8) belongs to $\mathcal{D}^*$. Consequently, $\mathcal{D}^*$ is a differential vector space with the derivative operation d/dt. It is the differential vector space uniquely defined by nonlinear system (1).

With respect to the derivative operations on $\mathcal{K}$ and $\mathcal{D}^*$, the differential operation d obeys the following commutative rule for any $\psi \in \mathcal{K}$:

$$\frac{d}{dt}(d\psi) = d\left(\frac{d\psi}{dt}\right). \tag{9}$$

Each element $\omega \in \mathcal{D}^*$ is called a *one-form*. A one-form $\omega \in \mathcal{D}^*$ is called an *exact one-form* if there exists a $\psi \in \mathcal{K}$ such that $\omega = d\psi$. Note that not every one-form $\omega \in \mathcal{D}^*$ is an exact one-form. If for l one-forms ω_i, with $0 \le i \le l-1$, there exist l functions $\psi_i \in \mathcal{K}$ with $0 \le i \le l-1$ such that $\sum_{i=0}^{l-1} \psi_i \omega_i = 0$, then the l one-forms ω_i are called *linearly dependent* one-forms over $\mathcal{K}$. Otherwise they are *linearly independent* one-forms over $\mathcal{K}$.

Let ω_i, $0 \le i \le l-1$, be l linearly independent one-forms in the differential vector space $\mathcal{D}^*$. These one-forms form an l-dimensional subspace, denoted by $\tilde{\mathcal{D}}^*$, of $\mathcal{D}^*$ written as

$$\tilde{\mathcal{D}}^* = \mathrm{span}_{\mathcal{K}}\{\omega_i \,|\, \omega_i \in \mathcal{D}^*, 0 \le i \le l-1\}. \tag{10}$$

Under the local condition (2), the subspace $\tilde{\mathcal{D}}^*$ is called *integrable* if there exist l linearly independent exact one-forms $d\psi_i \in \mathcal{D}^*$, with each $\psi_i \in \mathcal{K}$ in an open set of $\mathcal{R}^l$ such that

$$\tilde{\mathcal{D}}^* = \mathrm{span}_{\mathcal{K}}\{d\psi_i \,|\, \psi_i \in \mathcal{K}, 0 \le i \le l-1\}$$

A necessary and sufficient condition for integrability of the subspace $\tilde{\mathcal{D}}^*$ is given by *Frobenius Theorem* [10], [4], [5], as stated as follows. The subspace $\tilde{\mathcal{D}}^*$ in (10) is integrable if and only if

$$d\omega_i \wedge \omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{l-1} = 0, \quad \forall\, 0 \le i \le l-1 \tag{11}$$

where $d\omega_i$ denotes the exterior derivative of the differential form w_i and $\wedge$ denotes the exterior product.

We now define useful subspaces $\mathcal{D}_0^*$, $\mathcal{D}_1^*$ and $\mathcal{D}_\infty^*$ of the differential vector space $\mathcal{D}^*$ as follows:

$$\begin{aligned}\mathcal{D}_0^* &= \mathrm{span}_{\mathcal{K}}\left\{dy^{(i)}, du^{(j)} \,|\, 0 \le i \le n-1, 0 \le j \le m\right\}\\ \mathcal{D}_1^* &= \mathrm{span}_{\mathcal{K}}\left\{dy^{(i)}, du^{(j)} \,|\, 0 \le i \le n-1, 0 \le j \le m-1\right\}\\ \mathcal{D}_\infty^* &= \mathrm{span}_{\mathcal{K}}\left\{\omega \,|\, \omega^{(k)} \in \mathcal{D}_1^*, \forall k \ge 0\right\}.\end{aligned} \tag{12}$$

It is clear that $\mathcal{D}_0^*$, $\mathcal{D}_1^*$ and $\mathcal{D}_\infty^*$ are finite dimensional subspaces with $\mathcal{D}_\infty^* \subset \mathcal{D}_1^* \subset \mathcal{D}_0^*$. The subspace $\mathcal{D}_1^*$ is not necessarily closed under the derivative operation. For any $\omega \in \mathcal{D}_1^*$, it is possible that for some $k \ge 1$, $\omega^{(k)}$ becomes linearly dependent on $du^{(r)}$ for some $r \ge m$, i.e., $\omega^{(k)} \notin \mathcal{D}_1^*$.

We define the *relative degree* of the one-form $\omega \in \mathcal{D}_1^*$ to be the least integer k such that $\omega^{(k)} \notin \mathcal{D}_1^*$. If no such integer exists, i.e., if $\omega^{(k)} \in \mathcal{D}_1^*$ for all $k \ge 0$, we say that the relative degree of ω is infinity. Thus $\mathcal{D}_\infty^*$ is the subspace of $\mathcal{D}_1^*$ containing all the one-forms in $\mathcal{D}_1^*$ with relative degree infinity. $\mathcal{D}_\infty^*$ may be thought of as the largest subspace of $\mathcal{D}_1^*$ that is closed under the derivative operation. It may also be interpreted as a torsion submodule of $\mathcal{D}^*$.

Assume that the subspaces $\mathcal{D}_0^*$ and $\mathcal{D}_\infty^*$ satisfy the following local regularity condition.

Assumption 1: Over an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$, $\mathcal{D}_0^*$ and $\mathcal{D}_\infty^*$ are locally regular subspaces in the sense

$$\dim \mathcal{D}_0^* = n+m+1, \quad \dim \mathcal{D}_\infty^* = \text{constant} < n+m \tag{13}$$

for all $(y, y^{(1)}, \ldots, y^{(n-1)})^T \in Y$ and $(u, u^{(1)}, \ldots, u^{(m)})^T \in U$.

Lemma 2.1: If Assumption 1 is satisfied and $\dim \mathcal{D}_\infty^* \ne 0$, then $\mathcal{D}_\infty^*$ is an integrable subspace.

Proof: If $\mathcal{D}_\infty^* \subset \mathcal{D}_1^*$ satisfies $\dim \mathcal{D}_\infty^* = \text{constant} \ne 0$, let the dimension of $\mathcal{D}_\infty^*$ be l and let the one-forms ω_i, $0 \le i \le l-1$, be a set of bases for $\mathcal{D}_\infty^*$ such that

$$\mathcal{D}_\infty^* = \mathrm{span}_{\mathcal{K}}\{\omega_i \,|\, 0 \le i \le l-1\}$$

Following from the line of [1, Prop. 3.3], we obtain (11). Thus, $\mathcal{D}_\infty^*$ satisfies the condition of Frobenius Theorem and is an integrable subspace. $\square$

III. Polynomial Expression for Nonlinear Systems

The differential field $\mathcal{K}$ and the derivative operator d/dt induce the polynomial ring $\mathcal{K}[d/dt]$. A polynomial $G \in \mathcal{K}[d/dt]$ is written as $G(d/dt) = g_k(d/dt)^k + g_{k-1}(d/dt)^{k-1} + \cdots + g_1(d/dt) + g_0$, where $g_i \in \mathcal{K}$ for $0 \le i \le k$. The degree of the polynomial G is k if $g_k \ne 0$, and G is called monic if $g_k = 1$. Each polynomial $G \in \mathcal{K}[d/dt]$ is a mapping of $\mathcal{D}^*$ into itself. To evaluate G at any $\omega \in \mathcal{D}^*$, we use the following rules for the indeterminate d/dt:

$$\left(\frac{d}{dt}\right)^i dy = dy^{(i)} \quad \left(\frac{d}{dt}\right)^j du = du^{(j)} \tag{14}$$

$$\left(\frac{d}{dt}\right)\psi = \psi\left(\frac{d}{dt}\right) + \psi^{(1)} \tag{15}$$

for any $\psi \in \mathcal{K}$. The multiplication rule (15) satisfies the rule (4), and hence $\mathcal{K}[d/dt]$ is a noncommutative Ore ring. The rules (14) and (15) imply that $\omega^{(i)}$ satisfies $\omega^{(i)} = (d/dt)^i \omega$.

In the differential field $\mathcal{K}$, there are no zero divisors, in the sense that if $\psi_1, \psi_2 \in \mathcal{K}$ with $\psi_1, \psi_2 \ne 0$ then $\psi_1\psi_2 \ne 0$. Thus, for three polynomials $G, G_1, G_2 \in \mathcal{K}[d/dt]$ with $\deg G_1 = d_1 > 0$ and

$\deg G_2 = d_2 > 0$ such that $G(d/dt) = G_1(d/dt)G_2(d/dt)$, the degree of $G(d/dt)$ satisfies

$$\deg G = \deg G_1 + \deg G_2 = d_1 + d_2. \tag{16}$$

For $G(d/dt) = G_1(d/dt)G_2(d/dt)$, G_1 is called a *left divisor* and G_2 is called a *right divisor* of G, and G is called left divisible by G_1 and right divisible by G_2. If for $G_1, G_2 \in \mathcal{K}[d/dt]$ such that $G_c \in \mathcal{K}[d/dt]$ with $\deg G_c \geq 1$ is a left divisor of $(G_1 - G_2)$, then G_c is called a *left common factor* of G_1 and G_2. A polynomial G_c is called a *greatest left common factor* of G_1 and G_2 if $\deg G_c$ is the greatest over all left common factors of G_1 and G_2. Two polynomials G_1 and G_2 are called *left coprime* if they have no left common factor G_c with $\deg G_c \geq 1$.

Remark: Suppose $G_1, G_2 \in \mathcal{K}[d/dt]$ have a left common factor with degree $d_c \geq 1$. Then there, in general, exist multiple left common factors of G_1 and G_2 with degree d_c, i.e., there exist multiple solutions for G_c, $\tilde{G}_1$ and $\tilde{G}_2$ with $\deg G_c = d_c$ which satisfy $G_c(\tilde{G}_1 - \tilde{G}_2) = G_1 - G_2$. This is due to the noncommutative property of the polynomial ring $\mathcal{K}[d/dt]$, which is considerably different from the polynomial and common factor expressions for linear systems. Consequently, there in general exist multiple greatest left common factors for the polynomials G_1 and G_2.

We now represent the nonlinear system (1) in terms of polynomials in the Ore ring $\mathcal{K}[d/dt]$. Over an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ with (2) being satisfied, we apply the differential operation to (1) to obtain

$$dy^{(n)} - \sum_{i=0}^{n-1} \frac{\partial f}{\partial y^{(i)}}\, dy^{(i)} - \sum_{j=0}^{m} \frac{\partial f}{\partial u^{(j)}}\, du^{(j)} = 0. \tag{17}$$

Let

$$\begin{aligned} P\left(\frac{d}{dt}\right) &= \left(\frac{d}{dt}\right)^n - \sum_{i=0}^{n-1} \frac{\partial f}{\partial y^{(i)}} \left(\frac{d}{dt}\right)^i \\ Q\left(\frac{d}{dt}\right) &= \sum_{j=0}^{m} \frac{\partial f}{\partial u^{(j)}} \left(\frac{d}{dt}\right)^j. \end{aligned} \tag{18}$$

Since $(\partial f/\partial y^{(i)}), (\partial f/\partial u^{(j)}) \in \mathcal{K}$, we have $P, Q \in \mathcal{K}[d/dt]$. Using (14), we can write (17) as

$$P\left(\frac{d}{dt}\right) dy - Q\left(\frac{d}{dt}\right) du = 0. \tag{19}$$

Since $P(d/dt)$ in (19) is a monic polynomial, we call the polynomial equation in the form (19) a *monic polynomial equation*. The monic polynomial equation (19) is uniquely determined by nonlinear system (1).

IV. Common Factors and Controllability of Nonlinear Systems

This section presents the main result of this note on local controllability of nonlinear systems. Using the polynomial expression (19) of the nonlinear system (1), the local controllability of the nonlinear system is presented in terms of left common factors of the polynomials P and Q. This result is given in the following theorem.

Theorem 4.1: Under Assumption 1, the nonlinear system (1) is locally controllable in the sense of Definition 1 if and only if the polynomials P and Q defined by (18) have no left common factors.

Proof: Sufficiency: Assume that the nonlinear system (1) is not locally controllable. Then there exist functions $z, h \in \mathcal{K}$ such that (3) is satisfied. We apply the differential operation to the functions z and h in (3) and use (14) to obtain

$$\begin{aligned} dz &= \sum_{i=0}^{\hat{n}} \frac{\partial z}{\partial y^{(i)}}\, dy^{(i)} + \sum_{j=0}^{\hat{m}} \frac{\partial z}{\partial u^{(j)}} du^{(j)} \\ &= \tilde{P}\left(\frac{d}{dt}\right) dy - \tilde{Q}\left(\frac{d}{dt}\right) du \\ dh &= \sum_{k=0}^{l} \frac{\partial h}{\partial z^{(k)}}\, dz^{(k)} = H\left(\frac{d}{dt}\right) dz \\ &= H\left(\frac{d}{dt}\right)\left(\tilde{P}\left(\frac{d}{dt}\right) dy - \tilde{Q}\left(\frac{d}{dt}\right) du\right) = 0 \end{aligned} \tag{20}$$

where $(\partial z/\partial y^{(i)}), (\partial z/\partial u^{(j)}), (\partial h/\partial z^{(k)}) \in \mathcal{K}[d/dt]$, and $\tilde{P}(d/dt) = \sum_{i=0}^{\hat{n}} (\partial z/\partial y^{(i)})(d/dt)^i$, $\tilde{Q}(d/dt) = -\sum_{j=0}^{\hat{m}} (\partial z/\partial u^{(j)})(d/dt)^j$ and $H(d/dt) = \sum_{k=0}^{l} (\partial h/\partial z^{(k)})(d/dt)^k \in \mathcal{K}[d/dt]$. They are all in $\mathcal{K}[d/dt]$.

Let $\tilde{H}(d/dt) = ((\partial h/\partial z^{(l)})(\partial z/\partial y^{(\hat{n})}))^{-1} H(d/dt)$, it can be verified by the multiplication rule (15) that $\tilde{H}(d/dt)\tilde{P}(d/dt)$ is a monic polynomial. This yields the monic polynomial expression for (20)

$$\tilde{H}\left(\frac{d}{dt}\right)\left(\tilde{P}\left(\frac{d}{dt}\right) dy - \tilde{Q}\left(\frac{d}{dt}\right) du\right) = 0. \tag{21}$$

As monic expression (1) uniquely defines the differential field $\mathcal{K}$, the expressions (1) and (3), which carry operations in the unique differential field $\mathcal{K}$, must have the same monic expression in the form (1) in an open set in $\mathcal{R}^n \times \mathcal{R}^{m+1}$. Further, the monic expression (1) uniquely determines its monic polynomial equation in the form (19). Thus, under Assumption 1 on the regularity of $\mathcal{D}_0^*$, which implies that $\{dy^{(i)},\ du^{(j)} \mid 0 \leq i \leq n-1,\ 0 \leq j \leq m\}$ is a basis for $\mathcal{D}_0^*$, we can equate monic polynomial equations (21) and (19) to obtain

$$\begin{aligned} P\left(\frac{d}{dt}\right) &= \tilde{H}\left(\frac{d}{dt}\right) \tilde{P}\left(\frac{d}{dt}\right) \\ Q\left(\frac{d}{dt}\right) &= \tilde{H}\left(\frac{d}{dt}\right) \tilde{Q}\left(\frac{d}{dt}\right). \end{aligned}$$

This shows that the polynomials P and Q have a left common factor $\tilde{H}$. Hence, the nonlinear system (1) is locally controllable if P and Q have no left common factors.

Necessity: Assume that the polynomials P and Q with $\deg P = n$ and $\deg Q = m$ have a left common factor H with $\deg H = \tilde{l} \geq 1$. Then, the polynomial equation (19) can be written as

$$\begin{aligned} &P\left(\frac{d}{dt}\right) dy - Q\left(\frac{d}{dt}\right) du \\ &\qquad = H\left(\frac{d}{dt}\right)\left(\tilde{P}\left(\frac{d}{dt}\right) dy - \tilde{Q}\left(\frac{d}{dt}\right) du\right) = 0 \end{aligned} \tag{22}$$

where $\deg \tilde{P} = n - \tilde{l}$ and $\deg \tilde{Q} = m - \tilde{l}$, and the polynomial $H(d/dt)$ is written as

$$H\left(\frac{d}{dt}\right) = h_{\tilde{l}}\left(\frac{d}{dt}\right)^{\tilde{l}} + h_{\tilde{l}-1}\left(\frac{d}{dt}\right)^{\tilde{l}-1} + \cdots + h_1\left(\frac{d}{dt}\right) + h_0$$

with $h_i \in \mathcal{K}$ for $0 \leq i \leq \tilde{l}$. Let $\omega = \tilde{P}(d/dt)\, dy - \tilde{Q}(d/dt)\, du \in \mathcal{D}_1^*$ and

$$H\left(\frac{d}{dt}\right) \omega = 0. \tag{23}$$

Equation (23) can be equivalently expressed as

$$\sum_{i=0}^{\tilde{l}} h_i \omega^{(i)} = 0 \tag{24}$$

where each $h_i \in \mathcal{K}$. Define a nested sequence of subspaces $\{W_i \mid 0 \leq i \leq \tilde{l}\}$ of $\mathcal{D}^*$ with

$$W_i = \operatorname{span}_{\mathcal{K}} \left\{\omega, \omega^{(1)}, \ldots, \omega^{(i)}\right\}. \tag{25}$$

Note that $W_i \subseteq \mathcal{D}_1^*$, for $0 \leq i < \tilde{l}$ by the definition of ω. Also, the result (24) shows that the $\tilde{l}+1$ one-forms $\{\omega^{(i)} \mid 0 \leq i \leq \tilde{l}\}$ are linearly dependent over $\mathcal{K}$. This implies that for some $0 \leq k < \tilde{l}$

$$\omega^{(i)} \in W_k \subset \mathcal{D}_1^*, \quad \text{for all } i \geq 0$$

Thus $\omega \in \mathcal{D}_\infty^*$ by definition of $\mathcal{D}_\infty^*$ i.e., $\dim \mathcal{D}_\infty^* \geq 1$.

To complete the proof of Theorem 4.1, we use the following lemma whose proof is given in Appendix. □

Lemma 4.2: If $\dim \mathcal{D}_\infty^* \neq 0$, then under Assumption 1 over an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ there exist meromorphic functions $z : \mathcal{R}^{\hat{n}+1} \times \mathcal{R}^{\hat{m}+1} \to \mathcal{R}$ and $h : \mathcal{R}^{l+1} \to \mathcal{R}$ with $l \geq 1$, $\hat{n} + l = n$ and $\hat{m} + l = m$ such that

$$\begin{cases} h\left(z, z^{(1)}, \ldots, z^{(l)}\right) = 0 \\ z = z\left(y, y^{(1)}, \ldots, y^{(\hat{n})}, u, u^{(1)}, \ldots, u^{(\hat{m})}\right). \end{cases} \tag{26}$$

It follows from $\dim \mathcal{D}_\infty^* \geq 1$ and Lemma 4.2 that the nonlinear system (1) is not controllable. □

Remark: The solution for $H, \tilde{P}$ and $\tilde{Q}$ in (22) is in general nonunique, even in the case that H is a greatest left common factor. Further, the expression for the one-form $\omega = \tilde{P}(d/dt)\, dy - \tilde{Q}(d/dt)\, du$ is also nonunique. If ω is an exact one-form, i.e., there is a function $z \in \mathcal{K}$ such that $dz = \omega$, then it is straightforward to construct the functions h and z in (3) to obtain the necessity result of the theorem. However, this is not generally true since there is no guarantee that every common factor H yields an exact one-form ω such that (23) is satisfied. For this reason, the proof of the necessity part of Theorem 4.1 deals with the general case that ω in (23) is a nonexact one-form. The proof is established by showing $\omega \in \mathcal{D}_\infty^*$ and using the results of Lemmas 2.1 and 4.2.

Remark: There is a connection between Lemma 3.2 and Proposition 3.4 in [1] where controllability of nonlinear state space systems is defined in terms of autonomous no-exact one-forms. The result of Proposition 3.4 of [1] is to relate the nonlinear system controllability to the relative degree of the no-exact one-forms. There are two essential differences between Lemma 4.2 and Proposition 3.4 of [1], i.e., (i) Lemma 4.2 deals with the differential subspace of nonlinear input/output systems which is in a more general framework than that of the nonlinear state space systems in [1]; (ii) Lemma 4.2 deals with controllability defined in terms of autonomous variables in the form (26) rather than autonomous one-forms in [1]. The former case requires further substantial treatments on integrating the autonomous variable.

V. Computation of Left Common Factors

Theorem 4.1 provides a criterion for testing the controllability of nonlinear systems, which is to examine whether the polynomials $P, Q \in \mathcal{K}[d/dt]$ of the nonlinear system have a left common factor. We now present a procedure based on the Euclidean Algorithm for computing a greatest left common factor of two polynomials in $\mathcal{K}[d/dt]$. Specifically, for $G_1, G_2 \in \mathcal{K}[d/dt]$ with $\deg G_1 = d_1$, $\deg G_2 = d_2$, and $d_1 > d_2$, the Euclidean Algorithm firstly computes two polynomials $G_3, L_1 \in \mathcal{K}[d/dt]$, with $\deg G_3 \leq d_2 - 1$ and $\deg L_1 = d_1 - d_2$ to obtain

$$G_1\left(\frac{d}{dt}\right) = G_2\left(\frac{d}{dt}\right) L_1\left(\frac{d}{dt}\right) + G_3\left(\frac{d}{dt}\right). \tag{27}$$

Then it continues to compute polynomials $G_i(d/dt)$ and $L_j(d/dt)$ for $4 \leq i \leq k$ and $2 \leq j \leq k-1$ as follows:

$$\begin{aligned} G_2\left(\frac{d}{dt}\right) &= G_3\left(\frac{d}{dt}\right) L_2\left(\frac{d}{dt}\right) + G_4\left(\frac{d}{dt}\right) \\ &\vdots \\ G_{k-2}\left(\frac{d}{dt}\right) &= G_{k-1}\left(\frac{d}{dt}\right) L_{k-2}\left(\frac{d}{dt}\right) + G_k\left(\frac{d}{dt}\right) \\ G_{k-1}\left(\frac{d}{dt}\right) &= G_k\left(\frac{d}{dt}\right) L_{k-1}\left(\frac{d}{dt}\right) \end{aligned} \tag{28}$$

where $\deg G_i \leq \deg G_{i-1} - 1$ and $\deg L_j \geq 1$. The algorithm terminates in a finite number of, say k, steps. As a result, a greatest left common factor $G_k(d/dt)$ of $G_1(d/dt)$ and $G_2(d/dt)$ is obtained which yields that $G_1(d/dt) = G_k(d/dt)\tilde{G}_1(d/dt)$ and $G_2(d/dt) = G_k(d/dt)\tilde{G}_2(d/dt)$, where the two polynomials $\tilde{G}_1(d/dt)$ and $\tilde{G}_2(d/dt)$ are obtained by recursively substituting $G_i(d/dt)$ for $2 \leq i \leq k$ into (28) and (27). The polynomials G_1 and G_2 are left coprime polynomials if and only if $\deg G_k = 0$ and $G_k \neq 0$.

Consider the following nonlinear system as an illustrative example:

$$y^{(2)} - y^{(1)} - \frac{\left(y^{(1)}\right)^2 - y^{(1)}u}{y} - u^{(1)} + u = 0 \tag{29}$$

where P and Q of the system polynomial equation in the form (19) are monic polynomials as follows:

$$\begin{aligned} P\left(\frac{d}{dt}\right) &= \left(\frac{d}{dt}\right)^2 - \frac{2y^{(1)} + y - u}{y}\left(\frac{d}{dt}\right) + \frac{\left(y^{(1)}\right)^2 - y^{(1)}u}{y^2} \\ Q\left(\frac{d}{dt}\right) &= \left(\frac{d}{dt}\right) - \frac{y^{(1)} + y}{y} \end{aligned}$$

Applying the Euclidean algorithm directly yields $P(d/dt) = Q(d/dt)((d/dt) - (y^{(1)} - u)/y))$. Thus a greatest left common factor of P and Q is $H(d/dt) = Q(d/dt) = (d/dt) - ((y^{(1)} + y)/(y))$ and a monic polynomial equation of the system is

$$H\left(\frac{d}{dt}\right)\left(\tilde{P}\left(\frac{d}{dt}\right) dy - \tilde{Q}\left(\frac{d}{dt}\right) du\right) = 0$$

where $\tilde{P}(d/dt) = (d/dt) - ((y^{(1)} - u)/(y)), \tilde{Q}(d/dt) = 1$. Since P and Q have a left common factor H with $\deg H = 1$, the system is uncontrollable. It is noted that, in this example, $\omega = \tilde{P}(d/dt)\, dy - \tilde{Q}(d/dt)\, du$ is a nonexact one-form.

VI. Conclusion

In this note, we have developed a polynomial approach to nonlinear system controllability. It is shown that, with the definition of controllability as the nonexistence of autonomous variables, a differential nonlinear system is controllable if and only if two polynomials in an Ore ring have no left common factors. This result extends nonlinear system controllability to a broad class of nonlinear system beyond the state equation framework. We have also shown that the Euclidean

$$\left\{dz^{(i)} \,\middle|\, 0 \le i \le l-1\right\} \subseteq \mathrm{span}_{\mathcal{K}} \left\{dy^{(i)}, du^{(j)} \,\middle|\, 0 \le i \le \hat{n}+l-1, 0 \le j \le \hat{m}+l-1\right\}$$
$$dz^{(l)} \in \mathrm{span}_{\mathcal{K}} \left\{dz^{(i)} \,\middle|\, 0 \le i \le l-1\right\}. \quad (33)$$

$$dz^{(i)} \notin \mathrm{span}_{\mathcal{K}} \left\{dy, dy^{(1)}, \ldots, dy^{(\hat{n}-1)}, du, du^{(1)}, \ldots, du^{(m-1)}\right\}, \quad \forall 0 \le i \le l-1.$$

$$\dim \mathrm{span}_{\mathcal{K}} \left\{dy, dy^{(1)}, \ldots, dy^{(\hat{n}-1)}, du, du^{(1)}, \ldots, du^{(m-1)}\right\} + \dim \mathrm{span}_{\mathcal{K}} \left\{dz, dz^{(1)}, \ldots, dz^{(l-1)}\right\} = \hat{n}+m+l \le \dim \mathcal{D}_1^* = n+m.$$

---

algorithm provides an effective algorithm for verifying this condition. This leads to a computable procedure for testing nonlinear system controllability using computer algebra. An important question that is a logical follow-up is the implications of these results to the local stabilization of nonlinear systems around an equilibrium point.

Appendix

Proof of Lemma 4.2: If $\mathcal{D}_\infty^* \subset \mathcal{D}_1^*$ satisfies $\dim \mathcal{D}_\infty^* \neq 0$, it contains at least one nonzero exact one-form by Lemma 3.1. Let $z \in \mathcal{K}$ be such that $dz \in \mathcal{D}_\infty^*$, with $dz \neq 0$, and let $\hat{n}$ and $\hat{m}$, with $0 \le \hat{n} \le n-1$ and $0 \le \hat{m} \le m-1$, be the largest integers such that $(\partial z)/(\partial y^{(\hat{n})}) \neq 0, (\partial z)/(\partial u^{(\hat{m})}) \neq 0$. Thus, dz can be written as

$$dz = \sum_{i=0}^{\hat{n}} \frac{\partial z}{\partial y^{(i)}}\, dy^{(i)} + \sum_{j=0}^{\hat{m}} \frac{\partial z}{\partial u^{(j)}}\, du^{(j)}. \quad (30)$$

The integration of this equation yields the function z in the form (3) in an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ with (2) being satisfied.

Since $dz \in \mathcal{D}_\infty^*$, $dz^{(k)} \in \mathcal{D}_1^*$ $\forall k \ge 0$. As $\dim D_1^* = n+m$, there exists a largest integer l with $1 \le l \le n+m$ such that $\{dz^{(i)} \,|\, 0 \le i \le l-1\}$ is a set of linearly independent exact one-forms in $\mathcal{D}_1^*$ over $\mathcal{K}$. Thus, we write

$$dz^{(l)} = \sum_{i=0}^{l-1} \psi_{z_i}\, dz^{(i)}. \quad (31)$$

The integration of this equation yields a function h in the form (3) in an open set $Y \times U \subset \mathcal{R}^n \times \mathcal{R}^{m+1}$ with (2) being satisfied.

By (30) we obtain that for $0 \le i \le l$

$$dz^{(i)} = \frac{\partial z}{\partial y^{(\hat{n})}}\, dy^{(\hat{n}+i)} + \cdots + \frac{\partial z}{\partial u^{(\hat{m})}}\, du^{(\hat{m}+i)} + \cdots \quad (32)$$

where the dots indicate terms involving lower order derivatives of dy and du, and $(\partial z)/(\partial y^{(\hat{n})})\, dy^{(\hat{n}+i)} \neq 0$ and $(\partial z)/(\partial u^{(\hat{m})})\, du^{(\hat{m}+i)} \neq 0$ are satisfied.

Since $dz \in \mathrm{span}_{\mathcal{K}}\{dy^{(i)}, du^{(j)} \,|\, 0 \le i \le \hat{n}, 0 \le j \le \hat{m}\}$ by (30) (31), we have (33), as shown at the top of the page. By (32) and (33), we have $dy^{(\hat{n}+l)} \in \mathrm{span}_{\mathcal{K}}\{dy^{(i)}, du^{(j)} \,|\, 0 \le i \le \hat{n}+l-1,\ 0 \le j \le \hat{m}+l\}$. Under Assumption 1 on the regularity of $\mathcal{D}_0^*$, this is only possible if $\hat{n}+l \ge n$.

On the other hand, it follows from (32) that the first equation shown at the top of the next page holds true. This implies that

$$\mathrm{span}_{\mathcal{K}} \left\{dy, dy^{(1)}, \ldots, dy^{(\hat{n}-1)}, du, du^{(1)}, \ldots, du^{(m-1)}\right\} \cap \mathrm{span}_{\mathcal{K}} \left\{dz, dz^{(1)}, \ldots, dz^{(l-1)}\right\} = \{0\}. \quad (34)$$

Since $dz^{(i)} \in \mathcal{D}_1^*$, $0 \le i \le l-1$, we have

$$\mathrm{span}_{\mathcal{K}} \left\{dy, dy^{(1)}, \ldots, dy^{(\hat{n}-1)}, du, du^{(1)}, \ldots, du^{(m-1)}\right\} \oplus \mathrm{span}_{\mathcal{K}} \left\{dz, dz^{(1)}, \ldots, dz^{(l-1)}\right\} \subseteq \mathcal{D}_1^*.$$

It follows from this and (34) that the second equation at the top of the page holds true. This, together with $\hat{n}+l \ge n$, yields $\hat{n}+l = n$.

With $\hat{n}+l = n$, consider $i = l$ in (32). The term $dy^{(\hat{n}+l)} = dy^{(n)}$ is linearly dependent on $du^{(m)}$ by (17). In order that $dz^{(l)} \in \mathrm{span}_{\mathcal{K}}\{dz, dz^{(1)}, \ldots, dz^{(l-1)}\} \in \mathcal{D}_1^*$ be satisfied, the term $(\partial z)/(\partial u^{(\hat{m})}) du^{(\hat{m}+l)}$ with $(\partial z)/(\partial u^{(\hat{m})}) \neq 0$ in (32) with $i = l$ must cancel out the like term due to $(\partial \hat{z})/(\partial y^{(\hat{n})}) dy^{(n)}$. This leads to $\hat{m}+l = m$. □

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Robust H_∞ Filtering of Stationary Continuous-Time Linear Systems With Stochastic Uncertainties

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***Abstract*—The problem of applying H_∞-filters on stationary, continuous-time, linear systems with stochastic uncertainties in the state-space signal model is addressed. These uncertainties are modeled via white noise processes. The relevant cost function is the expected value of the standard H_∞ performance index with respect to the uncertain parameters. The solution is obtained via a stochastic bounded real lemma that results in a modified Riccati inequality. This inequality is expressed in the form of a linear matrix inequality whose solution provides the filter parameters. The method proposed is also applied to the case where, in addition to the stochastic uncertainty, other deterministic parameters of the system are not perfectly known and are assumed to lie in a given polytope. The problem of mixed H_2/H_∞ filtering for the above system is also treated. The theory developed is demonstrated by a practical example.**

***Index Terms*—Mixed H_2/H_∞ filtering, polytopic uncertainty, stochastic H_∞ filtering.**

I. Introduction

The analysis and design of controllers and estimators for systems with stochastic uncertainties, which ensure a worst case performance bound, have recently received a lot of attention [1]–[9]. An approach in which the parameter uncertainties are modeled as white noise processes in a linear system has been developed in [1], [3], [5], [8], and [9] for the discrete-time state-feedback problem, and in [2] and [4] for the continuous-time counterpart. The estimation problem of stochastic systems has been solved in [4], [8], [9], and [5] for the continuous-time and the discrete-time cases, respectively. Such models of uncertainty are encountered in many areas of applications (see [3] and the references therein).

Recently, the solution of the estimation problem in the stationary discrete-time case was solved, via linear matrix inequalities (LMIs), where in addition to the stochastic parameter uncertainties, the system matrices were allowed to lie in a convex-bounded domain [9]. The solution in [9] applied a bounded-real lemma (BRL) on a general-type filter structure.

Also recently, the solution of the output-feedback problem for continuous-time stochastic uncertain systems has been derived for the stationary case [7]. The solution is obtained by formulating a stochastic bounded-real lemma and a general-type controller. It results in two coupled nonlinear matrix inequalities which reduce to the standard H_∞ output-feedback problem in the nominal case. The solution in [7] does not include uncertainty in the measurement matrix.

An alternative approach to the treatment of real parameter uncertainties considers uncertainties that lie in a polytope. This approach has been adopted in [10]–[12]. In [10], the authors apply the H_∞ BRL [14] to the uncertain system, and a Riccati inequality is obtained whose solution guarantees the existence of a single solution (fixed filter) to the problem. This Riccati inequality has been expressed in a LMI form that is affine in the uncertain parameters. A single solution which covers all the vertices of the uncertainty polytope produces the desired result [10]. The mixed H_2/H_∞ problem has been solved in [10].

In the present paper we treat the continuous-time counter-part of [8] and [9]. In our case, a stochastic uncertainty appears in both the dynamic and the measurement matrices and correlations are allowed between the uncertain parameters. This case, where the uncertainties are restricted to the above matrices, is the most encountered in practice when one considers filtered estimation. Our solution is based on the BRL developed in [6] and [7]. In a manner similar to [9] we apply the techniques of [15] to the deterministic polytopic problem [11], [10]. Necessary and sufficient conditions are derived for the existence of a solution in terms of LMIs. The latter solution is extended to the case where the deterministic part of the system matrices lie in a convex bounded domain of polytopic-type. Our theory is also applicable to the case where the covariance matrices of the stochastic parameters are not perfectly known and lie in a polytopic domain. We also solve the mixed H_2/H_∞ problem where, of all the filters that solve the stochastic H_∞ filtering problem, the one that minimizes an upper-bound on the estimation error variance is found. The method developed here is demonstrated by a practical example.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. For a symmetric $P \in \mathcal{R}^{n\times n}$, $P > 0$ means that it is positive definite. We denote expectation by $\mathbf{E}\{\cdot\}$ and the trace of a matrix by $\mathrm{Tr}\{\cdot\}$. We denote by $L^2(\Omega, \mathcal{R}^k)$ the space of square-integrable $\mathcal{R}^k -$ valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_t)_{t>0}$ we denote an increasing family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. We also denote by $\tilde{L}^2([0,\infty); \mathcal{R}^k)$ the space of nonanticipative stochastic

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