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Dissipative differential systems and the state space $\mathscr{H}_\infty$ control problem

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SUMMARY

The purpose of this paper is to apply our very recent results on the synthesis of dissipative linear differential systems to the 'classical' state space $\mathscr{H}_\infty$ control problem. We first review our general problem set-up from Willems and Trentelman, *IEEE Transactions on Automatic Control*, Submitted, where the problem of rendering a given plant dissipative by general interconnection, is reformulated as the problem of finding a suitable dissipative behaviour wedged in between the 'hidden' behaviour and the 'manifest' behaviour of the plant. We review our main result from Willems and Trentelman, which states that a necessary and sufficient condition for the existence of such behaviour is, that the hidden behaviour and manifest behaviour are dissipative with respect to suitable supply rates, and have storage functions that satisfy a certain coupling condition. We then apply this result to the state-space $\mathscr{H}_\infty$ control problem. We show that our general result in this case reduces to the existence of solutions to certain algebraic Riccati equations, satisfying the well-known coupling condition. We also derive state-space formulas for the required controllers. Copyright © 2000 John Wiley and Sons, Ltd.

KEY WORDS: $\mathscr{H}_\infty$ control; behaviours; dissipativity; Riccati equations; coupling condition

1. INTRODUCTION

It is a privilege to contribute this article to this special issue dedicated to the memory of George Zames. His vision and views on control influenced our research in a deep way. For the second author, this influence has been a very direct one over more than three decades. I (JCW) wrote my doctoral dissertation on input/output stability [1], based on the setting of extended spaces introduced by Zames [2], and using his beautiful small gain and positive operator principles for obtaining stability results. It is with fond memories that I recall this early interaction with George, an interaction that continued off and on until his untimely passing away.

This article is about $\mathscr{H}_\infty$-control, a topic that was put to the foreground of control research by the seminal paper [3] by George Zames. Our recent research [4–7] in this area uses the behavioural approach [8] to control combined with the theory of quadratic differential forms [9] in order to come up with the controlled behaviour. While preparing an extensive paper [5] on this topic, we came to realize that our approach also simplifies and extends the results of Reference

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[10], where the solution to the state-space $\mathscr{H}_\infty$-problem in terms of the double Riccati equation with the remarkable coupling condition was developed. In Reference [5] a very general version of the $\mathscr{H}_\infty$-control problem is solved, which does not need to start with a state-space description of the plant, and has a cost functional that is a general quadratic differential form. Also, no non-singularity assumptions need to be made.

The aim of this article is to apply the very general results from Reference [5] to the 'classical' state-space suboptimal $\mathscr{H}_\infty$ control problem by measurement feedback, which, we argue, is merely a special case of our general problem formulation. Unlike most of the existing literature on the suboptimal $\mathscr{H}_\infty$ control problem, we address the problem of finding, for a given tolerance $\gamma > 0$, an internally stabilizing feedback controller that makes the closed-loop transfer matrix T satisfy the *non-strict* inequality $\|T\|_{\mathscr{H}_\infty} \leqslant \gamma$ (instead of $< \gamma$). It turns out that also this problem admits a solution in terms of two algebraic Riccati equations, together with a coupling condition. Unlike the strict suboptimal problem, the dimension of the state space of our controllers turns out to depend on the solutions of these Riccati equations.

The outline of this article is as follows. In Section 2 we review the basic material that we need on linear differential systems, dissipativity, and storage functions. In Section 3 we briefly discuss the general control problem of making a given differential system dissipative by interconnecting it with a controller. In Section 4 we apply these general results to the state-space $\mathscr{H}_\infty$ control problem by measurement feedback.

The following notation will be used in this paper. We denote $\mathbb{R}_- = (-\infty, 0)$, and $\mathbb{R}_+ = (0, \infty)$. $\mathbb{R}^{q \times q}$ denotes the space of $q \times q$ matrices with real coefficients. $\mathbb{R}^{p \times q}[\xi]$ denotes the set of $p \times q$ polynomial matrices with real coefficients. If the dimensions are not specified we use the notation $\mathbb{R}^{\bullet \times \bullet}[\xi]$. $\mathbb{R}^{p \times q}[\zeta, \eta]$ is the set of all $p \times q$ two-variable polynomial matrices with real coefficients. $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ is the space of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$. If the dimension of the codomain is not specified, we use the notation $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$. $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ denotes the subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ consisting of functions with compact support. $\mathfrak{D}(\mathbb{R}_-, \mathbb{R}^q)$ is the space of infinitely often differentiable functions from $\mathbb{R}_-$ to $\mathbb{R}^q$ that have compact support. Finally, $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^q)$ is the space of all square integrable functions from $\mathbb{R}$ to $\mathbb{R}^q$. For $S = S^{\mathrm{T}} \in \mathbb{R}^{q \times q}$ and $v \in \mathbb{R}^q$ we denote $|v|_S^2 := v^{\mathrm{T}} S v$.

2. LINEAR DIFFERENTIAL SYSTEMS AND DISSIPATIVITY

In this section we review the basics of linear differential systems. For a more detailed treatment, we refer to Reference [8]

A subset $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ (called a *behaviour*) is called a linear time-invariant differential system (briefly, a *differential system*) if there exists a polynomial matrix $R \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid R(\mathrm{d}/\mathrm{d}t)w = 0\}$. The set of all linear time-invariant differential systems is denoted by $\mathfrak{L}^\bullet$. We denote by $\mathfrak{L}^{\mathtt{w}}$ those with $\mathtt{w}$ real variables (in other words, with behaviours $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$). For a given differential system $\mathfrak{B}$, we denote by $\mathtt{m}(\mathfrak{B})$ its input cardinality. This is defined as the number of free variables in $\mathfrak{B}$, i.e. the number of components of the vector function w that are not constrained by the requirement that w is an element of the behaviour $\mathfrak{B}$.

We call $\mathfrak{B} \in \mathfrak{L}^\bullet$ *controllable* if for all $w_1, w_2 \in \mathfrak{B}$, there exists a $T \geqslant 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - T)$ for $t \geqslant T$. By $\mathfrak{L}^\bullet_{\mathrm{cont}}$, $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ we denote the controllable elements of $\mathfrak{L}^\bullet$, $\mathfrak{L}^{\mathtt{w}}$.

Whereas a differential system $\mathfrak{B}$ is *defined* as the solution space of a differential equation of the form $R(\mathrm{d}/\mathrm{d}t)w = 0$, there are other possible representations for such $\mathfrak{B}$. The ones used in this paper are the *driving variable representation*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bd, \quad w = Cx + Dd$$

with d a free, additional variable, called the driving variable, the *output nulling representation*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw, \quad 0 = Cx + Dw$$

and the *input/state/output representation*

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, \quad y = Cx + Du, \quad w = (u, y)$$

Every differential system $\mathfrak{B}$ admits a driving variable representation, an output nulling representation, and an input/state/output representation, the last one after possibly permuting the components of w. All these representations are a special case of the so-called *state representation*

$$E\frac{\mathrm{d}}{\mathrm{d}t}x + Fx + Gw = 0$$

In such representation of $\mathfrak{B}$, the variable x is called the *state variable*.

In order to be able to define the notions of dissipativity and storage function, we need the concepts of two-variable polynomial matrix and quadratic differential form. For a detailed treatment, we refer to References [9, 4]. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ be a two-variable polynomial matrix, i.e. a matrix whose entries are real polynomials in two indeterminates, say ζ and η. Written out in terms of its coefficient matrices such polynomial matrix is given by $\Phi(\zeta, \eta) = \Sigma_{k,\ell \in \mathbb{Z}_+} \Phi_{k,\ell}\zeta^k \eta^\ell$, where, of course, the sum is actually a finite one. This two-variable polynomial matrix induces the map $L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$L_\Phi(w_1, w_2) = \sum_{k,\ell \in \mathbb{Z}_+} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w_1\right)^{\mathrm{T}} \Phi_{k,\ell} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w_2\right)$$

This map is called the *bilinear differential form* induced by Φ. When $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$, this induces the map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{k,\ell \in \mathbb{Z}_+} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w\right)^{\mathrm{T}} \Phi_{k,\ell} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w\right), \quad \text{i.e. } Q_\Phi(w) = L_\Phi(w, w)$$

This map is called the *quadratic differential form* (QDF) induced by Φ.

We now define the notion of dissipative system. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. The system $\mathfrak{B}$ is said to be *dissipative with respect to* Q_Φ (or: Φ-*dissipative*) if $\int_{-\infty}^{+\infty} Q_\Phi(w)\,\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. The quadratic differential form Q_Φ is called the supply rate. Intuitively, we think of $Q_\Phi(w)$ as the power going into the physical system $\mathfrak{B}$. Dissipativity expresses the property that along trajectories w of $\mathfrak{B}$ that start at rest and bring the system back to rest, the net amount of energy flowing into the system is non-negative. The system $\mathfrak{B}$ is said to be *dissipative on* $\mathbb{R}_-$ *with respect to* Q_Φ (or Φ-*dissipative on* $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Phi(w)\,\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B}_- \cap \mathfrak{D}(\mathbb{R}_-, \mathbb{R}^{\mathtt{w}})$.

Here, $\mathfrak{B}_-$ denotes the behaviour $\mathfrak{B}$ restricted to the half-line $\mathbb{R}_- = (-\infty, 0)$, i.e. $\mathfrak{B}_- = \{w_{|\mathbb{R}_-|} w \in \mathfrak{B}\}$. Note that dissipativity on $\mathbb{R}_-$ implies dissipativity.

It is a basic fact (see e.g. Reference [11, Theorem 4.3] that a system is dissipative with respect to a given supply rate if and only if it has a storage function. The notion of storage function is defined as follows. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$, $\Phi \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$, and $\Psi \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$. Then Q_Ψ is said to be a *storage function for* $\mathfrak{B}$ *with respect to the supply rate* Q_Φ if the *dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_\Phi(w)$$

holds for all $w \in \mathfrak{B}$.

It was shown in Reference [11, Corollary 6.3], that if the system $\mathfrak{B}$ is dissipative with respect to the supply rate induced by a constant two-variable polynomial matrix, then every storage function is representable as a quadratic function of the state variable of the system:

Proposition 1

Let $E(\mathrm{d}/\mathrm{d}t)x + Fx + Gw = 0$ be a state representation of $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, with $\dim(x) = \mathrm{n}$. Let $S = S^{\mathrm{T}} \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}$ and assume that $\mathfrak{B}$ is dissipative with respect to the supply rate $|w|^2_S$. Let Q_Ψ be a storage function. Then there exists $K = K^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ such that for all (w, x) satisfying $E(\mathrm{d}/\mathrm{d}t)x + Fx + Gw = 0$ we have $Q_\Psi(w) = |x|^2_K$.

We also need the *orthogonal complement* of a controllable behaviour. This is defined as follows. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$. Then there exists a unique behaviour in $\mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, denoted by $\mathfrak{B}^\perp$, such that

$$\int_{-\infty}^{+\infty} w_1^{\mathrm{T}} w_2 \,\mathrm{d}t = 0$$

for all $w_1 \in \mathfrak{B} \cap \mathfrak{D}$ and $w_2 \in \mathfrak{B}^\perp$. It is easy to see that $(\mathfrak{B}^\perp)^\perp = \mathfrak{B}$. If $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, and $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}$, then we denote the orthogonal complement $(\Sigma\mathfrak{B})^\perp$ of $\Sigma\mathfrak{B}$ by $\mathfrak{B}^{\perp\Sigma}$. Note that $\mathfrak{B}^{\perp\Sigma} = \Sigma^{-1}\mathfrak{B}^\perp := \{w \mid \Sigma w \in \mathfrak{B}^\perp\}$.

In order to state our main result, we need the following fact that is proven in Reference [4]:

Proposition 2

Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$. Then there exists a two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\Phi(w_1, w_2) = w_1^{\mathrm{T}} w_2$$

for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$, if and only if $\mathfrak{B}_1 \subset \mathfrak{B}_2^\perp$. Moreover, if $\Phi_1, \Phi_2 \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ both satisfy this equality, then $L_{\Phi_1}(w_1, \mathrm{w}_2) = L_{\Phi_2}(w_1, \mathrm{w}_2)$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$.

If $L_\Phi(w_1, \mathrm{w}_2)$ is a bilinear differential form satisfying the condition of this proposition, then the (uniquely defined) restriction of $L_\Phi(w_1, w_2)$ to $\mathfrak{B}_1 \times \mathfrak{B}_2$ is called the $(\mathfrak{B}_1, \mathfrak{B}_2)$-*adapted* bilinear differential form.

Finally, we need the following fact that relates a driving variable representation of $\mathfrak{B}$ to an output-nulling representation of $\mathfrak{B}^\perp$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$. Then $(\mathrm{d}/\mathrm{d}t)x = Ax + Bd$, $w_1 = Cx + Dd$ is a driving variable representation of $\mathfrak{B}$ (with manifest variable w_1) if and only if $(\mathrm{d}/\mathrm{d}t)z = -A^{\mathrm{T}}z - C^{\mathrm{T}}w_2$, $0 = B^{\mathrm{T}}z + D^{\mathrm{T}}w_2$ is an output-nulling representation of $\mathfrak{B}^\perp$ (with manifest variable w_2). This fact is easily proven after observing that $(\mathrm{d}/\mathrm{d}t)z^{\mathrm{T}}x = w_2^{\mathrm{T}}w_1$ for $w_1 \in \mathfrak{B}$ and $w_2 \in \mathfrak{B}^\perp$.

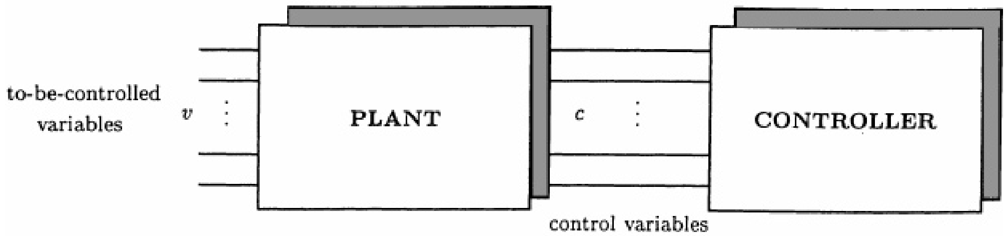

Figure 1. Plant and controller configuration.

3. THE CONTROL PROBLEM

Consider Figure 1. The plant is assumed to be a linear differential system with two types of terminals: terminals carrying the *to-be-controlled variables* v and terminals carrying the *control variables* c.

In the case of $\mathscr{H}_\infty$ control to be discussed later in this paper, we will have $v = (d, z)$, with d exogenous disturbance inputs, and z to be controlled outputs, and $c = (u, y)$, with u the control inputs and y the measured outputs. In the first part of this paper we aim at a somewhat higher level of generality, and therefore do not (yet) make these partitions of v and c. Assume that there are $\mathtt{v}$ to-be-controlled variables and $\mathtt{c}$ control variables.

Let $\mathscr{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$ be the full behaviour of the plant, i.e. the set of all trajectories (v, c) that satisfy the equations of the plant. Furthermore, let $\mathscr{P}$ (called the *manifest behaviour*) be the behaviour of the to-be-controlled variables v, with c eliminated. Hence

$$\mathscr{P}_{\text{full}} = \{(v, c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}+\mathtt{c}}) \mid (v, c) \text{ satisfies the plant equations}\}$$

$$\mathscr{P} := \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \mid \exists c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \text{ such that } (v, c) \in \mathscr{P}_{\text{full}}\}$$

A third relevant behaviour associated with the plant is the *hidden behaviour* $\mathscr{N}$ defined as

$$\mathscr{N} := \{v \in \mathscr{P} \mid (v, 0) \in \mathscr{P}_{\text{full}}\}$$

The controller restricts the control variables c and is described by a *controller behaviour* $\mathscr{C} \in \mathfrak{L}^{\mathtt{c}}$. Formally,

$$\mathscr{C} = \{c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid c \text{ satisfies the controller equations}\}$$

After the controller is attached, we obtain the *controlled behaviour* $\mathscr{K}$ defined by

$$\mathscr{K} := \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{v}}) \mid \exists c \in \mathscr{C} \text{ such that } (v, c) \in \mathscr{P}_{\text{full}}\} \tag{1}$$

We say that $\mathscr{C}$ *implements* $\mathscr{K}$ if the above relation holds between $\mathscr{C}$ and $\mathscr{K}$.

Now, our point of view is the following. We have been given a full plant behaviour $\mathscr{P}_{\text{full}}$, consisting of all time trajectories (v, c) that satisfy the plant equations. By restricting the control variables c to belong to the controller behaviour $\mathscr{C}$, the actual v trajectories will belong to the controlled behaviour $\mathscr{K}$ given by (1). It is a trivial fact that this implies that the hidden behaviour $\mathscr{N}$ must be contained in $\mathscr{K}$, and that $\mathscr{K}$ must be contained in the manifest behaviour $\mathscr{P}$, in other words, we must have $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$. We are however also very much interested in the converse question: if $\mathscr{K} \in \mathfrak{L}^{\mathtt{v}}$ is a given behaviour, under what conditions is it a controlled behaviour for

$\mathscr{P}_{\text{full}}$, i.e. under what conditions does there exist a controller $\mathscr{C}$ such that $\mathscr{K}$ is implemented by $\mathscr{C}$? It turns out that the inclusion $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$ is also *sufficient* for the existence of such $\mathscr{C}$:

Proposition 3

Let $\mathscr{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$ be the full plant behaviour, $\mathscr{P} \in \mathfrak{L}^{\mathtt{v}}$ the manifest plant behaviour, and $\mathscr{N}$ the hidden behaviour. Then $\mathscr{K} \in \mathfrak{L}^{\mathtt{v}}$ is implementable by a controller $\mathscr{C} \in \mathfrak{L}^{\mathtt{c}}$ acting on the control variables if and only if $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$.

Proposition 3 reduces control problems to finding the controlled behaviour directly. Of course, the problem of how to actually implement the controlled behaviour $\mathscr{K}$ (for example, as a signal processor from sensor outputs to actuator inputs), needs to be addressed at some point as well. Proposition 3 shows that $\mathscr{K}$ can be *any* behaviour that is wedged in between the given behaviours $\mathscr{N}$ and $\mathscr{P}$.

Supported by Proposition 3, we will now express the control specifications on the to-be-controlled variables v as properties of the controlled behaviour $\mathscr{K}$. We go back to Figure 1. Let $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$. The *weighting matrix* Σ defines the quadratic form $|v|^2_\Sigma$ that will serve as a control performance functional. Denote by $\operatorname{sign}(\Sigma) = (\sigma_-(\Sigma), \sigma_+(\Sigma))$ its signature, i.e. $\sigma_-(\Sigma)$ and $\sigma_+(\Sigma)$ equal, respectively, the number of negative and positive eigenvalues of Σ. We assume that Σ is non-singular, i.e. that $\operatorname{rank}(\Sigma) = \sigma_-(\Sigma) + \sigma_+(\Sigma) = \mathtt{v}$. We now formulate our control problem.

Problem formulation

Let $\mathscr{N}, \mathscr{P} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$, and $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be given, with $\mathscr{N} \subset \mathscr{P}$ and Σ nonsingular; $\mathscr{P}$ is called the *plant behaviour*, $\mathscr{N}$ the *hidden behaviour*, and Σ the *weighting matrix*. The problem is to find $\mathscr{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ (called the *controlled behaviour*) such that:

1. $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$ (implementability and realizability)
2. $\mathscr{K}$ is Σ-dissipative on $\mathbb{R}_-$ (dissipativity)
3. $\mathtt{m}(\mathscr{K}) = \sigma_+(\Sigma)$ (liveness)

We now discuss the interpretation of the conditions in this problem formulation. Condition 1. has already been discussed. The dissipativity condition 2 may be viewed as consisting of two parts: Σ-dissipativity, and Σ-dissipativity on $\mathbb{R}_-$. That the controlled behaviour $\mathscr{K}$ must be Σ-dissipative is the basic control design specification. As we shall see, by suitably choosing Σ, it implies $\mathscr{H}_\infty$ disturbance attenuation. The fact that Σ-dissipativity is required to hold on $\mathbb{R}_-$, and not just on $\mathbb{R}$, implies that the controlled behaviour is also required to be stable in a sense to be explained at the end of this section. The liveness condition 3 on the controlled behaviour requires that a certain number, $\sigma_+(\Sigma)$, of exogenous variables must remain free in the controlled behaviour.

The following necessary and sufficient conditions for the existence of a controlled behaviour $\mathscr{K}$ were obtained in Reference [4]:

Proposition 4

A controlled behaviour $\mathscr{K} \in \mathfrak{L}^{\mathtt{v}}_{\text{cont}}$ described in the problem formulation exists if and only if the following conditions are satisfied:

1. $\mathscr{N}$ is Σ-dissipative,
2. $\mathscr{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative,
3. there exist two-variable polynomial matrices

$$\Psi_{\mathscr{N}}, \Psi_{\mathscr{P}^{\perp_\Sigma}}, \Psi_{(\mathscr{N}, \mathscr{P}^{\perp})} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}[\zeta, \eta],$$

defining

- a storage function $Q_{\Psi_{\mathcal{N}}}$ for $\mathcal{N}$ as a Σ-dissipative system, i.e. $(d/dt)Q_{\Psi_{\mathcal{N}}}(v_1) \leqslant |v_1|^2_\Sigma$ for $v_1 \in \mathcal{N}$,
- a storage function $Q_{\Psi_{\mathscr{P}^{\perp\Sigma}}}$ for $\mathscr{P}^{\perp\Sigma}$ as a $(-\Sigma)$-dissipative system, i.e. $(d/dt)Q_{\Psi_{\mathscr{P}^{\perp\Sigma}}}(v_2) \leqslant -|v_2|^2_\Sigma$ for $v_2 \in \mathscr{P}^{\perp\Sigma}$,
- and the $(\mathcal{N}, \mathscr{P}^\perp)$-adapted bilinear differential form $L_{\Psi_{(\mathcal{N},\mathscr{P}^\perp)}}$, i.e. $(d/dt)L_{\Psi_{(\mathcal{N},\mathscr{P}^\perp)}}(v_1, \Sigma v_2) = v_1^T\Sigma v_2$, for $v_1 \in \mathcal{N}$, $v_2 \in \mathscr{P}^{\perp\Sigma}$ (note that by Proposition 2, $\Psi_{(\mathcal{N},\mathscr{P}^\perp)}$ is well-defined, since $\mathcal{N} \subset \mathscr{P}$)

such that the QDF

$$Q_{\Psi_{\mathcal{N}}}(v_1) - Q_{\Psi_{\mathscr{P}^{\perp\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N},\mathscr{P}^\perp)}}(v_1, \Sigma v_2) \tag{2}$$

is non-negative for all $v_1 \in \mathcal{N}$ and $v_2 \in \mathscr{P}^{\perp\Sigma}$.

Note that Theorem 4 is formulated completely in terms of properties of the plant behaviour $\mathscr{P}$ and the hidden behaviour $\mathcal{N}$. Nowhere in the conditions *representations* of these behaviours appear.

If the to-be-controlled variables v are given as $v = (d, f)$, and the weighting matrix Σ is equal to the signature matrix

$$\Sigma = \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{bmatrix}$$

where $\mathtt{d} = \dim(d)$ and $\mathtt{f} = \dim(f)$, then our general problem formulation is equivalent to an $\mathscr{H}_\infty$ control problem. Indeed, the following was proven in Reference [7]:

Proposition 5

Let $\mathscr{K} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\mathrm{cont}}$ be an arbitrary differential system, with manifest variable (d, z). Then (i) and (ii) below are equivalent

(i)
- $\int_{-\infty}^0 |d|^2 - |f|^2 \, dt \geqslant 0$ for all $(d, f) \in \mathscr{K}_- \cap \mathfrak{D}(\mathbb{R}_-, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$ (dissipativity on $\mathbb{R}_-$),
- $\mathtt{m}(\mathscr{K}) = \mathtt{d}$ (liveness).

(ii)
- $\int_{-\infty}^\infty |d|^2 - |f|^2 \, dt \geqslant 0$ for all $(d, f) \in \mathscr{K} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$ (disturbance attenuation with gain 1),
- $(0, f) \in \mathscr{K}$ implies $\lim_{t\to\infty} f(t) = 0$ (external stability),
- d is free in $\mathscr{K}$. More explicitly, for all $d \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$, there exists $f \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^f)$ such that $(d, f) \in \mathscr{K}$ (freedom of d).

4. APPLICATION TO THE STATE SPACE $\mathscr{H}_\infty$ CONTROL PROBLEM

In this section we will apply our general result of Proposition 4 to the special case that the full plant is given in state-space form. Thus, we will obtain necessary and sufficient conditions, in terms of solvability of two Riccati equations together with a coupling condition, for the existence of a feedback controller that achieves internal stability and a closed-loop transfer matrix with $\mathscr{H}_\infty$ norm less than or equal to one. We will also establish formulas for the actual controllers.

4.1. Problem formulation

Assume that $\mathscr{P}_{\mathrm{full}}$ is represented by

$$\begin{aligned} (d/dt)x &= Ax + Bu + Gd \\ y &= Cx \qquad + Dd \\ f &= Hx + Ju \end{aligned} \tag{3}$$

In these equations, u are the inputs to the actuators, y are the outputs of the sensors, d the exogenous disturbances, and f the endogenous to-be-controlled outputs. In terms of the notation used in the previous section we have $v = (d, f)$ as the to-be-controlled variables and $c = (u, y)$ as the control variables. The problem is to find a controller acting on the control variables (u, y) such that the controlled system meets certain specifications. We want this controller to be also in state representation, more exactly, in input/state/output representation, with y the input, u the output, and with the controller state denoted as x_c:

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)\, x_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y\end{aligned} \tag{4}$$

The control problem that we consider is to find a controller (4) such that the controlled system satisfies the following specifications:

1. *disturbance attenuation* with gain factor normalized to 1, i.e. for all $(d, f) \in \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}})$ for which there exist (u, y, x, x_c) satisfying both the plant and the controller equations, we have $\|f\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{f}})} \leqslant \|d\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{d}})}$;
2. *internal stability*, meaning that in the controlled system $d = 0$ implies that the signals (x, x_c, u, f) all go to zero as $t \to \infty$.

Clearly, this problem formulation involves the weighting matrix $\Sigma = \mathrm{diag}(I_{\mathrm{d}}, -I_{\mathrm{f}})$, in the sense that $|v|_\Sigma^2 = |d|^2 - |f|^2$. Note that these specifications are equivalent to internal stability and the condition that the $\mathscr{H}_\infty$ norm of the closed-loop transfer matrix T satisfies the (non-strict) inequality $\|T\|_{\mathscr{H}_\infty} \leqslant 1$, see also References [12–15]. This condition differs from the strict inequality $\|T\|_{\mathscr{H}_\infty} < 1$ usually studied in the literature, see References [4, 10, 15–22].

We want to apply the result of Proposition 4 to obtain necessary and sufficient conditions for the existence of a feedback controller that achieves the specifications 1 and 2 above. However, Proposition 4 is of course not concerned with internal stability, but only with external stability. In fact, by Proposition 5, Proposition 4 gives necessary and sufficient conditions for the existence of a behaviour $\mathscr{K} \in \mathfrak{L}^{\mathrm{d}+\mathrm{f}}$ that satisfies the conditions

1′. $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$,
2′. $\|f\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^f)} \leqslant \|d\|_{\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{d}})}$ for all $(d, f) \in \mathscr{K} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}})$,
3′. $(0, f) \in \mathscr{K}$ implies that $f(t)$ goes to zero as $t \to \infty$,
4′. the exogenous disturbances d are free in $\mathscr{K}$. More explicitly, for all $d \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{d}})$, there exists $f \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^f)$ such that $(d, f) \in \mathscr{K}$.

Here, $\mathscr{P}$ and $\mathscr{N}$ and are the manifest behaviour and hidden behaviour associated with the plant $\mathscr{P}_{\mathrm{full}}$ given by (3). Yet, it is immediately clear that if a controller (4) exists that achieves disturbance attenuation and internal stability, then the corresponding controlled behaviour $\mathscr{K}$ will satisfy 1′–4′. This simply follows from the facts that internal stability implies external stability, and that in the controlled system obtained by interconnecting (3) and (4), the disturbance d is automatically free. Thus, the conditions on $\mathscr{N}$ and $\mathscr{P}$ of proposition 4 are *necessary conditions* for the existence of a disturbance attenuating, internally stabilizing controller. In Section 4.3, we will therefore compute for $\mathscr{P}_{\mathrm{full}}$ in state-space representation (3) the manifest behaviour $\mathscr{P}$ and the hidden behaviour $\mathscr{N}$. The conditions of Proposition 4 will then turn out to reduce exactly to the existence of solutions to two Riccati equations that satisfy a coupling condition.

Conversely, if these two Riccati equations have solutions that satisfy the coupling condition, equivalently, if $\mathscr{N}$ and $\mathscr{P}$ satisfy the conditions of Proposition 4, then $\mathscr{K}$ exists such that $1'$–$4'$ are satisfied. From condition $1'$ we know that it is implementable by a controller acting on (u, y), achieving disturbance attenuation, *external* stability, and freedom of d. We will show in Sections 4.6 and 4.7 that in that case also a controlled behavior $\mathscr{K}$ exists that is implementable by an *internally* stabilizing *feedback* controller of the form (4).

In order to simplify formulas, we assume that the following conditions hold:

A.1. $DD^{\mathrm{T}} = I$ and $DG^{\mathrm{T}} = 0$,
A.2. $J^{\mathrm{T}}J = I$ and $J^{\mathrm{T}}H = 0$,
A.3. (A, G) is a controllable pair of matrices,
A.4. (H, A) is an observable pair of matrices.

The outline of this section is as follows. In the next subsection we state our main result, which gives necessary and sufficient conditions for the existence of a disturbance attenuating, internally stabilizing feedback controller, and formulas for such controller. Subsequently, in Sections 4.3–4.6, we derive this main result along the lines of the strategy outlined above.

4.2. Statement of the main result

In order to state our main result, we need to introduce the following Riccati inequalities and algebraic Riccati equations associated with the plant (3):

$$-A^{\mathrm{T}}K_{\mathscr{N}} - K_{\mathscr{N}}A - H^{\mathrm{T}}H - K_{\mathscr{N}}GG^{\mathrm{T}}K_{\mathscr{N}} + C^{\mathrm{T}}C \geqslant 0 \tag{5}$$

$$-A^{\mathrm{T}}K_{\mathscr{N}} - K_{\mathscr{N}}A - H^{\mathrm{T}}H - K_{\mathscr{N}}GG^{\mathrm{T}}K_{\mathscr{N}} + C^{\mathrm{T}}C = 0 \tag{6}$$

$$AK_{\mathscr{P}} + K_{\mathscr{P}}A^{\mathrm{T}} - GG^{\mathrm{T}} - K_{\mathscr{P}}H^{\mathrm{T}}HK_{\mathscr{P}} + BB^{\mathrm{T}} \geqslant 0 \tag{7}$$

$$AK_{\mathscr{P}} + K_{\mathscr{P}}A^{\mathrm{T}} - GG^{\mathrm{T}} - K_{\mathscr{P}}H^{\mathrm{T}}HK_{\mathscr{P}} + BB^{\mathrm{T}} = 0 \tag{8}$$

in the unknowns $K_{\mathscr{N}}$ and $K_{\mathscr{P}}$. Our main result is the following:

Theorem 6

Assume that the plant (3) satisfies A.1–A.4. Then the following statements are equivalent:

(i) There exists a feedback controller (4) such that the controlled system is internally stable, and the closed-loop transfer matrix T satisfies $\|T\|_{\mathscr{H}_\infty} \leqslant 1$,
(ii) there exist real symmetric solutions $K_{\mathscr{N}}$ and $K_{\mathscr{P}}$ of the Riccati inequalities (5) and (7), respectively, satisfying the conditions $K_{\mathscr{N}} > 0$, $K_{\mathscr{P}} < 0$, and $K_{\mathscr{N}} \geqslant (-K_{\mathscr{P}})^{-1}$.
(iii) there exist real symmetric solutions of the algebraic Riccati equations (6) and (8), and the largest real symmetric solution $K_{\mathscr{N}}^{+}$ of (6), and the smallest real symmetric solution $K_{\mathscr{P}}^{-}$ of (8) satisfy $K_{\mathscr{N}}^{+} > 0$, $K_{\mathscr{P}}^{-} < 0$, and $K_{\mathscr{N}}^{+} \geqslant (-K_{\mathscr{P}}^{-})^{-1}$.

In that case, one such feedback controller is given by the singular state-space representation

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_{\mathscr{N}}^{+} + (K_{\mathscr{P}}^{-})^{-1})\hat{x} = (K_{\mathscr{N}}^{+} + (K_{\mathscr{P}}^{-})^{-1})(A\hat{x} + Bu + G\hat{d}) + C^{\mathrm{T}}(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

$$\hat{d} = -G^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x}$$

$$u = B^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x}$$

Alternatively, one such feedback controller is given by the regular state-space representation

$$\frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{T}}(K_{\mathcal{N}}^{+} + (K_{\mathcal{P}}^{-})^{-1})R\hat{x}_1 = R^{\mathrm{T}}(K_{\mathcal{N}}^{+} + (K_{\mathcal{P}}^{-})^{-1})(A(P\hat{x}_1 + Qy) + Bu + G\hat{d}) + R^{\mathrm{T}}C^{\mathrm{T}}(y - \hat{y})$$

$$\hat{y} = C(P\hat{x}_1 + Qy)$$

$$\hat{d} = -G^{\mathrm{T}}K_{\mathcal{P}}^{-1}(P\hat{x}_1 + Qy)$$

$$u = B^{\mathrm{T}}K_{\mathcal{P}}^{-1}(P\hat{x}_1 + Qy)$$

Here, R is any injective matrix such that $\mathrm{im}(R) = \mathrm{im}(K_{\mathcal{N}}^{+} + (K_{\mathcal{P}}^{-})^{-1})$, N is any injective matrix such that $\mathrm{im}(N) = \ker(K_{\mathcal{N}}^{+} + (K_{\mathcal{P}}^{-})^{-1})$, $Q := N(CN)^{\#}$, and $P := (I - N(CN)^{\#}C)R$. Here, $(CN)^{\#}$ denotes the Moore–Penrose inverse of CN. The dimension of the state space of this controller is equal to $\mathrm{rank}(K_{\mathcal{N}}^{+} + (K_{\mathcal{P}}^{-})^{-1})$.

4.3. *Verification of the conditions*

Following the strategy outlined above, we first compute the manifest behaviour and hidden behaviour associated with the plant $\mathcal{P}_{\mathrm{full}}$. The equations for the plant immediately yield the following driving variable representation for the plant behaviour $\mathcal{P}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathcal{P}} = Ax_{\mathcal{P}} + [B \quad G]\begin{bmatrix} d'_{\mathcal{P}} \\ d''_{\mathcal{P}} \end{bmatrix}$$

$$v_{\mathcal{P}} = \begin{bmatrix} 0 \\ H \end{bmatrix} x_{\mathcal{P}} + \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} d'_{\mathcal{P}} \\ d''_{\mathcal{P}} \end{bmatrix}$$

and output nulling representation for the hidden behaviour $\mathcal{N}$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathcal{N}} = Ax_{\mathcal{N}} + [G \quad 0]\begin{bmatrix} v'_{\mathcal{N}} \\ v''_{\mathcal{N}} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C \\ H \end{bmatrix} x_{\mathcal{N}} + \begin{bmatrix} D & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} v'_{\mathcal{N}} \\ v''_{\mathcal{N}} \end{bmatrix}$$

Assumptions A.3 and A.4 ensure that the behaviours $\mathcal{N}$ and $\mathcal{P}$ are controllable. Moreover, their state-space representations obtained above are minimal, hence controllable and observable. These equations immediately yield the following output nulling representation for $\mathcal{P}^{\perp_\Sigma}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} z_{\mathcal{P}} = -A^{\mathrm{T}}z_{\mathcal{P}} + [0 \quad -H^{\mathrm{T}}]\begin{bmatrix} v'_{\mathcal{P}^{\perp_\Sigma}} \\ v''_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} B^{\mathrm{T}} \\ G^{\mathrm{T}} \end{bmatrix} z_{\mathcal{P}} + \begin{bmatrix} -I & 0 \\ 0 & J^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} v'_{\mathcal{P}^{\perp_\Sigma}} \\ v''_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix}$$

We will now check Σ-dissipativity of $\mathcal{N}$ and $(-\Sigma)$-dissipativity of $P^{\perp_\Sigma}$. We know that Σ-dissipativity of $\mathcal{N}$ is equivalent to the existence of a storage function. Furthermore, we know from Proposition 1 that a storage function is a state function. In the case under consideration, all the behaviours are represented in state form. Hence, $\mathscr{N}$ is Σ-dissipative if and only if there exists a matrix $K_{\mathscr{N}} = K_{\mathscr{N}}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ (with $n = \dim(x)$) such that the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_{\mathscr{N}}|^2_{K_{\mathscr{N}}} \leqslant |v'_{\mathscr{N}}|^2 - |v''_{\mathscr{N}}|^2$$

holds for all $(v'_{\mathscr{N}}, v''_{\mathscr{N}}, x_{\mathscr{N}})$ satisfying the equations for $\mathscr{N}$, equivalently,

$$|v'_{\mathscr{N}}|^2 - 2(v'_{\mathscr{N}})^{\mathrm{T}} G^{\mathrm{T}} K_{\mathscr{N}} x_{\mathscr{N}} - |Hx_{\mathscr{N}}|^2 - x_{\mathscr{N}}^{\mathrm{T}}(A^{\mathrm{T}} K_{\mathscr{N}} + K_{\mathscr{N}} A) x_{\mathscr{N}} \geqslant 0$$

for all $(v'_{\mathscr{N}}, x_{\mathscr{N}})$ satisfying $Cx_{\mathscr{N}} + Dv'_{\mathscr{N}} = 0$. To proceed, we need the following lemma, whose proof is left to the reader:

Lemma 7

Let $M = M^{\mathrm{T}}$, N, P, Q be real matrices of appropriate dimensions, and assume that Q is surjective. Then the quadratic form $x_1^{\mathrm{T}} M x_1 + 2x_1^{\mathrm{T}} N x_2 + x_2^{\mathrm{T}} x_2$ is non-negative on the subspace defined by $Px_1 + Qx_2 = 0$ if and only if

$$0 \leqslant M - NN^{\mathrm{T}} + (P - QN^{\mathrm{T}})^{\mathrm{T}}(QQ^{\mathrm{T}})^{-1}(P - QN^{\mathrm{T}}) =: L^{\mathrm{T}} L$$

in which case the quadratic form on the subspace equals

$$|Lx_1|^2 + |x_2 + N^{\mathrm{T}} x_1|^2_{(I - Q^{\mathrm{T}}(QQ^{\mathrm{T}})^{-1}Q)}$$

Using this, it follows that Σ-dissipativity of $\mathscr{N}$ is equivalent to the existence of a matrix $K_{\mathscr{N}} = K_{\mathscr{N}}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ such that the algebraic Riccati inequality

$$0 \leqslant -A^{\mathrm{T}} K_{\mathscr{N}} - K_{\mathscr{N}} A - H^{\mathrm{T}} H - K_{\mathscr{N}} G G^{\mathrm{T}} K_{\mathscr{N}} + C^{\mathrm{T}} C =: L_{\mathscr{N}}^{\mathrm{T}} L_{\mathscr{N}} \tag{9}$$

holds, in which case

$$|v'_{\mathscr{N}}|^2 - |v''_{\mathscr{N}}|^2 - \frac{\mathrm{d}}{\mathrm{d}t}|x_{\mathscr{N}}|^2_{K_{\mathscr{N}}} = |v'_{\mathscr{N}} - G^{\mathrm{T}} K_{\mathscr{N}} x_{\mathscr{N}}|^2_{(I - D^{\mathrm{T}} D)} + |L_{\mathscr{N}} x_{\mathscr{N}}|^2 \tag{10}$$

In a similar way, $(-\Sigma)$-dissipativity of $P^{\perp\Sigma}$ turns out to be equivalent to the existence of a matrix $K_{\mathscr{P}} = K_{\mathscr{P}}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z_{\mathscr{P}}|^2_{K_{\mathscr{P}}} \leqslant -|v'_{\mathscr{P}^{\perp\Sigma}}|^2 + |v''_{\mathscr{P}^{\perp\Sigma}}|^2$$

for all $(v'_{\mathscr{P}^{\perp\Sigma}}, v''_{\mathscr{P}^{\perp\Sigma}}, z_{\mathscr{P}})$ satisfying the equations for $P^{\perp\Sigma}$. Using lemma 7 again, it readily follows that $(-\Sigma)$-dissipativity of $P^{\perp\Sigma}$ is equivalent to the existence of $K_{\mathscr{P}} = K_{\mathscr{P}}^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ such that the algebraic Riccati inequality

$$0 \leqslant AK_{\mathscr{P}} + K_{\mathscr{P}} A^{\mathrm{T}} - GG^{\mathrm{T}} - K_{\mathscr{P}} H^{\mathrm{T}} H K_{\mathscr{P}} + BB^{\mathrm{T}} =: L_{\mathscr{P}}^{\mathrm{T}} L_{\mathscr{P}} \tag{11}$$

holds, in which case

$$-|v'_{\mathscr{P}^{\perp\Sigma}}|^2 + |v''_{\mathscr{P}^{\perp\Sigma}}|^2 - \frac{\mathrm{d}}{\mathrm{d}t}|z_{\mathscr{P}}|^2_{K_{\mathscr{P}}} = |v''_{\mathscr{P}^{\perp\Sigma}} + HK_{\mathscr{P}} z_{\mathscr{P}}|^2_{(I - JJ^{\mathrm{T}})} + |L_{\mathscr{P}} z_{\mathscr{P}}|^2 \tag{12}$$

Now apply Theorem 4, using the interpretation of $x_{\mathscr{N}}^{\mathrm{T}} K_{\mathscr{N}} x_{\mathscr{N}}$ and $z_{\mathscr{P}}^{\mathrm{T}} K_{\mathscr{P}} z_{\mathscr{P}}$ as storage functions for $\mathscr{N}$ and $\mathscr{P}^{\perp\Sigma}$, respectively, and the fact that the minimal states $(x_{\mathscr{N}}, z_{\mathscr{P}})$ appearing in the equations for $\mathscr{N}$ and $\mathscr{P}^{\perp\Sigma}$ satisfy $(\mathrm{d}/\mathrm{d}t)(x_{\mathscr{N}}^{\mathrm{T}} z_{\mathscr{P}}) = v_{\mathscr{N}}^{\mathrm{T}} \Sigma v_{\mathscr{P}}$. It follows that a necessary and sufficient condition for the existence of a controlled behaviour $\mathscr{K}$ satisfying $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$, Σ-dissipativity, external stability, and d free in $\mathscr{K}$, is that there exist real symmetric solutions $\mathscr{K}_{\mathscr{N}}$ and $\mathscr{K}_{\mathscr{P}}$ to the algebraic Riccati inequalities (9) and (11) such that

$$\mathscr{K} := \begin{bmatrix} K_{\mathscr{N}} & I \\ I & -K_{\mathscr{P}} \end{bmatrix} \geqslant 0 \tag{13}$$

This non-negativity is easily seen to be equivalent to the combined conditions

1. $K_{\mathscr{N}} > 0$
2. $K_{\mathscr{P}} < 0$,
3. $K_{\mathscr{N}} \geqslant (-K_{\mathscr{P}})^{-1}$

The last condition is easily seen to be equivalent to $\rho(K_{\mathscr{N}} K_{\mathscr{P}}) \geqslant 1$, where ρ denotes the spectral radius.

Using the theory of the algebraic Riccati equation and its relation with the algebraic Riccati inequalities and storage functions makes it possible to analyse the situation further. The final conclusion is that a necessary and sufficient condition for the existence of a required controlled behaviour $\mathscr{K}$ is that the two algebraic Riccati equations

$$-A^{\mathrm{T}} K_{\mathscr{N}} - K_{\mathscr{N}} A - H^{\mathrm{T}} H - K_{\mathscr{N}} G G^{\mathrm{T}} K_{\mathscr{N}} + C^{\mathrm{T}} C = 0 \tag{14}$$

$$A K_{\mathscr{P}} + K_{\mathscr{P}} A^{\mathrm{T}} - G G^{\mathrm{T}} - K_{\mathscr{P}} H^{\mathrm{T}} H K_{\mathscr{P}} + B B^{\mathrm{T}} = 0 \tag{15}$$

both have real symmetric solutions $K_{\mathscr{N}} = K_{\mathscr{N}}^{\mathrm{T}}$ and $K_{\mathscr{P}} = K_{\mathscr{P}}^{\mathrm{T}}$, and that the maximal real symmetric solution $K_{\mathscr{N}}^{+}$ of (14) combined with the minimal real symmetric solution $K_{\mathscr{P}}^{-}$ of (15) satisfies

$$\begin{bmatrix} K_{\mathscr{N}}^{+} & I \\ I & -K_{\mathscr{P}}^{-} \end{bmatrix} \geqslant 0$$

equivalently, $K_{\mathscr{N}}^{+} > 0$, $K_{\mathscr{P}}^{-} < 0$ and $K_{\mathscr{N}}^{+} \geqslant (-K_{\mathscr{P}}^{-})^{-1}$.

4.4. Construction of the controlled behaviour

In order to proceed, we need the following lemma, which states that an arbitrary $v \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}})$ can be represented as $v = v_{\mathscr{N}} + v_{\mathscr{P}^{\perp_\Sigma}} + v_{\mathscr{P} \cap \mathscr{N}^{\perp_\Sigma}}$, with $v_{\mathscr{N}} \in \mathscr{N}$, $v_{\mathscr{P}^{\perp_\Sigma}} \in \mathscr{P}^{\perp_\Sigma}$ and $v_{\mathscr{P} \cap N^{\perp_\Sigma}} \in \mathscr{P} \cap \boldsymbol{N}^{\perp_\Sigma}$.

Lemma 8

Let $\mathscr{N}$ and $\mathscr{P}$ be the hidden behaviour and manifest plant behaviour associated with $\mathscr{P}_{\text{full}}$ given by (3). Assume that the assumptions A.1–A.4 hold. Then we have

$$\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}}) = \mathscr{N} + \mathscr{P}^{\perp_\Sigma} + (\mathscr{P} \cap \mathscr{N}^{\perp_\Sigma})$$

Proof. First we prove that, in the case at hand, we actually have

$$\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}}) = \mathscr{P} + \mathscr{P}^{\perp_\Sigma} \text{ and } \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{d}+\mathrm{f}}) = \mathscr{N} + \mathscr{N}^{\perp_\Sigma} \tag{16}$$

We will only prove the first equality here. The second one is proven in a similar way. We claim that the system $\mathscr{P} \cap \mathscr{P}^{\perp_\Sigma}$ is autonomous, so for its number of inputs we have $\mathtt{m}(\mathscr{P} \cap \mathscr{P}^{\perp_\Sigma}) = 0$. Indeed, from the driving variable representation for $\mathscr{P}$ and the output nulling representation for

$P^{\perp\Sigma}$ derived in Section 4.3, we can deduce (using in an essential way assumption A.2) that $\mathscr{P}\cap\mathscr{P}^{\perp\Sigma}$ is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{\mathscr{P}} \\ z_{\mathscr{P}} \end{bmatrix} = \begin{bmatrix} A & -GG^{\mathrm{T}} + BB^{\mathrm{T}} \\ -H^{\mathrm{T}}H & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{\mathscr{P}} \\ z_{\mathscr{P}} \end{bmatrix}$$

$$v_{\mathscr{P}\cap\mathscr{P}^{\perp\Sigma}} = \begin{bmatrix} 0 & B^{\mathrm{T}} \\ H & -JG^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{\mathscr{P}} \\ z_{\mathscr{P}} \end{bmatrix}$$

which, indeed, represents an autonomous system. Using the general fact that for any pair of linear differential systems $\mathfrak{B}_1$ and $\mathfrak{B}_2$, we have $\mathtt{m}(\mathfrak{B}_1 + \mathfrak{B}_2) = \mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) - \mathtt{m}(\mathfrak{B}_1 \cap \mathfrak{B}_2)$ (see Reference [5, Proposition 7], from this we infer that $\mathtt{m}(\mathscr{P} + \mathscr{P}^{\perp\Sigma}) = \mathtt{m}(\mathscr{P}) + \mathtt{m}(\mathscr{P}^{\perp\Sigma})$, which can be easily seen to be equal to $\mathtt{d} + \mathtt{f}$. Since $\mathscr{P} + \mathscr{P}^{\perp\Sigma}$ and $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}})$ are both controllable differential systems, this implies that they must, in fact be equal.

To complete the proof, note that from the fact that $\mathscr{N} \subset \mathscr{P}$, whence $\mathscr{P}^{\perp\Sigma} \subset \mathscr{N}^{\perp\Sigma}$, we obtain $\mathscr{N}^{\perp\Sigma} = \mathscr{N}^{\perp\Sigma}\cap(\mathscr{P} + \mathscr{P}^{\perp\Sigma}) = \mathscr{P}^{\perp\Sigma} + (\mathscr{P}\cap\mathscr{N}^{\perp\Sigma})$. Obviously, this then yields $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{f}}) = \mathscr{N} + \mathscr{N}^{\perp\Sigma} = \mathscr{N} + \mathscr{P}^{\perp\Sigma} + (\mathscr{P}\cap\mathscr{N}^{\perp\Sigma})$. This completes the proof of the lemma. □

We now derive state representations for any of the behaviours in this decomposition. For the behaviours $\mathscr{N}$ and $\mathscr{P}^{\perp\Sigma}$ we already derived representations. From these it is easily seen that an output nulling representation for $\mathscr{N} + \mathscr{P}^{\perp\Sigma}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{\mathscr{N}} \\ z_{\mathscr{P}} \end{bmatrix} = \begin{bmatrix} A & -GG^{\mathrm{T}} \\ H^{\mathrm{T}}H & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{\mathscr{N}} \\ z_{\mathscr{P}} \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & -H^{\mathrm{T}} \end{bmatrix} v_{\mathscr{N}+\mathscr{P}^{\perp\Sigma}}$$

$$0 = \begin{bmatrix} C & 0 \\ 0 & B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{\mathscr{N}} \\ z_{\mathscr{P}} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & J^{\mathrm{T}} \end{bmatrix} v_{\mathscr{N}+\mathscr{P}^{\perp\Sigma}}$$

and hence that a driving variable representation for $\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} = \begin{bmatrix} -A^{\mathrm{T}} & -HH^{\mathrm{T}} \\ GG^{\mathrm{T}} & A \end{bmatrix} \begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \\ d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \end{bmatrix}$$

$$\begin{bmatrix} v'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \\ v''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \end{bmatrix} = \begin{bmatrix} G^{\mathrm{T}} & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} + \begin{bmatrix} -D^{\mathrm{T}} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \\ d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \end{bmatrix}$$

$$v_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} = (v'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}, v''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}})$$

Assume now that the Riccati inequalities (5) and (7) have solutions $K_{\mathscr{N}}$ and $K_{\mathscr{P}}$ such that K satisfies (13). We now introduce new variables $\theta, \chi \in \mathbb{R}^n$ and impose the constraint $\left[\begin{smallmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$. This equation merely expresses that the variables (θ, χ) must be well-defined, which is equivalent to stating that $\left[\begin{smallmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{smallmatrix}\right]$ must belong to $\mathrm{im}(K)$. A straightforward calculation using the algebraic Riccati inequalities yield

$$|v'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}|^2 - |v''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \theta \\ \chi \end{bmatrix}^{\mathrm{T}} K \begin{bmatrix} \theta \\ \chi \end{bmatrix} = |d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} - C\theta|^2 - |L_{\mathscr{N}}\theta|^2 - |d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} + B^{\mathrm{T}}\chi|^2 + |L_{\mathscr{P}}\chi|^2 \tag{17}$$

for all $(v'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}, v''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}, z_{\mathscr{N}}, x_{\mathscr{P}})$ satisfying the equations for $P\cap N^{\perp\Sigma}$, and with $\left[\begin{smallmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$.

Now, by combining (10), (12) and (17), we obtain that for *arbitrary* $v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{d+f})$, $v = v_{\mathcal{N}} + v_{\mathcal{P}^{\perp\Sigma}} + v_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}$, with $v_{\mathcal{N}} = (v'_{\mathcal{N}}, v''_{\mathcal{N}}) \in \mathcal{N}$, $v_{\mathcal{P}^{\perp\Sigma}} = (v'_{\mathcal{P}^{\perp\Sigma}}, v''_{\mathcal{P}^{\perp\Sigma}}) \in \mathcal{P}^{\perp\Sigma}$, and $v_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} = (v'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}, v''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}) \in \mathcal{P}\cap\mathcal{N}^{\perp\Sigma}$, with $x_{\mathcal{N}}, z_{\mathcal{P}}, z_{\mathcal{N}}, x_{\mathcal{P}}$, the variables introduced in the state representations of these behaviours, and $\left[\begin{smallmatrix} z_{\mathcal{N}} \\ x_{\mathcal{P}} \end{smallmatrix}\right] = K\left[\begin{smallmatrix} \theta \\ \chi \end{smallmatrix}\right]$, we have

$$
\begin{aligned}
&|v'_{\mathcal{N}} + v'_{\mathcal{P}^{\perp\Sigma}} + v'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 - |v''_{\mathcal{N}} + v''_{\mathcal{P}^{\perp\Sigma}} + v''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}}|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left\| \begin{bmatrix} x_{\mathcal{N}} \\ z_{\mathcal{P}} \end{bmatrix} + \begin{bmatrix} \theta \\ \chi \end{bmatrix} \right\|_K^2 \\
&\quad = |v'_{\mathcal{N}} - G^{\mathrm{T}} K_{\mathcal{N}} x_{\mathcal{N}}|^2_{(I - D^{\mathrm{T}} D)} + |L_{\mathcal{N}} x_{\mathcal{N}}|^2 - |v''_{\mathcal{P}^{\perp\Sigma}} + H K_{\mathcal{P}} z_{\mathcal{P}}|^2_{(I - J J^{\mathrm{T}})} - |L_{\mathcal{P}} z_{\mathcal{P}}|^2 \\
&\qquad + |d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} - C\theta|^2 - |L_{\mathcal{N}}\theta|^2 - |d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} + B^{\mathrm{T}}\chi|^2 + |L_{\mathcal{P}}\chi|^2.
\end{aligned} \tag{18}
$$

The crucial observation is that the above equation shows how the right-hand side can be made non-negative, thereby achieving a Σ-dissipative subbehaviour $\mathcal{K}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{d+f})$. Indeed, we should make sure that

1. $v_{\mathcal{P}^{\perp\Sigma}} = 0$, and $z_{\mathcal{P}} = 0$; this ensures that only $\mathcal{N}$, the Σ-dissipative part of $\mathcal{N} + \mathcal{P}^{\perp\Sigma}$, is incorporated in $\mathcal{K}$,
2. $L_{\mathcal{N}} = 0$; this is achieved by taking $K_{\mathcal{N}} = K^+_{\mathcal{N}}$, the largest real symmetric solution of the algebraic Riccati equation (14), yielding
$$K = K^+ := \begin{bmatrix} K^+_{\mathcal{N}} & I \\ I & -K_{\mathcal{P}} \end{bmatrix}$$
3. $d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} = -B^{\mathrm{T}}\chi$; this ensures that only a Σ-dissipative part of $\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}$ is incorporated in $\mathcal{K}$.

The resulting controlled behaviour $\mathcal{K}$ is obtained by adding these relations to the equations for $\mathcal{N} + \mathcal{P}^{\perp\Sigma}$ and $\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}$. This yields the following state-space representation for $\mathcal{K}$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathcal{N}} &= A x_{\mathcal{N}} + G v'_{\mathcal{N}}, \\
0 &= C x_{\mathcal{N}} + D v'_{\mathcal{N}} \\
\frac{\mathrm{d}}{\mathrm{d}t} K^+ \begin{bmatrix} \theta \\ \chi \end{bmatrix} &= \begin{bmatrix} -A^{\mathrm{T}} & -H^{\mathrm{T}} H \\ G G^{\mathrm{T}} & A \end{bmatrix} K^+ \begin{bmatrix} \theta \\ \chi \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} \\ d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} \end{bmatrix} \\
d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} &= -B^{\mathrm{T}}\chi \\
v_{\mathcal{K}} &= \begin{bmatrix} v'_{\mathcal{N}} \\ H x_{\mathcal{N}} \end{bmatrix} + \begin{bmatrix} G^{\mathrm{T}} K^+_{\mathcal{N}} & G^{\mathrm{T}} \\ H & -H K_{\mathcal{P}} \end{bmatrix} \begin{bmatrix} \theta \\ \chi \end{bmatrix} + \begin{bmatrix} -D^{\mathrm{T}} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} \\ d''_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} \end{bmatrix}
\end{aligned}
$$

Specializing the equality (18) to $\mathcal{K}$ leads to

$$|v_{K}|^2_{\Sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\left\| \begin{bmatrix} x_{\mathcal{N}} + \theta \\ \chi \end{bmatrix} \right\|^2_{K^+} = |v'_{\mathcal{N}} - G^{\mathrm{T}} K_{\mathcal{N}} x_{\mathcal{N}}|^2_{(I - D^{\mathrm{T}} D)} + |d'_{\mathcal{P}\cap\mathcal{N}^{\perp\Sigma}} - C\theta|^2 + |L\ P\chi|^2 \tag{19}$$

That $\mathcal{K}$ satisfies $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ follows immediately from the construction of $\mathcal{K}$. That $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$, equivalently that $\|f\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^f)} \leqslant \|d\|_{\mathcal{L}_2(\mathbb{R}, \mathbb{R}^d)}$ for all $(d, f) \in \mathcal{K} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{d+f})$ and $\mathcal{K}$ externally stable, follows from the equality (19) combined with the non-negative definiteness of K^+. Proving that in $\mathcal{K}$, d is free is the difficult part, especially in the case that $K^+ \geqslant 0$ is singular. We will give a proof in the next sections, together with the specification of the controller and a proof of internal stability.

4.5. Specification of the controller

Since $\mathscr{N} \subset \mathscr{K} \subset \mathscr{P}$, the controlled behaviour $\mathscr{K}$ can be implemented by means of a controller acting on the variables (u, y). However, the equations for $\mathscr{K}$ that we derived fail to make this apparent. In these equations, the controlled behaviour is given as $\mathscr{N}$ added to a suitable sub-behaviour of $\mathscr{P} \cap \mathscr{N}^{\perp\Sigma}$. What we need to do, is rewrite $\mathscr{K}$ as the manifest behaviour of the variables (d, f) of $\mathscr{P}_{\text{full}}$, interconnected with a suitable control law acting on the variables (u, y). Note that $\mathscr{K}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathscr{N}} = A x_{\mathscr{N}} + G v'_{\mathscr{N}}$$

$$0 = C x_{\mathscr{N}} + D v'_{\mathscr{N}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} = \begin{bmatrix} -A^{\mathrm{T}} & H^{\mathrm{T}}H \\ GG^{\mathrm{T}} & A \end{bmatrix} \begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \\ d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \end{bmatrix}$$

$$\begin{bmatrix} z_{\mathscr{N}} \\ x_{\mathscr{P}} \end{bmatrix} = K^{+} \begin{bmatrix} \theta \\ \chi \end{bmatrix}$$

$$d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} = -B^{\mathrm{T}} K_{\mathscr{P}} \chi$$

$$v_{\mathscr{K}} = \begin{bmatrix} v'_{\mathscr{N}} \\ H x_{\mathscr{N}} \end{bmatrix} + \begin{bmatrix} G^{\mathrm{T}} K^{+}_{\mathscr{N}} & G^{\mathrm{T}} \\ H & -HK_{\mathscr{P}} \end{bmatrix} \begin{bmatrix} \theta \\ \chi \end{bmatrix} + \begin{bmatrix} -D^{\mathrm{T}} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \\ d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} \end{bmatrix}$$

By introducing new state variables $x = x_{\mathscr{N}} + x_{\mathscr{P}}$, $z = z_{\mathscr{N}} - K^{+}_{\mathscr{N}} x_{\mathscr{P}}$, instead of $x_{\mathscr{N}}, z_{\mathscr{N}}$, and the new variables $u = d''_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}$, $d = G^{\mathrm{T}} z_{\mathscr{N}} + v'_{\mathscr{N}} - D^{\mathrm{T}} d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}$, $y = Cx + Du$, the equations for $\mathscr{K}$ become, after some straightforward calculations,

$$\frac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bu + Gd, \quad v_{\mathscr{K}} = (d, Hx + Ju)$$

$$y = Cx + Dd$$

$$\frac{\mathrm{d}}{\mathrm{d}t} z = -(A^{\mathrm{T}} + K^{+}_{\mathscr{N}} GG^{\mathrm{T}}) z - C^{\mathrm{T}} y - K^{+}_{\mathscr{N}} Bu$$

$$z = (I_{\mathrm{n}} + K^{+}_{\mathscr{N}} K_{\mathscr{P}}) \chi$$

$$u = -B^{\mathrm{T}} \chi$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathscr{P}} = GG^{\mathrm{T}} z + (A + GG^{\mathrm{T}} K^{+}_{\mathscr{N}}) x_{\mathscr{P}} + Bu$$

$$x_{\mathscr{P}} = \theta - K_{\mathscr{P}} \chi$$

$$d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}} = -Dd - C(x - x_{\mathscr{P}})$$

Note that the last three of these equations merely serve to define $x_{\mathscr{P}}$, θ, and $d'_{\mathscr{P}\cap\mathscr{N}^{\perp\Sigma}}$. The controlled behaviour $\mathscr{K}$ is hence given by the plant equations

$$\frac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bu + Gd, \quad v = (d, Hx + Ju), \quad y = Cx + Dd \tag{20}$$

combined with the control law

$$\frac{d}{dt}(I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}})\chi = -(A^T + K_{\mathscr{N}}^{+}GG^T)(I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}})\chi - C^T y - K_{\mathscr{N}}^{+}Bu$$

$$u = -B^T\chi \tag{21}$$

Note that when $I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}}$ is singular, this is a singular state system. This representation of the controller, while rather simple, has one major drawback: in the case that $I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}}$ is singular, it does not make apparent that the transfer function from y to u of the controller exists. In the case that $I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}}$ is non-singular, then obviously the transfer function from y to u exists, and is strictly proper. In the next section we will show that, surprisingly, also in the case that $I_n + K_{\mathscr{N}}^{+}K_{\mathscr{P}}$ is singular, the Equations (21) define a controller with a proper transfer matrix!

4.6. Properness of the controller

In the remainder of this paper, we take $K_{\mathscr{P}} = K_{\mathscr{P}}^{-}$, the smallest real symmetric solution of the algebraic Riccati Equation (15). In order to simplify the notation, we will write $K_{\mathscr{N}}$ instead of $K_{\mathscr{N}}^{+}$ and $K_{\mathscr{P}}$ instead of $K_{\mathscr{P}}^{-}$. A crucial role is played by the following relation, that is easily deduced from the algebraic Riccati equations for $K_{\mathscr{N}}$ and $K_{\mathscr{P}}$:

$$-(A^T + K_{\mathscr{N}}GG^T)(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}) = (K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})(A - GG^TK_{\mathscr{P}}^{-1}) + K_{\mathscr{P}}^{-1}BB^TK_{\mathscr{P}}^{-1} - C^TC \tag{22}$$

By substituting this relation into the controller Equations (21) and defining $\hat{x} = -K_{\mathscr{P}}\chi$ we obtain the following alternative representation of the controller:

$$\frac{d}{dt}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})\hat{x} = (K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})(A\hat{x} + Bu + G\hat{d}) + C^T(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

$$\hat{d} = -G^TK_{\mathscr{P}}^{-1}\hat{x}$$

$$u = B^TK_{\mathscr{P}}^{-1}\hat{x} \tag{23}$$

Note that this representation displays the controller both as an input/output system driven by the sensor outputs that returns the actuator inputs, and the structure of an observer driven by error feedback, with $\hat{y}$ the estimate of the sensor output, and $\hat{d}$ the estimate of the worst disturbance.

We now address the issue of the properness of the transfer function of the controller. Decompose the singular state system that specifies the controller in its regular and singular parts. Let $R \in \mathbb{R}^{n \times \dim(\mathrm{im}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}))}$ and $N \in \mathbb{R}^{n \times \dim(\ker(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}))}$ be matrices whose columns span $\mathrm{im}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})$ and $\ker(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}) = (\mathrm{im}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}))^{\perp}$, respectively. Note that the matrix $[R \;\; N]$ is non-singular. Introduce new state variables $(\hat{x}_1, \hat{x}_2)$ by

$$\hat{x} = [R \;\; N]\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$$

Now note that for all y, $\hat{y}$ compatible with the equations of the plant and the controller (23) we have

$$N^TC^T(y - \hat{y}) = 0$$

The idea is to use this relation to express $\hat{x}_2$ in terms of $\hat{x}_1$ and y. In order to do this, let $(CN)^{\#}$ be the Moore–Penrose inverse of CN. By definition we then have that $(CN)^{\#}$ is the projection onto $\ker((CN)^{\mathrm{T}})$ along $\mathrm{im}(CN)$. Consequently, $N^{\mathrm{T}}C^{\mathrm{T}}CN(CN)^{\#} = N^{\mathrm{T}}C^{\mathrm{T}}$ and hence $N^{\mathrm{T}}C^{\mathrm{T}}(y - CR\hat{x}_1 - CN\hat{x}_2) = 0$ implies that

$$N(CN)^{\#}(y - CR\hat{x}_1) - N\hat{x}_2 \in \ker(N^{\mathrm{T}}C^{\mathrm{T}}C) \cap \mathrm{im}(N)$$

As a consequence, for all $\hat{x}_2$, y, $\hat{x}_1$ compatible with the plant/controller equations, there exists a signal $a \in \ker(CN)$ such that

$$N\hat{x}_2 = N(CN)^{\#}(y - CR\hat{x}_1) + Na$$

The idea is now to substitute this expression for $N\hat{x}_2$ into the controller representation (23). We will prove now that the variable a does, in fact, not appear in these new equations for the controller. Indeed, by premultiplying (22) with N^{T} and postmultiplying with N, we get $N^{\mathrm{T}}K_{\mathscr{P}}^{-1}BB^{\mathrm{T}}K_{\mathscr{P}}^{-1}N = N^{\mathrm{T}}C^{\mathrm{T}}CN$. Since $CNa = 0$, this immediately yields $B^{\mathrm{T}}K_{\mathscr{P}}^{-1}a = 0$. Also, $(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})(A - GG^{\mathrm{T}}K_{\mathscr{P}}^{-1})Na = 0$. Hence, after substituting $N\hat{x}_2$ into (23), and premultiplying the resulting differential equation with the (non-singular) matrix $[R\ \ N]^{\mathrm{T}}$, we finally obtain that the controlled behaviour $\mathscr{K}$ is also represented by the plant equations (20) together with the controller represented by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})R\hat{x}_1 &= R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})(A(I - N(CN)^{\#}C)R\hat{x}_1 + Bu \\
&\quad + G\hat{d} + AN(CN)^{\#}y) + R^{\mathrm{T}}C^{\mathrm{T}}(y - \hat{y}) \\
\hat{y} &= C((I - N(CN)^{\#}C)R\hat{x}_1 + N(CN)^{\#}y) \\
\hat{d} &= -G^{\mathrm{T}}K_{\mathscr{P}}^{-1}((I - N(CN)^{\#}C)R\hat{x}_1 + N(CN)^{\#}y) \\
u &= B^{\mathrm{T}}K_{\mathscr{P}}^{-1}((I - N(CN)^{\#}C)R\hat{x}_1 + N(CN)^{\#}y)
\end{aligned} \tag{24}$$

These equations show that the transfer function of the controller is indeed proper. The feedthrough term is given by $B^{\mathrm{T}}K_{\mathscr{P}}^{-1}N(CN)^{\#}$, while the strictly proper part is given by the differential equation part of the above expression. This differential equation is a regular one, since $R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})R$ is a non-singular matrix.

4.7. *Internal stability*

In this subsection we show that the controlled system obtained by interconnecting the plant (3) with the controller (24) is internally stable, i.e. we will prove that for all x and $\hat{x}_1$ satisfying both the plant equations (3) and the controller equations (24), we have $(x(t), \hat{x}_1(t)) \to 0$ as $t \to \infty$, when $d = 0$. For any $\hat{x}_1$ and y satisfying the differential Equation in (24), the signal $\hat{x} := R\hat{x}_1 + N\hat{x}_2$, with $N\hat{x}_2 := N(CN)^{\#}(y - R\hat{x}_1)$, and y satisfy the differential equation in (23). Obviously, $R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})R\hat{x} = R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})R\hat{x}_1$, so to prove $(x(t), \hat{x}_1(t)) \to 0$ it suffices to prove $(x(t), \hat{x}(t)) \to 0$. To prove this, we need the following Lyapunov function argument for singular systems (see Reference [18, Theorem 4.3]):

Lemma 9

Consider the system $(\mathrm{d}/\mathrm{d}t)Ez = Fz$, where $E, F \in \mathbb{R}^{n \times n}$. Let $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ satisfy $P \geqslant 0$ and define $V(z) := |z|_P^2$. Assume that $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $Q \geqslant 0$, is such that for all z satisfying

$(\mathrm{d}/\mathrm{d}t)Ez = Fz$ we have

(i) $(\mathrm{d}/\mathrm{d}t)V(z) = -|z|_Q^2$, and
(ii) $(Qz = 0) \Rightarrow (z = 0)$

Then the system $(\mathrm{d}/\mathrm{d}t)Ez = Fz$ is asymptotically stable, i.e. all solutions z tend to 0 as $t \to \infty$.

We now apply this lemma to the controlled system obtained by interconnecting the plant with the controller (23). Consider

$$V(x, \hat{x}) := -|x|^2_{K_{\mathscr{P}}^{-1}} + |x - \hat{x}|^2_{K_{\mathscr{N}} + K_{\mathscr{P}}^{-1}}$$

Clearly $V(x, \hat{x}) \geqslant 0$ for all $(x, \hat{x})$. A straightforward computation shows that for all $(x, \hat{x})$ satisfying the state Equations (3), (23) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x, \hat{x}) = -|Hx + JB^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x}|^2 - |G^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x} - (G^{\mathrm{T}}K_{\mathscr{N}}^{+} - D^{\mathrm{T}}C)(x - \hat{x})|^2 \tag{25}$$

Hence, along solutions $(x, \hat{x})$ of the controlled system, the derivative of $V(x, \hat{x})$ coincides with a negative semi-definite quadratic form. This yields condition (i) of Lemma 9. Now turn to condition (ii). Clearly $(\mathrm{d}/\mathrm{d}t)V(x, \hat{x}) = 0$ if and only if $(x, \hat{x})$ satisfies the following two equations:

$$Hx + JB^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x} = 0 \tag{26}$$

and

$$G^{\mathrm{T}}K_{\mathscr{P}}^{-1}\hat{x} - (G^{\mathrm{T}}K_{\mathscr{N}}^{+} - D^{\mathrm{T}}C)(x - \hat{x}) = 0 \tag{27}$$

By premultiplying equation (26) with J^{T}, we obtain $B^{\mathrm{T}}K_P^{-1}\hat{x} = 0$ so $u = 0$. Combined with the Equation of the plant, this implies that x satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax, \quad Hx = 0$$

By regularity condition (A.4), i.e. observability of the pair (H, A), this implies $x = 0$.

By premultiplying the second Equation, (27), with D, we get $C\hat{x} = 0$. Again combining this with (27), we obtain $G^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})\hat{x} = 0$. Now note that $(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})\hat{x} = x_{\mathrm{c}}$, where x_{c} satisfies $(\mathrm{d}/\mathrm{d}t)x_{\mathrm{c}} = -(A^{\mathrm{T}} + K_{\mathscr{N}}^{+}GG^{\mathrm{T}})x_{\mathrm{c}}$, $G^{\mathrm{T}}x_{\mathrm{c}} = 0$. By the regularity assumption A.3 (observability of the pair $(G^{\mathrm{T}}, A^{\mathrm{T}})$), this yields $x_{\mathrm{c}} = 0$, so $(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})\hat{x} = 0$. Since $\hat{x} = R\hat{x}_1 + N\hat{x}_2$, this yields $R^{\mathrm{T}}(K_{\mathscr{N}} + K_{\mathscr{P}}^{-1})R\hat{x}_1 = 0$, so $\hat{x}_1 = 0$. Also $N\hat{x}_2 = N(CN)^{\#}(y - R\hat{x}_1) = 0$, since $y = 0$ and $\hat{x}_1 = 0$. Thus $\hat{x} = 0$. Now use Lemma 9 to conclude that the controlled system is indeed internally stable.

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