H_∞ Control in a Behavioral Context: The Full Information Case

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***Abstract*—In this paper the authors formulate the H_∞-control problem in a behavioral setting. Given a mathematical model, say a set of higher order differential equations together with some static equations, the vector of manifest variables (i.e., the variables to be modeled) is partitioned into yet to be controlled variables, unknown exogenous variables (called disturbances), and interconnection variables. The interconnection variables are available for interconnection, in the sense that they can be made to obey certain differential or static equations, to be specified by the designer. Such a system of differential equations and static equations is called a controller. The design problem that we consider is to find controllers such that (in the $\mathcal{L}_2$-sense) the size of the to be controlled variables is less than a given tolerance, for all disturbances in the unit ball, and such that the interconnection is a stable system. We find necessary and sufficient conditions for the existence of suitable controllers, under the hypothesis that we have a full information problem. These conditions involve indefinite factorizations of polynomial matrices and a test on a given Pick matrix.**

***Index Terms*—Behaviors, dissipativity, H_∞ control, linear systems, Pick matrices, quadratic differential forms, spectral factorization, storage functions.**

I. Introduction

PRESENT day control theory is centered around the problem of designing feedback loops around a given plant such that in the closed-loop system certain design specifications are satisfied. The plant under consideration typically has control inputs, exogenous inputs, measured outputs, and to be controlled outputs. The controller to be designed takes the measured outputs of the system as its inputs, and generates, on the basis of these inputs, control inputs for the plant. These controllers should be designed in such a way that the resulting closed-loop system meets the specifications. The above general scheme of approaching control design problems has been called *the intelligent control paradigm* (see [22]).

It is our conviction that in many cases it is more natural to view controller design as the problem of designing for a given plant an additional set of "laws" that the variables appearing in the system should obey. More specifically, if a plant is modeled as a set of "behavioral equations," then, from our point of view, the controller design question is to invent an additional set of equations—the controller equations—involving the variables appearing in the system. These additional equations should be such that the "controlled system" (i.e., the system consisting of those variables that are compatible with both set of equations) satisfies the given control specifications.

This point of view is, in our opinion, very natural. Suppose we have a mathematical model obtained from first principles modeling, say a set of higher order differential equations, together with some static equations. The collection of all (vector-valued) time trajectories satisfying these equations is called the behavior. In general, this vector of time trajectories (called the manifest variable) will consist of several types of components. Typically, certain components are variables that we want to keep small, as certain components represent unknown exogenous variables, and other components are variables that are still available for interconnection, in the sense that we can make them obey certain differential or static equations, to be specified by the control design. In the classical control framework one proceeds as follows. The mathematical model is put into some standard form, for example expressing the laws that are satisfied by the various variables in terms of a standard transfer matrix model or a standard state-space model. Inherent in this procedure is that the manifest variable is split up into input components and output components: some are labeled exogenous inputs, some to be controlled outputs, some control inputs, and some measured outputs. Next, one does a controller design. In the classical framework, this results in a controller description in the form of an input–output relation between the measured outputs and the control inputs. As in [22], in this paper we propose a more general way of looking at controller design. Instead of putting the original mathematical model into some standard form while specifying inputs and outputs, we prefer to leave the model as it is and not bother about the question which variables should be called inputs or outputs. Instead, we simply specify some of the components of the manifest variable to be *interconnection variables*, i.e., variables that we can make to satisfy certain equations. Then, depending on what properties one wants the controlled system to satisfy, we do a controller design. This controller design is now the determination of a set of additional equations involving the interconnection variables.

In this paper, we reformulate and study the H_∞-control problem from this vantage point. Starting from the dynamical model, some components of the manifest variable are assumed to be free, in the sense that they are not constrained by the model. Hence, such a component can in principle be any time trajectory. These components are the disturbances. Other components of the manifest variable are variables that we want to keep small (think of variables that measure the

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deviation from some desired time trajectory). These are called to be controlled variables. A third group of components are the interconnection variables (some of them are also free of course) as explained above. The control problem that we consider in this paper is to design a set of additional dynamic constraints on the interconnection variables (differential equations involving these variables) such that, roughly speaking, the to be controlled variables are "small" whatever the disturbance that occurs. We want to stress that this point of view generalizes the "classical" approach to H_∞. In that context, for the interconnection variable c one would take the composite vector (u, y) with u the control inputs and y the measured outputs. As also in some of the classical H_∞-theory, one feature of our theory is then that the dynamic constraints on (u, y) need not be described by a proper transfer matrix.

This paper is concerned with a detailed formulation of the problem and with a complete resolution of the full information version of the H_∞ problem. As other research in this area, we mention the work of d'Andrea [3], [4], where a similar problem formulation is considered from a state space point of view.

This paper is organized as follows. In addition to the main text, the paper contains an Appendix containing most of the proofs. In Section II of this paper we formulate the suboptimal and optimal H_∞-control problem in a representation-free, behavioral context. We define the notions of (strictly) γ-contracting controller and stabilizing controller. In Section III, we discuss some material on the class of linear differential systems, the class of systems that we will restrict ourselves to in this paper. In this paper, we heavily use two-variable polynomial matrices and quadratic differential forms (QDF's). These notions are briefly discussed in Section IV. For a more extensive treatment, we refer to [23]. In Sections V and VI, we study the H_∞-control problem for the class of linear differential systems. We also explain what is meant by a full information control problem. In Section VII we give conditions for a controller to be (strictly) γ-contracting and stabilizing. Before we formulate and prove the main results of this paper, in Section VIII, we discuss dissipative systems. Finally, in Sections IX and X we give a solution of the full information suboptimal H_∞-control problem.

A Few Words on Notation: In this paper, integers that refer to dimensions of linear spaces and/or sizes of matrices are always denoted in typewriter type style. For example, $\mathbb{R}^{\mathtt{w}}$ denotes the linear space of real column vectors with $\mathtt{w}$ components, $\mathbb{R}^{\mathtt{p}\times\mathtt{q}}$ ($\mathbb{C}^{\mathtt{p}\times\mathtt{q}}$) is the space of real (complex) $\mathtt{p}\times\mathtt{q}$ matrices, $I_{\mathtt{z}}$ denotes the identity matrix of size $\mathtt{z}\times\mathtt{z}$, etc. We also use the following convention: vectors w, d, z, ℓ, and x are always elements of $\mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{d}}$, $\mathbb{R}^{\mathtt{z}}$, $\mathbb{R}^{\mathtt{l}}$, and $\mathbb{R}^{\mathtt{x}}$, respectively. Given two column vectors x and y, the column vector obtained by stacking x over y is denoted by $\operatorname{col}(x, y)$. Likewise, for given matrices A and B with the same number of columns, $\operatorname{col}(A, B)$ denotes the matrix obtained by stacking A over B. For a given complex matrix M we denote by M^* the conjugate transpose of M. The complex conjugate of the complex number λ is denoted by $\overline{\lambda}$. For a given finite-dimensional Euclidean space X, we denote by $\mathcal{L}_2(\mathbb{R}, X)$ the space of all measurable functions f from $\mathbb{R}$ to X for which $\int_{-\infty}^{\infty} \|f(t)\|^2 \, dt$ is finite. We denote $\|f\|_2 := (\int_{-\infty}^{\infty} \|f(t)\|^2 \, dt)^{1/2}$. The space $\mathcal{C}^\infty(\mathbb{R}, X)$ consists as usual of all infinitely often differentiable functions from $\mathbb{R}$ to X, and $\mathcal{D}(\mathbb{R}, X)$ denotes the elements of $\mathcal{C}^\infty(\mathbb{R}, X)$ with compact support. If M is a Hermitian matrix, i.e., a square matrix with complex coefficients such that $M^* = M$, then we define its signature as the ordered triple $\operatorname{sign}(M) = (n_-(M), n_0(M), n_+(M))$, where $n_-(M)$ denotes the number (counting multiplicities) of negative eigenvalues of $M, n_0(M)$ the multiplicity of the eigenvalue zero, and $n_+(M)$ the number of positive eigenvalues of M. If G is a matrix of proper rational functions, then its L_∞-norm is defined by $\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(i\omega)\|$. If G has all its poles in the open left half of the complex plane, then $\|G\|_\infty = \sup_{\Re e(\lambda)\geq 0} \|G(\lambda)\|$, the H_∞-norm of G.

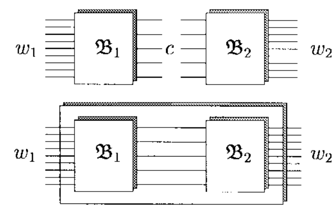

Fig. 1. Interconnection of $\mathcal{B}_1$ and $\mathcal{B}_2$.

II. H_∞ Control in a Behavioral Setting

In this section we first briefly recall our view of control in the context of the behavioral approach to dynamical systems. A dynamical system is a triple, $\Sigma = (T, W, \mathcal{B})$ with $T \subset \mathbb{R}$ the *time axis*, W a set called the *signal space*, and $\mathcal{B} \subset W^T$ the *behavior*. The behavior consists of a family of admissible functions w: $T \to W$. The variable w is called the *manifest variable* of the system. Since T and W are often apparent from the context (in the present paper $T = \mathbb{R}$ and $W = \mathbb{R}^{\mathtt{w}}$), we identify the system $\Sigma = (T, W, \mathcal{B})$ simply with its behavior $\mathcal{B}$. Let $\Sigma_1 = (T, W_1 \times C, \mathcal{B}_1)$ and $\Sigma_2 = (T, W_2 \times C, \mathcal{B}_2)$ be two dynamical systems with the same time axis. We assume that the signal spaces of Σ_1 and Σ_2 are Cartesian products, with the factor C in common. Correspondingly, trajectories of $\mathcal{B}_1$ are denoted by (w_1, c) and trajectories of $\mathcal{B}_2$ by (w_2, c). We define the *interconnection* of Σ_1 and Σ_2 as the dynamical system $\Sigma_1 \wedge \Sigma_2 := (T, W_1 \times W_2, \mathcal{B})$, with $\mathcal{B} = \{(w_1, w_2)$: $T \to W_1 \times W_2 |$ there exists c such that $(w_1, c) \in \mathcal{B}_1$ and $(w_2, c) \in \mathcal{B}_2\}$. The interconnection takes place via the variable c, which is called the *interconnection variable*. Often, we denote the interconnected system by $\mathcal{B}_1 \wedge \mathcal{B}_2$. This interconnection is illustrated in Fig. 1.

In this context, a control problem is formulated as follows. Assume that the *plant*, a dynamical system $\Sigma_p = (T, W_p \times C, \mathcal{B}_p)$ is given. The signal space W of the plant is given as the Cartesian product $W_p \times C$, where the second factor, C, denotes the space in which c, the interconnection variable, takes its values. C is called *the interconnection space* of Σ_p. Consider now a family $\mathcal{A}$ of dynamical systems, all with common time axis T and with common signal space C. An element $\Sigma_c = (T, C, \mathcal{B}_c)$ of $\mathcal{A}$ is called an *admissible controller*. The interconnected system $\Sigma_p \wedge \Sigma_c$ is called the *controlled system*.

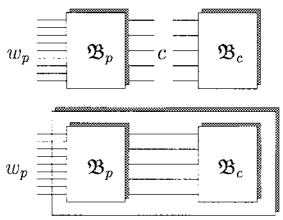

Fig. 2. $\mathcal{B}_p$ controlled by $\mathcal{B}_c$.

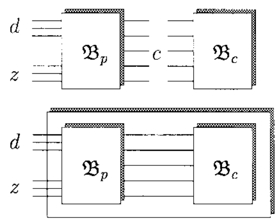

Fig. 3. $\mathcal{B}_p$ controlled by $\mathcal{B}_c$.

The control problem for the plant Σ_p is now to specify the set $\mathcal{A}$ of admissible controllers, to describe what desirable properties the controlled system should have, and, finally, to find an admissible controller Σ_c such that $\Sigma_p \wedge \Sigma_c$ has the desired properties. Thus control is nothing more than a special type of interconnection (see Fig. 2). This paper deals with the H_∞-control problem. In this context, the main desired property of the controlled system is that certain components (called the to be controlled variables) of the system's manifest variable are small (in an appropriate sense), regardless of the values that certain other components (called the disturbances) take. In addition, the controlled system should be stable, in the sense that if the disturbances happen to be zero, then the to be controlled variables should converge to zero as time runs off to infinity. Therefore, our starting point is that the manifest variable w of the plant Σ_p consist of three components, $w = (z, d, c)$. Here, z is the to be controlled variable, d is the disturbance, and c is the interconnection variable as referred to above. The variable c is available to attach a controller (see Fig. 3). Accordingly, the signal space of Σ_p is the Cartesian product $Z \times D \times C$, with Z, D, and C the sets in which respectively z, d, and c take their values. Thus, in the terminology used above, we take $W_p = Z \times D$. The component d is interpreted as a free unknown disturbance. This is modeled by assuming that "*any*" function d: $T \rightarrow D$ can occur as the second component of the manifest variable w of Σ_p. In order to formalize this, if, in general, we have a dynamical system $\Sigma = (T, W_1 \times W_2, \mathcal{B})$, with manifest variable (w_1, w_2) and if π: $W_1 \times W_2 \rightarrow W_2$ is the projection $\pi(w_1, w_2) = w_2$, then the variable w_2 is called *free* if $\pi(\mathcal{B}) = W_2^T$. Thus, for the plant Σ_p under consideration, we assume that the variable d is free. Of course, for mathematical reasons we will need to put some minor regularity conditions on $\pi(\mathcal{B})$ (see Section V).

We now specify the set of admissible controllers. Consider any dynamical system $\Sigma_c = (T, C, \mathcal{B}_c)$ with the same time axis as the plant Σ_p, whose signal space is equal to the interconnection space C of Σ_p. According to the above definition, the interconnection is $\Sigma_p \wedge \Sigma_c = (T, Z \times D, \mathcal{B}_p \wedge \mathcal{B}_c)$, with $\mathcal{B}_p \wedge \mathcal{B}_c := \{(z, d)$: $T \rightarrow Z \times D |$ there exists $c \in \mathcal{B}_c$ such that $(z, d, c) \in \mathcal{B}_p\}$. Now, in the controlled system, d is of course still interpreted as an unknown externally imposed disturbance. Hence, again, *any* d should be possible as the second component of the manifest variable (z, d) of the controlled system. If this requirement holds, then we call the controller *admissible*: Σ_c is admissible if in the controlled system $\Sigma_p \wedge \Sigma_c$ the variable d is free.

In the controlled system we want the signal z to be small, regardless of the disturbance d that occurs. This specification can of course be formalized in many ways, and in this paper we consider the H_∞ performance. We assume that T, the time axis, is equal to $\mathbb{R}$ and that the signal spaces Z, D, and C are finite-dimensional Euclidean spaces. The size of the signals z and d is measured by their $\mathcal{L}_2$-norms $\|z\|_2$ and $\|d\|_2$.

Definition 2.1: Let $\mathcal{B}_c$ be an admissible controller. The H_∞-performance of the controlled system $\mathcal{B}_p \wedge \mathcal{B}_c$ is defined as

$$J(\mathcal{B}_c) := \inf\{\gamma \geq 0 | \ \|z\|_2 \leq \gamma \|d\|_2 \quad \text{for all} \quad (z, d) \in (\mathcal{B}_p \wedge \mathcal{B}_c) \cap \mathcal{L}_2(\mathbb{R}, Z \times D)\}.$$

Given $\gamma > 0$, the controller $\mathcal{B}_c$ is called γ-*contracting* if $J(\mathcal{B}_c) \leq \gamma$, equivalently, if for all $(z, d) \in (\mathcal{B}_p \wedge \mathcal{B}_c) \cap \mathcal{L}_2(\mathbb{R}, Z \times D)$ we have $\|z\|_2 \leq \gamma \|d\|_2$, and *strictly* γ-*contracting* if $J(\mathcal{B}_c) < \gamma$, equivalently, if there exists $\epsilon > 0$ such that for all $(z, d) \in (\mathcal{B}_p \wedge \mathcal{B}_c) \cap \mathcal{L}_2(\mathbb{R}, Z \times D)$ we have $\|z\|_2 \leq (\gamma - \epsilon)\|d\|_2$.

Definition 2.2: An admissible controller $\mathcal{B}_c$ is called a *stabilizing* controller if in the controlled system the signal z converges to zero whenever $d = 0$, i.e., if $(z, 0) \in \mathcal{B}_p \wedge \mathcal{B}_c$ implies that $\lim_{t \to \infty} z(t) = 0$.

Example 2.3: As an example, suppose that the controlled system is given by a second-order linear differential equation $(d^2 z/t^2) + \alpha_1 (dz/dt) + \alpha_2 z = d$ (with α_1 and α_2 given constants). The behavior of this system consists of all (z, d) that satisfy this differential equation. Our notion of stability requires that for all solutions (z, d) with $d = 0$ we have $\lim_{t \to \infty} z(t) = 0$. Thus stability is equivalent to the requirement that all solutions of the homogeneous equation converge to zero as $t \to \infty$.

The H_∞-*optimal* control problem is to minimize the H_∞ performance of $\mathcal{B}_p \wedge \mathcal{B}_c$ over the class of all admissible stabilizing controllers, i.e., to calculate

$$\gamma^* := \inf\{J(\mathcal{B}_c) | \mathcal{B}_c \text{ admissible and stabilizing}\}$$

and to determine, if one exists, all optimal controllers, i.e., all admissible stabilizing controllers $\mathcal{B}_c^*$ such that $\gamma^* = J(\mathcal{B}_c^*)$. Given $\gamma > 0$ (the *tolerance*), the H_∞-suboptimal control problem is to determine, if one exists, all γ-contracting stabilizing controllers. The *strict* H_∞-suboptimal control problem is to determine all strictly γ-contracting stabilizing controllers. The present paper deals with the strict H_∞-suboptimal control problem.

III. Linear Time-Invariant Differential Systems

We restrict our attention to systems described by linear differential equations with constant coefficients. Let ξ denote an indeterminate, and let $\mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ be the set of all real polynomial matrices with $\mathtt{w}$ columns and any (finite) number of rows. An element $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ can be written explicitly as $R(\xi) = R_0 + R_1\xi + R_2\xi^2 + \cdots + R_N\xi^N$, for given real matrices $R_0, R_1, \cdots, R_N$. Consider now the system of differential equations $R_0 w + R_1(dw/dt) + \cdots + R_N(d^N w/dt^N) = 0$ or, in compact notation

$$R\left(\frac{d}{dt}\right)w = 0. \tag{1}$$

This defines a linear time-invariant differential system, i.e., a dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathcal{B})$ with time axis $\mathbb{R}$, signal space $\mathbb{R}^{\mathtt{w}}$, and behavior $\mathcal{B}$ equal to the solution set of (1): $\mathcal{B} := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | w \text{ satisfies (1)}\}$. The class of all such systems is denoted by $\mathcal{L}^{\mathtt{w}}$. Equation (1) is called a kernel representation of $\mathcal{B} \in \mathcal{L}^{\mathtt{w}}$.

Remark 3.1: In order to avoid irrelevant smoothness issues, in this paper we define the behavior of a linear differential system with kernel representation (1) to be the set of all $\mathcal{C}^\infty$-solutions (also called *strong* solutions) of this polynomial differential equation. We could also define the behavior to be the set of all $w \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ (i.e., all measurable w's for which $\int_{t_0}^{t_1} \|w\|^2 \, dt$ exists for all t_0 and t_1) that satisfy (1) in the sense of distributions (also called *weak* solutions). Temporarily denoting the set of strong solutions by $\mathcal{B}^\infty$ and the set of weak solutions by $\mathcal{B}$, it can be proven that $\mathcal{B}^\infty \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is dense in $\mathcal{B} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, i.e., for every $w \in \mathcal{B} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ there exists a sequence $\{w_n\}$ in $\mathcal{B}^\infty \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $w_n \to w$ $(n \to \infty)$ in $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$-sense. This implies that in the context of the H_∞ control problem there is no loss of generality in restricting oneself to strong solutions: $\|z\|_2 \leq \gamma\|d\|_2$ for all $(z, d) \in (\mathcal{B}_p \wedge \mathcal{B}_c) \cap \mathcal{L}_2(Z, D)$ iff this inequality holds for all $(z, d) \in (\mathcal{B}_p \wedge \mathcal{B}_c)^\infty \cap \mathcal{L}_2(Z, D)$.

We will make heavy use of image representations, that is, representations of the form $w = W(d/dt)\ell$. The image representation is called observable if ℓ is uniquely determined by w, i.e., if $w = W(d/dt)\ell_1 = W(d/dt)\ell_2$ implies $\ell_1 = \ell_2$. It can be shown that this image representation is observable iff $W(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ (see [14]). A system $\mathcal{B} \in \mathcal{L}^{\mathtt{w}}$ admits an image representation iff it is controllable (see [14] and [21]). Furthermore, such image representation can always be chosen to be observable.

For a given real polynomial matrix R, we define $\text{rank}(R)$ as the rank of R considered as a matrix with elements in the field $\mathbb{R}(\xi)$ of real rational functions. On the other hand, for a given $\lambda \in \mathbb{C}$, $\text{rank}(R(\lambda))$ denotes the rank of the complex matrix $R(\lambda)$. It is well known that $\text{rank}(R) = \max_{\lambda\in\mathbb{C}} \text{rank}(R(\lambda))$.

The following proposition gives conditions for given W and R, under which $R(d/dt)w = 0$ is a kernel representation of the system with image representation $w = W(d/dt)\ell$.

Proposition 3.2: Let $W \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$, $\text{rank}(W) = \mathtt{r}$, and let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. Then $R(d/dt)w = 0$ is a kernel representation of the system with image representation $w = W(d/dt)\ell$ iff

$$RW = 0 \quad \text{and} \quad \text{rank}(R(\lambda)) = \mathtt{w} - \mathtt{r} \quad \text{for all } \lambda \in \mathbb{C}. \tag{2}$$

The minimal number of rows over all R's that yield a kernel representation of the system with image representation $w = W(d/dt)\ell$ is thus equal to $\mathtt{w} - \mathtt{r}$. Hence, any R with $\mathtt{w} - \mathtt{r}$ rows that satisfies (2) yields a minimal kernel representation of the system with image representation $w = W(d/dt)\ell$.

Let $\mathcal{B} \in \mathcal{L}^{\mathtt{w}}$ be controllable and let $w = W(d/dt)\ell$ be an observable image representation. There exists a permutation matrix P such that $PM = \text{col}(U, Y)$, with YU^{-1} a matrix of proper rational functions (see [14] and [21]). This corresponds to permuting the components of w as $\Pi w = \text{col}(u, y)$, with $u = U(d/dt)\ell$ and $y = Y(d/dt)\ell$, such that u is an input and y is an output. The number of input components of $\mathcal{B}$, i.e., the size of u, is denoted by $m(\mathcal{B})$, and the number of output components of $\mathcal{B}$, i.e., the size of y, is denoted by $p(\mathcal{B})$. A polynomial matrix $X \in \mathbb{R}^{\mathtt{x}\times\mathtt{w}}[\xi]$ is said to define a *state map* for $\mathcal{B}$ if $x := X(d/dt)\ell$ is a state variable for $\mathcal{B}$ (see [16]). The dimension of the state space of a state-minimal representation of $\mathcal{B} \in \mathcal{L}^{\mathtt{w}}$ is to be called the *McMillan degree* of $\mathcal{B}$ and is denoted by $n(\mathcal{B})$. Often, $n(\mathcal{B})$ is denoted by $\mathtt{n}$. A state map X for $\mathcal{B}$ is called a minimal state map if its number of rows $\mathtt{x}$ is equal to $n(\mathcal{B})$.

We finally introduce the notion of duality for differential systems. Again consider a controllable system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathcal{B})$, in image representation given by $w = W(d/dt)\ell$ and in kernel representation by $R(d/dt)w = 0$. Assume that R has l' rows. We define *the dual of* Σ to be the system $\Sigma^\perp = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathcal{B}^\perp)$ with image representation $w' = R^T(-(d/dt))\ell'$ with latent variable $\ell' \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}'})$. Thus, the signal space of $\Sigma^\perp$ is equal to the signal space $\mathbb{R}^{\mathtt{w}}$ of Σ, and the behavior $\mathcal{B}^\perp$ of $\Sigma^\perp$ is equal to the image of $R^T(-(d/dt))$, i.e., $\mathcal{B}^\perp = R^T(-(d/dt))\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}'})$. The notation $\mathcal{B}^\perp$ is motivated by the fact that, in an appropriate sense, this is the set of trajectories orthogonal to $\mathcal{B}$; it can be shown that for all $w' \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ we have: $w' \in \mathcal{B}^\perp$ iff $\int_{-\infty}^{\infty} (w'(t))^T w(t) \, dt = 0$ for all $w \in \mathcal{B}$. Also, $(\mathcal{B}^\perp)^\perp = \mathcal{B}$. Since we will not use these facts in this paper, we omit the proof.

IV. Two-Variable Polynomial Matrices and Quadratic Differential Forms

An important role is played in this paper by two-variable polynomial matrices. An extensive treatment was given in [23]. In this section we give a brief review.

An $\mathtt{l}\times\mathtt{l}$ two-variable polynomial matrix in the (commuting) indeterminates ζ and η is an expression of the form $\Phi(\zeta, \eta) = \Sigma_{h,k=0}^N \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k}$ are real $\mathtt{l}\times\mathtt{l}$ matrices, and where $N \geq 0$ is an integer. With any such two-variable polynomial matrix we can associate a bilinear functional L_Φ: $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ by defining $L_\Phi(\ell_1, \ell_2) = \Sigma_{h,k=0}^N ((d^h\ell_1/dt^h))^T \Phi_{h,k}(d^k\ell_2/dt^k)$. The two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\Phi_{h,k} = \Phi_{k,h}^T$ for all h, k. In that case we also associate with $\Phi(\zeta, \eta)$ the QDF $Q_\Phi(\ell) := L_\Phi(\ell, \ell)$.

The properties of the two-variable polynomial matrix $\Phi(\zeta, \eta)$ are completely determined by the real constant $(N+1)\mathtt{l} \times (N+1)\mathtt{l}$ matrix $\tilde{\Phi}$ whose (h, k)th block is equal to $\Phi_{h,k}$. This matrix is called the *coefficient matrix* associated with $\Phi(\zeta, \eta)$. Note that $\Phi(\zeta, \eta)$ is symmetric if and only if its coefficient matrix is a symmetric matrix. Factorizations of the coefficient matrix immediately give rise to corresponding factorizations of the associated two-variable polynomial matrix and QDF. In fact, we have the following.

Proposition 4.1: Let $\Phi(\zeta,\eta)$ be a symmetric two-variable $\mathtt{l}\times\mathtt{l}$ polynomial matrix. Let $\tilde{\Phi}$ be its coefficient matrix. Let $\mathtt{l}_1,\mathtt{l}_2\geq 0$ be integers such that $\mathrm{rank}(\tilde{\Phi})=\mathtt{l}_1+\mathtt{l}_2$. Then the following statements are equivalent.

1) $\mathrm{sign}\ (\tilde{\Phi})=(\mathtt{l}_2,\mathtt{l}-\mathtt{l}_1-\mathtt{l}_2,\mathtt{l}_1)$.
2) There exist real matrices $\tilde{L}_1\in\mathbb{R}^{\mathtt{l}_1\times\bullet}$ and $\tilde{L}_2\in\mathbb{R}^{\mathtt{l}_2\times\bullet}$ such that $\tilde{\Phi}=\tilde{L}_1{}^T\tilde{L}_1-\tilde{L}_2{}^T\tilde{L}_2$.
3) There exist real polynomial matrices $L_1\in\mathbb{R}^{\mathtt{l}_1\times\mathtt{l}}[\xi]$ and $L_2\in\mathbb{R}^{\mathtt{l}_2\times\mathtt{l}}[\xi]$ such that $\Phi(\zeta,\eta)=L_1^T(\zeta)L_1(\eta)-L_2^T(\zeta)L_2(\eta)$.
4) There exist real polynomial matrices $L_1\in\mathbb{R}^{\mathtt{l}_1\times\mathtt{l}}[\xi]$ and $L_2\in\mathbb{R}^{\mathtt{l}_2\times\mathtt{l}}[\xi]$ such that for all $\ell\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$, for all $t\in\mathbb{R}$, $Q_\Phi(\ell)(t)=\|L_1(d/dt)\ell(t)\|^2-\|L_2(d/dt)\ell(t)\|^2$.

The QDF Q_Φ is called *nonnegative* if $Q_\Phi(\ell)\geq 0$, in the sense that $Q_\Phi(\ell)(t)\geq 0$ for all $t\in\mathbb{R}$. It is easily seen that Q_Φ is nonnegative iff the coefficient matrix $\tilde{\Phi}$ satisfies $\tilde{\Phi}\geq 0$. Let $\mathcal{B}\in\mathcal{L}^{\mathtt{w}}$ be a differential system as (1) and let $\Phi(\zeta,\eta)$ be a symmetric two-variable $\mathtt{w}\times\mathtt{w}$ polynomial matrix. Then Q_Φ is called *nonnegative on* $\mathcal{B}$ if $Q_L(w)\geq 0$ for all $w\in\mathcal{B}$. If $\mathcal{B}$ is controllable and given in image representation by $w=W(d/dt)\ell$, then it is easily seen that this holds iff the QDF Q_{Φ_1} associated with $\Phi_1(\zeta,\eta):=W^T(\zeta)\Phi(\zeta,\eta)W(\eta)$ is nonnegative (see also [23]). If $\Phi(\zeta,\eta)$ is constant, say $\Phi(\zeta,\eta)=S$, and if $W(\xi)=\Sigma_{k=0}^N W_k\xi^k$, then the coefficient matrix $\tilde{\Phi}_1$ of Φ_1 is equal to $\tilde{\Phi}_1=\tilde{W}^TS\tilde{W}$, with $\tilde{W}:=(W_0W_1\cdots W_N)$ the coefficient matrix of the (one variable) polynomial matrix $W(\xi)$. Hence, the QDF w^TSw is nonnegative on the system $\mathcal{B}$ given by $w=W(d/dt)\ell$ if and only if the matrix $\tilde{W}^TS\tilde{W}\geq 0$.

Any two-variable polynomial matrix $\Phi(\zeta,\eta)$ gives rise to an associated one-variable polynomial matrix in the indeterminate ξ by taking $\zeta=-\xi$ and $\eta=\xi$. The resulting polynomial matrix plays an important role in the sequel. It is denoted by $\partial\Phi:=\Phi(-\xi,\xi)$.

V. Linear Time-Invariant Differential Systems with Disturbances

As already mentioned, we deal with differential systems whose manifest variable w consists of three components: $w=\mathrm{col}(z,d,c)$, with z the to be controlled variables, d the disturbances, and c the interconnnection variables. Let $\mathcal{B}_p\in\mathcal{L}^{\mathtt{w}}$ (the plant) be such a system. We assume that z,d, and c take their values in $\mathbb{R}^{\mathtt{z}},\mathbb{R}^{\mathtt{d}}$, and $\mathbb{R}^{\mathtt{c}}$ respectively, so the signal space of the plant equals $\mathbb{R}^{\mathtt{w}}=\mathbb{R}^{\mathtt{z}}\times\mathbb{R}^{\mathtt{d}}\times\mathbb{R}^{\mathtt{c}}$. A standing assumption will be that the plant $\mathcal{B}_p$ is controllable. Therefore, it admits an image representation $w=W(d/dt)\ell$ for some real polynomial matrix W, say with $\mathtt{l}$ columns. Without loss of generality, we assume moreover that this image representation is observable, i.e., that $W(\lambda)$ has full column rank $\mathtt{l}$ for all $\lambda\in\mathbb{C}$. Partition W conformable the partition of w into $\mathrm{col}(z,d,c)$

$$W=\begin{pmatrix}Z\\D\\C\end{pmatrix}\tag{3}$$

with Z,D, and C real polynomial matrices of appropriate dimensions. $\mathcal{B}_p$ is therefore equal to the set of signals $w=\mathrm{col}(z,d,c)\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{z}}\times\mathbb{R}^{\mathtt{d}}\times\mathbb{R}^{\mathtt{c}})$ for which there exists a function $\ell\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ such that $z=Z(d/dt)\ell$, $d=D(d/dt)\ell$, and $c=C(d/dt)\ell$.

Recall from Section II that the signal d is interpreted as an unknown disturbance. We have formalized this by assuming that d is free. In the present context of linear differential systems this is formalized as follows. In general, if we have a dynamical system $\mathcal{B}\in\mathcal{L}^{\mathtt{w}}$ with signal space $\mathbb{R}^{\mathtt{w}_1}\times\mathbb{R}^{\mathtt{w}_2}$, and manifest variable (w_1,w_2), and if π: $\mathbb{R}^{\mathtt{w}_1}\times\mathbb{R}^{\mathtt{w}_2}\to\mathbb{R}^{\mathtt{w}_2}$ is the projection $\pi(w_1,w_2)=w_2$, then the variable w_2 is called $\mathcal{C}^\infty$*-free* if $\pi(\mathcal{B})=\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$. This is equivalent to saying that every $\mathcal{C}^\infty$ function can occur as the second component of a trajectory (w_1,w_2) of $\mathcal{B}$. Let us now examine how this notion translates into a property of an image representation. If $\mathcal{B}$ is given in image representation

$$\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}W_1\left(\dfrac{d}{dt}\right)\\W_2\left(\dfrac{d}{dt}\right)\end{pmatrix}\ell$$

then w_2 is $\mathcal{C}^\infty$-free iff the differential operator $W_2(d/dt)$: $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})\to\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ is surjective. This is the case if and only if the polynomial matrix W_2 has full row rank. This equivalence is easily proven, for example, via the Smith form of W_2.

We henceforth assume that in the plant $\mathcal{B}_p$ the variable d is $\mathcal{C}^\infty$-free. Thus, in (3) we assume that the polynomial matrix D has full row rank $\mathtt{d}$, equivalently, that the differential operator $D(d/dt)$ is surjective.

We now specify the set of admissible controllers in the context of linear differential systems. Any linear differential system $\mathcal{B}_c$ with manifest variable c and signal space equal to the interconnection space $\mathbb{R}^{\mathtt{c}}$ of the plant $\mathcal{B}_p$ is a candidate admissible controller. However, for obvious reasons, we require that in the interconnected system $\mathcal{B}_p\wedge\mathcal{B}_c$, the variable d (as an externally imposed disturbance) should still be free. In the context of linear differential systems we interpret this in the sense that d should remain $\mathcal{C}^\infty$-free.

Definition 5.1: The linear differential system $\mathcal{B}_c$ is called an *admissible controller* for our plant $\mathcal{B}_p$ if in $\mathcal{B}_p\wedge\mathcal{B}_c$ the variable d is $\mathcal{C}^\infty$-free.

We explain in the next section how the requirement of admissibility translates into a condition involving the polynomial matrices defining the plant and the controller.

VI. The Full Information H_∞-Control Problem

In this paper we restrict ourselves to a solution of the *full information* H_∞-control problem. Related material on this issue can be found in [5]. In the present section we explain the notion of full information control problem.

In general, if $\mathcal{B}$ is a dynamical system with manifest variable $w=\mathrm{col}(w_1,w_2)$, then we call w_1 *observable from* w_2 if w_1 is completely determined by w_2, in the sense that if $\mathrm{col}(w_1^1,w_2^1)$ and $\mathrm{col}(w_1^2,w_2^2)$ are in $\mathcal{B}$ and if $w_2^1=w_2^2$, then $w_1^1=w_1^2$. If w_1 is observable from w_2 then we call w_2 a *full information* *variable for* $\mathcal{B}$: in this case the whole manifest variable w can actually be determined from the component w_2, and observing w_2 alone still gives full information about w. Suppose that we have a linear differential system $\mathcal{B}$ given by an observable image representation

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} W_1\left(\frac{d}{dt}\right) \\ W_2\left(\frac{d}{dt}\right) \end{pmatrix} \ell.$$

We claim that w_2 is a full information variable for $\mathcal{B}$ if and only if the system $w_2 = W_2(d/dt)\ell$ is observable, equivalently, $W_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Indeed, let $W_2(d/dt)\ell_1 = W_2(d/dt)\ell_2$. Then, by observability of w_1 from w_2, we must have $W_1(d/dt)\ell_1 = W_1(d/dt)\ell_2$. Since the representation itself is observable, we conclude that $\ell_1 = \ell_2$. The converse is immediate.

Consider now, as before, a plant $\mathcal{B}_p \in \mathcal{L}^{\mathtt{w}}$ with manifest variable $w = \operatorname{col}(z, d, c)$. In this paper, we restrict attention to the case that the interconnection variable c is a full information variable for $\mathcal{B}_p$. In other words, (z, d) is observable from c. If this is the case, then we call the corresponding H_∞ optimal and suboptimal control problems *full information problems*. We now investigate how the property that c is a full information variable translates into conditions on the defining polynomial matrices in case that the plant is a linear differential system in image representation. Consider the plant $\mathcal{B}_p$ with image representation

$$\begin{pmatrix} z \\ d \\ c \end{pmatrix} = \begin{pmatrix} Z\left(\frac{d}{dt}\right) \\ D\left(\frac{d}{dt}\right) \\ C\left(\frac{d}{dt}\right) \end{pmatrix} \ell. \tag{4}$$

Then we arrive at the following.

Proposition 6.1: Assume that the representation (4) is observable. Then the interconnection variable c is a full information variable for $\mathcal{B}_p$ if and only if $c = C(d/dt)\ell$ is observable, equivalently, iff $\operatorname{rank}(C(\lambda)) = \mathtt{l}$ for all $\lambda \in \mathbb{C}$.

Thus we will henceforth assume that the plant is described by (4), with $C(\lambda)$ of full column rank for all $\lambda \in \mathbb{C}$.

We now specify the admissible controllers in the full information case. As a differential system, a controller imposes a restriction on the interconnection variables c of the form $K(d/dt)c = 0$. Such a controller can of course always also be viewed as imposing a condition on the latent variable ℓ of the plant (4). Indeed, imposing $K(d/dt)c = 0$ is equivalent to imposing $K'(d/dt)\ell = 0$ with $K' = KC$. However, in the full information case the converse also holds: any polynomial matrix K' with $\mathtt{l}$ columns can be written as $K' := KC$ (define $K := K'L$, with L a polynomial left inverse of C). Hence, if in the plant $\mathcal{B}_p$ the representation $c = C(d/dt)\ell$ is observable, i.e., in the full information case, the set of controllers of the form

$$K\left(\frac{d}{dt}\right)c = 0 \tag{5}$$

and the set of controllers of the form

$$c = C\left(\frac{d}{dt}\right)\ell, \qquad K'\left(\frac{d}{dt}\right)\ell = 0 \tag{6}$$

yield one and the same set of controlled systems. Therefore, we may without loss of generality restrict ourselves to the set of all controllers given by (6), where K' ranges over the set of all polynomial matrices with $\mathtt{l}$ columns. Without loss of generality we further restrict ourselves to polynomial matrices K' with full row rank.

In the following lemma we deal with the question under what conditions a controller (6) is admissible.

Lemma 6.2: Consider the plant $\mathcal{B}_p$ with observable image representation (6). Assume that c is a full information variable. Then the controller (6) with K' of full row rank is admissible if and only if $\begin{pmatrix} D \\ K' \end{pmatrix}$ has full row rank.

Proof: See the Appendix. □

In the sequel we simply write K instead of K'.

To summarize, we consider the plant $\mathcal{B}_p$ given by the observable image representation (4), with c a full information variable. This means that $c = C(d/dt)\ell$ is also observable. We consider controllers $\mathcal{B}_c$ given by

$$c = C\left(\frac{d}{dt}\right)\ell, \qquad K\left(\frac{d}{dt}\right)\ell = 0 \tag{7}$$

with K a polynomial matrix with $\mathtt{l}$ columns. Assuming, without loss of generality, that K has full row rank then such a controller is admissible iff $\operatorname{col}(D, K)$ has full row rank. The class of all admissible controllers $\mathcal{B}_c$ given by equations of the form (7) is be denoted by $\mathcal{A}$. Note that if $\mathcal{B}_c$ is admissible and K has full row rank, then K can at most have $\mathtt{l} - \mathtt{d}$ rows. Thus an admissible controller can impose at most $\mathtt{l} - \mathtt{d}$ differential relations on the latent variable ℓ.

In the following proposition, the properties of being stabilizing, γ-contracting, and strictly γ-contracting (as defined in Definitions 2.1 and 2.2) are formulated in terms of the polynomial matrices defining the plant and the controller. A proof follows immediately from the definitions.

Proposition 6.3: Let $\mathcal{B}_c$ be an admissible controller, i.e., $\mathcal{B}_c \in \mathcal{A}$. Then:

1) $\mathcal{B}_c$ is stabilizing iff for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ such that $K(d/dt)\ell = 0$ and $D(d/dt)\ell = 0$ we have $\lim_{t\to\infty} Z(d/dt)\ell(t) = 0$;
2) for a given $\gamma > 0, \mathcal{B}_c$ is γ-contracting iff for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ such that $K(d/dt)\ell = 0$, $D(d/dt)\ell \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$ and $Z(d/dt)\ell \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{z}})$ we have $\|Z(d/dt)\ell\|_2 \le \gamma\|D(d/dt)\ell\|_2$;
3) $\mathcal{B}_c$ is strictly γ-contracting iff there exists $\epsilon > 0$ such that for all ℓ as in 2) we have $\|Z(d/dt)\ell\|_2 \le (\gamma - \epsilon)\|D(d/dt)\ell\|_2$.

VII. When Is a Controller Stabilizing and Strictly Contracting?

In this section we derive conditions for a controller $\mathcal{B}_c \in \mathcal{A}$ to be stabilizing and strictly γ-contracting. Consider again the plant $\mathcal{B}_p$ with observable image representation (4), and with c

a full information variable (equivalently, $C(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$). In the sequel we denote

$$M := \begin{pmatrix} Z \\ D \end{pmatrix}. \tag{8}$$

Throughout this paper, as an additional assumption on the plant $\mathcal{B}_p$ we assume that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, equivalently that $\mathrm{col}(z,d) = M(d/dt)\ell$ is observable as an image representation (see Section III).

For a given $\gamma > 0$, define the $(\mathtt{z}+\mathtt{d}) \times (\mathtt{z}+\mathtt{d})$ diagonal matrix Σ_γ by

$$\Sigma_\gamma := \begin{pmatrix} I_{\mathtt{z}} & 0 \\ 0 & -\gamma^2 I_{\mathtt{d}} \end{pmatrix}. \tag{9}$$

Associated with the plant $\mathcal{B}_p$ and $\gamma > 0$, we consider the symmetric two-variable $\mathtt{l} \times \mathtt{l}$ polynomial matrix $\Phi_\gamma(\zeta,\eta)$ defined by

$$\Phi_\gamma(\zeta,\eta) := M^T(\zeta)\Sigma_\gamma M(\eta) = Z^T(\zeta)Z(\eta) - \gamma^2 D^T(\zeta)D(\eta). \tag{10}$$

This two-variable polynomial matrix induces a one-variable $\mathtt{l}\times\mathtt{l}$ polynomial matrix $\partial\Phi_\gamma$ (in the indeterminate ξ) defined as discussed in Section IV by $\partial\Phi_\gamma(\xi) = \Phi_\gamma(-\xi,\xi)$.

For $v = \mathrm{col}(z,d) \in \mathbb{C}^{\mathtt{z}+\mathtt{d}}$ define its squared Σ_γ-norm by $\|v\|^2_{\Sigma_\gamma} := v^*\Sigma_\gamma v = \|z\|^2 - \gamma^2\|d\|^2$, and for $v \in \mathcal{L}_2(\mathbb{R}, \mathbb{C}^{\mathtt{z}+\mathtt{d}})$, $v = \mathrm{col}(z,d)$, we define its squared $\mathcal{L}_2$ Σ_γ-norm by $\|v\|^2_{2,\Sigma_\gamma} := \int_{-\infty}^{\infty} \|v(t)\|^2_{\Sigma_\gamma}\, dt$. Of course, $\|v\|^2_{2,\Sigma_\gamma} = \|z\|^2_2 - \gamma^2\|d\|^2_2$.

Lemma 7.1: Let $\mathcal{B}_c \in \mathcal{A}$ be represented by $c = C(d/dt)\ell$, $K(d/dt)\ell = 0$, with K full row rank. Let $\gamma > 0$. Then the following statements are equivalent.

1) $\mathcal{B}_c$ is γ-contracting.
2) For all $\ell \in \mathcal{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ such that $K(d/dt)\ell = 0$ we have $\|M(d/dt)\ell\|^2_{2,\Sigma_\gamma} \le 0$.
3) For all $\omega \in \mathbb{R}$ such that $\mathrm{rank}(K(i\omega)) = \mathrm{rank}(K)$ and for all $v \in \ker K(i\omega)$ we have $\|M(i\omega)v\|^2_{\Sigma_\gamma} \le 0$.
4) The polynomial matrix $\binom{D}{K}$ is nonsingular, and the matrix of rational functions

$$G := Z\begin{pmatrix} D \\ K \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathtt{d}} \\ 0 \end{pmatrix} \tag{11}$$

is proper and satisfies $\|G\|_\infty \le \gamma$.

Proof: See the Appendix. □

Remark 7.2: The controlled system is governed by $z = Z(d/dt)\ell, d = D(d/dt)\ell$ and $K(d/dt)\ell = 0$. If $\mathcal{B}_c$ is γ-contracting, then because of the nonsingularity of $\binom{D}{K}$ we can (formally) solve for ℓ using standard transfer function notation, yielding $\ell = \binom{D}{K}^{-1}\binom{I_{\mathtt{d}}}{0}\, d$. Hence, the rational matrix (11) can be interpreted as the transfer matrix from d to z. Thus, as a consequence of the above result, $\mathcal{B}_c$ is a γ-contracting controller iff in the controlled system $\mathcal{B}_p \wedge \mathcal{B}_c$ the variables d and z are related by a proper rational matrix with L_∞-norm less than or equal to γ. In particular, this implies that in the controlled system the variables d and z must have the usual properties of input and output, respectively (see [21]).

Next, we derive the analogue of Lemma 7.1 for *strictly* γ-contracting controllers. If $\mathcal{B}_c$ is a strictly γ-contracting controller, then by Proposition 6.3 there exists $\epsilon > 0$ such that for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$, satisfying $K(d/dt)\ell = 0$ and $D(d/dt)\ell,\ Z(d/dt)\ell \in \mathcal{L}_2$ we have $\|Z(d/dt)\ell\|_2 \le (\gamma - \epsilon)\|D(d/dt)\ell\|_2$. By taking $\epsilon > 0$ sufficiently small and by taking $\epsilon_1 := 2\epsilon\gamma - \epsilon^2$, this is equivalent to $\|Z(d/dt)\ell\|^2_2 - \gamma^2\|D(d/dt)\ell\|^2_2 \le -\epsilon_1^2\|D(d/dt)\ell\|^2_2$. Next, by defining $\epsilon_2^2 := (\epsilon_1^2/1+\gamma^2-\epsilon_1^2)$ this, in turn, is equivalent to $\|Z(d/dt)\ell\|^2_2 - \gamma^2\|D(d/dt)\ell\|^2_2 \le -\epsilon_2^2(\|D(d/dt)\ell\|^2_2 + \|Z(d/dt)\ell\|^2_2)$. We can restate this in terms of $M(d/dt)\ell$ as follows.

Lemma 7.3: Let $\gamma > 0$. $\mathcal{B}_c \in \mathcal{A}$ is strictly γ-contracting iff there exists $\epsilon > 0$ such that for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ with $K(d/dt)\ell = 0$ and $M(d/dt)\ell \in \mathcal{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{z}+\mathtt{d}})$ we have

$$\|M\left(\frac{d}{dt}\right)\ell\|^2_{2,\Sigma_\gamma} \le -\epsilon^2\|M\left(\frac{d}{dt}\right)\ell\|^2_2. \tag{12}$$

By defining $\gamma_\epsilon := (\gamma^2 - \epsilon^2/1+\epsilon^2)^{1/2}$ and by noting that $\Sigma_{\gamma_\epsilon} = (1/1+\epsilon^2)(\Sigma_\gamma + \epsilon^2 I)$ we have that (12) is equivalent with $\|M(d/dt)\ell\|^2_{2,\Sigma_{\gamma_\epsilon}} \le 0$. Thus, we can immediately apply Lemma 7.1, to obtain the following.

Lemma 7.4: Let $\mathcal{B}_c \in \mathcal{A}$ be represented by $c = C(d/dt)\ell$, $K(d/dt)\ell = 0$, with K full row rank. Let $\gamma > 0$. Then the following statements are equivalent.

1) $\mathcal{B}_c$ is strictly γ-contracting.
2) There exists $\epsilon > 0$ such that for all $\ell \in \mathcal{D}(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ with $K(d/dt)\ell = 0$ we have $\|M(d/dt)\ell\|^2_{2,\Sigma_\gamma} \le -\epsilon^2\|M(d/dt)\ell\|^2_2$.
3) There exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ such that $\mathrm{rank}(K(i\omega)) = \mathrm{rank}(K)$ and for all $v \in \ker K(i\omega)$ we have $\|M(i\omega)v\|^2_{\Sigma_\gamma} \le -\epsilon^2\|M(i\omega)v\|^2$.
4) The polynomial matrix $\binom{D}{K}$ is nonsingular, and the matrix of rational functions

$$G := Z\begin{pmatrix} D \\ K \end{pmatrix}^{-1}\begin{pmatrix} I_{\mathtt{d}} \\ 0 \end{pmatrix} \tag{13}$$

is proper and satisfies $\|G\|_\infty < \gamma$.

Remark 7.5: A remark similar to Remark 7.2 holds. This time, however, the transfer matrix G from d to z has to have L_∞-norm *strictly less than* γ.

Remark 7.6: Consider a controller $\mathcal{B}_c$ represented by $c = C(d/dt)\ell,\ K(d/dt)\ell = 0$, with K full row rank. We can always factor $K = K^{\mathrm{un}}K^{\mathrm{cont}}$, with $\det(K^{\mathrm{un}}) \ne 0$ and $\mathrm{rank}(K^{\mathrm{cont}}(\lambda) = \mathrm{rank}(K)$ for all $\lambda \in \mathbb{C}$. The controller $\mathcal{B}_c^{\mathrm{cont}}$ given by $c = C(d/dt)\ell$, $K^{\mathrm{cont}}(d/dt)\ell = 0$ is the controllable part of $\mathcal{B}_c$. It is easily verified that $\mathcal{B}_c^{\mathrm{cont}}$ is admissible iff $\mathcal{B}_c$ is. It also follows easily from Lemmas 7.1 and 7.4 that $\mathcal{B}_c^{\mathrm{cont}}$ is (strictly) γ-contracting iff $\mathcal{B}_c$ is (strictly) γ-contracting. Thus, in this sense we may as well restrict our attention to controllable controllers.

A polynomial matrix F is called *Hurwitz* if it is square, if $\det(F) \ne 0$, and if $\det(F)$ has all its zeroes in the open left-half of the complex plane. As an easy consequence of Proposition 6.3, the following lemma gives a necessary and sufficient condition for a controller to be stabilizing.

Lemma 7.7: Let $\mathcal{B}_c \in \mathcal{A}$ be represented by $c = C(d/dt)\ell$, $K(d/dt)\ell = 0$, with K full row rank. Then $\mathcal{B}_c$ is stabilizing iff $\binom{D}{K}$ is Hurwitz.

Remark 7.8: If $\mathcal{B}_c$ is a stabilizing, strictly γ-contracting controller, then in the controlled system $\mathcal{B}_p \wedge \mathcal{B}_c$ the variables d and z are related by the proper rational matrix G with $\|G\|_\infty < \gamma$. Furthermore, G has now all its poles in $\Re e(\lambda) < 0$ so the L_∞-norm of G is in fact equal to the H_∞-norm of G. Thus, we see that $\mathcal{B}_c$ is a stabilizing and strictly γ-contracting controller iff $\binom{D}{K}$ is Hurwitz and in the controlled system the variables d and z are related by a proper rational matrix with H_∞-norm less than γ.

Remark 7.9: The assumption that $M(\lambda)$ should have full column rank for all $\lambda \in \mathbb{C}$ is made as a standing assumption in order to improve readability. However, at many places in this paper it is possible to relax this assumption. For example, Lemmas 7.1 and 7.4 already hold under the weaker assumption that $M(\lambda)$ has full column rank for $\Re e(\lambda) = 0$. Also, Lemma 7.7 already holds under the assumption that $M(\lambda)$ has full column rank for $\Re e(\lambda) \geq 0$. Indeed, already under this assumption an admissible controller $c = C(d/dt)\ell, K(d/dt)\ell = 0$, with K full row rank, is stabilizing iff $\mathrm{col}(D, K)$ is Hurwitz, or, equivalently, $D(d/dt)\ell = 0$ and $K(d/dt)\ell = 0$ implies $\ell(t) \to 0$ as $t \to \infty$. In particular, this implies that for any state variable $x = X(d/dt)\ell$ we have $\lim_{t\to\infty} x(t) = 0$.

VIII. Dissipative Systems and Storage Functions

In this paper, our aim is to establish conditions on the plant $\mathcal{B}_p$ for the existence of stabilizing, strictly γ-contracting controllers, and to provide algorithms for calculating such controllers. An important role in our development is played by the notions of dissipativeness, strict dissipativeness, and storage function. These notions have been studied before in [18], [8], and [17]. We also refer to [13]. In the present section we introduce and study these notions in the framework of linear differential systems.

Consider, in general, a controllable differential system $\mathcal{B}$ given by the observable image representation

$$w = W\left(\frac{d}{dt}\right)\ell \tag{14}$$

with $W \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$. In addition, let Q_Φ: $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$; $w \mapsto Q_\Phi(w)$, be the QDF associated with a given two-variable polynomial matrix $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Q_Φ is called the *supply rate*. System (14) is called *dissipative* with respect to the supply rate Q_Φ if for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ there holds

$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt \geq 0. \tag{15}$$

System (14) is called *strictly dissipative* with respect to the supply rate Q_Φ if there exists $\epsilon > 0$ such that for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$

$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|w(t)\|^2\, dt. \tag{16}$$

Given the image representation (14) and the polynomial matrix $\Phi(\zeta, \eta)$, define $\Phi' \in \mathbb{R}_s^{\mathtt{l}\times\mathtt{l}}[\zeta, \eta]$ by $\Phi'(\zeta, \eta) := W^T(\zeta)\Phi(\zeta, \eta)W(\eta)$. It is easily verified that if w and ℓ are related by (14), then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. Therefore, the system is dissipative iff for all $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ we have $\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)(t) \geq 0$, and strictly dissipative iff there exists $\epsilon > 0$ such that for all $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell)\, dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|W\left(\frac{d}{dt}\right)\ell\|^2\, dt.$$

These conditions are equivalent to

$$\Phi'(-i\omega, i\omega) \geq 0 \quad \text{for all } \omega \in \mathbb{R} \tag{17}$$

and

$$\Phi'(-i\omega, i\omega) \geq \epsilon^2 W^T(-i\omega)W(i\omega) \quad \text{for all } \omega \in \mathbb{R} \tag{18}$$

respectively (see [23]). It is well known (see [1], [2], [15], and [10]) that if (17) holds then we can factorize $\partial\Phi'(\xi) = \Phi'(-\xi, \xi) = F^T(-\xi)F(\xi)$, with $F \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\xi]$. If (18) holds, then F can be chosen Hurwitz, and also anti-Hurwitz (a polynomial matrix F is called *anti-Hurwitz* if it is square, if $\det(F) \neq 0$ and if $\det(F)$ has all its zeroes in the open right half of the complex plane). Introduce now the two-variable polynomial Δ defined by $\Delta(\zeta, \eta) := \Phi'(\zeta, \eta) - F^T(\zeta)F(\eta)$. Since $\partial\Delta = 0$, the two-variable polynomial Δ must contain a factor $\zeta + \eta$ (see [23, Th. 3.1]), and therefore we can define the new two-variable polynomial Ψ by

$$\Psi(\zeta, \eta) := (\zeta + \eta)^{-1}\Delta(\zeta, \eta). \tag{19}$$

Consider now the QDF's Q_Ψ and Q_Δ associated with Ψ and Δ, respectively. We have $Q_\Delta(\ell) = Q_{\Phi'}(\ell) - \|F(d/dt)\ell\|^2$. Furthermore, (19) is equivalent to $(dQ_\Psi(\ell)/dt) = Q_\Delta(\ell)$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$. Thus we obtain

$$\frac{dQ_\Psi(\ell)}{dt}(t) \leq Q_{\Phi'}(\ell)(t) \tag{20}$$

for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$, for all $t \in \mathbb{R}$. If we interpret $Q_\Psi(\ell)(t)$ as the amount of supply (e.g., energy) stored inside the system at time t, then (20) expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. Inequality (20) is called the *dissipation inequality*. Any QDF Q_Ψ: $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ that satisfies this inequality is called a *storage function*. It can be shown that $\mathcal{B}$ is dissipative if and only if there exists a symmetric two-variable polynomial matrix $\Psi(\zeta, \eta)$ such that the corresponding QDF Q_Ψ satisfies (20). In general, storage functions are not unique. In fact, we quote [23, Th. 5.7].

Proposition 8.1: Assume $\mathcal{B}$ is dissipative with respect to Q_Φ. Then there exist storage functions Q_{Ψ_-} and Q_{Ψ_+} such that any other storage function Q_Ψ satisfies $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$. If $\mathcal{B}$ is strictly dissipative then Ψ_- and Ψ_+ may be constructed as follows. Let H and A be, respectively, Hurwitz and anti-Hurwitz factorizations of $\partial\Phi'$. Then

$$\Psi_+(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta}$$

and

$$\Psi_-(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta}.$$

If a storage function Q_Ψ is a positive semidefinite (negative semidefinite) QDF, then we call it a positive semidefinite (negative semidefinite) storage function.

We now review a basic result from [23], which says that if the system $\mathcal{B}$ is dissipative, then storage functions can always be represented as quadratic functions of any state variable of $\mathcal{B}$. We now make this precise.

Theorem 8.2: Let $\mathcal{B}$ be dissipative with respect to Q_Φ and let Q_Ψ be a storage function, i.e., assume that (20) holds. Let $X \in \mathbb{R}^{\mathtt{x}\times\mathtt{l}}[\xi]$ define a state map of $\mathcal{B}$. Then there exists a real symmetric matrix $K \in \mathbb{R}^{\mathtt{x}\times\mathtt{x}}$ such that $\Psi(\zeta,\eta) = X^T(\zeta)KX(\eta)$. Equivalently, $L_\Psi(\ell_1,\ell_2) = (X(d/dt)\ell_1)^T KX(d/dt)\ell_2$ for all $\ell_1,\ell_2 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$. In particular this implies that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.

Proof: For a proof of this, we refer to [23]. □

A storage function Q_Ψ for $\mathcal{B}$ is called positive (negative) definite if there exists a state map X for $\mathcal{B}$ and a matrix $K>0$ $(K<0)$ such that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$.

In the remainder of this section we consider the case that $\Phi(\zeta,\eta)$ is constant, $\Phi(\zeta,\eta) = \Sigma$, with Σ the nonsingular signature matrix

$$\Sigma = \begin{pmatrix} I_{\mathtt{r}_+} & 0 \\ 0 & -I_{\mathtt{r}_-} \end{pmatrix}. \tag{21}$$

Here, $\mathtt{r}_-$ and $\mathtt{r}_+$ are given positive integers. The corresponding supply rate is then given by $Q_\Phi(w) = w^T\Sigma w$; $\mathtt{r}_+$ corresponds to the number of positive squares in Q_Φ and $\mathtt{r}_-$ to the number of negative squares. In this case the property that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$ is a storage function for $\mathcal{B}$ can be expressed in terms of nonnegativeness of a certain constant QDF on an auxiliary system associated with $\mathcal{B}$. We explain this now.

Let $X(d/dt)\ell$ be a state map of $\mathcal{B}$ given in image representation by (14). Define $W_e \in \mathbb{R}^{(\mathtt{w}+2\mathtt{x})\times\mathtt{l}}[\xi]$ by

$$W_e(\xi) := \begin{pmatrix} W(\xi) \\ X(\xi) \\ \xi X(\xi) \end{pmatrix}. \tag{22}$$

The system with image representation $w_e = W_e(d/dt)\ell$ is denoted by $\mathcal{B}_e$. This system will be called the *extension* of $\mathcal{B}$. Denote the coefficient matrix of W_e (see Section IV) by $\tilde{W}_e$. The following lemma holds.

Lemma 8.3: Let $K \in \mathbb{R}^{\mathtt{x}\times\mathtt{x}}$. Define $\Sigma_e \in \mathbb{R}^{(\mathtt{w}\times 2\mathtt{x})\times(\mathtt{w}\times 2\mathtt{x})}$ by

$$\Sigma_e := \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & 0 & -K \\ 0 & -K & 0 \end{pmatrix}. \tag{23}$$

Then $\|X(d/dt)\ell\|_K^2$ is a storage function for the system $\mathcal{B}$ with supply rate $w^T\Sigma w$ iff the QDF $w_e^T\Sigma_e w_e$ is nonnegative on $\mathcal{B}_e$, equivalently, iff $\tilde{W}_e{}^T\Sigma_e\tilde{W}_e \geq 0$.

Proof: See the Appendix. □

In addition to the notion of strict dissipativity of $\mathcal{B}$, which requires strict positivity of the integral $\int_{-\infty}^{+\infty} w^T\Sigma w\, dt$ over the whole real line, we also need the notion of strict *halfline* positivity. We call $\mathcal{B}$ *strictly* $\mathbb{R}^+$*-halfline* Σ*-positive* if there exists $\epsilon > 0$ such that for all $w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ we have

$$\int_0^\infty w^T\Sigma w\, dt \geq \epsilon^2 \int_0^\infty \|w\|^2\, dt. \tag{24}$$

Likewise we can define the notion of *strict* $\mathbb{R}^-$*-halfline* Σ*-positivity*, which requires the inequality over integrals from $-\infty$ to zero. The following theorem states that a controllable system $\mathcal{B}$ is strictly dissipative with respect to $w^T\Sigma w$ and has a negative definite storage function iff $\mathcal{B}$ is strictly $\mathbb{R}^+$-halfline Σ-positive.

Theorem 8.4: Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{l}}[\xi]$ define a minimal state map for $\mathcal{B}$. The following statements are equivalent.

1) $\mathcal{B}$ is strictly dissipative with respect to $w^T\Sigma w$ and there exists a negative definite matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$ is a storage function.
2) $\mathcal{B}$ is strictly $\mathbb{R}^+$-halfline Σ-positive.

Proof: The analogue of this theorem for the case of strict $\mathbb{R}^-$-halfline Σ-positivity was proven in [23, Th. 9.3]. The proof for the positive halfline counterpart is completely analogous. □

Again consider system (14). Let $R \in \mathbb{R}^{(\mathtt{w}-\mathtt{l})\times\mathtt{l}}[\xi]$ be such that $R(d/dt)w = 0$ is a kernel representation of it. Recall from Section III that the dual $\mathcal{B}^\perp$ of $\mathcal{B}$ is given in image representation by $w' = R^T(-(d/dt))\ell'$. The McMillan degree $n(\mathcal{B}^\perp)$ of $\mathcal{B}^\perp$ is equal to the McMillan degree $\mathtt{n}$ of $\mathcal{B}$. We now recall the following result from [23].

Lemma 8.5: Assume that $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{l}}[\xi]$ defines a minimal state map for $\mathcal{B}$, i.e., $x = X(d/dt)\ell$ defines a minimal state of $\mathcal{B}$. Then there exists a $Z \in \mathbb{R}^{\mathtt{n}\times(\mathtt{w}-\mathtt{l})}[\xi]$ defining a minimal state map $Z(d/dt)$ for $\mathcal{B}^\perp$, such that for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$ and $\ell' \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}-\mathtt{l}})$ we have

$$\frac{d}{dt}\left(\left(Z\left(\frac{d}{dt}\right)\ell'\right)^T X\left(\frac{d}{dt}\right)\ell\right) = \left(R^T\left(-\frac{d}{dt}\right)\ell'\right)^T W\left(\frac{d}{dt}\right)\ell. \tag{25}$$

If a pair of minimal state maps (X,Z) of $\mathcal{B}$ and $\mathcal{B}^\perp$ satisfies (25), then it is called a *matched pair* of state maps.

We also recall the following result from [23], which relates dissipativeness of $\mathcal{B}$ with that of $\mathcal{B}^\perp$.

Theorem 8.6: Assume that $\mathtt{r}_+ = m(\mathcal{B})$, i.e., the number of positive squares in Q_Φ is equal to $m(\mathcal{B})$, the number of inputs of $\mathcal{B}$. Then $\mathcal{B}$ is strictly dissipative with respect to $w^T\Sigma w$ iff $\mathcal{B}^\perp$ is strictly dissipative with respect to $-(w')^T\Sigma w'$. Assume this to be the case. Let (X,Z) be a matched pair of minimal state maps for $\mathcal{B}$ and $\mathcal{B}^\perp$, and let Q_Ψ define a storage function for $\mathcal{B}$. By Theorem 8.2, Q_Ψ is a quadratic state function, i.e., there exists a real symmetric matrix K such that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$. Assume that K is nonsingular. Then $Q_{\Psi'}(\ell') = \|Z(d/dt)\ell'\|_{-K^{-1}}^2$ is a storage function of $\mathcal{B}^\perp$.

Proof: See [23, Th. 10.2]. □

Remark 8.7: Theorem 8.6 can be easily extended to the case that the supply rate is given by $Q_\Phi(w) = w^TSw$, with $S = S^T$, $\det(S) \neq 0$ and $\mathrm{sign}(S) = (\mathtt{r}_-, 0, \mathtt{r}_+)$ with $\mathtt{r}_+ = m(\mathcal{B})$. Then $\mathcal{B}$ is dissipative w.r.t. w^TSw iff $\mathcal{B}^\perp$ is dissipative w.r.t. $-(w')^T S^{-1} w'$. If (X, Z) is a matched pair of minimal state maps for $\mathcal{B}$ and $\mathcal{B}^\perp$ and if $\|X(d/dt)\ell\|_K^2$ is a storage function of $\mathcal{B}$ w.r.t. $w^T S w$ (with $\det(K) \neq 0$) then $\|Z(d/dt)\ell'\|_{-K^{-1}}^2$ is a storage function of $\mathcal{B}^\perp$ w.r.t. $-(w')^T S^{-1} w'$.

To conclude this section, we discuss the relationship between the existence of positive semidefinite storage functions and stability.

Theorem 8.8: Assume that $\mathtt{r}_+ = m(\mathcal{B})$. Partition

$$W = \begin{pmatrix} P \\ N \end{pmatrix}$$

compatible with the partitioning of Σ. Then P square. Assume that $\det(P) \neq 0$. Let $X \in \mathbb{R}^{\mathtt{n} \times \mathtt{l}}[\xi]$ and define a minimal state map for $\mathcal{B}$. If $\mathcal{B}$ is dissipative with respect to $Q_\Phi(w) = w^T \Sigma w$ then NP^{-1} is proper, and the following statements are equivalent.

1) P is Hurwitz.
2) There exists a positive semidefinite storage function Q_Ψ.
3) There exists a positive definite matrix $K \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ such that $Q_\Psi(\ell) = \|X(d/dt)\ell\|_K^2$ is a storage function.

Proof: See [23, Th. 6.4]. □

IX. Existence of Stabilizing, Strictly Contracting Controllers

We now return to the plant $\mathcal{B}_p$ with image representation (4). Assume that $C(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. In the sequel, an important role is played by the system obtained by taking the z and d components of the plant $\mathcal{B}_p$

$$\begin{pmatrix} z \\ d \end{pmatrix} = \begin{pmatrix} Z\left(\frac{d}{dt}\right) \\ D\left(\frac{d}{dt}\right) \end{pmatrix} \ell. \tag{26}$$

This system is denoted simply by $\mathcal{B}$. Recall the Definition 7.1 of M $M = \mathrm{col}(Z, D)$. We assume that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, equivalently, that the representation (26) is observable. We also denote $\mathrm{col}(z, d)$ simply by w.

In this section we consider factorizations of the polynomial matrix $\partial\Phi_\gamma(\xi) = \Phi_\gamma(-\xi, \xi) = M^T(-\xi)\Sigma_\gamma M(\xi)$ [see (9) and (10)]. A factorization $\partial\Phi_\gamma(\xi) = F^T(-\xi)\Sigma_{\mathtt{r}_+, \mathtt{r}_-} F(\xi)$ with $F \in \mathbb{R}^{\mathtt{l} \times \mathtt{l}}[\xi]$, is called a *symmetric factorization* of $\partial\Phi_\gamma$. Here, $\mathtt{r}_+$ and $\mathtt{r}_-$ are nonnegative integers such that $\mathtt{r}_+ + \mathtt{r}_- = \mathtt{l}$ and $\Sigma_{\mathtt{r}_+, \mathtt{r}_-}$ denotes the signature matrix (21). If $\det(F) \neq 0$ then the factorization is said to be *nonsingular*. The integers $\mathtt{r}_+$ and $\mathtt{r}_-$ are called the *positivity index* and *negativity index*, respectively, of the factorization. A nonsingular factorization is called a *regular factorization* if MF^{-1} is a matrix of proper rational functions. The factorization is called *Hurwitz* if the factor F is Hurwitz.

In the following, in accordance with (9), let $\Sigma_{1/\gamma}$ be given by

$$\Sigma_{1/\gamma} = \begin{pmatrix} I_{\mathtt{z}} & 0 \\ 0 & -\frac{1}{\gamma^2} I_{\mathtt{d}} \end{pmatrix}.$$

We now formulate the main result of this section. It turns out that there exists a stabilizing, strictly γ-contracting controller for the plant $\mathcal{B}_p$ iff the dual $\mathcal{B}^\perp$ of $\mathcal{B}$ [given by (26)] is strictly dissipative with respect to the supply rate $(w')^T \Sigma_{1/\gamma} w'$ and, in addition, has a *negative definite* storage function.

We also show that the existence of stabilizing, strictly γ-contracting controllers is equivalent to the existence of certain regular Hurwitz factorizations of the polynomial matrix $\partial\Phi_\gamma$. These factorizations yield explicit formulas for the controllers that we are seeking. This result is strongly related to earlier work on the polynomial approach to H_∞ control by Meinsma (see [12] and [11]). In particular, the equivalence between the existence of a stabilizing, strictly γ-contracting controller and the existence of a regular Hurwitz J-spectral factor was already established in [12]. Related results can also be found in [7], [6], and [9].

Theorem 9.1: Let $\gamma > 0$. Then the following statements are equivalent.

1) There exists a stabilizing, strictly γ-contracting controller.
2) $\mathcal{B}^\perp$ is strictly dissipative with respect to the supply rate $(w')^T \Sigma_{1/\gamma} w'$, and there exists a negative definite storage function for it.
3) There exists a polynomial matrix $F \in \mathbb{R}^{\mathtt{l} \times \mathtt{l}}[\xi]$ such that
   a) $\partial\Phi_\gamma(\xi) = F^T(-\xi)\Sigma_{\mathtt{l}-\mathtt{d}, \mathtt{d}} F(\xi)$;
   b) MF^{-1} is proper;
   c) $\begin{pmatrix} D \\ F_+ \end{pmatrix}$ is Hurwitz;
   d) F is Hurwitz.

Here, F_+ is obtained by partitioning F into

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} \tag{27}$$

where F_+ has $\mathtt{l} - \mathtt{d}$ rows, and F_- has $\mathtt{d}$ rows. If F is a polynomial matrix such that (3) is satisfied, then F_+ has full row rank, and the controller $\mathcal{B}_c$ represented by $c = C(d/dt)\ell$, $F_+(d/dt)\ell = 0$ is admissible, stabilizing, and strictly γ-contracting.

In the remainder of this section we prove this theorem. We first show that the existence of a stabilizing, strictly γ-contracting controller implies that $\mathcal{B}^\perp$ has a negative definite storage function.

Lemma 9.2: Let $\gamma > 0$. If there exists a stabilizing, strictly γ-contracting controller, then $\mathcal{B}^\perp$ is strictly dissipative with respect to the supply rate $(w')^T \Sigma_{1/\gamma} w'$ and for every minimal state map $Z \in \mathbb{R}^{\mathtt{n} \times \mathtt{l}'}[\xi]$ of $\mathcal{B}^\perp$ there exists a negative definite matrix $K \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ such that $Q_\Psi(\ell') = \|Z(d/dt)\ell'\|_K^2$ is a storage function.

Proof: See the Appendix. □

Clearly, Lemma 9.2 immediately yields a proof of the implication (1) ⇒ (2) in Theorem 9.1. We now formulate a lemma that will enable us to prove the implication (2) ⇒ (3).

Lemma 9.3: Let $\gamma > 0$. Let (X, Z) be a matched pair of minimal state maps for $\mathcal{B}$ and $\mathcal{B}^\perp$. Let $\mathcal{B}^\perp$ be strictly dissipative with respect to $(w')^T \Sigma_{1/\gamma} w'$ and let $K_- = K_-^T \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ be such that $Q_\Psi(\ell') = \|Z(d/dt)\ell'\|_{K_-}^2$ is the smallest storage function of $\mathcal{B}^\perp$. Assume that $\det(K_-) \neq 0$. Then there exist

$F \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\xi]$ such that

$$-(\zeta+\eta)X^T(\zeta)K_-^{-1}X(\eta)+M^T(\zeta)\Sigma_\gamma M(\eta) = F^T(\zeta)\Sigma_{\mathtt{l-d},\mathtt{d}}F(\eta). \quad (28)$$

Under these assumptions, for every F such that (28) holds we have $\det(F)\neq 0$ and the rational matrix MF^{-1} is proper. If, in addition, $K_- < 0$, then for any F such that (28) holds, $\mathrm{col}(D, F_+)$ and F are Hurwitz. Here, F_+ is obtained by taking the first $\mathtt{l}-\mathtt{d}$ rows of F. Consequently, any such F yields a regular Hurwitz factorization of $\partial\Phi_\gamma$ with positivity index $\mathtt{l}-\mathtt{d}$

$$\partial\Phi_\gamma(\xi) = F^T(-\xi)\Sigma_{\mathtt{l-d},\mathtt{d}}F(\xi) \quad (29)$$

with $\mathrm{col}(D, F_+)$ Hurwitz.

Proof: See the Appendix. □

Proof of (2) ⇒ (3): Assume that $\mathcal{B}^\perp$ is strictly dissipative and has a negative definite storage function. Then there exists a minimal state map Z of $\mathcal{B}^\perp$ and $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, K<0$, such that $Q_\Psi(\ell') = \|Z(d/dt)\ell'\|_K^2$ is a storage function. Now let $Q_{\Psi_-}(\ell')$ be the smallest storage function of $\mathcal{B}^\perp$. According to Theorem 8.2, there exists $K_- = K_-^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $Q_{\Psi_-}(\ell') = \|Z(d/dt)\ell'\|_{K_-}^2$. Clearly, $K_- \leq K$, so $K_- < 0$. Now take a minimal state map X of $\mathcal{B}$ such that (X, Z) is a matched pair. Then according to Lemma 9.3 there exists $F \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\xi]$ such that (28) holds and such that $\det(F)\neq 0$. Any such F has the properties that are required in condition (3) of Theorem 9.1. □

The proof of the implication (3) ⇒ (1) is now straightforward.

Proof of (3) ⇒ (1): Since $\det(\mathrm{col}(D, F_+)) \neq 0$, F_+ has full row rank and yields an admissible controller. Since $\mathrm{col}(D, F_+)$ is Hurwitz, this controller is stabilizing (see Lemma 7.7). Next we show that $\mathcal{B}_c$ is a strictly γ-contracting controller. Consider the linear polynomial matrix equation $GF = M$ in the unknown G. Clearly, $G := MF^{-1}$ is a proper rational solution without poles on the imaginary axis. Hence $\sup_{\omega\in\mathbb{R}}\|G(i\omega)\| < \infty$. Now, for all $\omega \in \mathbb{R}$ and $v \in \ker\ F_+(i\omega)$ we have $M(i\omega)v = G(i\omega)F(i\omega)v = G(i\omega)\mathrm{col}(0\ \ F_-(i\omega))v$. Hence there exists $k>0$ such that $\|M(i\omega)v\|^2 \leq k\|F_-(i\omega)v\|^2$. This yields that for all $v\in \ker\ F_+(i\omega)$ we have $\|M(i\omega)v\|_{\Sigma_\gamma}^2 = v^*\partial\Phi_\gamma(i\omega)v = -\|F_-(i\omega)v\|^2 \leq -1/k\|M(i\omega)v\|^2$. Apply Lemma 7.4 to conclude that $\mathcal{B}_c$ is strictly γ-contracting. This completes the proof of Theorem 9.1. □

X. A Pick Matrix Test for the Existence of Stabilizing Strictly Contracting Controllers

In Section IX we have shown that there exists a stabilizing, strictly γ-contracting controller for the plant $\mathcal{B}_p$ iff the *dual* system $\mathcal{B}^\perp$ of $\mathcal{B}$ [given by (26)] is strictly dissipative and has a negative definite storage function. Of course, we would like to express these conditions in terms of the *original* system $\mathcal{B}$. In this section we obtain a test in terms of the original system $\mathcal{B}$ to decide whether the dual $\mathcal{B}^\perp$ is strictly dissipative. We also obtain a test in terms of the original system $\mathcal{B}$ to decide whether $\mathcal{B}^\perp$ has a negative definite storage function. It is shown that such negative definite storage function exists iff a certain Pick matrix associated with the system $\mathcal{B}$ is negative definite. At the end of the section these results are applied to the H_∞ control problem.

We first express strict dissipativity of $\mathcal{B}^\perp$ in terms of a condition on the original system. Recall the definition (10) of the two-variable polynomial matrix $\Phi_\gamma(\zeta,\eta)$. It turns out that $\mathcal{B}^\perp$ is strictly dissipative iff $\partial\Phi_\gamma(\xi)$ satisfies a strict signature condition along the imaginary axis.

Lemma 10.1: $\mathcal{B}^\perp$ is strictly dissipative with respect to the supply rate $(w')^T\Sigma_{1/\gamma}w'$ iff there exists $\epsilon>0$ such that

$$\mathrm{sign}(\partial\Phi_\gamma(i\omega)+\epsilon^2M^T(-i\omega)M(i\omega)) = (\mathtt{d}, 0, \mathtt{l}-\mathtt{d}) \quad \text{for all } \omega\in\mathbb{R}. \quad (30)$$

Proof: See the Appendix. □

Remark 10.2: It can be proven that there exists $\epsilon>0$ such that (26) holds iff there exists $\epsilon_0>0$ such that (26) holds *for all* $0\leq\epsilon\leq\epsilon_0$. In particular, this implies that a necessary condition for a stabilizing, strictly γ-contracting controller to exist is that already $\mathrm{sign}(\partial\Phi_\gamma(i\omega)) = (\mathtt{d}, 0, \mathtt{l}-\mathtt{d})$ for all $\omega\in\mathbb{R}$.

In the following definition we consider a general symmetric two-variable $\mathtt{l}\times\mathtt{l}$ polynomial matrix $\Phi(\zeta,\eta)$. We associate with Φ a Pick matrix to be defined below. Since the expression is much simpler in that case, we restrict ourselves here to the case that $\partial\Phi$ is semisimple. Due to space limitations, the general case will be omitted. However, the results below are still valid in that case. A polynomial matrix $P\in\mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$, $\det(P)\neq 0$, is called *semisimple* if for all $\lambda\in\mathbb{C}$ the dimension of $\ker(P(\lambda))$ is equal to the multiplicity of λ as a root of the polynomial $\det(P)$.

Definition 10.3 (Semisimple Case): Assume that $\det(\partial\Phi)$ has no roots on the imaginary axis. Let $\lambda_1,\lambda_2,\cdots,\lambda_{\mathtt{k}}\in\mathbb{C}$ be the roots (counting multiplicity) of $\det(\partial\Phi)$ with negative real part and let $a_1, a_2,\cdots,a_{\mathtt{k}}\in\mathbb{C}^{\mathtt{l}}$ be such that $\partial\Phi(\lambda_i)a_i = 0$, and such that the a_j's associated with the same λ_i form a basis of $\ker(\partial\Phi(\lambda_i))$. Then we define $T_\Phi\in\mathbb{C}^{\mathtt{k}\times\mathtt{k}}$ to be the Hermitian matrix $T_\Phi = (T_{i,j})$ with

$$T_{i,j} := \frac{a_i^*\Phi(\bar\lambda_i,\lambda_j)a_j}{\bar\lambda_i+\lambda_j}. \quad (31)$$

Next, we express the existence of a negative definite storage function for $\mathcal{B}^\perp$ as a condition on the Pick matrix associated with the original problem data Φ_γ.

Lemma 10.4: Assume that $\mathcal{B}^\perp$ is strictly dissipative with respect to the supply rate $(w')^T\Sigma_{1/\gamma}w'$. Then it has a negative definite storage function iff the Pick matrix T_{Φ_γ} is negative definite.

Proof: See the Appendix. □

Thus, summarizing, we immediately obtain the following theorem giving necessary and sufficient conditions (in terms of the original problem data summarized in the two-variable polynomial matrix Φ_γ) for the existence of a stabilizing, strictly γ-contracting controller.

Theorem 10.5: Let $\gamma>0$. There exists a stabilizing, strictly γ-contracting controller iff the following two conditions are satisfied.

1) There exists $\epsilon > 0$ such that $\text{sign}(\partial\Phi_\gamma(i\omega) + \epsilon^2 M^T(-i\omega)M(i\omega)) = (\mathtt{d}, 0, \mathtt{l} - \mathtt{d})$ for all $\omega \in \mathbb{R}$.
2) $T_{\Phi_\gamma} < 0$.

This theorem yields a test for the existence of stabilizing, strictly γ-contracting controllers and an algorithm to calculate one. We start with a plant $\mathcal{B}_p$ in image representation (4) and a required tolerance $\gamma > 0$. Hence, our data are the polynomial matrices Z, D, and C and $\gamma > 0$. We assume that $C(\lambda)$ and $M(\lambda) = \binom{Z(\lambda)}{D(\lambda)}$ have full column rank for all $\lambda \in \mathbb{C}$.

Step 1: Calculate $\Phi_\gamma(\zeta, \eta)$. Check if there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ we have $\text{sign}(\partial\Phi_\gamma(i\omega) + \epsilon^2 M^T(-i\omega)M(i\omega)) = (\mathtt{d}, 0, \mathtt{l} - \mathtt{d})$. If not, then stop: no stabilizing, strictly γ-contracting controller exists. If yes, then go to Step 2.

Step 2: Calculate the roots of $\det(\partial\Phi_\gamma)$ in the open left-half of the complex plane, and calculate the Pick matrix T_{Φ_γ}. Check if $T_{\Phi_\gamma} < 0$. If not, then stop: no stabilizing strictly γ-contracting controller exists. If yes, a stabilizing strictly γ-contracting controller exists. Go to Step 3.

Step 3: Factor $\partial\Phi_\gamma(\xi) = F^T(-\xi)\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}F(\xi)$ with F Hurwitz, $\binom{D}{F_+}$ Hurwitz, and $\binom{Z}{D}F^{-1}$ proper. Here, F_+ is obtained by taking the first $\mathtt{l} - \mathtt{d}$ rows of F. The controller $\mathcal{B}_c$ represented by $c = C(d/dt)\ell$, $F_+(d/dt)\ell = 0$ is admissible, stabilizing, and strictly γ-contracting.

XI. Conclusions

In this paper we formulated the H_∞-control problem from a behavioral perspective. We focused on the strictly suboptimal, full information case. It was shown that a stabilizing, strictly γ-contracting controller exists for the plant under consideration, iff a given one-variable polynomial matrix associated with the plant has a certain regular, indefinite spectral factorization. The required controller can be obtained directly from the spectral factor. We also showed that such a regular, indefinite spectral factorization exists iff the polynomial matrix associated with the plant satisfies a given strict signature condition along the imaginary axis, and a given Pick matrix is negative definite. Future research will be dedicated to a treatment of the general, not full-information problem. Also, in a forthcoming paper we develop algorithms to obtain the required indefinite spectral factorizations.

Appendix
Proofs

Proof of Lemma 6.2: We need to prove that in (4) combined with (6), d is $\mathcal{C}^\infty$-free iff $\binom{D}{K'}$ has full row rank. If this is the case, then the differential operator $\binom{D(d/dt)}{K(d/dt)}$ is surjective. Let $d \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$ be arbitrary. There exists $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ such that $\binom{d}{0} = \binom{D(d/dt)}{K'(d/dt)}\ell$. Define $z = Z(d/dt)\ell$, $d = D(d/dt)\ell$, $c = C(d/dt)\ell$. Then $\text{col}(z, d, c) \in \mathcal{B}_p \wedge \mathcal{B}'_c$. This proves that in $\mathcal{B}_p \wedge \mathcal{B}_c$, d is indeed $\mathcal{C}^\infty$-free. Conversely, if $\binom{D}{K'}$ does not have full row rank then there exists a polynomial row vector $0 \neq (p_1 p_2)$ such that $p_1 D + p_2 K' = 0$. Since K' has full row rank, $p_1 \neq 0$. Now let $\text{col}(z, d, c) \in \mathcal{B}_p \wedge \mathcal{B}'_c$ be a $\mathcal{C}^\infty$ function, $z = Z(d/dt)\ell$, $d = D(d/dt)\ell$ $c = C(d/dt)\ell$, and $K'(d/dt)\ell = 0$. Then there must hold $p_1(d/dt)d = 0$. In other words, d necessarily satisfies a nontrivial differential equation, so it can not be $\mathcal{C}^\infty$-free. □

Proof of Lemma 7.1: Factor $K = K^{\text{un}}K^{\text{cont}}$, with $\det(K^{\text{un}}) \neq 0$ and $\text{rank}(K^{\text{cont}}(\lambda)) = \mathtt{r}$ for all $\lambda \in \mathbb{C}$. Note that $\text{rank}(K(\lambda)) = \mathtt{r}$ iff $\det(K^{\text{un}}(\lambda)) \neq 0$. Furthermore, $\text{rank}(K(\lambda)) < \mathtt{r}$ for at most finitely many λ. Also, there exists $N \in \mathbb{R}^{\mathtt{l}\times(\mathtt{l}-\mathtt{r})}[\xi]$ such that $K^{\text{cont}}N = 0$ and such that $\text{im}(N(\lambda)) = \ker(K^{\text{cont}}(\lambda))$ for all $\lambda \in \mathbb{C}$. Hence $N(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

(1)⇒ (2): This implication follows immediately from Proposition 6.3.

(2)⇒ (3): Let $\omega \in \mathbb{R}$ be such that $\text{rank}(K(i\omega)) = \mathtt{r}$. Let $v \in \ker(K(i\omega))$. Then $v \in \ker(K^{\text{cont}}(i\omega))$ so there exist v' such that $v = N(i\omega)v'$. Define $\ell'_n \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{\mathtt{l}-\mathtt{r}})$ by

$$\ell'_n(t) = \begin{cases} e^{i\omega t}v', & |t| \leq \dfrac{2\pi n}{\omega} \\ \tilde{\ell}\left(t + \dfrac{2\pi n}{\omega}\right), & t < -\dfrac{2\pi n}{\omega} \\ \tilde{\ell}\left(t - \dfrac{2\pi n}{\omega}\right), & t > \dfrac{2\pi n}{\omega}. \end{cases}$$

Here $\tilde{\ell}$ is chosen such that $\ell'_n \in \mathcal{D}(\mathbb{R}, \mathbb{C}^{\mathtt{l}-\mathtt{r}})$. Note that $\tilde{\ell}$ is (and can) be chosen to be independent of n. Define $\ell_n := N(d/dt)\ell'$. Then $K(d/dt)\ell_n = 0$. Now, an easy calculation shows $\int_{-\infty}^{\infty} \|M(d/dt)\ell_n\|^2_{\Sigma_\gamma}\, dt = (4\pi n/\omega)\|M(i\omega)v\|^2_{\Sigma_\gamma} + \int_{-\infty}^{\infty} \|M(d/dt)N(d/dt)\tilde{\ell}(t)\|^2_{\Sigma_\gamma}\, dt$. Since the integral on the left is ≤ 0 we must have $\|M(i\omega)v\|^2_{\Sigma_\gamma} \leq 0$.

(3) ⇒ (4): First note that $\binom{D}{K}$ has full row rank. We show that it also has full column rank. Let $\omega \in \mathbb{R}$ be such that $\text{rank}(K(i\omega)) = \mathtt{r}$ and assume $\binom{D(i\omega)}{K(i\omega)}v = 0$. From (3) we get $Z(i\omega)v = 0$. By observability of M this yields $v = 0$. Hence $\binom{D(i\omega}{K(i\omega)}$ has full column rank. Define a matrix of rational functions $R := \binom{D}{K}^{-1}\binom{I}{0}$, where I denotes the $\mathtt{d} \times \mathtt{d}$ identity matrix. Note that $DR = I$ and $KR = 0$. Again let ω be such that $\text{rank}(K(i\omega)) = \mathtt{r}$. Then $\binom{D(i\omega)}{K(i\omega)}$ is nonsingular. For all $v \in \mathbb{C}^{\mathtt{d}}$ we have $R(i\omega)v \in \ker(K(i\omega))$ and hence $v^*R^T(-i\omega)Z^T(-i\omega)Z(i\omega)R(i\omega)v \leq \gamma^2 v^*$. Thus $G^T(-i\omega)G(i\omega) \leq \gamma^2 I$. This inequality holds for all but finitely many $\omega \in \mathbb{R}$. We conclude that G is proper and $\|G\|_\infty \leq \gamma$.

(4) ⇒ (1): Consider the system $z = Z(d/dt)\ell$, $d = D(d/dt)\ell$, $K(d/dt)\ell = 0$. Using that $G = Z\binom{D}{K}^{-1}\binom{I}{0}$ is a proper rational matrix, it can be shown that d is an input and z an output. Hence, there exist constant matrices F, G, H, and J such that z and d satisfy $(dx/dt) = Fx + G_1 d$, $z = Hx + Jd$, for some absolutely continuous function x. Also, $G(\xi) = J + H(\xi I - F)^{-1}G_1$. We can choose F and H such that the pair (H, F) is observable. Using that G has no poles on the imaginary axis, we can choose F such that $\Re e(\lambda) \neq 0$ for every eigenvalue λ. Our proof is now completed by applying the following well-known result.

Lemma 12.1: Assume that $z \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{z}})$ and $d \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{d}})$ are related by $(dx/dt) = Fx + G_1 d$, $z = Hx + Jd$, where (H, F) is observable with F such that $\Re e(\lambda) \neq 0$ for every eigenvalue λ. Then $\|z\|_2 \leq \|G\|_\infty \|d\|_2$, with $\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|J + H(i\omega I - F)^{-1}G_1\|$.

Proof of Lemma 8.3: $\|X(d/dt)\ell\|_K^2$ is a storage function for $\mathcal{B}$ with supply rate $w^T\Sigma w$ iff $(d/dt)\|X(d/dt)\ell\|_K^2 - \|W(d/dt)\ell\|_\Sigma^2 \leq 0$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$. This can be seen to hold iff $(W_e(d/dt)\ell)^T\Sigma_e W_e(d/dt)\ell \geq 0$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}})$, equivalently iff the QDF $w_e^T\Sigma_e w_e$ is nonnegative on the extension $\mathcal{B}_e$.

Proof of Lemma 9.2: Let R be such that $RM = 0$ and $\mathrm{rank}(R(\lambda)) = \mathtt{z}+\mathtt{d}-\mathtt{l}$ for all $\lambda \in \mathbb{C}$, so that $R(d/dt)w = 0$ is a minimal kernel representation of $\mathcal{B}$. Assume that there exists a stabilizing, strictly γ-contracting controller $c = C(d/dt)\ell$, $K(d/dt)\ell = 0$. We may assume that K has full row rank and that $K(d/dt)\ell = 0$ is controllable. Let N be such that $KN = 0$ and $\mathrm{rank}(N(\lambda)) = \mathtt{d}$ for all $\lambda \in \mathbb{C}$. Then $\ell = N(d/dt)\ell'$ is an observable image representation of $K(d/dt)\ell = 0$. Denote the system $w = M(d/dt)N(d/dt)\ell'$ by $\mathcal{B}_N$. Obviously, $\mathcal{B}_N \subset \mathcal{B}$. It is easily verified that DN is Hurwitz since $\binom{D}{K}$ is Hurwitz, and that the system $\mathcal{B}_N$ is strictly dissipative with respect to the supply rate $-w^T\Sigma_\gamma w$. Let X define a minimal state map for $\mathcal{B}_N$, and let Z define a minimal state map of $\mathcal{B}_N^\perp$ such that (X,Z) is a matched pair. Note that the number of positive eigenvalues of $-\Sigma_\gamma$ is equal to $m(\mathcal{B}_N)$, the number of inputs of $\mathcal{B}_N$. Since DN is Hurwitz, it follows from Theorem 8.8 that there exists a positive definite matrix K such that $\|X(d/dt)\ell'\|_K^2$ is a storage function for $\mathcal{B}_N$. By applying Theorem 8.6, $\mathcal{B}_N^\perp$ is strictly dissipative with respect to $(w')^T\Sigma_{1/\gamma}w'$, and $\|Z(d/dt)\ell''\|_{-K^{-1}}^2$ is a storage function (ℓ'' which denotes the latent variable of $\mathcal{B}_N^\perp$). Now apply Theorem 8.4 to find that $\mathcal{B}_N^\perp$ is strictly $\mathbb{R}^+$-halfine $\Sigma_{1/\gamma}$-positive. Since, however, $\mathcal{B}^\perp \subset \mathcal{B}_N^\perp$, the same holds for $\mathcal{B}^\perp$, which, again by Theorem 8.4, completes our proof.

Proof of Lemma 9.3: In the proof of Lemma 9.3 we need the following general matrix theoretical lemma.

Lemma 12.2: Let $Q \in \mathbb{C}^{\mathtt{r}\times\mathtt{r}}$ be nonsingular and Hermitian. For given integers $\mathtt{m}_1$ and $\mathtt{m}_2$, let $L_1 \in \mathbb{C}^{\mathtt{r}\times\mathtt{m}_1}$, and $L_2 \in \mathbb{C}^{\mathtt{r}\times\mathtt{m}_2}$ be such that $\mathrm{rank}(L_1)+\mathrm{rank}(L_2) = \mathtt{r}$, and $L_2^*L_1 = 0$. Define the $(\mathtt{m}_1+\mathtt{m}_2)\times(\mathtt{m}_1+\mathtt{m}_2)$ matrix P by

$$P := \begin{pmatrix} L_1^* \\ L_2^*Q^{-1}\end{pmatrix} Q(L_1 \quad Q^{-1}L_2) = \begin{pmatrix} L_1^*QL_1 & 0 \\ 0 & L_2^*Q^{-1}L_2\end{pmatrix}. \tag{32}$$

Then the following holds.

1) $\mathrm{rank}(L_1^*QL_1) \leq \mathrm{rank}(L_1)$ and $\mathrm{rank}(L_2^*Q^{-1}L_2) \leq \mathrm{rank}(L_2)$.
2) For any integer $\mathtt{k} \geq 0$ we have $\mathrm{rank}(L_1^*QL_1) = \mathrm{rank}(L_1) - \mathtt{k}$ iff $\mathrm{rank}(L_2^*Q^{-1}L_2) = \mathrm{rank}(L_2) - \mathtt{k}$. Consequently, there exists an integer $\mathtt{k} \geq 0$ such that $\mathrm{rank}(P) = \mathtt{r} - 2\mathtt{k}$, i.e., in forming the product matrix P the loss of rank (compared to the rank of the nonsingular $\mathtt{r}\times\mathtt{r}$ matrix Q) is always even.
3) Let $\mathtt{k}$ be such that $\mathrm{rank}(P) = \mathtt{r}-2\mathtt{k}$. Then the signature of P is given by

$$\begin{aligned}(n_-(P), n_0(P), n_+(P)) \\ = (n_-(Q)-\mathtt{k}, \mathtt{m}_1+\mathtt{m}_2-\mathtt{r}+2\mathtt{k}, n_+(Q)-\mathtt{k}).\end{aligned} \tag{33}$$

In particular this implies that if in forming the product matrix P the loss of rank is $2\mathtt{k}$, then $\mathtt{k}$ negative eigenvalues and $\mathtt{k}$ positive eigenvalues will be lost.

Proof: Statement 1) is obvious. Now first assume that L_1 and L_2 have full column rank. Then $\mathtt{m}_1+\mathtt{m}_2 = \mathtt{r}$. To prove 2), use that $L_2^*L_1 = 0$. Using this it can easily be verified that $QL_1\mathrm{ker}(L_1^*QL_1) = L_2\mathrm{ker}(L_2^*Q^{-1}L_2)$. Since QL_1 and L_2 have full column rank, this implies that $\dim(\mathrm{ker}(L_1^*QL_1)) = \dim(\mathrm{ker}(L_2^*Q^{-1}L_2))$, which proves 2). Next we prove 3). Let $\mathrm{sign}(L_1^*QL_1) = (\nu_1,\mathtt{k},\pi_1)$ and $\mathrm{sign}(L_2^*Q^{-1}L_2) = (\nu_2,\mathtt{k},\pi_2)$. Then clearly $\mathrm{sign}(P) = (\nu_1+\nu_2, 2\mathtt{k}, \pi_1+\pi_2)$. We show now that $\nu_1+\nu_2 = n_-(Q)-\mathtt{k}$ and $\pi_1+\pi_2 = n_+(Q)-\mathtt{k}$. This will be done using a perturbation argument. For $\epsilon > 0$ sufficiently small we have $\mathrm{sign}(L_1^*(Q+\epsilon I)L_1) = (\nu_1, 0, \pi_1+\mathtt{k})$. Also, for $\epsilon > 0$ sufficiently small we have $\mathrm{sign}(L_2^*(Q+\epsilon I)^{-1}L_2) = (\nu_2+\mathtt{k}, 0, \pi_2)$. Also

$$\begin{aligned} P_\epsilon &:= \begin{pmatrix} L_1^* \\ L_2^*(Q+\epsilon I)^{-1}\end{pmatrix}(Q+\epsilon I)(L_1 (Q+\epsilon I)^{-1}L_2) \\ &= \begin{pmatrix} L_1^*(Q+\epsilon I)L_1 & 0 \\ 0 & L_2^*(Q+\epsilon I)^{-1}L_2\end{pmatrix}.\end{aligned}$$

The right-hand side of this equation is nonsingular, so all three factors on the left-hand side are nonsingular as well. For $\epsilon > 0$ sufficiently small this implies that $(\nu_1+\nu_2+\mathtt{k}, 0, \pi_1+\pi_2+\mathtt{k}) = \mathrm{sign}(P_\epsilon) = \mathrm{sign}(Q+\epsilon I) = (n_+(Q), 0, n_-(Q))$. This proves the claim.

To prove the lemma for general L_1 and L_2 (not necessarily of full column rank), let T_1 and T_2 be such that $\det(T_1) \neq 0$ and $\det(T_2) \neq 0$ and such that $L_1T_1 = (\tilde{L}_1 \quad 0)$, $L_2T_2 = (\tilde{L}_2 0)$ with $\tilde{L}_1$ and $\tilde{L}_2$ full column rank. Then apply the previous data using $\tilde{L}_1$ and $\tilde{L}_2$, and note that rank and signature are invariant under premultiplication by T_i and postmultiplication by T_i^*. □

We now proceed with the proof of Lemma 9.3. In addition to the extension $\mathcal{B}_e$ of $\mathcal{B}$, we consider the extension $\mathcal{B}_e^\perp$ of $\mathcal{B}^\perp$. Let Z be a state map of $\mathcal{B}^\perp$. We define $\mathcal{B}_e^\perp$ to be the system with image representation $w_e' = R_e^T(-(d/dt)\ell'$, with $R_e^T(-\xi)$ defined by

$$R_e^T(-\xi) := \begin{pmatrix} R^T(-\xi) \\ -\xi Z(\xi) \\ -Z(\xi)\end{pmatrix}. \tag{34}$$

Lemma 12.3: Let (X,Z) be a matched pair of minimal state maps for $\mathcal{B}$ and $\mathcal{B}^\perp$. Define subspaces $\mathcal{M} \subset \mathbb{R}^{\mathtt{z}+\mathtt{d}+2\mathtt{n}}$, $\mathcal{R} \subset \mathbb{R}^{\mathtt{z}+\mathtt{d}+2\mathtt{n}}$ by

$$\begin{aligned} \mathcal{M} := \Big\{ a \in \mathbb{R}^{\mathtt{z}+\mathtt{d}+2\mathtt{n}} \,|\, \exists \ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \text{ such that} \\ a = M_e\left(\frac{d}{dt}\right)\ell(0)\Big\} \end{aligned} \tag{35}$$

$$\begin{aligned} \mathcal{R} := \Big\{ b \in \mathbb{R}^{\mathtt{z}+\mathtt{d}+2\mathtt{n}} \,|\, \exists \ell' \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}'}) \text{ such that} \\ b = R_e^T\left(-\frac{d}{dt}\right)\ell'(0)\Big\}. \end{aligned} \tag{36}$$

Then $\dim(\mathcal{M}) = \mathtt{n}+\mathtt{l}$ and $\mathcal{M}^\perp = \mathcal{R}$.

Proof: For a proof, we refer to [23, Lemma 12.4]. □

We now give a proof of Lemma 9.3. (X, Z) is a matched pair of minimal state maps. Consider the extensions $M_e(\xi)$ of $M(\xi)$ and $R_e^T(-\xi)$ of $R^T(-\xi)$. Define $\Sigma_e \in \mathbb{R}^{(\mathtt{z}+\mathtt{d}+2\mathtt{n})\times(\mathtt{z}+\mathtt{d}+2\mathtt{n})}$ by

$$\Sigma_e := \begin{pmatrix} \Sigma_{1/\gamma} & 0 & 0 \\ 0 & 0 & -K_- \\ 0 & -K_- & 0 \end{pmatrix}. \tag{37}$$

Then condition (28) is equivalent to $M_e^T(\zeta)\Sigma_e^{-1}M_e(\eta) = F^T(\zeta)\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}F(\eta)$. Let $\tilde{M}_e$ and $\tilde{R}_e^T$ be the coefficient matrices associated with the extensions $M_e(\xi)$ and $R_e^T(-\xi)$. According to Lemma 12.3 $\tilde{R}_e\tilde{M}_e = 0$. Also, $\operatorname{rank}(\tilde{M}_e) = \mathtt{n}+\mathtt{l}$ and $\operatorname{rank}(\tilde{R}_e^T) = \mathtt{n}+\mathtt{z}+\mathtt{d}-\mathtt{l}$. Let $\mathtt{m}_1$ and $\mathtt{m}_2$ denote the number of columns of $\tilde{M}_e$ and $\tilde{R}_e{}^T$, respectively. Note that $n_+(\Sigma_e^{-1}) = \mathtt{n}+\mathtt{z}$ and $n_-(\Sigma_e^{-1}) = \mathtt{n}+\mathtt{d}$. Now consider the product

$$P = \begin{pmatrix} \tilde{M}_e{}^T \\ \tilde{R}_e\Sigma_e \end{pmatrix}\Sigma_e^{-1}(\tilde{M}_e \quad \Sigma_e\tilde{R}_e{}^T) = \begin{pmatrix} \tilde{M}_e{}^T\Sigma_e^{-1}\tilde{M}_e & 0 \\ 0 & \tilde{R}_e\Sigma_e\tilde{R}_e{}^T \end{pmatrix}.$$

Since $\operatorname{rank}(\tilde{R}_e{}^T) = \mathtt{n}+\mathtt{z}+\mathtt{d}-\mathtt{l}$, we have $\operatorname{rank}(\tilde{R}_e\Sigma_e\tilde{R}_e{}^T) = \mathtt{n}+\mathtt{z}+\mathtt{d}-\mathtt{l}-\mathtt{k}$ for some integer $\mathtt{k} \geq 0$. We prove that, in fact, $\mathtt{k} = \mathtt{n}$. Indeed, we have $\mathtt{k} = \mathtt{n}+\mathtt{z}+\mathtt{d}-\mathtt{l}-\operatorname{rank}(\tilde{R}_e\Sigma_e\tilde{R}_e{}^T)$. Now, $Q_{\Psi_-}(\ell') = \|Z(d/dt)\ell'\|_{K_-}^2$ is the smallest storage function of $\mathcal{B}^\perp$ with respect to the supply rate $(w')^T\Sigma_{1/\gamma}w'$. Since there exists $\epsilon > 0$ such that $R^T(i\omega)\Sigma_{1/\gamma}R(i\omega) \geq \epsilon R^T(i\omega)R(i\omega)$ for all $\omega \in \mathbb{R}, Q_{\Psi_-}(\ell')$ is obtained from a Hurwitz spectral factorization, in the sense that if we factor $R(\xi)\Sigma_{1/\gamma}R^T(-\xi) = H^T(-\xi)H(\xi)$ with H Hurwitz, then

$$\Psi_-(\zeta,\eta) = Z^T(\zeta)K_-Z(\eta) = \frac{R(-\zeta)\Sigma_{1/\gamma}R^T(-\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta}$$

(see Proposition 8.1). Evidently, this is equivalent to $R_e(-\zeta)\Sigma_eR_e^T(-\eta) = H^T(\zeta)H(\eta)$ by definition. Thus we find $\tilde{R}_e\Sigma_e\tilde{R}_e{}^T = \tilde{H}^T\tilde{H}$, where $\tilde{H}$ is the coefficient matrix of H. Since $\det(H) \neq 0, \tilde{H}$ must have full row rank $\mathtt{z}+\mathtt{d}-\mathtt{l}$. This implies that $\tilde{R}_e\Sigma_e\tilde{R}_e{}^T$ has rank $\mathtt{z}+\mathtt{d}-\mathtt{l}$, so that $\mathtt{k} = \mathtt{n}$ as claimed. According to Lemma 12.2 we then have $\operatorname{rank}(P) = \mathtt{z}+\mathtt{d}$ and $\operatorname{sign}(P) = (\mathtt{d}, \mathtt{m}_1+\mathtt{m}_2-\mathtt{z}-\mathtt{d}, \mathtt{z})$. We also have $\operatorname{rank}(\tilde{M}_e{}^T\Sigma_e^{-1}\tilde{M}_e) = \mathtt{l}$.

Since $\mathcal{B}^\perp$ has storage function $\|Z(d/dt)\ell'\|_{K_-}^2$, the QDF $(w_e')^T\Sigma_ew_e'$ is nonnegative on $\mathcal{B}_e^\perp$. Equivalently, $\tilde{R}_eQ_e\tilde{R}_e{}^T \geq 0$. Of course, this is equivalent to $\operatorname{sign}(\tilde{R}_e\Sigma_e\tilde{R}_e{}^T) = (0, \mathtt{m}_2-\mathtt{z}-\mathtt{d}+\mathtt{l}, \mathtt{z}+\mathtt{d}-\mathtt{l})$. According to Lemma 12.2, this is equivalent to $\operatorname{sign}(\tilde{M}_e{}^T\Sigma_e^{-1}\tilde{M}_e) = (\mathtt{d}, \mathtt{m}_1-\mathtt{l}, \mathtt{l}-\mathtt{d})$. It then follows from Proposition 4.1 that there exists $F \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}(\xi]$ such that (28) holds. It is easily seen that if F satisfies (28) then $\det(F) \neq 0$. Note that its coefficient matrix $\tilde{F}$ then automatically has full row rank.

Next we prove that if F satisfies 9.3, then MF^{-1} is a proper rational matrix. There exists a permutation matrix P such that $PM = \operatorname{col}(U, Y)$ with $U \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\xi]$, $\det(U) \neq 0$, and such that YU^{-1} is a proper rational matrix. Define $M_\infty := \lim_{|\lambda|\to\infty} M(\lambda)U(\lambda)^{-1}$. We first show that $\det(M_\infty^T\Sigma_\gamma M_\infty) \neq 0$. Note that, by construction, M_∞ has full column rank. Next, as before, let $R \in \mathbb{R}^{(\mathtt{z}+\mathtt{d}-\mathtt{l})\times(\mathtt{z}+\mathtt{d})}[\xi]$ be such that $RM = 0$ and such that $\operatorname{rank}(R(\lambda)) = \mathtt{z}+\mathtt{d}-\mathtt{l}$ for all λ. Since $\mathcal{B}^\perp$ is strictly dissipative, there exists $\epsilon > 0$ such that $R(i\omega)\Sigma_{1/\gamma}R^T(-i\omega) \geq \epsilon^2R(i\omega)R^T(-i\omega)$ for all $\omega \in \mathbb{R}$. Let $\Lambda \in \mathbb{R}^{(\mathtt{z}+\mathtt{d}-\mathtt{l})\times(\mathtt{z}+\mathtt{d}-\mathtt{l})}[\xi]$ be such that $\lim_{|\lambda|\to\infty} \Lambda(\lambda)^{-1}R(\lambda) =: R_\infty$ has full row rank. Then we have $R_\infty\Sigma_{1/\gamma}R_\infty^T \geq \epsilon^2R_\infty R_\infty^T > 0$, so $R_\infty\Sigma_{1/\gamma}R_\infty^T$ is nonsingular. We claim that this implies that $M_\infty^T\Sigma_\gamma M_\infty$ is nonsingular. Indeed, consider the square matrix $(M_\infty R_\infty^T)$. Obviously, since $R_\infty M_\infty = 0$, this matrix is nonsingular. Consider the product

$$\begin{pmatrix} M_\infty^T \\ R_\infty \end{pmatrix}(M_\infty \quad \Sigma_{1/\gamma}R_\infty^T) = \begin{pmatrix} M_\infty^TM_\infty & * \\ 0 & R_\infty\Sigma_{1/\gamma}R_\infty^T \end{pmatrix}.$$

Since $M_\infty^TM_\infty > 0$, we have that $R_\infty\Sigma_{1/\gamma}R_\infty^T$ is nonsingular iff $(M_\infty\Sigma_{1/\gamma}R_\infty^T)$ is nonsingular. This holds iff

$$\begin{pmatrix} M_\infty^T \\ R_\infty\Sigma_{1/\gamma} \end{pmatrix}\Sigma_\gamma(M_\infty \quad \Sigma_{1/\gamma}R_\infty^T) = \begin{pmatrix} M_\infty^T\Sigma_\gamma M_\infty & 0 \\ 0 & R_\infty\Sigma_{1/\gamma}R_\infty^T \end{pmatrix}$$

is nonsingular. This implies that $M_\infty^T\Sigma_\gamma M_\infty$ is nonsingular as claimed.

We now show that FU^{-1} is a proper rational matrix. To prove this, postmultiply (28) with $U(\eta)^{-1}$ to obtain

$$\begin{aligned} &-(\zeta+\eta)X^T(\zeta)K^{-1}X(\eta)U(\eta)^{-1} \\ &\quad + M^T(\zeta)\Sigma_\gamma M(\eta)U(\eta)^{-1} \\ &\quad = F^T(\zeta)\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}F(\eta)U(\eta)^{-1}. \end{aligned} \tag{38}$$

Assume that $L_i\eta^i$ is the term of degree i in the polynomial part of the rational matrix $M(\eta)U(\eta)^{-1}$. Since the left-hand side of (38) is proper in η and equating powers of η yields $F^T(\zeta)\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}L_i = 0$ $(i \geq 1)$. Expressing $F(\zeta)$ as $F(\zeta) = \tilde{F}\operatorname{col}(I, I\zeta, \cdots, I\zeta^N)$ we obtain $\tilde{F}^T\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}L_i = 0$. Since $\tilde{F}$ has full row rank, this yields $L_i = 0$. Thus we proved that FU^{-1} is proper. Define $F_\infty := \lim_{|\lambda|\to\infty} F(\lambda)U(\lambda)^{-1}$. By putting in (28), $\zeta = -\lambda$, and $\eta = \lambda$, premultiplying the result by $U(-\lambda)^{-T}$ and postmultiplying by $U(\lambda)^{-1}$, and by finally letting $|\lambda| \to \infty$, we obtain $M_\infty^T\Sigma_\gamma M_\infty = F_\infty^T\Sigma_{\mathtt{l}-\mathtt{d},\mathtt{d}}F_\infty$. This implies that F_∞ is nonsingular. Therefore, FU^{-1} is in fact biproper, i.e., has a proper inverse. Thus $MF^{-1} = MU^{-1}(FU^{-1})^{-1}$ is proper as claimed.

Now assume that $K_- < 0$. We first prove that $\operatorname{col}(D, F_+)$ is Hurwitz. Clearly, it is square. To prove that it is nonsingular, we show that $\det(\operatorname{col}(D(i\omega), F_+(i\omega))) \neq 0$ for all ω. Indeed, assume to the contrary that $v \neq 0$ satisfies $D(i\omega)v = 0$ and $F_+(i\omega)v = 0$. Then it follows from (29) that also $Z(i\omega)v = 0$. This contradicts observability of M. We conclude from this that $\det(\operatorname{col}(D, F_+)) \neq 0$. Consider now the system $\tilde{w} = W(d/dt)\ell$, with W defined by $W = \operatorname{col}(\gamma D, F_+, Z)$. Since (28) is equivalent with

$(d/dt)\|X(d/dt)\ell\|^2_{-K_-^{-1}} + \|Z(d/dt)\ell\|^2 - \gamma^2\|D(d/dt)\ell\|^2 = \|F_+(d/dt)\ell\|^2 - \|F_-(d/dt)\ell\|^2$ for all $\ell \in C^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$, we have

$$\frac{d}{dt}|X(d/dt)\ell|^2_{-K_-^{-1}} \le \left\| \begin{pmatrix} \gamma D\left(\frac{d}{dt}\right) \\ F_+\left(\frac{d}{dt}\right) \end{pmatrix} \ell \right\|^2 - \left\|Z\left(\frac{d}{dt}\right)\ell\right\|^2$$

for all $\ell \in C^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$. Thus $Q_\Psi(\ell) := \|X(d/dt)\ell\|^2_{-K_-^{-1}}$ is a storage function for the system $\tilde{w} = W(d/dt)\ell$. This storage function is positive semidefinite. It follows from Theorem 8.8 that $\mathrm{col}(\gamma D, F_+)$, so also $\mathrm{col}(D, F_+)$, is Hurwitz.

Finally, we prove that the factor F itself is Hurwitz. As before, let $\tilde{R}_e{}^T$ and $\tilde{M}_e$ be the coefficient matrices of the extensions $R_e^T(-\xi)$ and $M_e(\xi)$, respectively. Let $\mathcal{R}$ and $\mathcal{M}$ be the subspaces defined by (35) and (36). Note that $\mathcal{R} = \mathrm{im}(\tilde{R}_e{}^T)$ and $\mathcal{M} = \mathrm{im}(\tilde{M}_e)$. According to Lemma 12.3, $\mathcal{R} = \ker(\tilde{M}_e{}^T)$ and $\mathcal{M} = \ker(\tilde{R}_e)$. From this, it is easily verified that

$$\mathcal{M} \cap \ker(\tilde{M}_e{}^T \Sigma_e^{-1}) = \Sigma_e(\mathcal{R} \cap \ker(\tilde{R}_e \Sigma_e)). \qquad (39)$$

Note that (29) is equivalent with $M_e^T(\zeta)\Sigma_e^{-1}M_e(\eta) = F^T(\zeta)\Sigma_{\mathtt{l-d,d}}F(\eta)$. In terms of the coefficient matrices this reads $\tilde{M}_e{}^T\Sigma_e^{-1}\tilde{M}_e = \tilde{F}^T\Sigma_{\mathtt{l-d,d}}\tilde{F}$. As before, let H be a Hurwitz factor of $R(\xi)\Sigma_{1/\gamma}R^T(-\xi)$ so that $R_e(-\zeta)\Sigma_e R_e^T(-\eta) = H^T(\zeta)H(\eta)$. This is equivalent with $\tilde{R}_e\Sigma_e\tilde{R}_e{}^T = \tilde{H}^T\tilde{H}$.

We claim that the following inclusion holds:

$$M_e\left(\frac{d}{dt}\right)\ker\left(F\left(\frac{d}{dt}\right)\right) \subset R_e^T\left(-\frac{d}{dt}\right)\ker\left(H\left(\frac{d}{dt}\right)\right). \qquad (40)$$

To prove this, let $F(d/dt)\ell = 0$. Clearly, $\tilde{M}_e{}^T\Sigma_e^{-1} M_e(d/dt)\ell = 0$, so for all t we have $(M_e(d/dt)\ell)(t) \in \mathcal{M} \cap \ker(\tilde{M}_e{}^T\Sigma_e^{-1})$. By (39) this yields $\Sigma_e^{-1}(M_e(d/dt)\ell)(t) \in \mathcal{R}$. Since $\mathcal{R} = \mathrm{im}(\tilde{R}_e{}^T)$, there exists $\ell_1 \in \mathcal{C}^\infty$ such that $\Sigma_e^{-1}M_e(d/dt)\ell = R_e^T(-(d/dt))\ell_1$. Also $\tilde{R}_e\Sigma_e R_e^T(-(d/dt))\ell_1 = \tilde{R}_e\, M_e(d/dt)\ell = 0$, so $H(d/dt)\ell_1 = 0$. This proves the inclusion (40).

The proof that F is Hurwitz is now completed by noting that $F(d/dt)\ell = 0$ implies that there exists $\ell_1 \in \ker(H(d/dt))$ such that $M_e(d/dt)\ell = R_e^T(-(d/dt))\ell_1$. Since H is Hurwitz, ℓ_1 is a linear combination of products t^k with stable exponentials $e^{\lambda t}$. The same thus holds for $M_e(d/dt)\ell$ so for $M(d/dt)\ell$. Since M is observable, there exists a polynomial matrix L such that $LM = I$. Thus, $\ell = L(d/dt)M(d/dt)\ell$ is a linear combination of products t^k with stable exponentials $e^{\lambda t}$. This completes the proof. □

Proof of Lemma 10.1: The proof is based on the following matrix theoretical result, which is a special case of Lemma 12.2.

Lemma 12.4: Let $Q \in \mathbb{C}^{\mathtt{r}\times\mathtt{r}}$ be a nonsingular Hermitian matrix. Let $\mathcal{L} \subset \mathbb{C}^{\mathtt{r}}$ be a linear subspace. Let L_1 and L_2 be full column rank matrices such that $\mathcal{L} = \mathrm{im}(L_1)$ and $\mathtt{L}^\perp = \mathrm{im}(L_2)$. Then $L_1^*QL_1$ is nonsingular iff $L_2^*Q^{-1}L_2$ is nonsingular. In that case $\mathrm{sign}(Q) = \mathrm{sign}(L_1^*QL_1) + \mathrm{sign}(L_2^*Q^{-1}L_2)$.

Recall from Proposition 3.2 that $R(d/dt)w = 0$ is a kernel representation of $\mathcal{B}$ iff $RM = 0$ and $\mathrm{rank}(R(\lambda)) = \mathtt{z} + \mathtt{d} - \mathtt{l}$ for all $\lambda \in \mathbb{C}$. Choose such R with $z + \mathtt{d} - \mathtt{l}$ rows. Then $w' = R^T(-(d/dt))\ell'$ is an observable image representation of $\mathcal{B}^\perp$. The idea of the proof is to apply Lemma 12.4 with $Q = \Sigma_\gamma + \epsilon^2 I$, $L_1 = M(i\omega)$, and $L_2 = R^T(-i\omega)$. For each $\omega \in \mathbb{R}$ define $\mathcal{L}(i\omega) \subset \mathbb{C}^{\mathtt{z+d}}$ by $\mathcal{L}(i\omega) := \mathrm{im}(M(i\omega))$. Since $\dim(\mathrm{im}(R^T(-i\omega))) = \mathtt{z} + \mathtt{d} - \mathtt{l} = \mathtt{z} + \mathtt{d}\ \dim(\mathrm{im}(M(i\omega)))$, we have $\mathcal{L}(i\omega)^\perp = \mathrm{im}(R^T(-i\omega))$. Both $M(i\omega)$ and $R^T(-i\omega)$ have full column rank for all $\omega \in \mathbb{R}$. By applying the previous lemma, we find that if either $M^T(-i\omega)(\Sigma_\gamma + \epsilon^2 I)M(i\omega)$ or $R(i\omega)(\Sigma_\gamma + \epsilon^2 I)^{-1}R^T(-i\omega)$ is nonsingular for all ω, then for all $\omega \in \mathbb{R}$ we have $\mathrm{sign}(\Sigma_\gamma + \epsilon^2 I) = \mathrm{sign}(M^T(-i\omega)(\Sigma_\gamma + \epsilon^2 I)M(i\omega)) + \mathrm{sign}(R(i\omega)(\Sigma_\gamma + \epsilon^2 I)^{-1}R^T(-i\omega))$. Note that, for $\epsilon \ge 0$ sufficiently small, $\mathrm{sign}(\Sigma_\gamma + \epsilon^2 I) = (\mathtt{d}, 0, \mathtt{z})$. Hence, for all $\omega \in \mathbb{R}$ we have $\mathrm{sign}(M^T(-i\omega)(\Sigma_\gamma + \epsilon^2 I)M(i\omega)) = (\mathtt{d}, 0, \mathtt{l} - \mathtt{d})$ iff for all $\omega \in \mathbb{R}$ we have $\mathrm{sign}(R(i\omega)(\Sigma_\gamma + \epsilon^2 I)^{-1}R^T(-i\omega)) = (0, 0, \mathtt{z} - \mathtt{l} + \mathtt{d})$ or, equivalently,

$$R(i\omega)(\Sigma_\gamma + \epsilon^2 I)^{-1}R^T(-i\omega) > 0. \qquad (41)$$

Finally, there exists $\epsilon > 0$ such that (41) holds iff there exists $\epsilon > 0$ such that $R(i\omega)(\Sigma_{1/\gamma} - \epsilon^2 I)R^T(-i\omega) > 0$ Indeed,

$$(\Sigma_\gamma + \epsilon^2 I)^{-1} = \begin{pmatrix} \frac{1}{1+\epsilon^2} I_{\mathtt{z}} & 0 \\ 0 & \frac{1}{-\gamma^2 + \epsilon^2} I_{\mathtt{d}} \end{pmatrix}$$

so (41) is equivalent to $R(i\omega)\Sigma_{1/\gamma(\epsilon)}R^T(-i\omega) > 0$, with $\gamma(\epsilon) := (\gamma^2 - \epsilon^2/1 + \epsilon^2)^{1/2}$. By defining $\epsilon_1 := (\epsilon/\gamma)$, this is equivalent to $R(i\omega)(\Sigma_{1/\gamma} - \epsilon_1^2 I)R^T(-i\omega) > 0$. The proof is completed by noting that such $\epsilon_1 > 0$ exists iff $\mathcal{B}^\perp$ is strictly dissipative with respect to $(w')^T\Sigma_{1/\gamma}w'$.

Proof of Lemma 10.4: As in the proof of Lemma 10.1, let R be such that $w' = R^T(-(d/dt))\ell'$ is an observable image representation of $\mathcal{B}^\perp$. Define $\Gamma_\gamma(\zeta, \eta) := R(-\zeta)\Sigma_{1/\gamma}R^T(-\eta)$. If $\mathcal{B}^\perp$ is strictly dissipative, then $\partial\Gamma_\gamma$ has no roots on the imaginary axis. Let T_{Γ_γ} be the Pick matrix associated with Γ_γ. It was shown in [23, Th 9.1] that the smallest storage function of $\mathcal{B}^\perp$ is negative definite iff $T_{\Gamma_\gamma} < 0$. Now, it can be shown that, in fact, the Pick matrices associated with Φ_γ and Γ_γ are the same. The proof is omitted due to space limitations.

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