shown in the top figure (the disturbance vector is actually a randomly generated vector multiplied by the displayed variable). Note that the system states are affected by the disturbances (bottom), but the state vector remains inside the sliding surface (center). The sliding mode controller performs optimal $\mathcal{H}_2$-guaranteed cost attenuation of the nonmatching disturbances.

V. CONCLUSION

The design of sliding mode controllers for nominal systems may lead to an unpredictable behavior of the closed loop in the case of nonmatching disturbances. In order to take such disturbances into account, an $\mathcal{H}_2$ guaranteed cost design of sliding surfaces has been developed in this note for convex-bounded model uncertainties of polytope type. The quadratic stability of the closed loop is guaranteed by the method, and a performance level (defined in terms of an $\mathcal{H}_2$ norm) is assured. The design is performed via a convex optimization method, using highly efficient algorithms, with assured convergence to the global optimum.

REFERENCES

[1] R. A. De Carlo, S. H. Zak, and G. P. Matthews, "Variable structure control of nonlinear multivariable systems: a tutorial," *Proc. IEEE,* vol. 76, pp. 212–232, Mar. 1988.
[2] J. Y. Hung, W. Gao, and J. C. Hung, "Variable structure control: A survey," *IEEE Trans. Ind. Electron.,* vol. 40, no. 1, pp. 2–22, 1993.
[3] V. I. Utkin, *Sliding Modes and Their Application in Variable Structure Systems.*   Moscow, Russia: Mir, 1978.
[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.*   Philadelphia, PA: SIAM, 1994.
[5] J. C. Geromel, P. L. D. Peres, and S. R. Souza, "$\mathcal{H}_2$ guaranteed cost control for uncertain continuous-time linear systems," *Syst. Contr. Lett.,* vol. 19, pp. 23–27, 1992.
[6] V. I. Utkin and K. D. Yang, "Methods for constructing discontinuity planes in multidimensional variable structure systems," *Autom. Remote Contr.,* vol. 39, pp. 1466–1470, 1978.
[7] C. M. Dorling and A. S. I. Zinober, "Two approaches to hyperplane design in multivariable variable structure control systems," *Int. J. Contr.,* vol. 44, no. 1, pp. 65–82, 1986.
[8] B. Drazenovic, "The invariance conditions in variable structure systems," *Automatica,* vol. 5, pp. 287–295, 1969.
[9] S. K. Spurgeon and R. Davies, "A nonlinear control strategy for robust sliding mode performance in the presence of unmatched uncertainty," *Int. J. Contr.,* vol. 57, no. 5, pp. 1107–1123, 1993.
[10] C. M. Kwan, "Sliding mode control of linear systems with mismatched uncertainties," *Automatica,* vol. 31, no. 2, pp. 303–307, 1995.
[11] R. H. C. Takahashi and P. L. D. Peres, "$\mathcal{H}_\infty$ design of switching surfaces for sliding mode control with nonmatching disturbances," *Proc. Inst. Elect. Eng.,* vol. 145, no. 4, pp. 435–441, 1998.
[12] ———, "Sliding modes solution for the $\mathcal{H}_2$ singular problem," in *Proc. 35th IEEE Conf. Decision Contr.,* Kobe, Japan, 1996, vol. 1, pp. 243–248.

The Kharitonov Theorem with Degree Drop

Jan C. Willems and Roberto Tempo

Abstract— **The purpose of this note is to present a proof of the Kharitonov theorem based on Bezoutians. An interesting consequence of this proof is that it shows the validity of Kharitonov's result in the presence of a degree drop.**

Index Terms—**Bezoutians, Kharitonov theorem.**

I. INTRODUCTION

In the early proofs of the Kharitonov theorem [1]–[4], [11], it is generally assumed that the degree of the interval polynomial family is constant. The question thus arises if the Kharitonov result remains valid with degree drop. In order to put the problem clearly in perspective, it is convenient to estabish first the notation.

Let $I \subset \mathbb{R}[\xi]$ denote the interval family of polynomials defined by the coefficient intervals $[\underline{a}_k, \overline{a}_k]$, $k = 0, 1, \cdots, n$, i.e.,

$$I = \{ p \in \mathbb{R}[\xi]: \quad p(\xi) = p_0 + p_1\xi + p_2\xi^2 + \cdots + p_n\xi^n,$$
$$\text{with } \underline{a}_k \leq p_k \leq \overline{a}_k \}.$$

The problem is to find conditions so that all of the elements of I are Hurwitz (a polynomial is said to be *Hurwitz* if it has all of its roots in the open left half of the complex plane). Define

$$E_1(\xi) = \underline{a}_0 + \overline{a}_2\xi^2 + \underline{a}_4\xi^4 + \overline{a}_6\xi^6 + \underline{a}_8\xi^8 + \overline{a}_{10}\xi^{10} + \cdots$$
$$E_2(\xi) = \overline{a}_0 + \underline{a}_2\xi^2 + \overline{a}_4\xi^4 + \underline{a}_6\xi^6 + \overline{a}_8\xi^8 + \underline{a}_{10}\xi^{10} + \cdots$$
$$O_1(\xi) = \underline{a}_1\xi + \overline{a}_3\xi^3 + \underline{a}_5\xi^5 + \overline{a}_7\xi^7 + \underline{a}_9\xi^9 + \overline{a}_{11}\xi^{11} + \cdots$$
$$O_2(\xi) = \overline{a}_1\xi + \underline{a}_3\xi^3 + \overline{a}_5\xi^5 + \underline{a}_7\xi^7 + \overline{a}_9\xi^9 + \underline{a}_{11}\xi^{11} + \cdots.$$

Define the polynomials

$$k_1(\xi) = \underline{a}_0 + \underline{a}_1\xi + \overline{a}_2\xi^2 + \overline{a}_3\xi^3 + \underline{a}_4\xi^4 + \underline{a}_5\xi^5$$
$$+ \overline{a}_6\xi^6 + \overline{a}_7\xi^7 + \cdots$$
$$k_2(\xi) = \underline{a}_0 + \overline{a}_1\xi + \overline{a}_2\xi^2 + \underline{a}_3\xi^3 + \underline{a}_4\xi^4 + \overline{a}_5\xi^5$$
$$+ \overline{a}_6\xi^6 + \underline{a}_7\xi^7 + \cdots$$
$$k_3(\xi) = \overline{a}_0 + \underline{a}_1\xi + \underline{a}_2\xi^2 + \overline{a}_3\xi^3 + \overline{a}_4\xi^4 + \underline{a}_5\xi^5$$
$$+ \underline{a}_6\xi^6 + \overline{a}_7\xi^7 + \cdots$$
$$k_4(\xi) = \overline{a}_0 + \overline{a}_1\xi + \underline{a}_2\xi^2 + \underline{a}_3\xi^3 + \overline{a}_4\xi^4 + \overline{a}_5\xi^5$$
$$+ \underline{a}_6\xi^6 + \underline{a}_7\xi^7 + \cdots.$$

The polynomials k_1, k_2, k_3, k_4 are called the *Kharitonov polynomials* associated with I. In the classic paper [1], Kharitonov proved the remarkable result that, under the assumption that the highest degree coefficient of the interval family does not vanish (from which either $\underline{a}_n > 0$ or $\overline{a}_n < 0$), then all elements of I are Hurwitz if and only if the four Kharitonov polynomials k_1, k_2, k_3, k_4 associated with I are Hurwitz.

In this paper, we study the following question: What is the situation if the highest degree coefficient is allowed to vanish (from which, if

either $\underline{a}_n = 0$ or $\overline{a}_n = 0$)? In order to examine this, observe that it is easy to see that a real polynomial p with degree$(p) = n - 1$ and positive coefficients is Hurwitz *only if* $\epsilon\xi^n + p(\xi)$ is Hurwitz for $\epsilon > 0$ sufficiently small. The *if* version of this statement (which, if correct, would prove the Kharitonov result with a vanishing leading term) is, unfortunately, not correct. Indeed, as observed before in [5], $\epsilon\xi^n + p(\xi)$ may be Hurwitz for $\epsilon > 0$, but its roots may hit the imaginary axis for $\epsilon = 0$. The following polynomial is an example of such a situation: $p(\xi) = 0.5\xi^5 + 0.5\xi^4 + 5\xi^3 + 3\xi^2 + 8\xi + 4$. It has roots on the imaginary axis, but $\epsilon\xi^6 + 0.5\xi^5 + 0.5\xi^4 + 5\xi^3 + 3\xi^2 + 8\xi + 4$ is Hurwitz for $\epsilon > 0$ sufficiently small. To verify this, apply the Routh test. (In [5], it is also shown that $n = 5$ provides the lowest order such example.)

This is relevant for the generalization of Kharitonov's result in the case of a vanishing leading term. Assume that $[0, \overline{a}_n]$ is the coefficient range of the highest coefficient of the interval poynomial, and that the four usual Kharitonov polynomials k_1, k_2, k_3, k_4 are Hurwitz. Does it follow that a polynomial p in the interval family with vanishing leading coefficient is Hurwitz? Of course, it follows from the usual Kharitonov theorem that $\epsilon\xi^n + p(\xi)$ will be Hurwitz for $\epsilon > 0$ sufficiently small, but, as we have just seen, this simply does not imply that p itself is Hurwitz. One way around this is to consider, in addition to the four usual Kharitonov polynomials k_1, k_2, k_3, k_4, the polynomials

$$
\begin{aligned}
k_5(\xi) = {} & \underline{a}_{n-1}\xi^{n-1} + \underline{a}_{n-2}\xi^{n-2} + \overline{a}_{n-3}\xi^{n-3} + \overline{a}_{n-4}\xi^{n-4} \\
& + \underline{a}_{n-5}\xi^{n-5} + \underline{a}_{n-6}\xi^{n-6} + \overline{a}_{n-7}\xi^{n-7} \\
& + \overline{a}_{n-8}\xi^{n-8} + \cdots \\
k_6(\xi) = {} & \overline{a}_{n-1}\xi^{n-1} + \underline{a}_{n-2}\xi^{n-2} + \underline{a}_{n-3}\xi^{n-3} + \overline{a}_{n-4}\xi^{n-4} \\
& + \overline{a}_{n-5}\xi^{n-5} + \underline{a}_{n-6}\xi^{n-6} + \underline{a}_{n-7}\xi^{n-7} \\
& + \overline{a}_{n-8}\xi^{n-8} + \cdots.
\end{aligned}
$$

Notice that k_5 and k_6 are not Kharitonov polynomials. It now follows directly from the classical Kharitonov result (using, in the case where the leading coefficient is zero, k_1, k_3, k_5, k_6 for n odd and k_3, k_4, k_5, k_6 for n even) that the interval family is Hurwitz if and only if the six extreme polynomials k_1, k_2, k_3, k_4, k_5, k_6 are Hurwitz. This is indeed the result obtained in [5].

However, surprisingly, it turns out that k_1, k_2, k_3, k_4 Hurwitz implies that k_5, k_6 are also Hurwitz, and so there is no need to modify Kharitonov's theorem in any way in the case where the leading term vanishes! This is what we will prove in this note. This result is not new. In fact, it was demonstrated in [6], using Nyquist-like complex function analysis. There are also other, shorter proofs [7] of this result. Our method of proof, however, is new. It provides a short proof that encompasses the vanishing leading coefficient case without having to pay special attention to it. Moreover, this proof is based on a Lyapunov function argument, an aspect that by itself has some merit in its own right.

The result that we want to prove is the following.

Theorem 1: All elements of I are Hurwitz if and only if the four Kharitonov polynomials k_1, k_2, k_3, k_4 associated with I are Hurwitz.

Assume, without loss of generality, that $\underline{a}_n \geq 0$. Since, obviously, k_1, k_2, k_3, $k_4 \subset I$, it suffices to prove the "if" part. The "if" part is the classical Kharitonov result [1]–[4] when $\underline{a}_n > 0$. In Section III, we prove that this result also holds for the case $\underline{a}_n = 0$ and $\overline{a}_n > 0$. We remark that a degree drop of more than one immediately implies that one of the Kharitonov polynomials is not Hurwitz, and therefore $\overline{a}_{n-1} > 0$.

II. THE BEZOUTIAN

Let $p \in \mathbb{R}[\xi]$. Define $p^* \in \mathbb{R}[\xi]$ by $p^*(\xi) = p(-\xi)$. The *Bezoutian* (see, e.g., [8]) associated with p is a two-variable polynomial $B_p \in \mathbb{R}[\zeta, \eta]$ defined by

$$
B_p(\zeta, \eta) = \frac{p(\zeta)p(\eta) - p^*(\zeta)p^*(\eta)}{\zeta + \eta}.
$$

It is easily verified that, since $p(\xi)p(-\xi) - p^*(\xi)p^*(-\xi) = 0$, $p(\zeta)p(\eta) - p^*(\zeta)p^*(\eta)$ has a factor $\zeta + \eta$, and so B_p is indeed polynomial. In terms of the even and odd parts of p, $E_p = (p + p^*)/2$ and $O_p = (p - p^*)/2$, B_p is equal to

$$
B_p(\zeta, \eta) = 2\,\frac{E_p(\zeta)O_p(\eta) + O_p(\zeta)E_p(\eta)}{\zeta + \eta}.
$$

Now, B_p is of the form

$$
B_p(\zeta, \eta) = \sum_{k, \ell = 0}^{\mathrm{degree}(p) - 1} A_{k\ell}\zeta^k\eta^\ell
$$

with $A_{k\ell} = A_{\ell k}$. Denote the real symmetric matrix $[A_{k\ell}]_{k, \ell = 0}^{\mathrm{degree}(p) - 1}$ by $\tilde{B}_p$. Define the rank of B_p to be equal to that of $\tilde{B}_p$, and define B_p to be positive definite if $\tilde{B}_p$ is.

The following classical result relates the stability of $p(d/dt)w = 0$ with the Bezoutian B_p.

Proposition 2 [8]: The polynomial $p \in \mathbb{R}[\xi]$ is Hurwitz if and only if B_p is positive definite.

In order to make this note self contained, a Lyapunov proof of this proposition has been included in the Appendix.

An interesting immediate consequence of Proposition 2 is the following result on the stability of linear systems whose defining polynomial is a convex combination of even and odd polynomials. Let $E_1, E_2, \cdots, E_{N'} \in \mathbb{R}[\xi]$ be even polynomials ($E_k(\xi) = E_k(-\xi)$; hence, only even powers appear). Let $O_1, O_2, \cdots, O_{N''} \in \mathbb{R}[\xi]$ be odd polynomials ($O_k(\xi) = -O_k(-\xi)$; hence, only odd powers appear).

Proposition 3 [9]: Assume that the polynomials $p_{k\ell} = E_k + O_\ell$, $k = 1, 2, \cdots, N'; \ell = 1, 2, \cdots, N''$ are Hurwitz. Assume that $\alpha_k > 0$, $k = 1, 2, \cdots, N'$ and $\beta_\ell > 0$, $\ell = 1, 2, \cdots, N''$. Then the polynomial

$$
p = \sum_{k=1}^{N'}\alpha_k E_k + \sum_{\ell=1}^{N''}\beta_\ell O_\ell
$$

is also Hurwitz.

Proof: Note that

$$
B_p = \sum_{k=1}^{N'}\sum_{\ell=1}^{N''}\alpha_k\beta_\ell B_{p_{k\ell}}.
$$

Hence, B_p is nonnegative definite, and its rank is equal to the maximum of the degrees of the polynomials $p_{k\ell}$. Since this maximum is, in fact, also the degree of p, it follows from Proposition 2 that p is Hurwitz.

III. PROOF OF THEOREM 1

Assume that k_1, k_2, k_3, k_4 are Hurwitz and, without loss of generality, assume that $\overline{a}_n > 0$. Note that this implies that $a_k > 0$ for $k = 0, 1, \cdots, n - 1$. Hence, $p \in I$ implies that all of its coefficients are positive.

1) First, we prove that every convex combination $k = \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4$, with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ and $\sum_{\ell=1}^4 \alpha_i = 1$, is also Hurwitz. In order to see this, write k as

$$
k = (\alpha_1 + \alpha_2)E_1 + (\alpha_3 + \alpha_4)E_2 + (\alpha_1 + \alpha_3)O_1 + (\alpha_2 + \alpha_4)O_2
$$

and use Proposition 3. Thus, all polynomials in the convex hull of $\{k_1, k_2, k_3, k_4\}$ are Hurwitz.

2) Next, observe (see [10] and [11]) that any $p \in I$ satisfies the following relations for $\omega \geq 0$:

$$\mathrm{Re}(k_1(i\omega)) = \mathrm{Re}(k_2(i\omega)) = E_1(i\omega)$$
$$E_2(i\omega) = \mathrm{Re}(k_3(i\omega)) = \mathrm{Re}(k_4(i\omega))$$
$$\mathrm{Im}(k_1(i\omega)) = \mathrm{Im}(k_3(i\omega)) = -iO_1(i\omega)$$
$$-iO_2(i\omega) = \mathrm{Im}(k_2(i\omega)) = \mathrm{Im}(k_4(i\omega))$$

and

$$E_1(i\omega) \leq \mathrm{Re}(p(i\omega)) \leq E_2(i\omega)$$
$$-iO_1(i\omega) \leq \mathrm{Im}(p(i\omega)) \leq -iO_2(i\omega).$$

From these relations, it follows that $p(i\omega)$ belongs to the convex hull in $\mathbb{C}$ of

$$\{k_1(i\omega), k_2(i\omega), k_3(i\omega), k_4(i\omega)\}.$$

3) By 1), no convex combination k of k_1, k_2, k_3, k_4 can have a root on the imaginary axis. By 2), this implies that, also, no element of I can have a root on the imaginary axis.

4) Note that since k_1, k_2, k_3, k_4 are Hurwitz and $\overline{a}_n > 0$, all of the coefficients of k_1, k_2, k_3, k_4 are positive. Hence, two situations can occur: either $\underline{a}_n > 0$, in which case all elements of I have degree n, or $\underline{a}_n = 0$, in which case two elements of $\{k_1, k_2, k_3, k_4\}$ have degree n and two elements have degree $n - 1$ and, moreover, all elements of I have degree n or $n - 1$.

5) Let $p \in I$. Let $k \in \{k_1, k_2, k_3, k_4\}$ be such that $\mathrm{degree}(k) = \mathrm{degree}(p)$. Now, consider the convex combination $p_\alpha = \alpha p + (1 - \alpha)k$, $0 \leq \alpha \leq 1$, of p and k. Obviously, $p_\alpha \in I$. Hence, by 3), p_α has no root on the imaginary axis for $0 \leq \alpha \leq 1$. Furthermore, $p_0 = k$ is Hurwitz and $\mathrm{degree}(p_\alpha) = \mathrm{degree}(k)$ for $0 \leq \alpha \leq 1$. Hence, $p_1 = p$ is also Hurwitz.

This ends the proof of Theorem 1. Note that we proved the theorem both for $\underline{a}_n > 0$ and for $\underline{a}_n = 0$.

In closing, we pose as an open problem the question to provide a direct, matrix proof of the implication that positive definiteness of the four matrices $\tilde{B}_{k_1}, \tilde{B}_{k_2}, \tilde{B}_{k_3}, \tilde{B}_{k_4}$ implies positive definiteness of $\tilde{B}_p$.

APPENDIX
PROOF OF PROPOSTION 2

1) (*if*): Let $\mathrm{degree}(p) = n$. Consider the (Lyapunov) function V induced by B_p:

$$V\left(w, \frac{dw}{dt}, \cdots, \frac{d^{n-1}w}{dt^{n-1}}\right) = \sum_{k,\ell=0}^{n-1} A_{k\ell} \frac{d^k w}{dt^k} \frac{d^\ell w}{dt^\ell}.$$

Using the definition of the Bezoutian, we obtain

$$\frac{d}{dt}V\left(w, \frac{dw}{dt}, \cdots, \frac{d^{n-1}w}{dt^{n-1}}\right)$$
$$= \left|p\left(\frac{d}{dt}\right)w\right|^2 - \left|p^*\left(\frac{d}{dt}\right)w\right|^2.$$

Hence, along solutions of $p(d/dt)w = 0$, this derivative is equal to $-|p^*(d/dt)w|^2$. Since $\tilde{B}_p$ is symmetric and positive-definite, V is a positive-definite Lyapunov function for $p(d/dt)w = 0$ with a nonpositive-definite derivative. Hence, by the invariance principle, all solutions of $p(d/dt)w = 0$ approach, for $t \rightarrow \infty$, the set where $p^*(d/dt)w = 0$

or, in other words, the set where both $p(d/dt)w = 0$ and $p^*(d/dt)w = 0$. We claim that, since $\mathrm{rank}(B_p) = \mathrm{rank}(\tilde{B}_p) = \mathrm{degree}(p)$, p and p^* are coprime. If $p = fq$ and $p^* = fq^*$ with $f = f^*$, then $B_p(\zeta, \eta) = f(\zeta)B_q(\zeta, \eta)f(\eta)$, which is easily seen to imply that $\mathrm{rank}(B_p) \leq \mathrm{rank}(B_q) \leq \mathrm{degree}(q)$. This allows us to conclude that all solutions of $p(d/dt)w = 0$ converge to zero as $t \rightarrow \infty$.

2) (*only if*): Assume that p is Hurwitz. From the definition of the Bezoutian, it follows that, for any solution w of $p(d/dt)w = 0$, there holds

$$\sum_{k,\ell=0}^{n-1} A_{k\ell} \frac{d^k w}{dt^k}(0) \frac{d^\ell w}{dt^\ell}(0) = \int_0^\infty \left|p^*\left(\frac{d}{dt}\right)w\right|^2 dt.$$

This equation shows that B_p is nonnegative definite. Furthermore, since p is Hurwitz, p and p^* are coprime, from which the right-hand side cannot be zero unless $w = 0$, which shows that the rank of B_p is indeed equal to $\mathrm{degree}(p)$.

ACKNOWLEDGMENT

REFERENCES

[1] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential'nye Uraveniya,* vol. 14, pp. 1483–1485, 1978.

[2] B. R. Barmish, *New Tools for Robustness of Linear Systems.* New York: Macmillan, 1994.

[3] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[4] J. Kogan, *Robust Stability and Convexity (Lecture Notes in Contr. Inform. Sci.).* New York: Springer-Verlag, 1995.

[5] T. Mori and H. Kokame, "Stability of interval polynomials with vanishing extreme coefficients," in *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing I.* Tokyo, Japan: Mita, 1992, pp. 409–414.

[6] R. Hernández and S. Dormido, "Kharitonov's theorem estension to interval polynomials which can drop in degree: A Nyquist approach," *IEEE Trans. Automat. Contr.,* vol. 41, pp. 1009–1012, 1996.

[7] A. L. Tits, private communication.

[8] P. Lancaster and M. Tismenetsky, *The Theory of Matrices.* New York: Academic, 1985.

[9] S. Bialas and J. Garloff, "Convex combinations of stable polynomials," *J. Franklin Inst.,* vol. 319, pp. 373–377, 1985.

[10] S. Dasgupta, "Kharitonov's theorem revisited," *Syst. Contr. Lett.,* vol. 11, pp. 381–384, 1988.

[11] R. J. Minnichelli, J. J. Anagnost, and C. A. Desoer, "An elementary proof of Kharitonov's theorem," *IEEE Trans. Automat. Contr.,* vol. 34, pp. 995–998, 1989.