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Dead beat observer synthesis

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Abstract

In the paper the observer design problem is investigated in the context of linear left shift invariant discrete behaviors, whose trajectories have support on the positive axis. Observability and reconstructibility properties of certain manifest variables from certain others, in the presence of latent variables, are defined and fully characterized. Necessary and sufficient conditions for the existence of either a dead-beat or an exact observer are introduced, and a complete parametrization of all dead-beat observers is given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The last decade has witnessed an increasingly broader interest in the behavioral approach to dynamics systems modeling [3,6,7], which is now generally recognized as a natural setting for describing and analyzing the trajectories that a system produces. Indeed, modeling from first principles generally leads to a set of differential (or difference) equations, relating the time-evolution of the various variables involved in the plant description. The collection of all time trajectories satisfying these equations is called the *behavior*.

Typically, the variables appearing in the mathematical model of the plant can be conceptually partitioned into two groups: *manifest variables* and *latent variables*. While the former are fundamental to the system description, and endowed with a physical interpretation, the latter play a somehow "auxiliary role", and often, have only a mathematical meaning [3,6]. So, it is natural to assume that latent variables are not measurable.

In the last few years, important aspects of classical system theory have been translated and solved in the behavioral setting, thus leading to powerful generalizations of well-known results obtained within the input/output or state-space contexts. In particular, the relevant issue of estimating some system variables, not available for measurements, from others which are measured, has been recently investigated for linear time invariant (continuous time) systems [5].

The aim of this paper is that of addressing and solving the observer design problem for linear left shift invariant systems whose trajectories are defined on $\mathbb{Z}_+$. We first provide necessary and sufficient conditions for the existence of dead-beat (or exact) observers, and then derive a complete parametrization of all possible dead-beat observers, based on the polynomial matrices involved in the system description.

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In this paper, all trajectories will be assumed defined on the set $\mathbb{Z}_+$ of nonnegative integers. The right (forward) and the left (backward) shift operators on $(\mathbb{F}^q)^{\mathbb{Z}^+}$, the set of trajectories defined on $\mathbb{Z}_+$ and taking values in $\mathbb{F}^q$, are defined as

$$\tau : (\mathbb{F}^q)^{\mathbb{Z}_+} \to (\mathbb{F}^q)^{\mathbb{Z}_+} : (\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots) \mapsto (0, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots),$$
$$\sigma : (\mathbb{F}^q)^{\mathbb{Z}_+} \to (\mathbb{F}^q)^{\mathbb{Z}_+} : (\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots) \mapsto (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots).$$

As we will deal with left shift invariant behaviors, we will restrict our attention to the left shift operator σ, and correspondingly define certain matrix shift operators. In fact, if $P(\xi) = \sum_{i=0}^{L} P_i \xi^i \in \mathbb{F}[\xi]^{q\times m}$ is a polynomial matrix, we associate with it the polynomial matrix operator $P(\sigma) = \sum_{i=0}^{L} P_i \sigma^i$. It can be proved that $P(\sigma)$ describes an injective map from $(\mathbb{F}^m)^{\mathbb{Z}_+}$ to $(\mathbb{F}^q)^{\mathbb{Z}_+}$ if and only if P is a right prime matrix, and a surjective map if and only if P is of full row rank.

2. Basic results about infinite support behaviors in $(\mathbb{F}^q)^{\mathbb{Z}_+}$

Before proceeding, it is convenient to briefly summarize some basic definitions and results about linear left shift invariant behaviors, whose trajectories have support in $\mathbb{Z}_+$. Further details on the subject can be found in [4,8].

A behavior $\mathfrak{B} \subseteq (\mathbb{F}^q)^{\mathbb{Z}_+}$ is said to be *linear* if it is a vector subspace (over $\mathbb{F}$) of $(\mathbb{F}^q)^{\mathbb{Z}_+}$, and *left shift invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$. A linear left shift invariant behavior $\mathfrak{B} \subseteq (\mathbb{F}^q)^{\mathbb{Z}_+}$ is *complete* if for every sequence $\tilde{\boldsymbol{w}} \in (\mathbb{F}^q)^{\mathbb{Z}_+}$, the condition $\tilde{\boldsymbol{w}}|_{\mathscr{S}} \in \mathfrak{B}|_{\mathscr{S}}$ for every finite set $\mathscr{S} \subset \mathbb{Z}_+$ implies $\tilde{\boldsymbol{w}} \in \mathfrak{B}$, where $\tilde{\boldsymbol{w}}|_{\mathscr{S}}$ denotes the restriction to $\mathscr{S}$ of the trajectory $\tilde{\boldsymbol{w}}$ and $\mathfrak{B}|_{\mathscr{S}}$ the set of all restrictions to $\mathscr{S}$ of behavior trajectories.

Linear left shift invariant complete behaviors are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\boldsymbol{w} = \{\boldsymbol{w}(t)\}_{t\in\mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{F}^q)^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\boldsymbol{w}(t) + R_1\boldsymbol{w}(t+1) + \cdots + R_L\boldsymbol{w}(t+L) = 0, \quad t \in \mathbb{Z}_+, \tag{2.1}$$

with $R_i \in \mathbb{F}^{p\times q}$, and hence described by the equation

$$R(\sigma)\boldsymbol{w} = 0, \tag{2.2}$$

where $R(\xi) := \sum_{i=0}^{L} R_i \xi^i$ belongs to $\mathbb{F}[\xi]^{p\times q}$. In the sequel, a behavior $\mathfrak{B}$ described as in Eq. (2.2) will be denoted, for short, as $\mathfrak{B} = \ker(R(\sigma))$.

It can be shown that $\ker(R_1(\sigma)) \subseteq \ker(R_2(\sigma))$ if and only if $R_2 = PR_1$ for some polynomial matrix P.

A complete behavior $\mathfrak{B} = \ker(R(\sigma)) \subseteq (\mathbb{F}^q)^{\mathbb{Z}_+}$ is said to be *autonomous* if there exists $m \in \mathbb{Z}_+$ such that if $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathfrak{B}$ and $\boldsymbol{w}_1(t) = \boldsymbol{w}_2(t)$ for $t \in [0, m]$, then $\boldsymbol{w}_1 = \boldsymbol{w}_2$. Of course, this amounts to saying that $\mathfrak{B}$ is a finite dimensional vector subspace of $(\mathbb{F}^q)^{\mathbb{Z}_+}$, whose dimension is not greater than $q(m+1)$.

Proposition 2.1. *A complete behavior $\mathfrak{B} = \ker(R(\sigma))$, with $R \in \mathbb{F}[\xi]^{p\times q}$, is autonomous if and only if R has rank q.*

Proof. Consider first the case $p = q = 1$. Given any nonzero polynomial $r \in \mathbb{F}[\xi]$, $\ker(r(\sigma))$ is an autonomous behavior. Indeed, if $r \in \mathbb{F}\setminus\{0\}$ then it is the zero behavior. If $r = \sum_{i=0}^{L} r_i \xi^i$, with $r_L \neq 0$, $L > 0$, then for every choice of the first L samples, i.e. $\boldsymbol{w}(0), \boldsymbol{w}(1), \ldots, \boldsymbol{w}(L-1)$, the remaining samples of $\boldsymbol{w}$ are uniquely determined by means of the recursive scheme $\boldsymbol{w}(t+L) = -(1/r_L)\sum_{i=0}^{L-1} r_i \boldsymbol{w}(t+i)$, $t \geqslant 0$. This implies that $\mathfrak{B}$ is autonomous.

Consider, next, the matrix case. Let r be the rank of R and let U and V be unimodular matrices, of suitable sizes, such that

$$URV = \Gamma = \left[\begin{array}{cccc|c} \gamma_1 & & & & \\ & \gamma_2 & & & 0 \\ & & \ddots & & \\ & & & \gamma_r & \\ \hline & & 0 & & 0 \end{array}\right],$$

$\gamma_i \in \mathbb{F}[\xi]\setminus\{0\}$ and $\gamma_i | \gamma_{i+1}$, is the Smith form of R over $\mathbb{F}[\xi]$. Of course, as $U(\sigma)$ defines an injective map, $\ker(R(\sigma)) = \ker(\Gamma(\sigma)V^{-1}(\sigma))$. On the other hand, due to the unimodularity of V, $\ker(\Gamma(\sigma)V^{-1}(\sigma))$ and $\ker(\Gamma(\sigma))$ are isomorphic. By the analysis carried on in the scalar case, $\ker(\Gamma(\sigma))$ (and hence $\mathfrak{B}$) is finite-dimensional if and only if $r = q$. □

Every autonomous behavior in $(\mathbb{F}^q)^{\mathbb{Z}_+}$ can be expressed as $\ker(R(\sigma))$ for some nonsingular square polynomial matrix R. Among all possible autonomous behaviors, special attention deserve the *nilpotent autonomous* behaviors, namely autonomous behaviors for which there exists some $\delta \in \mathbb{N}$ such that all their trajectories have (compact) supports included in $[0, \delta-1]$. Nilpotent autonomous behaviors are kernels of polynomial matrix operators $R(\sigma)$ corresponding to full column rank matrices, with a nonzero monomial as g.c.d. of their maximal (i.e., qth) order minors.

Proposition 2.2. *Let* $\mathfrak{B} = \ker(R(\sigma))$, *with* $R \in \mathbb{F}[\xi]^{p\times q}$ *of rank* q *, be an autonomous behavior.* $\mathfrak{B}$ *is nilpotent if and only if the g.c.d. of the qth-order minors of* R *is a nonzero monomial.*

Proof. If $p = q = 1$, the result is obvious. Consider, now, the matrix case. If the g.c.d. of the maximal order minors of R is a nonzero monomial, then there exists $P \in \mathbb{F}[\xi]^{q\times p}$ such that $PR = c\xi^N I_q$, for some $c \in \mathbb{F}\backslash\{0\}$ and some $N \in \mathbb{Z}_+$. As an immediate extension of the scalar case, it is easily seen that $\ker(c\sigma^N I_q)$ is a nilpotent autonomous behavior. So, being included in a nilpotent autonomous behavior, $\mathfrak{B}$ is nilpotent autonomous, too.

Conversely, suppose that $\mathfrak{B}$ is nilpotent, and let

$$\Gamma = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_q \\ \hline & & 0 & \end{bmatrix} = URV,$$

with U and V unimodular matrices, $\gamma_i \in \mathbb{F}[\xi]\backslash\{0\}$ and $\gamma_i | \gamma_{i+1}$, be the Smith form of R over $\mathbb{F}[\xi]$. If the g.c.d. of the maximal order minors of R (which coincides with $\gamma_1 \cdots \gamma_q$) would not be a monomial, neither would be γ_q. But then, there would be some sequence $v_q \in \mathbb{F}^{\mathbb{Z}_+}$, whose support extends indefinitely in $\mathbb{Z}_+$, satisfying $r_q(\sigma)v_q = 0$. Consequently, $\boldsymbol{w} := V^{-1}(\sigma)\boldsymbol{e}_q v_q$, with $\boldsymbol{e}_q$ the qth canonical vector, would be an infinite support sequence in $\mathfrak{B}$, thus contradicting the nilpotency assumption on $\mathfrak{B}$. □

A complete behavior $\mathfrak{B} = \ker(R(\sigma)) \subseteq (\mathbb{F}^q)^{\mathbb{Z}_+}$ is said to be *controllable* if there exists some positive integer L such that for every $t \in \mathbb{Z}_+$ and every pair of trajectories $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathfrak{B}$, there exists $\boldsymbol{w} \in \mathfrak{B}$ such that $\boldsymbol{w}|_{[0,t]} = \boldsymbol{w}_1|_{[0,t]}$ and $\boldsymbol{w}|_{[t+L,+\infty)} = \boldsymbol{w}_2|_{[t,+\infty)}$. Controllable behaviors are endowed with very strong properties. In particular, for a controllable behavior $\mathfrak{B}$ there exist an $m \in \mathbb{N}$, an $\ell \in \mathbb{Z}_+$, and matrices $G_i \in \mathbb{F}^{q\times m}$, for $i = 0, 1, \ldots, \ell$, such that $\mathfrak{B}$ coincides with the set of all trajectories $\boldsymbol{w} \in (\mathbb{F}^q)^{\mathbb{Z}_+}$ generated by the difference equation

$$\boldsymbol{w}(t) = G_0\boldsymbol{u}(t) + G_1\boldsymbol{u}(t+1) + \cdots + G_\ell \boldsymbol{u}(t+\ell), \quad t \in \mathbb{Z}_+, \tag{2.3}$$

where $\boldsymbol{u} \in (\mathbb{F}^q)^{\mathbb{Z}_+}$ is an arbitrary driving sequence [8]. This amounts to saying that there is a polynomial matrix $G \in \mathbb{F}[\xi]^{q\times m}$, $G(\xi) := \sum_{i=0}^{\ell} G_i\xi^i$, such that $\boldsymbol{w} \in \mathfrak{B}$ if and only if $\boldsymbol{w} = G(\sigma)\boldsymbol{u}$, for some $\boldsymbol{u} \in (\mathbb{F}^m)^{\mathbb{Z}_+}$. The set of trajectories, will support in $\mathbb{Z}_+$, thus obtained is denoted by $\mathrm{im}(G(\sigma))$. Also, it is possible to prove, by resorting to the Smith form and to the fact that full row rank matrices define surjective operators (on the set of sequences with support in $\mathbb{Z}_+$), that the polynomial matrix G involved in the image description of $\mathfrak{B}$ can always be chosen to be of full column rank and even right prime.

3. Observability and reconstructibility

Given a linear left shift invariant system with latent variables $\Sigma_\ell = (\mathbb{Z}_+, \mathbb{F}^{\mathtt{w}}, \mathbb{F}^{\mathtt{d}}, \mathfrak{B}_f)$, every trajectory in $\mathfrak{B}_f$ can be viewed as a pair, $(\boldsymbol{w}, \boldsymbol{d}) = (\{\boldsymbol{w}(t)\}_{t\in\mathbb{Z}_+}, \{\boldsymbol{d}(t)\}_{t\in\mathbb{Z}_+})$, where $\boldsymbol{w}$ ($\dim \boldsymbol{w} = \mathtt{w}$) represents the *manifest variable* vector and $\boldsymbol{d}$ ($\dim \boldsymbol{d} = \mathtt{d}$) the *latent variable* vector. Often the manifest variable vector can be split into two subvectors, i.e. $\boldsymbol{w} = \left[\begin{smallmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{smallmatrix}\right]$, $\dim \boldsymbol{w}_i = \mathtt{w}_i$, $\mathtt{w}_1 + \mathtt{w}_2 = \mathtt{w}$, where $\boldsymbol{w}_1$ consists of those variables which can be (exactly) measured, while $\boldsymbol{w}_2$ is completely unknown. In this situation, it is natural to investigate under what conditions the knowledge of $\boldsymbol{w}_1$ is sufficient to obtain a "good estimate" of $\boldsymbol{w}_2$. The preliminary step toward this goal is that of introducing the notions of observability and reconstructibility.

Definition 3.1. Given a (linear left shift invariant) dynamic system with latent variables $\Sigma_\ell = (\mathbb{Z}_+, \mathbb{F}^{\mathtt{w}_1} \times \mathbb{F}^{\mathtt{w}_2}, \mathbb{F}^{\mathtt{d}}, \mathfrak{B}_f)$, with trajectories $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d})$, we say that $\boldsymbol{w}_2$ is *δ-reconstructible* from $\boldsymbol{w}_1$, if $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d}), (\boldsymbol{w}_1, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{d}}) \in \mathfrak{B}_f$ implies $\boldsymbol{w}_2(t) - \bar{\boldsymbol{w}}_2(t) = 0$, $\forall t \geqslant \delta$. Also, $\boldsymbol{w}_2$ is *reconstructible* from $\boldsymbol{w}_1$ if it is *δ-reconstructible* for some $\delta \geqslant 0$. In particular, when $\delta = 0$, $\boldsymbol{w}_2$ is said to be *observable* from $\boldsymbol{w}_1$.

Consider a dynamic system Σ_ℓ endowed with a linear left shift invariant complete behavior $\mathfrak{B}_f$, and suppose that $\mathfrak{B}_f$ is expressed as the set of all trajectory triples $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d})$ satisfying the difference equation

$$R_2(\sigma)\boldsymbol{w}_2 = R_1(\sigma)\boldsymbol{w}_1 + D(\sigma)\boldsymbol{d}, \tag{3.1}$$

for suitable polynomial matrices R_1, R_2 and D. In order to obtain a characterization of reconstructibility and observability for a system Σ_ℓ endowed with these properties, it is convenient to modify the above description (3.1) and assume that $\mathfrak{B}_f$ is represented in the form

$$R_{22}(\sigma)\boldsymbol{w}_2 = R_{21}(\sigma)\boldsymbol{w}_1 + D_2(\sigma)\boldsymbol{d}, \tag{3.2}$$

$$0 = R_{11}(\sigma)\boldsymbol{w}_1 + D_1(\sigma)\boldsymbol{d}, \tag{3.3}$$

with $R_{22}, R_{21}, R_{11}, D_2$ and D_1 polynomial matrices, R_{22} and $[R_{11}|D_1]$ of full row rank. We can always reduce a first-principles model of the behavior to this form by means of suitable elementary transformations, performed on the set of difference equations.

Proposition 3.2. *Let* $\Sigma_\ell = (\mathbb{Z}_+, \mathbb{F}^{\mathtt{w}_1} \times \mathbb{F}^{\mathtt{w}_2}, \mathbb{F}^{\mathtt{d}}, \mathfrak{B}_f)$ *be a dynamic system with latent variables, whose behavior* $\mathfrak{B}_f$ *is described by* (3.2)–(3.3). *Then*

(i) $\boldsymbol{w}_2$ *is reconstructible from* $\boldsymbol{w}_1$ *if and only if* R_{22} *is nonsingular square with* $\det(R_{22})$ *a nonzero monomial, and there exists some nonsingular square matrix* T, *with* $\det T$ *a monomial, such that*

$$\ker(D_1(\sigma)) \subseteq \ker(T(\sigma)D_2(\sigma)). \tag{3.4}$$

(ii) $\boldsymbol{w}_2$ *is observable from* $\boldsymbol{w}_1$ *if and only if* R_{22} *is unimodular and* $\ker(D_1(\sigma)) \subseteq \ker(D_2(\sigma))$.

Proof. (i) Assume that $\boldsymbol{w}_2$ is reconstructible from $\boldsymbol{w}_1$. Clearly, $\ker(R_{22}(\sigma))$ must be a nilpotent autonomous behavior, otherwise for every infinite support sequence $\boldsymbol{w}_2 \in \ker(R_{22}(\sigma))$, both $(0,0,0)$ and $(0,\boldsymbol{w}_2,0)$ would be behavior trajectories, thus contradicting the reconstructibility assumption. Since R_{22} is of full row rank, this implies that it must be nonsingular square with $\det R_{22}$ a nonzero monomial. Also, $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d}), (\boldsymbol{w}_1, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{d}}) \in \mathfrak{B}_f$ implies

$$R_{22}(\sigma)\boldsymbol{e}_2 = D_2(\sigma)\boldsymbol{v}, \tag{3.5}$$

$$0 = D_1(\sigma)\boldsymbol{v}, \tag{3.6}$$

where $\boldsymbol{e}_2 := \boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2$ and $\boldsymbol{v} := \boldsymbol{d} - \bar{\boldsymbol{d}}$. By the reconstructibility assumption, the projection of the behavior $\mathfrak{B}^* := \{(\boldsymbol{e}_2, \boldsymbol{v})$ satisfying (3.5)–(3.6)$\}$ over the variable $\boldsymbol{e}_2$ [6] must be a nilpotent autonomous behavior. Let U be a unimodular matrix that reduces $\left[\begin{smallmatrix} D_2 \\ D_1 \end{smallmatrix}\right]$ to its (column) Hermite form [2], i.e.

$$U \begin{bmatrix} D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} D^* \\ 0 \end{bmatrix}, \tag{3.7}$$

where D^* is of full row rank. Correspondingly, we get

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} R_{22} \\ 0 \end{bmatrix} = \begin{bmatrix} U_{11}R_{22} \\ U_{21}R_{22} \end{bmatrix}$$

and hence $\mathfrak{B}^*$ is equivalently described by the set of equations

$$(U_{11}R_{22})(\sigma)\boldsymbol{e}_2 = D^*(\sigma)\boldsymbol{v},$$
$$(U_{21}R_{22})(\sigma)\boldsymbol{e}_2 = 0.$$

Clearly, the projection of $\mathfrak{B}^*$ over the variable $\boldsymbol{e}_2$ coincides with $\ker((U_{21}R_{22}(\sigma))$, and therefore U_{21} must be a full column rank matrix with g.c.d. of its maximal order minors which is a nonzero monomial. Moreover, by (3.7), one gets $U_{21}D_2 = -U_{22}D_1$. So, once we factor U_{21} as $U_{21} = \bar{U}_{21}T$, with $\bar{U}_{21}$ right factor prime and T nonsingular square, $\det T$ a monomial, and we introduce a left polynomial inverse $\bar{U}_{21}^{-1}$ of $\bar{U}_{21}$, we get $TD_2 = -\bar{U}_{21}^{-1}U_{22}D_1$, thus proving that (3.4) holds. The converse is easily proved along the same lines.

(ii) Suppose that $\boldsymbol{w}_2$ is observable from $\boldsymbol{w}_1$. By the same kind of reasonings we resorted to in part (i), R_{22} has to be right prime and hence, being of full row rank, unimodular. On the other hand, if $\ker(D_1(\sigma))$ would not be included in $\ker(D_2(\sigma))$, there would be some nonzero sequence $\boldsymbol{d}$ such that $D_1(\sigma)\boldsymbol{d} = 0$ but $D_2(\sigma)\boldsymbol{d} \neq 0$, and hence we would have $(0,0,0) \in \mathfrak{B}_f$ and $(0, \boldsymbol{w}_2, \boldsymbol{d}) \in \mathfrak{B}_f$, for some $\boldsymbol{w}_2 \neq 0$. This contradicts observability.

The proof of the converse follows using the same reasonings. □

Since reconstructibility, and hence condition (3.4), is very relevant in the observer analysis that follows, it is convenient to search for further insights into the inclusion relation (3.4). To this end, we can resort to a different but equivalent description of the behavior $\mathfrak{B}_f$ with respect to the one given in (3.2)–(3.3). Assuming, without loss of generality,

$$[R_{11}\ D_1] = \begin{bmatrix} R_{1A} & D_{1A} \\ R_{1B} & 0 \end{bmatrix},$$

with R_{1B} and D_{1A} full row rank matrices, $\mathfrak{B}_f$ can be represented by means of the following set of difference equations:

$$R_{22}(\sigma)\boldsymbol{w}_2 = R_{21}(\sigma)\boldsymbol{w}_1 + D_2(\sigma)\boldsymbol{d}, \tag{3.8}$$

$$0 = R_{1A}(\sigma)\boldsymbol{w}_1 + D_{1A}(\sigma)\boldsymbol{d}, \tag{3.9}$$

$$0 = R_{1B}(\sigma)\boldsymbol{w}_1, \tag{3.10}$$

with R_{22}, R_{1B} and D_{1A} of full row rank. From now on, we will assume that the behavior is given by this representation. Within this setting, the inclusion relation (3.4), which can be equivalently stated as $TD_2 = MD_1$, with M a suitable polynomial matrix [6], becomes

$$TD_2 = MD_1 = [M_1\ M_2]\begin{bmatrix} D_{1A} \\ 0 \end{bmatrix} = M_1 D_{1A},$$

with M_1 a polynomial matrix.

We are, now, in a position to derive a necessary and sufficient condition for the existence of a nonsingular matrix T, with $\det T$ a monomial, such that (3.4) holds.

Lemma 3.3. *Consider the dynamic system with latent variables Σ_ℓ, whose behavior $\mathfrak{B}_f$ described by* (3.8)–(3.10), *with R_{22} nonsingular square, R_{1B} and D_{1A} of full low rank. Assume that $\left[\begin{smallmatrix} D_2 \\ D_{1A} \end{smallmatrix}\right]$ has rank $\bar{\mathtt{d}}$ and is expressed as*

$$\begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = \begin{bmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{bmatrix} L,$$

with $\left[\begin{smallmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{smallmatrix}\right]$ of full column rank $\bar{\mathtt{d}}$ and L left prime. Let r be a g.c.d. of the $\bar{\mathtt{d}}$th order minors of $\left[\begin{smallmatrix} D_2 \\ D_{1A} \end{smallmatrix}\right]$. A necessary and sufficient condition for the existence of a nonsingular square matrix T, with $\det T$ *a monomial, such that*

$$\ker(D_{1A}(\sigma)) = \ker(D_1(\sigma)) \subseteq \ker(T(\sigma)D_2(\sigma)), \quad (3.11)$$

is that $\bar{D}_{1A}$ is nonsingular square of size $\bar{\mathtt{d}}$ with $\det \bar{D}_{1A} = r(\xi)\xi^\ell$, *for some $\ell \in \mathbb{Z}_+$.*

Proof. Let $\mathtt{r}_A$ denote the rank of D_{1A}, which coincides, by the full row rank assumption, with the number of its rows. Let $[V_2\ \ -V_A]$ be a left prime $(\mathtt{w}_2 + \mathtt{r}_A - \bar{\mathtt{d}}) \times (\mathtt{w}_2 + \mathtt{r}_A)$ polynomial matrix satisfying

$$[V_2\ \ -V_A]\begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = 0,$$

and hence being a minimal left annihilator of $\left[\begin{smallmatrix} D_2 \\ D_{1A} \end{smallmatrix}\right]$. Notice that $[V_2\ -V_A]$ is also minimal left annihilator of $\left[\begin{smallmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{smallmatrix}\right]$. As we have seen, the existence of some $\mathtt{w}_2 \times \mathtt{w}_2$ matrix T, with $\det T$ a monomial, such that condition (3.11) holds, is equivalently restated by saying that there are polynomial matrices T and M_1, $\det T$ a monomial, such that $TD_2 = M_1 D_{1A}$. This amounts to saying that $[T\ \ -M_1] = P[V_2\ \ -V_A]$, for some matrix $P \in \mathbb{R}[\xi]^{\mathtt{w}_2 \times (\mathtt{w}_2 + \mathtt{r}_A - \bar{\mathtt{d}})}$. Consequently, (3.11) holds if and only if there exists a polynomial matrix P, of suitable size, such that $T = PV_2$ is nonsingular square, with determinant which is a monomial.

As $\bar{\mathtt{d}} = \operatorname{rank}\left[\begin{smallmatrix} D_2 \\ D_{1A} \end{smallmatrix}\right] \geqslant \mathtt{r}_A$, then $\mathtt{w}_2 + \mathtt{r}_A - \bar{\mathtt{d}} \leqslant \mathtt{w}_2$. So, $T = PV_2$ is the nonsingular square if and only if $\mathtt{r}_A = \bar{\mathtt{d}}$ and P and V_2 are both nonsingular square. Furthermore, $\det T$ is a monomial if and only if both $\det P$ and $\det V_2$ are. Finally, it is well known [1] that when two matrices are mutually related as $[\,V_2\quad -V_A\,]$ and $\left[\begin{smallmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{smallmatrix}\right]$ are, then every maximal order minor of $[\,V_2\quad -V_A\,]$ coincides with the "complementary" maximal order minor of $\left[\begin{smallmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{smallmatrix}\right]$, (obtained by selecting rows of complementary indices with respect to the indices of the columns selected if $[\,V_2\quad -V_A\,]$), divided by r. In particular, $\det V_2 = \det \bar{D}_{1A}/r$. Therefore, $\det V_2$ can be a monomial if and only if $\det \bar{D}_{1A} = r(\xi)\xi^\ell$, $\exists \ell \in \mathbb{Z}_+$. This completes the proof. □

4. Dead-beat observers design

Consider the dynamic system described by (3.8)–(3.10), with $\boldsymbol{w}_1$ the measured variable, $\boldsymbol{w}_2$ the to-be-estimated variable and $\boldsymbol{d}$ the latent one. The problem we now address is that of introducing a sound definition of "observer" of $\boldsymbol{w}_2$ from $\boldsymbol{w}_1$ for system Σ_ℓ. As a first natural requirement, an observer should "accept" every sequence $\boldsymbol{w}_1$, which is part of a trajectory $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d})$ of the plant, and correspondingly produce some (in general, not unique) estimated trajectory $\hat{\boldsymbol{w}}_2$. This means that an observer of Σ_ℓ should not introduce additional constraints on the $\boldsymbol{w}_1$ components of the system trajectories in addition to those already imposed by the plant. As a further requirement, it is reasonable to assume that the output of an observer should be "consistent" whilst tracking $\boldsymbol{w}_2$. By this we mean that when the trajectories $\hat{\boldsymbol{w}}_2$ and $\boldsymbol{w}_2$ coincide for a sufficiently long time interval $[0, m]$, then they should coincide over all of $[0, +\infty)$. Therefore, an observer for Σ_ℓ is a system that, corresponding to every $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d})$ in $\mathfrak{B}_f$, produces an estimate $\hat{\boldsymbol{w}}_2$ of the trajectory $\boldsymbol{w}_2$, and does not lose track of the correct trajectory once it has followed it over a sufficiently long time interval. Such an observer is said to be "dead-beat" if any estimate $\hat{\boldsymbol{w}}_2$ that it provides coincides with the sequence $\boldsymbol{w}_2$ except, possibly, for a finite number of time instances, or, equivalently, if the sequence $\boldsymbol{w}_2(t) - \hat{\boldsymbol{w}}_2(t)$ equals zero after a finite number of steps. In particular, an observer for Σ_ℓ which produces an estimate $\hat{\boldsymbol{w}}_2$ of $\boldsymbol{w}_2$ which coincides with $\boldsymbol{w}_2$ at each time instant $t \in \mathbb{Z}_+$ (and hence is not affected by any "estimation error") is an "exact" observer. These notions are formalized in the following definition.

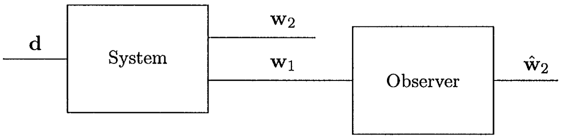

Fig. 1. Observer-system connection.

Definition 4.1. Consider the dynamic system with latent variables $\Sigma_\ell = (\mathbb{Z}_+, \mathbb{F}^{w_1} \times \mathbb{F}^{w_2}, \mathbb{F}^{d}, \mathfrak{B}_f)$, whose behavior $\mathfrak{B}_f$ is described by (3.8)–(3.10). The system represented by the difference equations

$$Q(\sigma)\hat{w}_2 = P(\sigma)w_1, \tag{4.1}$$

with P and Q polynomial matrices of suitable dimensions, is said to define

- an *observer of* w_2 *from* w_1 *for* Σ_ℓ if (a) for every $(w_1, w_2, d) \in \mathfrak{B}_f$ there exists $\hat{w}_2$ such that $(w_1, \hat{w}_2)$ satisfies (4.1), and moreover (b) there exists $m \in \mathbb{Z}_+$ such that whenever (w_1, w_2, d) is in $\mathfrak{B}_f$ and $(w_1, \hat{w}_2)$ satisfies (4.1) with $\hat{w}_2(t) = w_2(t)$ for $t \in [0, m]$, then $\hat{w}_2(t) = w_2(t)$ for every $t \in \mathbb{Z}_+$.

The observer (4.1) of w_2 from w_1 for Σ_ℓ is said to be

- a *dead-beat observer* if there exists $\delta \in \mathbb{Z}_+$ such that for every (w_1, w_2, d) in $\mathfrak{B}_f$ and $(w_1, \hat{w}_2)$ satisfying (4.1), we have $w_2(t) - \hat{w}_2(t) = 0$ for every $t \geqslant \delta$, and
- an *exact observer* if for every (w_1, w_2, d) in $\mathfrak{B}_f$ and $(w_1, \hat{w}_2)$ satisfying (4.1), we have $w_2(t) = \hat{w}_2(t)$ for every $t \geqslant 0$ (see Fig 1).

In the sequel, as the roles of w_1, w_2 and d will always be the same, we will refer to the observers of w_2 from w_1 for Σ_ℓ simply as to the observers for Σ_ℓ.

Given an observer, described by (4.1), its behavior $\hat{\mathfrak{B}}_f$ is the set of all solutions $(w_1, \hat{w}_2)$ of the difference equation (4.1), and, by definition, satisfies the constraint $\mathscr{P}_1\mathfrak{B}_f := \{w_1 : \exists (w_1, w_2, d) \in \mathfrak{B}_f\} \subseteq \{w_1 : \exists (w_1, \hat{w}_2) \in \hat{\mathfrak{B}}_f\} =: \mathscr{P}_1\hat{\mathfrak{B}}_f$. Among the trajectories of $\hat{\mathfrak{B}}_f$, however, we will be interested only in those produced corresponding to the trajectories of $\mathfrak{B}_f$, namely in the set $\{(w_1, \hat{w}_2) \in \hat{\mathfrak{B}}_f : w_1 \in \mathscr{P}_1\mathfrak{B}_f\}$. So, it is reasonable to regard two observers for the system Σ_ℓ as *equivalent*, provided that their behaviors $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ (not necessarily coinciding) satisfy the identity $\{(w_1, \hat{w}_2) \in \hat{\mathfrak{B}}_1 : w_1 \in \mathscr{P}_1\mathfrak{B}_f\} = \{(w_1, \hat{w}_2) \in \hat{\mathfrak{B}}_2 : w_1 \in \mathscr{P}_1\mathfrak{B}_f\}$.

For an observer described by (4.1), the difference variable $e_2 := w_2 - \hat{w}_2$ represents the *estimation error*. So, the previous definitions can be paraphrased by saying that an observer is dead-beat (exact) if the set of its estimation error trajectories constitutes an autonomous behavior, denoted by $\mathfrak{B}_e$, which is nilpotent (the zero behavior). As we have seen in Section 2, this amounts to saying that there exists some matrix $\Delta \in \mathbb{F}[\xi]^{w_2 \times w_2}$, whose determinant is a nonzero monomial $c\xi^n$ (a nonzero constant term), such that $\mathfrak{B}_e = \ker(\Delta(\sigma))$. Δ will be called *error-dynamics matrix*.

Necessary and sufficient conditions for the existence of dead-beat or exact observers are given in Proposition 4.2.

Proposition 4.2. *Consider a dynamic system with latent variables* $\Sigma_\ell = (\mathbb{Z}_+, \mathbb{F}^{w_1} \times \mathbb{F}^{w_2}, \mathbb{F}^{d}, \mathfrak{B}_f)$, *whose behavior* $\mathfrak{B}_f$ *is described by* (3.8)–(3.10), *with* R_{22}, R_{1B} *and* D_{1A} *full row rank polynomial matrices.*

(i) *A necessary and sufficient condition for the existence of a dead-beat observer for* Σ_ℓ *is that* w_2 *is reconstructible from* w_1;

(ii) *a necessary and sufficient condition for the existence of an exact observer for* Σ_ℓ *is that* w_2 *is observable from* w_1.

Proof. (i) Assume, first, that there exists a dead-beat observer for Σ_ℓ, whose estimation error goes to zero within δ steps. If w_2 were not δ-reconstructible from w_1, then there would be two behavior sequences (w_1, w_2, d) and $(w_1, \bar{w}_2, \bar{d})$ such that $w_2(t) - \bar{w}_2(t)$ is not identically zero for $t \geqslant \delta$. If $\hat{w}_2$ is an estimate provided by the dead-beat observer corresponding to w_1, then it should be, at the same time, $w_2(t) - \hat{w}_2(t) = 0$ and $\bar{w}_2(t) - \hat{w}_2(t) = 0, \forall t \geqslant \delta$, and, consequently, $w_2(t) - \bar{w}_2(t) = [w_2(t) - \hat{w}_2(t)] - [\bar{w}_2(t) - \hat{w}_2(t)]$ should be zero for $t \geqslant \delta$, a contradiction.

To prove the converse, assume that w_2 is reconstructible from w_1, or, equivalently, by Proposition 3.2, that R_{22} is nonsingular square, with $\det R_{22}$ a nonzero monomial, and there exist polynomial matrices T and M_1, with $\det T$ a monomial, such that $TD_2 = M_1 D_{1A}$. We aim to show that $(TR_{22})(\sigma)\hat{w}_2 = (TR_{21} - M_1 R_{1A})(\sigma)w_1$ is a dead-beat observer for Σ_ℓ. Actually, by applying (3.8)–(3.10), we get

$$\begin{aligned}(TR_{22})(\sigma)(w_2 - \hat{w}_2) &= (TR_{21})(\sigma)w_1 + (TD_2)(\sigma)d \\ &\quad -(TR_{21} - M_1 R_{1A})(\sigma)w_1 \\ &= (M_1 D_{1A})(\sigma)d + (M_1 R_{1A})(\sigma)w_1 = 0,\end{aligned}$$

thus proving that $e = w_2 - \hat{w}_2$ belongs to some nilpotent autonomous behavior,

(ii) Follows the same lines as the proof of (i). □

From now on, we will restrict our attention to dead-beat observers. The analogous results for the case of exact observers can be easily derived by suitably modifying those obtained for the dead-beat ones. In order to obtain a complete parametrization of the dead-beat observers of Σ_ℓ, we need a technical result. Lemma 4.3 shows that, given any observer for Σ_ℓ, it is possible to obtain an equivalent one, (i.e. producing the same set of trajectories $(\boldsymbol{w}_1, \hat{\boldsymbol{w}}_2)$ for every $\boldsymbol{w}_1$ in $\mathscr{P}_1\mathfrak{B}_f$), for which Q is of full row rank.

Lemma 4.3. *If $Q(\sigma)\hat{\boldsymbol{w}}_2 = P(\sigma)\boldsymbol{w}_1$ is an observer for Σ_ℓ, there exists an equivalent observer $\bar{Q}(\sigma)\hat{\boldsymbol{w}}_2 = \bar{P}(\sigma)\boldsymbol{w}_1$ with $\bar{Q}$ of full row rank.*

Proof. Let U be a unimodular matrix that reduces Q to its (column) Hermite form $\left[\begin{smallmatrix}\bar{Q}\\ 0\end{smallmatrix}\right]$, with $\bar{Q}$ of full row rank. Then we get

$$U[Q \quad -P] = \begin{bmatrix} \bar{Q} & -\bar{P} \\ 0 & -V \end{bmatrix},$$

and hence the observer can be equivalently described by the set of equations

$$\bar{Q}(\sigma)\hat{\boldsymbol{w}}_2 = \bar{P}(\sigma)\boldsymbol{w}_1, \tag{4.2}$$

$$0 = V(\sigma)\boldsymbol{w}_1. \tag{4.3}$$

As an observer must accept every sequence $\boldsymbol{w}_1$ which is part of a behavioral trajectory, for every $\boldsymbol{w}_1 \in \ker(R_{1B}(\sigma))$ equations (4.2)–(4.3) have to be fulfilled for some sequence $\hat{\boldsymbol{w}}_2$, which implies, in particular, $\ker(R_{1B}(\sigma)) \subseteq \ker(V(\sigma))$. So, the observer can be equivalently described by (4.2). □

If it is assumed that the matrix Q appearing in the observer equation is of full row rank, we can obtain deeper insight into the algebraic properties of the polynomial matrices P and Q involved in the observer description, and explicitly relate them to the matrices appearing in (3.8)–(3.10).

Theorem 4.4. *Consider a system Σ_ℓ, with latent variables, whose behavior $\mathfrak{B}_f$ is described by* (3.8)–(3.10), *with* R_{22}, R_{1B} and D_{1A} *of full row rank. Assume that $\boldsymbol{w}_2$ is reconstructible from $\boldsymbol{w}_1$,* (*and hence, in particular, that*

$$\begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = \begin{bmatrix} \bar{D}_2 \\ \bar{D}_{1A} \end{bmatrix} L,$$

with $\left[\begin{smallmatrix}\bar{D}_2\\ \bar{D}_{1A}\end{smallmatrix}\right]$ of full column rank $\mathtt{\bar{d}}, L$ left prime, and $\bar{D}_{1A}$ nonsingular square of size $\mathtt{\bar{d}}$ with $\det \bar{D}_{1A} = r(\xi)\xi^\ell, r$:=g.c.d. $\{\mathtt{\bar{d}}th-order\ minors\ of\ \left[\begin{smallmatrix}D_2\\ D_{1A}\end{smallmatrix}\right]\}$ and $l \in \mathbb{Z}_+$). Let $[T \ \ {-M_1}]$ be a minimal left annihilator of $\left[\begin{smallmatrix}D_2\\ D_{1A}\end{smallmatrix}\right]$, i.e. a left prime $\mathtt{w}_2 \times (\mathtt{w}_2 + \mathtt{\bar{d}})$ matrix such that

$$[T \quad -M_1] \begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = 0. \tag{4.4}$$

If P and Q are polynomial matrices, with Q of full row rank, then $Q(\sigma)\hat{\boldsymbol{w}}_2 = P(\sigma)\boldsymbol{w}_1$ is a dead-beat observer for Σ_ℓ if and only if there exist polynomial matrices $Y \in \mathbb{F}[\xi]^{\mathtt{w}_2 \times \mathtt{w}_2}$, with $\det Y$ *a monomial, and X, of suitable size, such that*

$$[Q \quad -P] = [Y \quad X] \begin{bmatrix} TR_{22} & -TR_{21} + M_1 R_{1A} \\ 0 & -R_{1B} \end{bmatrix}. \tag{4.5}$$

Moreover, the set $\mathfrak{B}_e$ of error trajectories coincides with $\ker(Q(\sigma))$, *which is equivalent to state that we can take Q as error-dynamics matrix.*

Proof. Assume, first, that P and Q satisfy (4.5) for suitable polynomial matrices X and Y, with $\det Y$ a monomial. Notice that, by the reconstructibility and the assumption on Y, $Q = YTR_{22}$ is nonsingular square, with $\det Q$ a monomial. If $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d})$ is any trajectory in $\mathfrak{B}_f$, and $\hat{\boldsymbol{w}}_2$ is an estimate of $\boldsymbol{w}_2$ correspondingly provided by the observer, by exploiting (3.9), (3.10) and (4.4), one gets

$$\begin{aligned} Q(\sigma)(\boldsymbol{w}_2 - \hat{\boldsymbol{w}}_2) &= (YTR_{22})(\sigma)\boldsymbol{w}_2 - P(\sigma)\boldsymbol{w}_1 = (YTR_{21})(\sigma)\boldsymbol{w}_1 \\ &\quad + (YM_1D_{1A})(\sigma)\boldsymbol{d} - [Y(TR_{21} - M_1R_{1A}) \\ &\quad + XR_{1B}](\sigma)\boldsymbol{w}_1 \\ &= (YM_1D_{1A})(\sigma)\boldsymbol{d} + (YM_1R_{1A} - XR_{1B})(\sigma)\boldsymbol{w}_1 = 0. \end{aligned}$$

This immediately proves that $\mathfrak{B}_e$ is included in $\ker(Q(\sigma))$, a nilpotent autonomous behavior, and hence the observer is a dead-beat one.

Suppose, now, that $Q(\sigma)\hat{\boldsymbol{w}}_2 = P(\sigma)\boldsymbol{w}_1$, with Q of full row rank, is a dead-beat observer for Σ_ℓ. We aim to show that Q and P satisfy (4.5) for suitable matrices X and Y, with $\det Y$ a monomial. To this end, observe, first, that if U is a unimodular matrix that reduces $\left[\begin{smallmatrix}D_2\\ D_{1A}\end{smallmatrix}\right]$ to its (column) Hermite form, i.e.

$$U \begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D_2 \\ D_{1A} \end{bmatrix} = \begin{bmatrix} D^* \\ 0 \end{bmatrix}, \tag{4.6}$$

with D^* of full row rank, correspondingly, we get

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} R_{22} & -R_{21} \\ 0 & -R_{1A} \end{bmatrix} = \begin{bmatrix} U_{11}R_{22} & -(U_{11}R_{21} + U_{12}R_{1A}) \\ U_{21}R_{22} & -(U_{21}R_{21} + U_{22}R_{1A}) \end{bmatrix}.$$

Thus, a description of the external behavior $\mathfrak{B}$ of $\mathfrak{B}_f$ (the projection of $\mathfrak{B}_f$ over its external variables $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$), is easily obtained from (3.8)–(3.10) as

$$(U_{21}R_{22})(\sigma)\boldsymbol{w}_2 = (U_{21}R_{21} + U_{22}R_{1A})(\sigma)\boldsymbol{w}_1, \tag{4.7}$$

$$0 = R_{1B}(\sigma)\boldsymbol{w}_1. \tag{4.8}$$

Since, $\boldsymbol{w}_2$ is reconstructible from $\boldsymbol{w}_1$, by the same reasonings adopted in the proof of Proposition 3.2, part (i), U_{21} must be of full column rank with a monomial as g.c.d. of its maximal order minors. Moreover, by (4.6), one has $U_{21}D_2 = -U_{22}D_{1A}$.

Corresponding to the behavior trajectory $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{d}) = (0,0,0)$, the trajectory $(0, \hat{\boldsymbol{w}}_2)$, $\hat{\boldsymbol{w}}_2$ arbitrarily selected in $\ker(Q(\sigma))$, must be admissible for the observer, and hence $\boldsymbol{e}_2 = 0 - \hat{\boldsymbol{w}}_2$ must be in $\mathfrak{B}_e$. This proves that $\ker(Q(\sigma)) \subseteq \mathfrak{B}_e$ and that the full row rank matrix Q is also of full column rank, and hence nonsingular square. Moreover, as the observer is a dead-beat one, $\mathfrak{B}_e$ must be a nilpotent behavior, which implies that $\det Q$ is a monomial. Also, as the zero error trajectory possibly occurs, every pair of trajectories $(\boldsymbol{w}_1, \boldsymbol{w}_2)$ belonging to the external behavior $\mathfrak{B}$ of $\mathfrak{B}_f$ must satisfy the observer equations. Consequently, matrices P and Q are related to the matrices of the behavior $\mathfrak{B}$ in (4.7)–(4.8) as

$$[Q \quad -P] = [\bar{Y} \;\; \bar{X}] \begin{bmatrix} U_{21}R_{22} & -U_{21}R_{21} + U_{22}R_{1A} \\ 0 & -R_{1B} \end{bmatrix},$$

for suitable polynomial matrices $\bar{Y}$ and $\bar{X}$. Since, of course, $[U_{21} \quad -U_{22}] = P[T \quad -M_1], \exists P$ polynomial, then (4.5) holds for $Y = \bar{Y}P$ and $X = \bar{X}P$. The fact that $\det Q$ is a monomial ensures that $\det Y$ is a monomial too. This also implies $\mathfrak{B}_e \subseteq \ker(Q(\sigma))$, and hence, by the first part of the proof, $\mathfrak{B}_e = \ker(Q(\sigma))$. □

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