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Systems & Control Letters 32 (1997) 249–259

SYSTEMS & CONTROL LETTERS

Every storage function is a state function

H.L. Trentelman*, J.C. Willems

Institute for Mathematics and Computing Science, University of Groningen, P.O. Box 800, 9700 AV Groningen, Netherlands

Received 11 November 1996; revised version received 27 March 1997; 14 April 1997

Abstract

It is shown that for linear dynamical systems with quadratic supply rates, a storage function can always be written as a quadratic function of the state of an associated linear dynamical system. This dynamical system is obtained by combining the dynamics of the original system with the dynamics of the supply rate. © 1997 Elsevier Science B.V.

Keywords: Dissipative dynamical systems; State; Storage function; Quadratic differential forms; Two-variable polynomial matrices

1. Introduction

The concept of dissipativeness is of much interest in physics and engineering. Whereas dynamical systems are used to model physical phenomena that evolve with time, dissipative dynamical systems can be used as models for physical phenomena in which also the energy or entropy exchanged with the environment plays a role. Typical examples of dissipative dynamical systems are electrical circuits, in which part of the electric and magnetic energy is dissipated in the resistors in the form of heat, and visco-elastic mechanical systems in which friction causes a similar loss of energy. For earlier work on dissipative systems, we refer to [8, 4, 7].

In a dissipative dynamical system, the book-keeping of energy is done via the *supply rate* and a *storage function*. The supply rate is the rate at which energy flows into the system, and a storage function is a function that measures the amount of energy that is stored inside the system. These functions are related via the *dissipation inequality*, which states that along time trajectories of the dynamical system the supply rate is not less than the increase in storage. This expresses the assumption that a system cannot store more energy than is supplied to it from the outside. The difference between the supplied and the internally stored energy is the dissipated energy.

The storage function measures the amount of energy that is stored inside the system at any instant of time. It is reasonable to expect that the value of the storage function at a particular time instant depends only on the past of the time trajectories through the memory of the system. A standard way to express the memory of a time trajectory of a system is by using the notion of *state*. Thus, we should expect that storage functions are functions of the *state variable* of the system.

In this paper, we prove the general statement that for linear dynamical systems with quadratic supply rates, *any* quadratic storage function can be represented as a quadratic function of *any* state variable of a linear dynamical system whose dynamics are obtained by combining the dynamics of the original system and the dynamics of the supply rate.

A few words on notation. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ denotes the set of all infinitely often differentiable functions

* Corresponding author. E-mail: h.l.trentelman@math.rug.nl.

PII S0167-6911(97)00081-9

$w : \mathbb{R} \to \mathbb{R}^q$; $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ denotes the subset of those $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ that have compact support; given two column vectors x and y, the column vector obtained by stacking x over y is denoted by $\mathrm{col}(x, y)$; likewise, for given matrices A and B with the same number of columns, $\mathrm{col}(A, B)$ denotes the matrix obtained by stacking A over B.

2. Linear differential systems

We will first introduce some basic facts from the behavioral approach to linear dynamical systems. For more details we refer to [9–11].

In this paper we consider dynamical systems described by a system of linear constant coefficient differential equations

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \tag{2.1}$$

in the real variables $w_1, w_2, \ldots, w_q$, arranged as the column vector w; R is a real polynomial matrix with, of course, q columns. This is denoted as $R \in \mathbb{R}^{\bullet \times q}[\xi]$, where ξ denotes the indeterminate. Thus, if $R(\xi) = R_0 + R_1\xi + \cdots + R_N\xi^N$, then Eq. (2.1) denotes the system of differential equations

$$R_0 w + R_1 \frac{\mathrm{d}w}{\mathrm{d}t} + \cdots + R_N \frac{\mathrm{d}^N w}{\mathrm{d}t^N} = 0. \tag{2.2}$$

Formally, Eq. (2.1) defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, with $\mathbb{R}$ the time axis, $\mathbb{R}^q$ the signal space, and $\mathfrak{B}$ the behavior, i.e., the solution set of Eq. (2.1). It is usually advisable to consider weak solutions. Since smoothness plays no role for the results of this paper, we will consider only infinitely differentiable solutions:

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,\middle|\, R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

The family of dynamical systems Σ obtained in this way is denoted by $\mathfrak{L}^q$. Instead of writing $\Sigma \in \mathfrak{L}^q$, we often write $\mathfrak{B} \in \mathfrak{L}^q$. For obvious reasons we refer to Eq. (2.1) as a *kernel representation* of $\mathfrak{B}$. In this paper we will also meet other ways to represent a given $\mathfrak{B} \in \mathfrak{L}^q$, in particular using *latent variable representations* and *image representations*. We will now briefly introduce these. The system of differential equations

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \ell \tag{2.3}$$

is said to be a *latent variable model*. We will call w the *manifest* and ℓ the *latent* variable. We assume that there are q manifest and d latent variables. R and M are polynomial matrices of appropriate dimension. Of course, Eq. (2.3), being a differential equation as Eq. (2.1), defines the behavior

$$\mathfrak{B}_\mathrm{f} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) \mid \text{Eq. (2.3) holds}\}.$$

$\mathfrak{B}_\mathrm{f}$ will be called the *full behavior*, in order to distinguish it from the *manifest behavior* which will be introduced next. Consider the projection of $\mathfrak{B}_\mathrm{f}$ on the manifest variable space, i.e., the set

$$\{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^d) \text{ such that } (w, \ell) \in \mathfrak{B}_\mathrm{f}\}. \tag{2.4}$$

This set is called the *manifest behavior* of Eq. (2.3). If, for a given $\mathfrak{B} \in \mathfrak{L}^q$, the manifest behavior, Eq. (2.4) of Eq. (2.3) equals $\mathfrak{B}$, then Eq. (2.3) is called a *latent variable representation* of $\mathfrak{B}$. The latent variable representation is called *observable* if the latent variable is uniquely determined by the manifest variable, i.e., if $(w, \ell_1), (w, \ell_2) \in \mathfrak{B}_\mathrm{f}$ implies that $\ell_1 = \ell_2$. It can be shown that Eq. (2.3) is observable iff $\mathrm{rank}(M(\lambda)) = d$ for all $\lambda \in \mathbb{C}$.

A system $\mathfrak{B} \in \mathfrak{L}^q$ is said to be *controllable* if for each $w_1, w_2 \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ and a $t' \geqslant 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - t')$ for $t \geqslant t'$. It can be shown that $\mathfrak{B}$ is controllable iff its kernel representation satisfies $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ for all $\lambda \in \mathbb{C}$. Controllable systems are exactly those that admit *image representations*. More concretely, $\mathfrak{B} \in \mathfrak{L}^q$ is controllable iff there exists an $M \in \mathbb{R}^{q \times \bullet}[\xi]$ such that $\mathfrak{B}$ is the manifest behavior of a latent variable model of the form

$$w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \ell. \tag{2.5}$$

For obvious reasons, Eq. (2.5) is called an *image representation* of $\mathfrak{B}$. An image representation is called observable if it is observable as a latent variable representation. Hence, the image representation, Eq. (2.5), is observable iff $\mathrm{rank}(M(\lambda)) = d$ for all $\lambda \in \mathbb{C}$. A controllable system always has an observable image representation.

3. Quadratic differential forms

An important role in this paper is played by quadratic differential forms and two-variable polynomial matrices. These are studied extensively in [12]. In this section we give a brief review.

We denote by $\mathbb{R}^{q\times q}[\zeta,\eta]$ the set of $q\times q$, real polynomial matrices in the indeterminates ζ and η, i.e., expressions of the form

$$\Phi(\zeta,\eta)=\sum_{k,\ell}\Phi_{k\ell}\zeta^k\eta^\ell. \tag{3.1}$$

The sum in Eq. (3.1) ranges over the non-negative integers and is assumed to be finite, and $\Phi_{k\ell}\in\mathbb{R}^{q\times q}$. Such a Φ induces a *quadratic differential form* (QDF) $Q_\Phi:\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)\to\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by

$$Q_\Phi(w)(t):=\sum_{k,\ell}\left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}(t)\right)^{\mathrm{T}}\Phi_{k\ell}\left(\frac{\mathrm{d}^\ell w}{\mathrm{d}t^\ell}(t)\right). \tag{3.2}$$

If $\Phi\in\mathbb{R}^{q\times q}[\zeta,\eta]$ satisfies $\Phi(\zeta,\eta)=\Phi^*(\zeta,\eta):=\Phi(\eta,\zeta)^{\mathrm{T}}$ then Φ will be called *symmetric*. The symmetric elements of $\mathbb{R}^{q\times q}[\zeta,\eta]$ will be denoted by $\mathbb{R}_s^{q\times q}[\zeta,\eta]$. Clearly, $Q_\Phi=Q_{\Phi^*}=Q_{(1/2)(\Phi+\Phi^*)}$. This shows that when considering quadratic differential forms we can restrict attention to Φ's in $\mathbb{R}_s^{q\times q}[\zeta,\eta]$. It is easily seen that $\Phi\in\mathbb{R}^{q\times q}[\zeta,\eta]$ is symmetric iff $\Phi_{k\ell}^{\mathrm{T}}=\Phi_{\ell k}$ for all k and ℓ.

Associated with $\Phi\in\mathbb{R}_s^{q\times q}[\zeta,\eta]$ we form the symmetric matrix

$$\tilde{\Phi}=\begin{pmatrix}\Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots\\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots\\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi_{k\ell} & \cdots\\ \vdots & \vdots & & \vdots & \end{pmatrix}. \tag{3.3}$$

Note that, although $\tilde{\Phi}$ is an infinite matrix, all but a finite number of its elements are zero. We can factor $\tilde{\Phi}$ as $\tilde{\Phi}=\tilde{M}^{\mathrm{T}}\Sigma_M\tilde{M}$, with $\tilde{M}$ an infinite matrix having a finite number of rows and all but a finite number of elements equal to zero, and Σ_M a signature matrix, i.e., a matrix of the form

$$\Sigma_M=\begin{pmatrix}I_{r_+} & 0\\ 0 & -I_{r_-}\end{pmatrix}.$$

This factorization leads, after pre-multiplication by $(I_q\ I_q\zeta\ I_q\zeta^2\cdots)$ and post-multiplication by $\mathrm{col}(I_q\ I_q\eta\ I_q\eta^2\cdots)$, to a factorization of Φ as $\Phi(\zeta,\eta)=M^{\mathrm{T}}(\zeta)\Sigma_M M(\eta)$. This decomposition is not unique but if we take $\tilde{M}$ full row rank, then Σ_M will be unique. Denote this Σ_M as Σ_Φ. In this case, the resulting r_+ is the number of positive eigenvalues and r_- the number of negative eigenvalues of $\tilde{\Phi}$. Any factorization $\Phi(\zeta,\eta)=M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$ will be called a *canonical factorization* of Φ. In such a factorization, the rows of the polynomial matrix $M(\xi)$ are linearly independent over $\mathbb{R}$. Of course, a canonical factorization is not unique. However, they can all be obtained from one by replacing $M(\xi)$ by $UM(\xi)$ with $U\in\mathbb{R}^{\mathrm{rank}(\tilde{\Phi})\times\mathrm{rank}(\tilde{\Phi})}$ such that $U^{\mathrm{T}}\Sigma_\Phi U=\Sigma_\Phi$. Also note that if $\Phi(\zeta,\eta)=M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$ is a canonical factorization, and $\Phi(\zeta,\eta)=M_1^{\mathrm{T}}(\zeta)\Sigma_{M_1}M_1(\eta)$ any other factorization, then there exists a real constant matrix H such that $M(\xi)=HM_1(\xi)$.

The main motivation for identifying QDF's with two-variable polynomial matrices is that they allow a very convenient calculus. One example of this is differentiation. If Q_Φ is a QDF, so will be $(\mathrm{d}/\mathrm{d}t)Q_\Phi$ defined by $((\mathrm{d}/\mathrm{d}t)Q_\Phi)(w):=\mathrm{d}Q_\Phi(w)/\mathrm{d}t$. It is easily checked that $(\mathrm{d}/\mathrm{d}t)Q_\Phi=Q_{\dot\Phi}$ with $\dot\Phi(\zeta,\eta):=(\zeta+\eta)\Phi(\zeta,\eta)$. Suppose now that $\Phi\in\mathbb{R}_s^{q\times q}[\zeta,\eta]$ is given. An important question is: does there exist $\Psi\in\mathbb{R}_s^{q\times q}[\zeta,\eta]$ such that $\dot\Psi=\Phi$, equivalently $(\mathrm{d}/\mathrm{d}t)Q_\Psi=Q_\Phi$? Obviously, such Ψ exists iff Φ contains a factor $\zeta+\eta$. Under this condition we can simply take $\Psi(\zeta,\eta)=(1/(\zeta+\eta))\Phi(\zeta,\eta)$. It was shown in [12] that Φ contains a factor $\zeta+\eta$ iff $\partial\Phi=0$, where $\partial\Phi$ is the one-variable polynomial matrix defined by $\partial\Phi(\xi):=\Phi(-\xi,\xi)$. It was also proven in [12] that $\partial\Phi=0$ iff $\int_{-\infty}^{\infty}Q_\Phi(w)\,\mathrm{d}t=0$ for all $w\in\mathfrak{D}(\mathbb{R},\mathbb{R}^q)$.

For $\Phi\in\mathbb{R}_s^{q\times q}[\zeta,\eta]$, we call Φ *nonnegative* (denoted $\Phi\geqslant0$) if $Q_\Phi(w)\geqslant0$ for all $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$. It was shown in [12] that $\Phi\geqslant0$ iff there exists $D\in\mathbb{R}^{\bullet\times q}[\xi]$ such that $\Phi(\zeta,\eta)=D^{\mathrm{T}}(\zeta)D(\eta)$, equivalently $Q_\Phi(w)=\|D(\mathrm{d}/\mathrm{d}t)w\|^2$ for all $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$. In addition, we need the concept of *average non-negativity* (denoted as $\int Q_\Phi\geqslant0$). This is defined as $\int_{-\infty}^{\infty}Q_\Phi(w)\,\mathrm{d}t\geqslant0$ for all $w\in\mathfrak{D}(\mathbb{R},\mathbb{R}^q)$. Again, it was shown in [12] that $\int Q_\Phi\geqslant0$ iff $(\partial\Phi)(\mathrm{i}\omega)\geqslant0$ for all $\omega\in\mathbb{R}$. In turn, this condition is equivalent with the existence of polynomial spectral factorizations of $\partial\Phi$: $(\partial\Phi)(\mathrm{i}\omega)\geqslant0$ iff there exists $D\in\mathbb{R}^{q\times q}[\xi]$ such that $(\partial\Phi)(\xi)=D^{\mathrm{T}}(-\xi)D(\xi)$ (see [1, 2]).

4. Dissipative systems

Let $\mathfrak{B}\in\mathfrak{L}^q$ be a controllable linear differential system. Let $R(\mathrm{d}/\mathrm{d}t)w=0$ and $w=M(\mathrm{d}/\mathrm{d}t)\ell$ be a kernel and an observable image representation, respectively, of $\mathfrak{B}$, with $R\in\mathbb{R}^{g\times q}[\xi]$ and $M\in\mathbb{R}^{q\times d}$. In addition, consider the quadratic differential form $Q_\Phi:\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)\to\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ induced by the symmetric two-variable polynomial matrix Φ; Q_Φ is called the *supply rate*. Intuitively, we think of $Q_\Phi(w)$ as the power going into the physical system $\mathfrak{B}$. In many applications, the power will indeed be a quadratic expression involving the system variables and their derivatives. For example, in mechanical systems, it is $\sum_k F_k\,\mathrm{d}q_k/\mathrm{d}t$ with F_k the external force acting on, and q_k the position of the kth pointmass; in electrical circuits it is $\sum_k V_k I_k$ with V_k the potential and I_k the current into the circuit at the kth terminal. The system $\mathfrak{B}$ is called *dissipative* with respect to the supply rate Q_Φ if along trajectories that start at rest and bring the system back to rest, the net amount of energy flowing into the system is non-negative: the system dissipates energy.

Definition 4.1. $(\mathfrak{B}, Q_\Phi)$ is called *dissipative* if $\int_{-\infty}^{\infty} Q_\Phi(w)\,\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$.

Of course, at some times t the power $Q(w)(t)$ might be positive: energy is flowing into the system; at other times, it might be negative, energy is flowing out of the system. This outflow is possible because energy is stored. However, because of dissipation, the rate of increase of the storage cannot exceed the supply. The interaction between supply, storage, and dissipation is formalized as follows:

Definition 4.2. The QDF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ is called a *storage function* for $(\mathfrak{B}, Q_\Phi)$ if

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_\Phi(w) \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q). \tag{4.1}$$

The QDF Q_Δ induced by $\Delta \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ is called a *dissipation function* for $(\mathfrak{B}, Q_\Phi)$ if $Q_\Delta(w) \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and

$$\int_{-\infty}^{\infty} Q_\Phi(w)\,\mathrm{d}t = \int_{-\infty}^{\infty} Q_\Delta(w)\,\mathrm{d}t \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q).$$

If the supply rate Q_Φ, the dissipation function Q_Δ, and the storage function Q_Ψ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w) \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q), \tag{4.2}$$

then we call the triple $(Q_\Phi, Q_\Psi, Q_\Delta)$ *matched along* $\mathfrak{B}$.

Theorem 4.3. *The following conditions are equivalent*:
1. $(\mathfrak{B}, Q_\Phi)$ *is dissipative.*
2. $M(-\mathrm{i}\omega)^{\mathrm{T}}\Phi(-\mathrm{i}\omega, \mathrm{i}\omega)M(\mathrm{i}\omega) \geqslant 0$ *for all* $\omega \in \mathbb{R}$.
3. $(\mathfrak{B}, Q_\Phi)$ *admits a storage function.*
4. $(\mathfrak{B}, Q_\Phi)$ *admits a dissipation function.*

Furthermore, for any dissipation function Q_Δ *there exists a storage function* Q_Ψ, *and for any storage function* Q_Ψ *there exists a dissipation function* Q_Δ *such that* $(Q_\Phi, Q_\Psi, Q_\Delta)$ *is matched along* $\mathfrak{B}$.

Proof. See the Appendix.

Example 4.4. Consider the system

$$M\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + D\frac{\mathrm{d}q}{\mathrm{d}t} + Kq = F \tag{4.3}$$

with $K, D, M \in \mathbb{R}^{k\times k}$, $K = K^{\mathrm{T}} \geqslant 0$, $D + D^{\mathrm{T}} \geqslant 0$, and $M = M^{\mathrm{T}} \geqslant 0$. The position vector q and force vector F take their values in $\mathbb{R}^k$. Such second order equations occur frequently as models of (visco-)elastic mechanical systems. As manifest variable take $w = \mathrm{col}(q, F)$, and as supply rate take $Q_\Phi(q, F) = F^{\mathrm{T}}\,\mathrm{d}q/\mathrm{d}t$. This corresponds to

$$\Phi(\zeta,\eta) = \frac{1}{2}\begin{pmatrix} 0 & \zeta I \\ \eta I & 0 \end{pmatrix}.$$

An image representation of the system is given by $\mathrm{col}(q, F) = M(\mathrm{d}/\mathrm{d}t)\ell$, with M equal to

$$M(\xi) = \begin{pmatrix} I \\ M\xi^2 + D\xi + K \end{pmatrix}.$$

Obviously, due to damping, the system is dissipative. This indeed follows from the fact that $M^{\mathrm{T}}(-\mathrm{i}\omega)\Phi(-\mathrm{i}\omega, \mathrm{i}\omega)M(\mathrm{i}\omega) = \frac{1}{2}(D + D^{\mathrm{T}})\omega^2 \geqslant 0$. A storage function is given by $Q_\Psi(q, F) = \frac{1}{2}(\mathrm{d}q/\mathrm{d}t)^{\mathrm{T}} M(\mathrm{d}q/\mathrm{d}t) + \frac{1}{2}q^{\mathrm{T}}Kq$. This corresponds to

$$\Psi(\zeta,\eta) = \frac{1}{2}\begin{pmatrix} K + \zeta\eta M & 0 \\ 0 & 0 \end{pmatrix}.$$

Indeed, for all (q, F) satisfying Eq. (4.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^{\mathrm{T}} M\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{2}q^{\mathrm{T}}Kq\right)$$

$$= F^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^{\mathrm{T}}(D + D^{\mathrm{T}})\frac{\mathrm{d}q}{\mathrm{d}t} \leqslant F^{\mathrm{T}}\frac{\mathrm{d}q}{\mathrm{d}t}.$$

It also follows that a dissipation function is given by

$$Q_\Delta(q, F) = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^{\mathrm{T}}(D + D^{\mathrm{T}})\frac{\mathrm{d}q}{\mathrm{d}t}.$$

This corresponds to taking

$$\Delta(\zeta,\eta) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \zeta\eta(D + D^{\mathrm{T}}) \end{pmatrix}.$$

Obviously, the triple $(Q_\Phi, Q_\Psi, Q_\Delta)$ is matched on the behavior $\mathfrak{B}$ of Eq. (4.3).

5. State representations

A latent variable model $R'(\mathrm{d}/\mathrm{d}t)w = M(\mathrm{d}/\mathrm{d}t)x$ (with the latent variable denoted by x this time) is said to be a *state model* if whenever (w_1, x_1) and (w_2, x_2) are elements of the full behavior $\mathfrak{B}_\mathrm{f}$, and $x_1(0) = x_2(0)$, then the concatenation $(w, x) := (w_1, x_1) \wedge (w_2, x_2)$ will also satisfy $R'(\mathrm{d}/\mathrm{d}t)w = M(\mathrm{d}/\mathrm{d}t)x$. Since this concatenation need not be $\mathfrak{C}^\infty$, we only require it to be a weak solution, that is, a solution in the sense of distributions.

Let $\mathfrak{B} \in \mathfrak{L}^q$. A latent variable representation of $\mathfrak{B}$ is called a *state representation* of $\mathfrak{B}$ if it is a state model. Given $w_1, w_2 \in \mathfrak{B}$, to decide whether $w_1 \wedge w_2 \in \mathfrak{B}$, we can look at the value of the state variables x_1 and x_2 at time $t = 0$. If $x_1(0) = x_2(0)$, then $w_1 \wedge w_2 \in \mathfrak{B}$. In other words, in order to decide whether a future continuation is possible within $\mathfrak{B}$, not the whole past needs to be remembered, but only the *present* value of the state is relevant. Thus x parametrizes the memory of the system.

An important role is played by latent variable models of the form

$$Gw + Fx + E\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{5.1}$$

Here, E, F, and G are real constant matrices. The important feature of Eq. (5.1) is that it is an (implicit) differential equation containing derivatives of order at most one in x and zero in w. It was shown in [6] that any latent variable model of the form Eq. (5.1) is a state model. Conversely, every state model $R'(\mathrm{d}/\mathrm{d}t)w = M(\mathrm{d}/\mathrm{d}t)x$ is equivalent to a representation of the form Eq. (5.1) in the sense that their full behaviors $\mathfrak{B}_\mathrm{f}$ coincide. This means that state representations of a given $\mathfrak{B}$ of the form Eq. (5.1) are in fact *all* state representations of $\mathfrak{B}$: given a state representation $\mathfrak{B}_\mathrm{f}$ of $\mathfrak{B}$, it will have a kernel representation of the type Eq. (5.1) and hence, without loss of generality, we can assume that the associated differential equation is of this form. In the case of state models, we call x the *state* or the vector of *state variables*. The number of state variables, i.e., the size of x, is called the *dynamic order* of the model. This number is denoted by $n(\mathfrak{B})$, or when $\mathfrak{B}$ is obvious from the context, by n.

Let $\mathfrak{B}$ be the manifest behavior of any (not necessarily observable) state representation, Eq. (5.1). It turns out that there exists an *observable* state representation of $\mathfrak{B}$ with smaller dynamic order, such that the respective state variables are related by a linear map:

Lemma 5.1. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be the manifest behavior of Eq. (5.1). Then there exists an observable state representation $G'w + F'x' + E'\,\mathrm{d}x'/\mathrm{d}t = 0$ of $\mathfrak{B}$ (its full behavior denoted by $\mathfrak{B}'_\mathrm{f}$) with dynamic order $n' \leqslant n$, and a linear map $L : \mathbb{R}^n \to \mathbb{R}^{n'}$ such that $(w, x) \in \mathfrak{B}_\mathrm{f}$, $(w, x') \in \mathfrak{B}'_\mathrm{f}$ implies $x' = Lx$.*

Proof. See the Appendix.

If $Gw + Fx + E\,\mathrm{d}x/\mathrm{d}t = 0$ is a state representation of $\mathfrak{B}$, then it is observable (i.e., x is observable from w, see Section (2)) iff there exists $X \in \mathbb{R}^{\bullet \times q}[\xi]$ such that for all $w \in \mathfrak{B}$ we have $(w, x) \in \mathfrak{B}_\mathrm{f} \Leftrightarrow x = X(\mathrm{d}/\mathrm{d}t)w$ (see [11]). The differential operator $X(\mathrm{d}/\mathrm{d}t)$ is called a *state map* for $\mathfrak{B}$. In general, if $R(\mathrm{d}/\mathrm{d}t)w = 0$ is a kernel representation of $\mathfrak{B}$, then $X(\mathrm{d}/\mathrm{d}t)$ is called a state map for $\mathfrak{B}$ if

$$\begin{pmatrix} R(\mathrm{d}/\mathrm{d}t) \\ X(\mathrm{d}/\mathrm{d}t) \end{pmatrix} w = \begin{pmatrix} 0 \\ I \end{pmatrix} x$$

is a state representation of $\mathfrak{B}$.

Assume now that $\mathfrak{B}$ is controllable and let $w = M(\mathrm{d}/\mathrm{d}t)\ell$ be an observable image representation. Let Π be a permutation matrix such that $\Pi M = \mathrm{col}(U, Y)$, with YU^{-1} a matrix of proper rational functions (such Π always exists, see [11]). This corresponds to permuting the components of w as $\Pi w = \mathrm{col}(u, y)$, with $u = U(\mathrm{d}/\mathrm{d}t)\ell$ and $y = Y(\mathrm{d}/\mathrm{d}t)\ell$, such that u is an input and y is an output. The number of input components of $\mathfrak{B}$, i.e., the size of u, is denoted by $m(\mathfrak{B})$, or when $\mathfrak{B}$ is obvious from the context, by m. Consider the set of real polynomial row vectors

$$\mathfrak{F} := \{f \in \mathbb{R}^{\bullet \times q}[\xi] \mid fU^{-1} \text{ is strictly proper}\}.$$

It is easily seen that $\mathfrak{F}$ is a linear vector space over $\mathbb{R}$. Let $X \in \mathbb{R}^{\bullet \times q}[\xi]$. It was shown in Ref. [6] that $X(\mathrm{d}/\mathrm{d}t)$ is a state map for $\mathfrak{B}$ iff the rows of the polynomial matrix XM span $\mathfrak{F}$, i.e., every element of $\mathfrak{F}$ is a real linear combination of the rows of XM.

Suppose now that we have a system $\mathfrak{B} \in \mathfrak{L}^q$, and suppose a state representation of this system is given, with state variable, say x. Assume that to the manifest variable w we add an extra component, say f, i.e., we consider a new system $\mathfrak{B}_\mathrm{ext}$ with the property that $w \in \mathfrak{B}$ iff there exists f such that $\mathrm{col}(w, f) \in \mathfrak{B}_\mathrm{ext}$. In the following theorem we establish conditions under which f can be written as a linear function of the state variable of the original system $\mathfrak{B}$, and as a linear function of the state variable and input variable of $\mathfrak{B}$.

Theorem 5.2. *Let $\mathfrak{B} \in \mathfrak{L}^q$ and let*

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} U(\mathrm{d}/\mathrm{d}t) \\ Y(\mathrm{d}/\mathrm{d}t) \end{pmatrix} \ell$$

be an observable image representation with YU^{-1} a matrix of proper rational functions. Let

$$G\begin{pmatrix} u \\ y \end{pmatrix} + Fx + E\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

be an observable state representation of $\mathfrak{B}$ (with full behavior denoted by $\mathfrak{B}_{\mathrm{f}}$). Let $F \in \mathbb{R}^{\bullet \times d}[\xi]$ and let $\mathfrak{B}_{\mathrm{ext}}$ be the system with image representation

$$\begin{pmatrix} u \\ y \\ f \end{pmatrix} = \begin{pmatrix} D(\mathrm{d}/\mathrm{d}t) \\ N(\mathrm{d}/\mathrm{d}t) \\ F(\mathrm{d}/\mathrm{d}t) \end{pmatrix} \ell.$$

Then there exists a real constant matrix $H \in \mathbb{R}^{\bullet \times n}$ such that $f = Hx$ for all f and x for which there exists $\mathrm{col}(u, y) \in \mathfrak{B}$ such that $\mathrm{col}(u, y, f) \in \mathfrak{B}_{\mathrm{ext}}$ and $\mathrm{col}(u, y, x) \in \mathfrak{B}_{\mathrm{f}}$, iff FD^{-1} is a matrix of strictly proper rational functions. There exist real constant matrices $H \in \mathbb{R}^{\bullet \times n}$ and $J \in \mathbb{R}^{\bullet \times m}$ such that $f = Hx + Ju$ for all f, x and u for which there exists y such that $\mathrm{col}(u, y, f) \in \mathfrak{B}_{\mathrm{ext}}$ and $\mathrm{col}(u, y, x) \in \mathfrak{B}_{\mathrm{f}}$, iff FD^{-1} is a matrix of proper rational functions.

Proof. See the Appendix.

6. Main results

In this section we show that storage functions can always be represented as quadratic functions of a state variable, and that dissipation functions can always be represented as quadratic functions of a state variable, jointly with the manifest variable of a given system.

We first treat the case that $\mathfrak{B}$ is unconstrained, i.e., $\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^q)$. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Assume that $(\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$ is dissipative. It turns out that every storage function is a quadratic function of any state variable of a particular system $\mathfrak{B}_\Phi$ obtained from the dynamics of Φ. Also, every dissipation function is a quadratic function of any state variable, jointly with the manifest variable of this system $\mathfrak{B}_\Phi$. We now explain what we mean by $\mathfrak{B}_\Phi$. The system $\mathfrak{B}_\Phi$ is defined as follows. Let $\Phi(\zeta, \eta) = M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$ be a canonical factorization of Φ, with $\Sigma_\Phi \in \mathbb{R}^{r \times r}$. Now, consider the system $\mathfrak{B}_\Phi \in \mathfrak{L}^r$ (with manifest variable $v \in \mathbb{R}^r$) with image representation

$$v = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w. \tag{6.1}$$

Theorem 6.1. *Let $Gv + Fx + E\,\mathrm{d}x/\mathrm{d}t = 0$ be a state representation of $\mathfrak{B}_\Phi$, with full behavior $\mathfrak{B}_{\mathrm{f}}$. Let Q_Ψ be a storage function for $(\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$. Then there exists $K = K^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ such that $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$ implies $Q_\Psi(w) = x^{\mathrm{T}}Kx$. Furthermore, if Q_Δ is a dissipation function for $(\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$, then there exists $L = L^{\mathrm{T}} \in \mathbb{R}^{(n+q) \times (n+q)}$ such that $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ v \end{pmatrix}^{\mathrm{T}} L \begin{pmatrix} x \\ v \end{pmatrix}.$$

Proof. See the Appendix.

Next, we treat the general case. Let $\mathfrak{B} \in \mathfrak{L}^q$ be an arbitrary controllable system. Let $\Phi \in \mathbb{R}_s^{q \times q}[\xi]$. Assume that $(\mathfrak{B}, Q_\Phi)$ is dissipative. Also in this case, every storage function turns out to be a quadratic function of any state variable, and every dissipation function a quadratic function of any state variable, jointly with the manifest variable of a system $\mathfrak{B}_\Phi$. This time, however, the system $\mathfrak{B}_\Phi$ is obtained by combining the dynamics of $\mathfrak{B}$ and Φ. Again, let $\Phi(\zeta, \eta) = M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$ be a canonical factorization of Φ, with $\Sigma_\Phi \in \mathbb{R}^{r \times r}$. Now, consider the system $\mathfrak{B}_\Phi \in \mathfrak{L}^r$ (with manifest variable $v \in \mathbb{R}^r$) represented by

$$v = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w, \quad w \in \mathfrak{B}. \tag{6.2}$$

Theorem 6.2. *Let $Gv + Fx + E\,\mathrm{d}x/\mathrm{d}t = 0$ be a state representation of $\mathfrak{B}_\Phi$, with full behavior $\mathfrak{B}_{\mathrm{f}}$. Let Q_Ψ be a storage function for $(\mathfrak{B}, Q_\Phi)$. Then there exists $K = K^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ such that $w \in \mathfrak{B}$ and $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$ implies $Q_\Psi(w) = x^{\mathrm{T}}Kx$. If Q_Δ is a dissipation function for $(\mathfrak{B}, Q_\Phi)$, then there exists $L = L^{\mathrm{T}} \in \mathbb{R}^{(n+d) \times (n+d)}$ such that $w \in \mathfrak{B}$ and $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ v \end{pmatrix}^{\mathrm{T}} L \begin{pmatrix} x \\ v \end{pmatrix}.$$

Proof. See the Appendix.

Finally, we discuss the special case that the supply rate Q_Φ is of order zero in w, i.e., $Q_\Phi(w) = w^\mathrm{T}Pw$, with $P = P^\mathrm{T} \in \mathbb{R}^{q\times q}$. Let $\mathfrak{B} \in \mathfrak{L}^q$ be controllable, and assume that $(\mathfrak{B}, Q_\Phi)$ is dissipative. In this case every storage function is simply a quadratic function of any state variable of $\mathfrak{B}$, and every dissipation function is a quadratic function of any state variable of $\mathfrak{B}$, jointly with the manifest variable of $\mathfrak{B}$.

Corollary 6.3. *Let $Gw + Fx + E\,\mathrm{d}x/\mathrm{d}t = 0$ be a state representation of $\mathfrak{B}$, with full behavior $\mathfrak{B}_\mathrm{f}$. Let Q_Ψ be a storage function for $(\mathfrak{B}, Q_\Phi)$. Then there exists $K = K^\mathrm{T} \in \mathbb{R}^{n\times n}$ such that $\mathrm{col}(w,x) \in \mathfrak{B}_\mathrm{f}$ implies $Q_\Psi(w) = x^\mathrm{T}Kx$. If Q_Δ is a dissipation function for $(\mathfrak{B}, Q_\Phi)$, then there there exists $L = L^\mathrm{T} \in \mathbb{R}^{(n+d)\times(n+d)}$ such that $\mathrm{col}(w,x) \in \mathfrak{B}_\mathrm{f}$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ w \end{pmatrix}^\mathrm{T} L \begin{pmatrix} x \\ w \end{pmatrix}.$$

Proof. Follows immediately from Theorem 6.2. □

Example 6.4. Consider the mechanical system, Eq. (4.3), together with the supply rate Q_Φ. A canonical factorization of $\Phi(\zeta, \eta)$ is given by

$$\Phi(\zeta,\eta) = \frac{1}{2}\begin{pmatrix} \zeta I & -\zeta I \\ I & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\frac{1}{2}\begin{pmatrix} \eta I & I \\ -\eta I & I \end{pmatrix}.$$

The corresponding system $\mathfrak{B}_\Phi$ (with manifest variable $v = \mathrm{col}(v_1, v_2)$) is represented by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{d}q/\mathrm{d}t + F \\ -\mathrm{d}q/\mathrm{d}t + F \end{pmatrix},$$

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + D\frac{\mathrm{d}q}{\mathrm{d}t} + Kq = F.$$

It is easily seen that $\mathrm{col}(\mathrm{d}q/\mathrm{d}t, q)$ is a state variable for $\mathfrak{B}_\Phi$. It was indeed shown that a storage function is given by

$$Q_\Psi(q,F) = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^\mathrm{T} M\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{2}q^\mathrm{T}Kq$$

and that a dissipation function is given by

$$Q_\Delta(q,F) = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^\mathrm{T}(D + D^\mathrm{T})\frac{\mathrm{d}q}{\mathrm{d}t}.$$

Example 6.5. The relation between force F and position q due to a potential field $V(q)$ is given by $F = (\nabla V)(q)$. This defines a (in general nonlinear) system $\mathfrak{B}$ with manifest variable $w = \mathrm{col}(q, F)$. This system is dissipative (even lossless) with respect to the supply rate $Q_\Phi(q,F) = F^\mathrm{T}\,\mathrm{d}q/\mathrm{d}t$, and $V(q)$ defines a storage function

$$\frac{\mathrm{d}}{\mathrm{d}t}V(q) = (\nabla V)(q)^\mathrm{T}\frac{\mathrm{d}q}{\mathrm{d}t} = F^\mathrm{T}\frac{\mathrm{d}q}{\mathrm{d}t}.$$

The storage function $V(q)$ is a function of the position q. The question is now: in what sense is $V(q)$ a function of the state? For the case that $\mathfrak{B}$ is linear, equivalently $V(q) = \frac{1}{2}q^\mathrm{T}Kq$, $(\nabla V)(q) = Kq$, with $K = K^\mathrm{T}$, the answer is provided by Theorem 6.2: storage functions of $\mathfrak{B}$ are quadratic functions of state variables of the system $\mathfrak{B}_\Phi$ represented by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{d}q/\mathrm{d}t + F \\ -\mathrm{d}q/\mathrm{d}t + F \end{pmatrix}, \quad F = Kq$$

It is easily seen that q is indeed a state variable for $\mathfrak{B}_\Phi$.

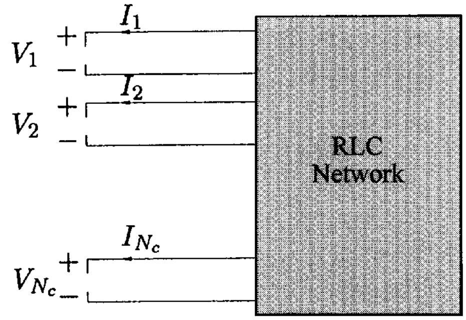
Example 6.6. Consider a linear time-invariant RLC-circuit with N_e external ports with currents $I_1, I_2, \ldots, I_{N_\mathrm{e}}$ and voltages $V_1, V_2, \ldots, V_{N_\mathrm{e}}$. Denote $I = \mathrm{col}(I_1, I_2, \ldots, I_{N_\mathrm{e}})$ and $V = \mathrm{col}(V_1, V_2, \ldots, V_{N_\mathrm{e}})$. The circuit contains resistors $R_1, R_2, \ldots, R_{N_\mathrm{r}}$. The current through and voltage across the kth resistor are I_{R_k} and V_{R_k}, respectively. Denote by I_R and V_R the vectors of resistor currents and voltages. The network contains N_c capacitors with capacitances $C_1, C_2, \ldots, C_{N_\mathrm{c}}$. The current through and voltage across the ℓth capacitor are I_{C_ℓ} and V_{C_ℓ}, respectively; the vectors I_C and V_C are defined in the obvious way. Finally, the network contains N_i inductors $L_1, L_2, \ldots, L_{N_i}$. The current through and voltage across the mth inductor are I_{L_m} and V_{L_m}, respectively; the vectors I_L and V_L are defined in the obvious way.

The network defines a system $\mathfrak{B}$ with manifest behavior

$$\mathfrak{B} = \{(V,I) \mid \text{there exists } (V_R, I_R, V_C, I_C, V_L, I_L) \text{ such that the laws of the constitutive elements, together with Kirchoff's laws are satisfied}\}. \tag{6.3}$$

If one writes down the system equations explicitly, Eq. (6) yields a latent variable representation of $\mathfrak{B}$, with latent variable $\mathrm{col}(V_R, I_R, V_C, I_C, V_L, I_L)$. Denote by $\mathfrak{B}_\mathrm{f}$ the full behavior of this representation. By applying Tellegens's theorem (see e.g. [3]), we obtain

$$V_R^\mathrm{T}I_R + V_C^\mathrm{T}I_C + V_L^\mathrm{T}I_L = V^\mathrm{T}I.$$



This, together with the constitutive laws

$$V_R = RI_R,\ C\frac{dV_C}{dt} = I_C \text{ and } L\frac{dI_L}{dt} = V_L$$

yields

$$\frac{d}{dt}\left(\frac{1}{2}V_C^{\mathrm{T}}CV_C + \frac{1}{2}V_L^{\mathrm{T}}LV_L\right) = V^{\mathrm{T}}I - \frac{1}{2}I_R^{\mathrm{T}}RI_R. \quad (6.4)$$

Here, the matrix C is defined by $C := \mathrm{diag}(C_1, C_2, \ldots, C_{N_c})$, and R and L are defined similarly. Eq. (6.4) shows that $\mathfrak{B}$ is dissipative with respect to the supply rate $Q_\Phi(V,I) = V^{\mathrm{T}}I$. It also follows from Eq. (6.4) that any QDF $Q_\Psi(V,I)$ such that $(V,I,V_R,I_R,V_C,I_C,V_L,I_L) \in \mathfrak{B}_f$ implies $Q_\Psi(V,I) = \frac{1}{2}V_C^{\mathrm{T}}CV_C + \frac{1}{2}V_L^{\mathrm{T}}LV_L$, is a storage function of $(\mathfrak{B}, Q_\Phi)$. Note that $\frac{1}{2}V_C^{\mathrm{T}}CV_C + \frac{1}{2}V_L^{\mathrm{T}}LV_L$ is the total electric energy stored in the capacitors plus the total magnetic energy stored in the inductors in the circuit. It can be shown that $\mathrm{col}(V_C, I_L)$ is a state variable for the system $\mathfrak{B}$. Thus, this storage function is indeed a quadratic function of a state variable of the system, illustrating the result of Theorem 6.1. It follows from Eq. (6.4) that any QDF $Q_\Delta(V,I)$ such that $(V,I,V_R,I_R,V_C,I_C,V_L,I_L) \in \mathfrak{B}_f$ implies $Q_\Delta(V,I) = \frac{1}{2}I_R^{\mathrm{T}}RI_R$, is a dissipation function of $(\mathfrak{B}, Q_\Phi)$. Note that this is exactly the electric energy dissipated in the resistors. According to Corollary 6.3, $\frac{1}{2}I_R^{\mathrm{T}}RI_R$ can be written as a quadratic function of the variables V_C, I_L, V and I.

7. Conclusions

We have shown (in the context of linear systems and quadratic functionals) that any storage function of a dissipative system is a function of the state. This state involves the dynamics of the dissipative system as well as those of the supply rate.

Appendix A. Proofs

Proof of Theorem 4.3. We will prove (4) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (4). To show that (4) $\Rightarrow$ (3), let Q_Δ be a dissipation function. Define $\Delta'(\zeta,\eta) := M^{\mathrm{T}}(\zeta)\Delta(\zeta,\eta)M(\eta)$ and $\Phi'(\zeta,\eta) := M^{\mathrm{T}}(\zeta)\Phi(\zeta,\eta)M(\eta)$. For all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q)$ we have $\int_{-\infty}^{\infty} Q_\Delta(w)\,dt = \int_{-\infty}^{\infty} Q_\Phi(w)\,dt$, and hence for all $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^d)$ we have $\int_{-\infty}^{\infty} Q_{\Delta'-\Phi'}(\ell)\,dt = 0$. This is equivalent with the condition that $\partial(\Delta' - \Phi') = 0$. Thus, $\Phi' - \Delta'$ contains a factor $\zeta + \eta$. Define $\Psi'(\zeta,\eta) := (1/(\zeta+\eta))(\Phi'(\zeta,\eta) - \Delta'(\zeta,\eta))$, and let $\Psi(\zeta,\eta) := M^{\dagger T}(\zeta,\eta)\Psi'(\zeta,\eta)M^\dagger(\eta)$. Here $M^\dagger$ is any polynomial left-inverse of the polynomial matrix M: $M^\dagger M = I$. It is easily checked that $(d/dt)Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$ for all $w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$, so Q_Ψ is a storage function. To prove (3) $\Rightarrow$ (1), Let Q_Ψ be a storage function. Then for all $w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ we have $(d/dt)Q_\Psi(w) \leqslant Q_\Phi(w)$. Taking w to have compact support and integrating this inequality, we get $\int_{-\infty}^{\infty} Q_\Phi(w)\,dt \geqslant 0$, proving that $(\mathfrak{B}, Q_\Phi)$ is dissipative. Next we prove (1) $\Rightarrow$ (2). We will silently switch from $\mathbb{R}^q$ as signal space to $\mathbb{C}^q$. Assume that there exists $a \in \mathbb{C}^q$ and $\omega_0 \in \mathbb{R}$ such that $\bar{a}^{\mathrm{T}}\Phi'(-i\omega_0, i\omega_0)a < 0$. Now consider the function $\ell_N \in \mathfrak{D}(\mathbb{R},\mathbb{C}^d)$ for $N = 1,2,\ldots$, defined by

$$\ell_N(t) = \begin{cases} e^{i\omega_0 t}a, & |t| \leqslant \dfrac{2\pi N}{\omega_0}, \\ \tilde{\ell}\left(t + \dfrac{2\pi N}{\omega_0}\right), & t < -\dfrac{2\pi N}{\omega_0}, \\ \tilde{\ell}\left(t - \dfrac{2\pi N}{\omega_0}\right), & t > \dfrac{2\pi N}{\omega_0}, \end{cases} \quad (A.1)$$

where $\tilde{\ell}$ is chosen such that $\ell_N \in \mathfrak{D}(\mathbb{R},\mathbb{C}^d)$. Note that $\tilde{\ell}$ is and can be chosen to be independent of N. Next evaluate $\int_{-\infty}^{+\infty} Q_{\Phi'}(\ell_N)\,dt$ and observe that this integral can be made negative by taking N sufficiently large. Finally, we prove (2) $\Rightarrow$ (4). (2) holds iff there exists $D_1 \in \mathbb{R}^{d\times d}[\xi]$ such that $\Phi'(-\xi,\xi) = D_1^{\mathrm{T}}(-\xi)D_1(\xi)$. Define $\Delta'(\zeta,\eta) := D_1^{\mathrm{T}}(\zeta)D_1(\eta)$ and $\Delta(\zeta,\eta) := M^{\dagger T}(\zeta)\Delta'(\zeta,\eta)M^\dagger(\eta)$. Then we have $\partial(\Phi' - \Delta') = 0$, and hence $\int_{-\infty}^{\infty} Q_{\Delta'-\Phi'}(\ell)\,dt = 0$. This implies $\int_{-\infty}^{\infty} Q_{\Delta-\Phi}(w)\,dt = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q)$. Since also $Q_\Delta(w) \geqslant 0$ for all $w \in \mathfrak{B}$, Q_Δ is a dissipation function. Note that in this proof, for any dissipation function Q_Δ we constructed a storage function Q_Ψ, and for any storage function Q_Ψ we constructed a dissipation function Q_Δ such that $(Q_\Phi, Q_\Psi, Q_\Delta)$ is matched. □

Proof of Lemma 5.1. Consider the state representation, Eq. (5.1), of $\mathfrak{B}$. Consider the matrix pencil $\xi E + F$. It was shown in Ref. [5] that there exist nonsingular matrices S and T such that

$$S(\xi E + F)T = \begin{pmatrix} \xi E_{11} + F_{11} & \xi E_{12} + F_{12} \\ 0 & \xi E_{22} + F_{22} \end{pmatrix}$$

with $\xi E_{22} + F_{22}$ a full column polynomial matrix, and E_{11} full row rank. Let $\bar{x} := T^{-1}x$ and partition $\bar{x} = \mathrm{col}(x_1, x_2)$. Partition $SG = \mathrm{col}(G_1, G_2)$. Clearly, $SGw + S(F + E\,\mathrm{d}/\mathrm{d}t)T\bar{x} = 0$ is a state representation of $\mathfrak{B}$, which, written out in components, becomes

$$G_1 w + F_{11}x_1 + F_{12}x_2 + E_{11}\frac{\mathrm{d}x_1}{\mathrm{d}t} + E_{12}\frac{\mathrm{d}x_2}{\mathrm{d}t} = 0, \quad \text{(A.2)}$$

$$G_2 w + F_{22}x_2 + E_{22}\frac{\mathrm{d}x_2}{\mathrm{d}t} = 0. \quad \text{(A.3)}$$

Using the fact that E_{11} has full row rank, it is easily seen that the state model, Eq. (A.3) is already a state representation of $\mathfrak{B}$. Denote its full behavior by $\mathfrak{B}'_{\mathrm{f}}$. Since $\xi E_{22} + F_{22}$ has full column rank, this state representation is observable. Now define $L := (0 \;\; I)T^{-1}$. Now assume that $(w,x) \in \mathfrak{B}_{\mathrm{f}}$ and $(w,x_2) \in \mathfrak{B}'_{\mathrm{f}}$. Then $(w, Lx) \in \mathfrak{B}'_{\mathrm{f}}$, and by observability we must have $x_2 = Lx$. □

Proof of Theorem 5.2. Denote $\mathrm{col}(u,y)$ by w, and $\mathrm{col}(U,Y)$ by M. Since the state representation is observable, there exists $X \in \mathbb{R}^{n\times q}[\xi]$ such that $x = X(\mathrm{d}/\mathrm{d}t)w$. Thus, X defines a state map so the rows of XM span the linear space $\mathfrak{F}$.

($\Leftarrow$) Assume that FU^{-1} is strictly proper. Let f_i be the ith row of F. Then $f_i \in \mathfrak{F}$, so $f_i = h_i XM$ for some constant row vector h_i. Define H to be the constant matrix whose ith row is equal to h_i. Then we have $F = HXM$. Thus, if $f = F(\mathrm{d}/\mathrm{d}t)\ell$, $u = U(\mathrm{d}/\mathrm{d}t)\ell$, $y = Y(\mathrm{d}/\mathrm{d}t)\ell$, and $x = X(\mathrm{d}/\mathrm{d}t)w$, then we have $f = Hx$.

($\Rightarrow$) If $f = F(\mathrm{d}/\mathrm{d}t)\ell$, $u = U(\mathrm{d}/\mathrm{d}t)\ell$, $y = Y(\mathrm{d}/\mathrm{d}t)\ell$, and $x = X(\mathrm{d}/\mathrm{d}t)w$, then we have $f = Hx$. Thus, for all ℓ we have $F(\mathrm{d}/\mathrm{d}t)\ell = HX(\mathrm{d}/\mathrm{d}t)M(\mathrm{d}/\mathrm{d}t)\ell$ so $F = HXM$. This implies that every row of F is in $\mathfrak{F}$, so FU^{-1} is strictly proper.

We now prove the second part of the theorem.

($\Leftarrow$) Assume that FU^{-1} is proper. Define $J := \lim |\lambda| \to \infty F(\lambda)U^{-1}(\lambda)$. Then $(F - JU)U^{-1}$ is strictly proper, so the rows of $F - JU$ are elements of $\mathfrak{F}$. This implies that there exists a constant matrix H such that $F - JU = HXM$. Thus, if $f = F(\mathrm{d}/\mathrm{d}t)\ell$, $u = U(\mathrm{d}/\mathrm{d}t)\ell$, $y = Y(\mathrm{d}/\mathrm{d}t)\ell$, and $x = X(\mathrm{d}/\mathrm{d}t)w$, then we have $f = Hx + Ju$.

($\Rightarrow$) If $f = F(\mathrm{d}/\mathrm{d}t)\ell$, $u = D(\mathrm{d}/\mathrm{d}t)\ell$, $y = Y(\mathrm{d}/\mathrm{d}t)\ell$, and $x = X(\mathrm{d}/\mathrm{d}t)w$, then we have $f = Hx + Ju$. Thus, for all ℓ we have $F(\mathrm{d}/\mathrm{d}t)\ell = HX(\mathrm{d}/\mathrm{d}t)M(\mathrm{d}/\mathrm{d}t)\ell + JU(\mathrm{d}/\mathrm{d}t)\ell$ so $F = HXM + JU$. This implies that every row of $F - JU$ is in $\mathfrak{F}$, so $(F - JU)U^{-1}$ is strictly proper. Hence FU^{-1} is proper. □

Proof of Theorem 6.1. Assume that the statement about storage functions has been proven for observable state representations. Assume now we have an arbitrary one. According to Lemma 5.2, there exists an observable one with state variable, say x', and a constant matrix L such that $(v,x) \in \mathfrak{B}_{\mathrm{f}}$ and $(v,x') \in \mathfrak{B}'_{\mathrm{f}}$ implies $x' = Lx$. Now, there exists $K = K^{\mathrm{T}}$ such that $(M(\mathrm{d}/\mathrm{d}t)w, x') \in \mathfrak{B}'_{\mathrm{f}}$ implies $Q_\Psi(w) = x'^{\mathrm{T}}Kx'$. Assume now $(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$. Let $(M(\mathrm{d}/\mathrm{d}t)w, x') \in \mathfrak{B}'_{\mathrm{f}}$. We have $x' = Lx$. Hence $Q_\Psi(w) = x^{\mathrm{T}}L^{\mathrm{T}}KLx$. Thus, in the rest of this proof we will assume that we have an observable state representation. The proof is split up into two parts. First we give a proof for the lossless case, and next for the general case.

The lossless case. First assume $(\mathrm{d}/\mathrm{d}t)Q_\Psi = Q_\Phi$, equivalently $\dot{\Psi} = \Phi$.

1. *M observable.* Assume that in the canonical factorization $\Phi(\zeta,\eta) = M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$, $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. This means that $v = M(\mathrm{d}/\mathrm{d}t)w$ is an observable image representation of $\mathfrak{B}_\Phi$. After permuting the components of v, if need be, $M = \mathrm{col}(U,Y)$, with $\det(U) \neq 0$ and YU^{-1} a matrix of proper rational functions. Accordingly, write $v = \mathrm{col}(u,y)$. Let $\Psi(\zeta,\eta) = F^{\mathrm{T}}(\zeta)\Sigma_\Psi F(\eta)$ be a canonical factorization of Ψ. We have

$$\begin{aligned}(\zeta + \eta)&F^{\mathrm{T}}(\zeta)\Sigma_\Psi F(\eta)U^{-1}(\eta) \\ &= M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)U^{-1}(\eta). \quad \text{(A.4)}\end{aligned}$$

Interpreted as a matrix of rational functions in the indeterminate η, the right-hand side of Eq. (A.4) is proper. Now, we claim that FU^{-1} is a matrix of strictly proper rational functions. Suppose it is not. Let $F_k\eta^k$ be the term of degree k in the polynomial part of FU^{-1}. By equating powers of η in Eq. (A.4), we obtain $F^{\mathrm{T}}(\zeta)\Sigma_\Psi F_k = 0$. Since the columns of the polynomial matrix F^{T} are linearly independent over $\mathbb{R}$, this implies that $\Sigma_\Psi F_k = 0$, so $F_k = 0$. This proves the claim. According to Theorem 5.1, there exists a constant matrix H such that if $\mathrm{col}(v,x) \in \mathfrak{B}_{\mathrm{f}}$, $v = M(\mathrm{d}/\mathrm{d}t)w$, and $f = F(\mathrm{d}/\mathrm{d}t)w$, then $f = Hx$. This implies that $Q_\Psi(w) = \|F(\mathrm{d}/\mathrm{d}t)w\|^2_{\Sigma_\Psi} = x^{\mathrm{T}}Kx$ with $K := H^{\mathrm{T}}\Sigma_\Psi H$.

2. *General M*. In general, the representation $v=M(\mathrm{d}/\mathrm{d}t)w$ need not be observable. There exist however polynomial matrices M_1 and N, with $M_1(\lambda)$ full column rank for all $\lambda\in\mathbb{C}$, and with N full row rank, such that $M=M_1N$. This amounts to representing $\mathfrak{B}_\Phi$ as $v=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$, $\ell_1=N(\mathrm{d}/\mathrm{d}t)w$. In fact, $v=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$ is already an (observable) image representation of $\mathfrak{B}_\Phi$. Define $\Phi_1(\zeta,\eta):=M_1^{\mathrm{T}}(\zeta)\Sigma_\Phi M_1(\eta)$. Then Φ_1 is observable and we have $\Phi(\zeta,\eta)=N^{\mathrm{T}}(\zeta)\Phi_1(\zeta,\eta)N(\eta)$. Clearly, Φ_1 contains a factor $\zeta+\eta$. Indeed, $0=\Phi(-\xi,\xi)=N^{\mathrm{T}}(-\xi)\Phi_1(-\xi,\xi)N(\xi)$, so $\Phi_1(-\xi,\xi)=0$. Define $\Psi_1(\zeta,\eta)=(1/(\zeta+\eta))\Phi_1(\zeta,\eta)$. Then we have $\dot{\Psi}_1=\Phi_1$ and $\Psi(\zeta,\eta)=N^{\mathrm{T}}(\zeta)\Psi_1(\zeta,\eta)N(\eta)$. According to part 1 of this proof, there exists a matrix $K=K^{\mathrm{T}}$ such that $(M_1(\mathrm{d}/\mathrm{d}t)\ell_1,x)\in\mathfrak{B}_{\mathrm{f}}$ implies $Q_{\Psi_1}(\ell_1)=x^{\mathrm{T}}Kx$. Now let $(M(\mathrm{d}/\mathrm{d}t)w,x)\in\mathfrak{B}_{\mathrm{f}}$. Define $\ell_1:=N(\mathrm{d}/\mathrm{d}t)w$. Then $M(\mathrm{d}/\mathrm{d}t)w=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$. We conclude that $Q_\Psi(w)=Q_{\Psi_1}(\ell_1)=x^{\mathrm{T}}Kx$.

The general case. We now treat the general, possibly non-lossless, case.

1. *M full column rank.* We first assume that in the canonical factorization $\Phi(\zeta,\eta)=M^{\mathrm{T}}(\zeta)\Sigma_\Phi M(\eta)$, M has full column rank. After permuting the components of v, if need be, $M=\mathrm{col}(U,Y)$, with $\det(U)\neq0$ and YU^{-1} a matrix of proper rational functions. Accordingly, write $v=\mathrm{col}(u,y)$. The dissipation inequality says that for all $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ we have $(\mathrm{d}/\mathrm{d}t)Q_\Psi(w)\leqslant Q_\Phi(w)$, equivalently $\Phi-\dot{\Psi}\geqslant0$. Thus there exists $D\in\mathbb{R}^{\bullet\times q}[\xi]$ such that $(\zeta+\eta)\Psi(\zeta,\eta)=\Phi(\zeta,\eta)-D^{\mathrm{T}}(\zeta)D(\eta)$. This can be restated as

$$(\zeta+\eta)\Psi(\zeta,\eta)=\begin{pmatrix}M(\zeta)\\D(\eta)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\Sigma_\Phi&0\\0&-I\end{pmatrix}\begin{pmatrix}M(\eta)\\D(\eta)\end{pmatrix}.$$

Introduce the new system $\mathfrak{B}_{\mathrm{ext}}$ with image representation $v=M(\mathrm{d}/\mathrm{d}t)w$, $d=D(\mathrm{d}/\mathrm{d}t)w$. Since $M^{\mathrm{T}}(-\xi)\Sigma_\Phi M(\xi)=D^{\mathrm{T}}(-\xi)D(\xi)$, DU^{-1} is a matrix of proper rational functions. Thus, there exist constant matrices H and J such that $d=Hx+Ju=Hx+J_1v$ (take $J_1:=(J\ \ 0)$). It is then easily seen that

$$\begin{pmatrix}G&0\\-J_1&I\end{pmatrix}\begin{pmatrix}v\\d\end{pmatrix}+\begin{pmatrix}F\\-H\end{pmatrix}x+\begin{pmatrix}E\\0\end{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}t}=0$$

is a state representation of $\mathfrak{B}_{\mathrm{ext}}$ with full behavior, say, $\mathfrak{B}_{\mathrm{ext,f}}$. We are now back in the lossless case. There exists $K=K^{\mathrm{T}}$ such that $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w, D(\mathrm{d}/\mathrm{d}t)w,x)\in\mathfrak{B}_{\mathrm{ext,f}}$ implies $Q_\Psi(w)=x^{\mathrm{T}}Kx$. Hence, $\mathrm{col}(M(\mathrm{d}/\mathrm{d}t)w,x)\in\mathfrak{B}_{\mathrm{f}}$ implies $Q_\Psi(w)=x^{\mathrm{T}}Kx$.

2. *General M*. Finally, we treat the general case. In general, M need not have full column rank, but there exist a unimodular V and a full column rank M_1 such that $M=(M_1\ \ 0)V$. This amounts to representing $\mathfrak{B}_\Phi$ as $v=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$, $\ell_1=(I\ \ 0)V(\mathrm{d}/\mathrm{d}t)w$. In fact, $v=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$ is already an image representation of $\mathfrak{B}_\Phi$. Let $\dot{\Psi}\leqslant\Phi$. Let D be such that $(\zeta+\eta)\Psi(\zeta,\eta)=\Phi(\zeta,\eta)-D^{\mathrm{T}}(\zeta)D(\eta)$. Partition $DV^{-1}=(D_1\ \ D_2)$. It is easily verified that $D_2^{\mathrm{T}}(-\xi)D_2(\xi)=0$ so $D_2=0$. Now define $\Phi_1(\zeta,\eta):=m_1^{\mathrm{T}}(\zeta)\Sigma_\Phi M_1(\eta)$. Then

$$(\zeta+\eta)V^{-T}(\zeta)\Psi(\zeta,\eta)V^{-1}(\eta)=\begin{pmatrix}\Phi_1(\zeta,\eta)-D_1^{\mathrm{T}}(\zeta)D_1(\eta)&0\\0&0\end{pmatrix}$$

Consequently,

$$V^{-T}(\zeta)\Psi(\zeta,\eta)V^{-1}(\eta)=\begin{pmatrix}\Psi_1(\zeta,\eta)&0\\0&0\end{pmatrix}$$

for some Ψ_1, with $\dot{\Phi}_1\leqslant\Psi_1$. Since $\det(\Phi_1)\neq0$, we are back in the situation of part 2 above. Hence there exists $K=K^{\mathrm{T}}$ such that $(M_1(\mathrm{d}/\mathrm{d}t)\ell_1,x)\in\mathfrak{B}_{\mathrm{f}}$ implies $Q_{\Psi_1}(\ell_1)=x^{\mathrm{T}}Kx$. Now let $(M(\mathrm{d}/\mathrm{d}t)w,x)\in\mathfrak{B}_{\mathrm{f}}$. Define $\ell_1:=(I\ \ 0)V(\mathrm{d}/\mathrm{d}t)w$. Then $M(\mathrm{d}/\mathrm{d}t)w=M_1(\mathrm{d}/\mathrm{d}t)\ell_1$. Hence $Q_\Psi(w)=Q_{\Psi_1}(\ell_1)=x^{\mathrm{T}}Kx$.

The proof of the statement about dissipation functions is much easier. Again, we may as well assume that we have an observable state representation. We will only do the case that M is observable. Let Q_Δ be a dissipation function. There exists D such that $\Delta(\zeta,\eta)=D^{\mathrm{T}}(\zeta)D(\eta)$. Since $\partial(\Phi-\Delta)=0$ we have $M^{\mathrm{T}}(-\xi)\Sigma_\Phi M(\xi)=D^{\mathrm{T}}(-\xi)D(\xi)$. Since MU^{-1} is a matrix of proper rational functions, the same holds for DU^{-1}. Consequently, if $d=D(\mathrm{d}/\mathrm{d}t)w$, we have $d=Hx+Ju=Hx+J_1v$ (take $J_1:=(J\ \ 0)$). This yields $Q_\Delta(w)=\|D(\mathrm{d}/\mathrm{d}t)w\|^2=\|Hx+J_1v\|^2$. The case that M is not observable is left to the reader. □

Proof of Theorem 6.2. Let $w=W(\mathrm{d}/\mathrm{d}t)\ell$ be any image representation of $\mathfrak{B}$, with $\ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^d)$. Define $\Phi'(\zeta,\eta):=W^{\mathrm{T}}(\zeta)\Phi(\zeta,\eta)W(\eta)$ and $\Psi'(\zeta,\eta):=W^{\mathrm{T}}(\zeta)\Psi(\zeta,\eta)W(\eta)$. Clearly we are then back in the situation of Theorem 6.1: $(\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^d),Q_{\Phi'})$ is dissipative and $Q_{\Psi'}$ is a storage function. According to Theorem 6.1, $Q_{\Psi'}$ is a quadratic function of any state variable of the system $\mathfrak{B}_{\Phi'}$ obtained from a canonical factorization of Φ':

$$\Phi'(\zeta,\eta)=M'(\zeta)\Sigma_{\Phi'}M'(\eta). \tag{A.5}$$

The system $\mathfrak{B}_{\Phi'}$ is represented by $v'=M'(\mathrm{d}/\mathrm{d}t)\ell$. The idea of the proof is now, that the state of $\mathfrak{B}_\Phi$ (given by

Eq. (6.2)) is also a state of $\mathfrak{B}_{\Phi'}$. We first investigate the relation between $\mathfrak{B}_{\Phi'}$ and $\mathfrak{B}_{\Phi}$. Clearly, the canonical factorization $\Phi(\zeta,\eta) = M^{\mathrm{T}}(\zeta)\Sigma_{\Phi}M(\eta)$ yields a (in general non-canonical) factorization

$$\Phi'(\zeta,\eta) = W^{\mathrm{T}}(\zeta)M^{\mathrm{T}}(\zeta)\Sigma_{\Phi}M(\eta)W(\eta) \tag{A.6}$$

of Φ'. Combining Eqs. (A.5) and (A.6), there exists a real contant matrix, say H, such that $M' = HMW$ (see Section (3)). In terms of the behaviors $\mathfrak{B}_{\Phi'}$ and $\mathfrak{B}_{\Phi}$, this says that $\mathfrak{B}_{\Phi'} = H\mathfrak{B}_{\Phi}$. Consider now the equations $Gv + Fx + E\,\mathrm{d}x/\mathrm{d}t = 0$, $v' = Hv$, equivalently

$$\begin{pmatrix} E\,\mathrm{d}/\mathrm{d}t + F & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} x \\ v' \end{pmatrix} = \begin{pmatrix} -G \\ H \end{pmatrix} v \tag{A.7}$$

We interpret this as a system with manifest variable $\mathrm{col}(x, v')$ and latent variable v. It is easily seen that after eliminating the variable v, the manifest behavior of Eq. (A.7) is represented by a first-order model of the form

$$G'v' + F'x + E'\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{A.8}$$

in the sense that (v', x) satisfies Eq. (A.7) for some v, iff it satisfies Eq. (A.8). This shows that Eq. (A.8) is a state representation of $\mathfrak{B}_{\Phi'}$. Denote the full behavior of Eq. (A.8) by $\mathfrak{B}'_{\mathrm{f}}$. Now apply Theorem 6.1: there exists $K = K^{\mathrm{T}}$ such that $(M'(\mathrm{d}/\mathrm{d}t)\ell, x) \in \mathfrak{B}'_{\mathrm{f}}$ implies $Q_{\Psi'}(\ell) = x^{\mathrm{T}}Kx$. Now let $w \in \mathfrak{B}$ and $(M(\mathrm{d}/\mathrm{d}t)w, x) \in \mathfrak{B}_{\mathrm{f}}$. There exists ℓ such that $M(\mathrm{d}/\mathrm{d}t)w = M(\mathrm{d}/\mathrm{d}t)W(\mathrm{d}/\mathrm{d}t)\ell$. Hence,

$$M'(\mathrm{d}/\mathrm{d}t)\ell = HM(\mathrm{d}/\mathrm{d}t)W(\mathrm{d}/\mathrm{d}t)\ell,$$

so $(M'(\mathrm{d}/\mathrm{d}t)\ell, x) \in \mathfrak{B}'_{\mathrm{f}}$.

This implies $Q_{\Psi}(w) = Q_{\Psi'}(\ell) = x^{\mathrm{T}}Kx$. The claim on dissipation functions is proven along the same lines.

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