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BEHAVIORAL CONTROLLABILITY OF DELAY-DIFFERENTIAL SYSTEMS*

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Abstract. In this paper we will prove that the system described by the delay-differential equation $R(d/dt, \Delta)w = 0$ (with Δ the unit delay operator) is controllable if and only if the rank of $R(\lambda, e^{-\lambda})$ is constant for all $\lambda \in \mathbb{C}$. This condition is compared with the existing results obtained both by the analytic approach and by the algebraic approach to delay-differential systems.

Key words. behavioral systems, controllability, delay-differential systems, polynomial matrices

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1. Introduction.

The aim of this paper is to analyze controllability for delay-differential (d-d) systems. We will derive a concrete necessary and sufficient condition for controllability of d-d systems in kernel representation. We will use the behavioral approach to dynamical systems [12]. Thus a continuous-time dynamical system is a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B})$ with *behavior* $\mathcal{B}$ being a set of trajectories $w : \mathbb{R} \to \mathbb{R}^q$. We will assume that $\mathcal{B}$ is shift invariant, i.e., that $(w(.) \in \mathcal{B}) \Rightarrow (w(t + .) \in \mathcal{B} \ \ \forall t \in \mathbb{R})$. Since the behavior is the most intrinsic feature of a system, it is logical to define the system properties in terms of the set $\mathcal{B}$, i.e., at an external level. This applies in particular for the notion of controllability.

DEFINITION 1.1. *The system Σ is said to be* controllable *if for all $w_1, w_2 \in \mathcal{B}$ there exist a $w \in \mathcal{B}$ and a $T \geq 0$ such that*

$$w(t) = \begin{cases} w_1(t) & for \quad t < 0, \\ w_2(t-T) & for \quad t \geq T. \end{cases}$$

Note that for shift-invariant behaviors the controllability condition of Definition 1.1 is equivalent to the following property. For all $w_1, w_2 \in \mathcal{B}$, w_1 is *$\mathcal{B}$-compatible* with w_2, i.e., for all $t_1 \in \mathbb{R}$ there exist $t_2 \geq t_1$ and $w \in \mathcal{B}$ such that $w^* = w_1 \wedge_{t_1} w \wedge_{t_2} w_2 \in \mathcal{B}$. Here $w^* = w_1 \wedge_{t_1} w \wedge_{t_2} w_2$ stands for the successive concatenation of w_1, w, and w_2, respectively, at times t_1 and t_2 and is defined as follows: $w^*(t) = w_1(t)$ for $t < t_1$, $w^*(t) = w(t)$ for $t_1 \leq t < t_2$, and $w^*(t) = w_2(t)$ for $t \geq t_2$. In other words, controllability requires that every past trajectory can be transferred to any future trajectory. In order to distinguish this property from the classical state controllability and to emphasize the fact that it concerns the system behavior we will refer to it as behavioral controllability.

Behavioral controllability has been widely studied for both continuous- and discrete-time systems, respectively, described by differential and difference equations, see [12, 8]. In this paper we consider continuous-time systems described by differential

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equations with delays, i.e., d-d systems. More concretely, we will be concerned with systems whose behavior $\mathcal{B}$ can be described as the kernel of a d-d operator $R(d/dt, \Delta)$ (where $R(z_1, z_2)$ is a two-dimensional (2D) polynomial matrix in z_1 and z_2 and Δ is the delay). This is a very general description which can comprise both the polynomial input-output equations and the pseudostate representations considered in the literature [11], [3]. We will show that $\mathcal{B} = \ker R(d/dt, \Delta)$ is controllable if and only if (iff) $R(\lambda, e^{-\lambda})$ has constant rank for all $\lambda \in \mathbb{C}$. It turns out that this condition reduces to spectral controllability if one considers pseudostate representations as in [3], [6], and [9].

This characterization of behavioral controllability has also been independently obtained in [1], where the author develops an elegant theory for d-d systems in a behavioral framework based on the properties of a suitable ring of entire functions. Here we follow a different approach based on the analysis of the exponential-polynomial trajectories in the system.

2. Delay-differential systems. Let $R(z_1, z_2)$ be a 2D polynomial matrix having g rows and q columns. Now consider the equation

$$R\left(\frac{d}{dt}, \Delta\right) w = 0, \tag{1}$$

where Δ denotes the unit delay operator: $(\Delta f)(t) := f(t-1)$. Equation (1) defines the dynamical system $(\mathbb{R}, \mathbb{R}^q, \mathcal{B})$ with $\mathcal{B} = \ker(R(d/dt, \Delta))$ and $R(d/dt, \Delta)$ viewed as a map from $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ into $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^g)$. In other words, the behavior consists of the $\mathcal{C}^\infty$-solutions of (1). We will call (1) a d-d system (even though it would be more appropriate to refer to it as a d-d system in kernel representation).

Note that this kernel representation is more general than the polynomial input-output descriptions considered in [11], as well as than the pseudostate descriptions of [3]. Indeed, any polynomial input-output d-d equation $Py = Qu$ can be regarded as a kernel representation with $R(d/dt, \Delta) = [P(d/dt, \Delta) \mid -Q(d/dt, \Delta)]$ and with $w = col(y, u)$. In turn, the pseudostate description $\{dx/dt = A(\Delta)x + Bu \;\; y = Cx\}$ can also be viewed as a kernel representation with $w = col(x, y, u)$ and $R(d/dt, \Delta) = col([d/dt - A(\Delta) \mid 0 \mid -B], [-C \mid I \mid 0])$. Observe, however, that (1) is a broader class of systems than those mentioned. For example, both the systems defined by

$$w_1 = \frac{d}{dt} \Delta w_2$$

and by

$$\Delta w_1 = (1 + \Delta^2) w_2$$

fit (1) but not the classical input/state/output frameworks.

3. Behavioral controllability of d-d systems. Our problem is to find conditions on the 2D polynomial matrix $R(z_1, z_2)$ such that (1) defines a system which is controllable in the sense of Definition 1.1. The following is the main result of this paper.

THEOREM 3.1. (1) *defines a controllable d-d system iff the rank of the complex matrix* $R(\lambda, e^{-\lambda})$ *is constant for* $\lambda \in \mathbb{C}$.

The above theorem is a natural generalization of the well-known identical result for differential systems $R(d/dt)w = 0$. However, the proof will show that from a mathematical point of view Theorem 3.1 is a much deeper result.

As an alternative to systems (1), consider the following d-d systems. Let $M(z_1, z_2)$ be a 2D polynomial matrix with q rows and l columns. Consider the equation

$$w = M\left(\frac{d}{dt}, \Delta\right) a, \tag{2}$$

where $a \in C^\infty(\mathbb{R}, \mathbb{R}^l)$ corresponds to an auxiliary variable. Equation (2) defines a dynamical system $(\mathbb{R}, \mathbb{R}^q, \mathrm{im}(M(d/dt, \Delta)))$ with $M(d/dt, \Delta)$ viewed as an operator from $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ into $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. We will call (2) a d-d system in image representation. It is easy to prove that (2) defines a dynamical system which is automatically controllable. For differential systems, a system is controllable if and *only if* it admits an image representation. This is, in fact, also the case for d-d systems (1).

THEOREM 3.2. *A d-d system* (1) *is a controllable system iff there exists a* 2D *polynomial matrix* $M(z_1, z_2)$ *such that*

$$\ker R\left(\frac{d}{dt}, \Delta\right) = \mathrm{im} M\left(\frac{d}{dt}, \Delta\right). \tag{3}$$

In order to give a further insight, it is useful to compare our result with the existing results on state controllability for d-d systems. We will first focus on the class of retarded d-d systems Σ considered in [3] which have a pseudostate description of the form

$$\begin{cases} dx/dt &= A(\Delta)x + Bu, \\ y &= Cx, \end{cases}$$

where x is the (n-dimensional) pseudostate, u is the input, y is the output, and $A(z) = A_N z^N + \cdots + A_1 z + A_0$ is a polynomial matrix in z. For the system Σ, the state at time t is defined in [3] as being $z(t) = col(x(t), x_t)$, where $x_t \in L_2[(-N, 0], \mathbb{R}^n]$ is given by $x_t(\tau) = x(t + \tau)$ for all $\tau \in (-N, 0]$. This yields the infinite-dimensional state space $Z = \mathbb{R}^n \times L_2[(-N, 0), \mathbb{R}^n]$. Define, in this state space, the set K_t of all attainable states in time t, and let $K_\infty := \cup_{t>0} K_t$. Then Σ is said to be approximately controllable if K_∞ is dense in Z. The next theorem, providing a characterization of approximate controllability, has been derived in [3].

THEOREM 3.3. Σ *is approximately controllable iff* (1) $\mathrm{rank}[(\lambda I - A(e^{-\lambda}) \mid B] = n \ \ \forall \lambda \in \mathbb{C}$ *and* (2) $\mathrm{rank}[A_N \mid B] = n$.

The first condition of the theorem is known as *spectral controllability*.

Note that the pseudostate description that we have considered here can be regarded as a kernel representation with $R(d/dt, \Delta) = col([d/dt - A(\Delta) \mid 0 \mid -B], [-C \mid I \mid 0])$ if Σ is viewed as a system with external variable vector $w = col(x, y, u)$ and with smooth signals. It turns out from Theorem 3.1 that the behavior of Σ is controllable iff $\mathrm{rank}[(\lambda I - A(e^{-\lambda}) \mid B] = n$ for all $\lambda \in \mathbb{C}$. So behavioral controllability seems to correspond to spectral rather than to approximate controllability. The situation can be illustrated by the following example.

EXAMPLE 3.4. Let $A(z) = A_0 + A_1 z$ with $A_0 = col([0 \mid 1], [0 \mid 0])$, $A_1 = col([0 \mid 0], [-1 \mid 0])$, and $B = col(0, -1)$. Then the corresponding system Σ is not approximately controllable since $\mathrm{rank}[A_1 \mid B] = 1 < 2$. However, it is easy to check that $[\lambda I - A(e^{-\lambda}) \mid B]$ has rank 2 $\forall \lambda \in \mathbb{C}$ and hence the behavior of Σ is controllable. What happens in this case is that the pseudostate components x_1 and x_2 are related by $dx_1/dt = x_2$. This holds in particular in the interval $[-1, 0)$; therefore, not all the elements in the state space $\mathbb{R}^2 \times L_2([-1,0),\mathbb{R}^2)$ are feasible, which prevents approximate controllability. This obstacle does not arise for behavioral controllability since this property exclusively regards admissible system signals (and hence one does not take into account the signals which do not satisfy $dx_1/dt = x_2$).

The characterization of approximate controllability has been extended to neutral d-d systems in [6] and [9] and later generalized in [13] to the case of (possibly) noncommensurable delays. For systems with a pseudostate description of the form

$$\left\{ \begin{array}{lcl} dx/dt & = & A(\Delta_1,\dots,\Delta_N,D)x + Bu, \\ y & = & Cx \end{array} \right. \tag{4}$$

(where $A(z_1,\dots,z_N,z_{N+1}) = A_0 + \sum_{i=1}^N A_i(z_{N+1})z_i$, $A_i(z_{N+1})z_i = E_i + F_i z_{N+1}$, and Δ_i represents the delay by h_i units of time, $i = 1,\dots,N$), the following result has been derived (and formulated in slightly different terms).

THEOREM 3.5 (see [13]). *The system described by* (4) *is approximately controllable iff* (a) $\operatorname{rank}[A(e^{h_1\lambda},\dots,e^{h_N\lambda},\lambda),B] = n \;\; \forall\lambda \in \mathbb{C}$ *and* (b) $\operatorname{rank}[A_N(\lambda),B] = n$ *for some* $\lambda \in \mathbb{C}$.

As before, the first condition corresponds to spectral controllability and coincides with our characterization of behavioral controllability if the delays are commensurable.

Another interesting issue is the comparison of our notion of controllability with the ones which have been studied in [5] and [2] within an algebraic approach. Here the authors consider systems Σ with pseudostate-space representations of the form

$$\left\{ \begin{array}{lcl} dx/dt & = & A(\Delta)x + B(\Delta)u, \\ y & = & C(\Delta)x + D(\Delta)u, \end{array} \right. \tag{5}$$

where $A(z_2), B(z_2), C(z_2), D(z_2)$ are polynomial matrices in z_2. For such systems the following two notions of controllability are introduced. Let $\mathcal{R}(z_2) := [B(z_2) \mid A(z_2)B(z_2) \mid \dots \mid (A(z_2))^{n-1}B(z_2)]$, where n is the size of $A(z_2)$. Σ is said to be *weakly controllable* if $\mathcal{R}(z_2)$ has full row rank over the field of fractions $\mathbb{R}(z_2)$. If $\mathcal{R}(\lambda_2)$ has full row rank $\forall\lambda_2 \in \mathbb{C}$, Σ is said to be *strictly controllable.* Theorem 3.6 is shown in [2].

THEOREM 3.6. *With the previous notation,* (1) Σ *is weakly controllable iff* $[z_1 - A(z_2) \mid B(z_2)]$ *is left prime, and* (2) Σ *is strictly controllable iff* $\operatorname{rank}[\lambda_1 - A(\lambda_2) \mid B(\lambda_2)] = n \;\forall(\lambda_1,\lambda_2) \in \mathbb{C}\times\mathbb{C}$.

Regarding the pseudostate representation (5) as a kernel representation, it follows from Theorem 3.1 that the behavior of Σ is controllable iff $\operatorname{rank}[\lambda - A(e^{-\lambda}) \mid B(e^{-\lambda})] = n \;\forall\lambda \in \mathbb{C}$. Thus strict controllability implies behavioral controllability. On the other hand, if $[z_1 - A(z_2) \mid B(z_2)]$ has a left factor $\Phi(z_1,z_2)$ with nontrivial determinant $f(z_1,z_2)$, then $\Phi(\lambda,e^{-\lambda})$ will be a left factor of $[\lambda - A(e^{-\lambda}) \mid B(e^{-\lambda})]$, implying that this matrix drops in rank when λ is a zero of $f(\lambda,e^{-\lambda})$. Therefore we can conclude that behavioral controllability implies weak controllability.

Summarizing the preceding considerations, we have that strict controllability implies behavioral controllability, which in its turn implies weak controllability. The next examples show that the converse implications do not hold true.

EXAMPLE 3.7. Consider the delay-differential system Σ described by

$$\left\{ \begin{array}{lcl} dx/dt & = & (-\Delta+1)x + (2-\Delta)u, \\ y & = & x. \end{array} \right.$$

Letting $w := col(u,y,x)$ and $R(z_1,z_2) := col([z_2 - 2 \mid 0 \mid z_1 + (z_2 - 1)], [0 \mid 1 \mid -1])$ this description becomes $R(d/dt,\Delta)w = 0$. Since $R(\lambda,e^{-\lambda}) = col([e^{-\lambda} - 2 \mid 0 \mid$

$\lambda + (e^{-\lambda} - 1)], [0 \mid 1 \mid -1])$ has rank 2 $\forall \lambda \in \mathbb{C}$, the behavior of Σ is controllable. However, $[\lambda_1 - (1 - \lambda_2) \mid 2 - \lambda_2]$ clearly drops in rank for $(\lambda_1, \lambda_2) = (-1, 2)$, and hence Σ is not strictly controllable.

EXAMPLE 3.8. Let Σ be described by the following equations:

$$\begin{cases} dx/dt &= (-\Delta + 1)u, \\ y &= x. \end{cases}$$

Proceeding as in the previous example, we have that $R(\lambda, e^{-\lambda}) = col([1 - e^{-\lambda} \mid 0 \mid \lambda], [0 \mid 1 \mid -1])$, which drops in rank for $\lambda = 0$. So the behavior of Σ is not controllable. However $[z_1 \mid z_2 - 1]$ is left prime and hence Σ is weakly controllable.

EXAMPLE 3.9. Consider the system described in image representation by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 - \Delta \\ \frac{d}{dt} \end{bmatrix} a. \tag{6}$$

This is a system with transfer function $w_2 \to w_1$:

$$\frac{\hat{w}_1}{\hat{w}_2} = \frac{1 - e^{-s}}{s}. \tag{7}$$

Obviously, since it is an image representation, it defines a controllable system. The logical candidate for the kernel representation is

$$\frac{d}{dt} w_1 = (1 - \Delta) w_2. \tag{8}$$

However, (8) is not controllable and hence not a faithful representation of (6). This shows that the d-d system (6) *cannot*, in fact, be represented as a kernel representation (1). In particular, this implies that what we call the latent variable elimination theorem [12] does not hold for d-d systems!

4. Proofs. We will show Theorems 3.1 and 3.2 in three main steps, respectively, corresponding to Propositions 4.1, 4.5, and 4.6 below. In the first step we prove that the rank constancy of $R(\lambda, e^{-\lambda})$ implies that (1) has an image representation. In the second step we prove that the existence of an image representation implies controllability. Finally, in the third step we show that if (1) defines a controllable system then $R(\lambda, e^{-\lambda})$ must have constant rank over $\mathbb{C}$. For a question of simplicity in the notation, in this section we will write $D = d/dt$ for the differentiator.

PROPOSITION 4.1. *With the previous notation, if* $\operatorname{rank} R(\lambda, e^{-\lambda}) = r \ \forall \lambda \in \mathbb{C}$ *then there exists a* $2D$ *polynomial matrix* $M(z_1, z_2)$ *such that* $\mathcal{B} := \ker R(D, \Delta) = \operatorname{im} M(D, \Delta)$*, with the operator* $M(D, \Delta)$ *acting on* $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ *for a certain integer* l.

Proof. Under the hypothesis, the 2D polynomial matrix $R(z_1, z_2)$ has rank r (over the field of fractions $\mathbb{R}(z_1, z_2)$). Suppose first that $R(z_1, z_2)$ has $q = r$ columns. Then $R(\lambda, e^{-\lambda})$ has full column rank $\forall \lambda \in \mathbb{C}$, and hence $\ker R(D, \Delta)$ does not contain any element with components of the form $t^k e^{\lambda t}$. By [10, Theorem 5] this implies that $\ker R(D, \Delta) = \{0\}$, and the equality $\ker R(D, \Delta) = \operatorname{im} M(D, \Delta)$ is trivially satisfied with $M(z_1, z_2)$ being the $q \times 1$ zero matrix. Suppose now that $R(z_1, z_2)$ has $q > r$ columns. Then Lemma 4.2 follows.

LEMMA 4.2. $R(z_1, z_2)$ *can be factored as* $F(z_1, z_2)\bar{R}(z_1, z_2)$*, where* F *and* $\bar{R}$ *are* $2D$ *polynomial matrices of sizes* $g \times r$ *and* $r \times q$*, respectively, such that* $F(z_1, z_2)$ *has full column rank (over* $\mathbb{R}(z_1, z_2)$*) and* $\mathbb{R}(z_1, z_2)$ *is left prime (i.e.,* $\bar{R}$ *has full row rank and all its left factors are invertible in* $\mathbb{R}^{r \times r}[z_1, z_2]$*).*

Proof. Without loss of generality we may assume that $R(z_1, z_2) = [-Q(z_1, z_2) \mid P(z_1, z_2)]$, where $P(z_1, z_2)$ is a full rank matrix with r columns. Moreover, there exists a rational matrix $G(z_1, z_2)$ such that $PG = Q$. Let $G = \hat{Q}\hat{P}^{-1}$ and $G = \bar{P}^{-1}\bar{Q}$ be, respectively, a right coprime and a left coprime factorization of G [4]. Then the matrix $\bar{R} = [-\bar{Q} \mid \bar{P}]$ is a minimal left annihilator of $H := col(\hat{P}, \hat{Q})$ (cf. [7]); i.e., $\bar{R}H = 0$ and for every 2D polynomial matrix $S(z_1, z_2)$ such that $SH = 0$ there exists a 2D polynomial matrix $L(z_1, z_2)$ satisfying $S = L\bar{R}$. Since, obviously, also $RH = 0$, there exists a polynomial matrix $F(z_1, z_2)$ such that $R = F\bar{R}$. Further, since $\bar{R}$ is a full rank polynomial matrix with r rows, F must have column rank. ◻

Let then F and $\bar{R}$ be as in the previous lemma. Note that due to the fact that $\text{rank} R(\lambda, e^{-\lambda}) = r \ \forall \lambda \in \mathbb{C}$ neither $F(z_1, z_2)$ nor $\bar{R}(z_1, z_2)$ can have zeros of the form $(z_1, z_2) = (\lambda, e^{-\lambda})$. Now, since $\bar{R}$ is left prime, there exists a polynomial matrix W such that

$$\bar{R}(z_1, z_2)W(z_1, z_2) = N(z_1),$$

with $N(z_1) = \text{diag}(d(z_1), \ldots, d(z_1))$ for a suitable (nonzero) 1D polynomial $d(z_1)$. Let $M(z_1, z_2)$ be a right-prime 2D polynomial matrix such that $\bar{R}M = 0$ (we can take $M = H$ as in Lemma 4.2) and define the matrix $U(z_1, z_2) := [W(z_1, z_2) \mid M(z_1, z_2)]$.

LEMMA 4.3. *The operator* $U(D, \Delta) : C^\infty(\mathbb{R}, \mathbb{R}^q) \to C^\infty(\mathbb{R}, \mathbb{R}^q)$ *is surjective.*

Proof. We start by showing that $\det U = \det N = d^r(z_1) =: n(z_1)$. Without loss of generality we may assume that $\bar{R}(z_1, z_2)$ can be partitioned as $\bar{R}(z_1, z_2) = [P(z_1, z_2) \mid -Q(z_1, z_2)]$, with $P(z_1, z_2)$ square and nonsingular. Consider the corresponding partitions $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] =: W$ and $\left[\begin{smallmatrix} \bar{Q} \\ \bar{P} \end{smallmatrix}\right] =: M$ of W and M. It is well known (see [4]) that $\det \bar{P}(z_1, z_2) = \det P(z_1, z_2)$. Now,

$$\begin{aligned} \det U = \det \begin{bmatrix} X & \bar{Q} \\ Y & \bar{P} \end{bmatrix} &= \det \left(\begin{bmatrix} X & \bar{Q} \\ Y & \bar{P} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\bar{P}^{-1}Y & I \end{bmatrix} \right) \\ = \det \begin{bmatrix} X - \bar{Q}\bar{P}^{-1}Y & \bar{Q} \\ 0 & \bar{P} \end{bmatrix} &= \det \bar{P} \cdot \det(X - \bar{Q}\bar{P}^{-1}Y) \\ &= \det \bar{P} \cdot \det(X - P^{-1}QY), \end{aligned}$$

since $\bar{Q}\bar{P}^{-1} = P^{-1}Q$ (due to the fact that M is a dual basis of $\bar{R}$). Thus,

$$\begin{aligned} \det U &= \det \bar{P} \cdot \det(P^{-1}(PX - QY) \\ &= \det \bar{P}. \det P^{-1} \cdot \det(PX - QY) \\ &= \det \bar{P}.(\det P)^{-1} \cdot \det(PX - QY) \\ &= \det(PX - QY), \end{aligned}$$

and as $N = PX - QY$, we conclude that

$$\det U = \det N = \det(\text{diag}(d(z_1), \ldots, d(z_1))) = d^r(z_1) =: n(z_1).$$

Consider now the equation

$$U(D, \Delta)\alpha = \beta.$$

Given $\beta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, define $\bar{\alpha}$ such that

$$\bar{N}(D)\bar{\alpha} = \beta,$$

with $\bar{N}(z_1) := \operatorname{diag}(n(z_1), \dots, n(z_1))$. Note that $\bar{N}(D)$ is a surjective operator in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. Define $\alpha := \bar{V}(D, \Delta)\bar{\alpha} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, where $\bar{V}(z_1, z_2)$ is such that

$$U(z_1, z_2)\bar{V}(z_1, z_2) = \bar{N}(z_1).$$

Then

$$U(D, \Delta)\alpha = U(D, \Delta)\bar{V}(D, \Delta)\bar{\alpha} = \bar{N}(D)\bar{\alpha} = \beta,$$

showing that $U(D, \Delta)$ is a surjective operator in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. ∎

This implies that $\forall w \in \mathcal{B} = \ker R(D, \Delta)$ there exists $\bar{w}$ such that $w = U(D, \Delta)\bar{w}$ and hence $R(D, \Delta)U(D, \Delta)\bar{w} = 0$, i.e., $\bar{w} \in \ker R(D, \Delta)U(D, \Delta)$. So,

$$\mathcal{B} \subseteq U(D, \Delta) \ker(R(D, \Delta)U(D, \Delta)).$$

On the other hand, if $w = U(D, \Delta)\bar{w}$ and $R(D, \Delta)U(D, \Delta)\bar{w} = 0$, then $R(D, \Delta)w = 0$, i.e.,

$$\mathcal{B} \supseteq U(D, \Delta) \ker(R(D, \Delta)U(D, \Delta)).$$

Therefore $\mathcal{B} = U(D, \Delta) \ker(R(D, \Delta)U(D, \Delta))$. Taking into account that $U = [W \mid M]$ and that $RU = [FN \mid 0]$, this yields $\mathcal{B} = [W(D, \Delta) \mid M(D, \Delta)](\ker[F(D, \Delta)N(D, \Delta) \mid 0])$. Thus

$$\mathcal{B} = W(D, \Delta) \ker(F(D, \Delta)N(D, \Delta)) + \operatorname{im} M(D, \Delta).$$

Finally, it turns out that Lemma 4.4 follows.

LEMMA 4.4. *$W(\ker FN) \subseteq \operatorname{im} M$.*

Proof. Recall that $F(\lambda, e^{-\lambda})$ has full column rank $\forall \lambda \in \mathbb{C}$. This implies that $\ker F(D, \Delta) = \{0\}$, and hence $\ker FN = \ker N$. Therefore, in order to prove the lemma we will show that $W(\ker N) \subseteq \operatorname{im} M$. As is well known,

$$\ker N(D) = \operatorname{span}\{t^j e^{\lambda_i t} e_k : i = 1, \dots, p, j = 0, \dots, \mu(\lambda_i) - 1, k = 1, \dots, r\},$$

where $\lambda_1, \dots, \lambda_p$ are the distinct roots of $d(z_1)$, $\mu(\lambda_i) (i = 1, \dots, p)$ are the corresponding multiplicities, and e_k is the kth vector in the canonical basis of $\mathbb{R}^r$. So, $W(\ker N) \subseteq \operatorname{im} M$ iff for every root λ of $d(z_1)$, for every m subject to $0 \le m \le \mu(\lambda) - 1$, and for every $k \in \{1, \dots, r\}$, $W(t^m e^{\lambda t} e_k) \in \operatorname{im} M$; i.e., there is a $\mathcal{C}^\infty$ trajectory x such that

$$Mx(t) = W(t^m e^{\lambda t} e_k). \tag{9}$$

Let then λ be a root of $d(z_1)$ and let m be a positive integer not greater than $\mu(\lambda) - 1$. Without loss of generality we may assume that $\bar{R} = [P \mid -Q]$ with $P(\lambda, e^{-\lambda})$ invertible. Consider the corresponding partitions of M and W as defined in the proof of Lemma 4.3. Note that in this case, $\bar{P}(\lambda, e^{-\lambda})$ is also invertible. Now, (9) can be rewritten as

$$\bar{Q}\, x = X(t^m e^{\lambda t} e_k), \tag{10}$$

$$\bar{P}\, x = Y(t^m e^{\lambda t} e_k). \tag{11}$$

It is not difficult to see that

$$Y(D, \Delta)(t^m e^{\lambda t} e_k) \quad = \quad Y(\lambda, e^{-\lambda}) e_k t^m e^{\lambda t} + (Y_{m-1} t^{m-1} + \dots + Y_0) e_k e^{\lambda t}$$

for some suitable matrices $Y_{m-1}, \dots, Y_0$. Take x to be of the form $x(t) = (\xi_m t^m + \cdots + \xi_0)e^{\lambda t}$. Then

$$(\bar{P}(D,\Delta)x)(t) = \{\bar{P}(\lambda, e^{-\lambda})\xi_m t^m + [\bar{P}(\lambda, e^{-\lambda})\xi_{m-1} + G_m^{m-1}\xi_m]t^{m-1} \\ + \cdots + [\bar{P}(\lambda, e^{-\lambda})\xi_0 + G_1^0\xi_1 + \cdots + G_m^0\xi_m]\}e^{\lambda t},$$

and x satisfies (11) iff

$$(12) \qquad \begin{cases} \bar{P}(\lambda, e^{-\lambda})\xi_m &= Y(\lambda, e^{-\lambda})e_k, \\ \bar{P}(\lambda, e^{-\lambda})\xi_{m-1} &= Y_{m-1}e_k - G_m^{m-1}, \\ \vdots & \\ \bar{P}(\lambda, e^{-\lambda})\xi_0 &= Y_0 e_k - (G_1^0\xi_1 + \cdots + G_1^0\xi_1). \end{cases}$$

As $\bar{P}(\lambda, e^{-\lambda})$ is invertible, there is a (unique) solution $(\xi_m, \dots, \xi_0)$ to (12), showing that (11) has a $\mathcal{C}^\infty$ solution $x(t) = (\xi_m t^m + \cdots + \xi_0)e^{\lambda t}$. It remains to prove that this solution $x(t)$ also satisfies (10). It follows from (11) that

$$(13) \qquad \begin{aligned} Q\bar{P}x &= QY(t^m e^{\lambda t} e_k) \quad \Leftrightarrow \\ P\bar{Q}x &= QY(t^m e^{\lambda t} e_k) \quad \Leftrightarrow \\ P\bar{Q}x &= (PX - N)(t^m e^{\lambda t} e_k) \quad \Leftrightarrow \\ 0 &= P(\bar{Q}x - Xt^m e^{\lambda t} e_k), \end{aligned}$$

since $Q\bar{P} = P\bar{Q}$, $PX + QY = N$, and $t^m e^{\lambda t} e_k \in \ker N$. Note that $\bar{Q}(D,\Delta)x - X(D,\Delta)t^m e^{\lambda t} e_k = E(t)e^{\lambda t}$, where $E(t)$ is a polynomial column in t containing powers of t of order not greater than m. Assume that $E(t) = E_{\bar{m}} t^{\bar{m}} + \cdots + E_0$, where $E_{\bar{m}}$ is a nonzero column and $\bar{m} \le m$. Then (13) becomes

$$[P(\lambda, e^{-\lambda})E_{\bar{m}} t^{\bar{m}} + (G_{\bar{m}-1} t^{m-1} + \cdots + G_0)]e^{\lambda t} = 0,$$

which implies that $P(\lambda, e^{-\lambda})E_{\bar{m}} = 0$. This is absurd, since $P(\lambda, e^{-\lambda})$ is an invertible matrix and $E_{\bar{m}}$ is assumed to be nonzero. Thus, $E(t)$ must be zero, i.e.,

$$\bar{Q}x - Xt^m e^{\lambda t} e_k = 0,$$

which shows that x satisfies equation (10) and hence also (9). □

As a consequence of this lemma we have that $\mathcal{B} = \mathrm{im} M(D,\Delta)$, i.e., (1) has an image representation, proving the proposition. □

Now, it is not difficult to come to the following conclusion.

PROPOSITION 4.5. *If* (1) *has an image representation, then it defines a controllable system.*

Proof. Suppose that (1) has an image representation, i.e., $\mathcal{B} = \mathrm{im} M(D,\Delta)$, and let w_1 and w_2 be two arbitrary signals in $\mathcal{B}$. Then, there exist a_1 and a_2 in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ such that $w_i = Ma_i, (i = 1, 2)$. Now, it is possible to construct a smooth signal a^* which coincides with a_1 in the past and with a_2 in the (sufficiently far) future. Such signal yields an element $w^* = Ma^*$ in $\mathcal{B}$ which coincides with w_1 in the past and with w_2 in the future. Thus w_1 is $\mathcal{B}$-compatible, with w_2 showing that $\mathcal{B}$ is controllable. □

Finally, if $R(z_1, z_2)$ is a 2D polynomial matrix of rank r and $\mathrm{rank} R(\lambda, e^{-\lambda}) < r$ for some $\lambda_0 \in \mathbb{C}$ we can show that there exists a signal associated with the frequency λ_0 which is not $\mathcal{B}$-compatible with the identical zero signal and hence $\mathcal{B}$ is not controllable.

PROPOSITION 4.6. *Let $\mathcal{B} := \ker R(D, \Delta)$, where $R(z_1, z_2)$ is a 2D polynomial matrix of rank r. If $\mathcal{B}$ is controllable then* $\operatorname{rank} R(\lambda, e^{-\lambda}) = r$ $\forall \lambda \in \mathbb{C}$.

Proof. We start by noting that, formally, $e^{-z_1} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} z_1^k$. Thus, if $z_2 = e^{-z_1}$, $R(z_1, z_2) = \tilde{\Pi}(z_1)$, where $\tilde{\Pi}(z_1)$ is a matrix over the ring $\mathbb{R}[[z_1]]$ of formal power series in z_1. Suppose now that $\operatorname{rank} R(\lambda, e^{-\lambda}) < \operatorname{rank} R(z_1, z_2) = \operatorname{rank} R(z_1, e^{-z_1}) = r$ for a certain $\lambda_0 \in \mathbb{C}$. We consider first the case where $\lambda_0 = 0$; so $\operatorname{rank} R(0, 1) < \operatorname{rank} R(z_1, e^{-z_1})$. This means that $\operatorname{rank} \tilde{\Pi}(0) < \operatorname{rank} \tilde{\Pi}(z_1) = r$, and therefore we may assume without loss of generality that

$$\tilde{\Pi}(z_1) = \operatorname{diag}(z_1^{k_1}, \ldots, z_1^{k_r}, z_1^{k_{r+1}}, \ldots, z_1^{k_s}) \tilde{\Gamma}(z_1),$$

where $k_1, \ldots, k_s$ are integers, $k_1 \geq 1$, and

$$\tilde{\Gamma}(0) = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix}$$

(with the zero rows possibly void).

Let

$$\Gamma(z_1) = \begin{bmatrix} I_{r \times r} & 0 \\ 0 & 0 \end{bmatrix} + z_1 \Gamma_1 + z_1^2 \Gamma_2 + \cdots + z_1^{k_1 - 1} \Gamma_{k_1 - 1}$$

be such that

$$\tilde{\Gamma}(z_1) = \Gamma(z_1) + \text{higher-order terms}.$$

Then, using the same kind of arguments as in the proof of Lemma 4.4, it is possible to show that there exists a trajectory $w^*(t) = \alpha_{k_1 - 1} t^{k_1 - 1} + \cdots + \alpha_0$ such that

$$\Gamma(D) w^*(t) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \frac{t^{k_1 - 1}}{(k_1 - 1)!}.$$

Now, this trajectory w^* is clearly such that

$$R(D, \Delta) w^* = \operatorname{diag}(D^{k_1}, \ldots, D^{k_s}) \Gamma(D) w = D^{k_1} t^{k_1 - 1} = 0,$$

and hence it belongs to $\mathcal{B}$.

LEMMA 4.7. *With the previous notation, w^* is not $\mathcal{B}$-compatible with the zero trajectory.*

Proof. Suppose that w^* is $\mathcal{B}$-compatible with the zero trajectory, yielding a trajectory $v^* \in \mathcal{B}$ such that $v^*|_{(-\infty, \tau_1)} = w^*|_{(-\infty, \tau_1)}$ and $v^*|_{[\tau_2, +\infty)} = 0|_{[\tau_2, +\infty)}$ for some $\tau_1 < \tau_2$. Since $v^* \in \mathcal{B}$, $R(D, \Delta) v^* = 0$ and therefore also

$$\int_{T_1}^{T_2 + 1} [R(D, \Delta) v^*] dt = 0.$$

In particular, if $r(z_1, z_2)$ denotes the first row of $R(z_1, z_2)$, we have that

$$\int_{T_1}^{T_2 + 1} [r(D, \Delta) v^*] dt = 0. \tag{14}$$

Note further that $r(\lambda, e^{-\lambda})$ becomes zero for $\lambda = 0$, and hence $r(z_1, z_2) = r(z_1, e^{-z_1})$ must be of the form

$$r(z_1, z_2) = r(z_1, e^{-z_1}) = z_1 r_0(z_1) + (e^{-z_1} - 1) r_1(z_1, e^{-z_1}),$$

where $r_0(z_1)$ and $r_1(z_1, z_2)$, respectively, are 1D and 2D polynomial rows. So equation (14) is of the form

$$\int_{T_1}^{T_2+1} [(\Delta - 1) r_1(D, \Delta) + D r_0(D)] v^* dt = 0.$$

This is equivalent to

$$\int_{T_1-1}^{T_1} r_1(D, \Delta) v^* dt + \int_{T_2}^{T_2+1} r_1(D, \Delta) v^* dt + [r_0(D) v^*] \Big|_{T_1}^{T_2+1} = 0,$$

which is still equivalent to

$$\int_{T_1-1}^{T_1} r_1(D, \Delta) w^* dt + \int_{T_2}^{T_2+1} r_1(D, \Delta) 0 dt$$
$$+ (r_0(D) 0)(T_2 + 1) - (r_0(D) w^*)(T_1) = 0$$

if $T_2 + 1 \gg \tau_2$ and $T_1 \ll \tau_1$ so that in a sufficiently big interval around $T_2 + 1$, v^* coincides with the zero trajectory, and in a sufficiently big interval around T_1, it coincides with w^*. This yields

$$\int_{T_1-1}^{T_1} r_1(D, \Delta) w^* dt - (r_0(D) w^*)(T_1) = 0. \tag{15}$$

Let $\eta = col(\eta_1, \ldots, \eta_q)$ be a trajectory such that $\eta(t) = \frac{\alpha_{k_1-1}}{k_1} t^{k_1} + \cdots + \alpha_0 t$; then $D\eta = w^*$ and we may write (15) as

$$\int_{T_1-1}^{T_1} r_1(D, \Delta) D\eta \, dt - (r_0(D) D\eta)(T_1) = 0,$$

which is equivalent to having

$$[((\Delta - 1) r_1(D, \Delta) + r_0(D) D)\eta](T_1) = 0$$

or still

$$(r(D, \Delta)\eta)(T_1) = 0.$$

Now, it follows from our previous considerations that

$$r(D, \Delta)\eta)(T_1) = (D^{k_1} [1 \;\; 0 \;\; \ldots \;\; 0] \eta)(T_1) = [1 \;\; 0 \;\; \ldots \;\; 0] \alpha_{k_1-1} (k_1 - 1)! = 1,$$

since $[1 \;\; 0 \;\; \ldots \;\; 0] \Gamma(D) w^* = \frac{t^{k_1-1}}{(k_1-1)!}$. In this way we obtain that $0 = 1$, which is absurd. Consequently the hypothesis that w^* is $\mathcal{B}$-concatenable with the zero trajectory cannot hold true. ∎

It follows from this result that if $R(\lambda, e^{-\lambda})$ drops in rank for $\lambda = 0$, then $\mathcal{B}$ is not controllable. It remains to show that if $\text{rank} R(\lambda, e^{-\lambda}) < \text{rank} R(z_1, z_2)$ for $\lambda = \lambda^* \neq 0$ then $\mathcal{B}$ is not controllable.

Assume now that $R(\lambda, e^{-\lambda})$ drops in rank for $\lambda = \lambda^* \neq 0$, and consider the system Σ^* with behavior $\mathcal{B}^* := \exp_{\lambda^*} \mathcal{B}$, where $\exp_{\lambda^*}$ is defined by $\exp_{\lambda^*}(t) = e^{-\lambda^* t}$ $\forall t \in \mathbb{R}$. Then $\mathcal{B}^*$ is described by a polynomial matrix $R^*(z_1, z_2)$ such that $R^*(\lambda, e^{-\lambda}) = R(\lambda + \lambda^*, e^{-(\lambda+\lambda^*)})$. As $\text{rank}R(\lambda, e^{-\lambda})$ drops for $\lambda = \lambda^*$, $R^*(\lambda, e^{-\lambda})$ drops for $\lambda = 0$. Thus, by the foregoing arguments, $\mathcal{B}^*$ is not controllable. This implies that $\mathcal{B}$ is also not controllable, completing the proof of the proposition. $\square$

5. Conclusion. We have presented a necessary and sufficient condition for the controllability of the behavior of d-d systems with kernel representations. Moreover, we have compared the notion of behavioral controllability with the notions of approximate and spectral controllability considered in [3] as well as with other controllability properties (namely, weak and strict controllability) that have been introduced within an algebraic approach to d-d systems [5]. Contrary to what happens with the results of [3], [6], [9], and [5], our results hold for all types of systems with commensurable delays and not only for retarded or neutral systems in pseudostate form.

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