On Interconnections, Control, and Feedback

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***Abstract*—The purpose of this paper is to study interconnections and control of dynamical systems in a behavioral context. We start with an extensive physical example which serves to illustrate that the familiar input–output feedback loop structure is not as universal as we have been taught to believe. This leads to a formulation of control problems in terms of interconnections. Subsequently, we study interconnections of linear time-invariant systems from this vantage point. Let us mention two of the results obtained. The first one states that any polynomial can be achieved as the characteristic polynomial of the interconnection with a given plant, provided the plant is not autonomous. The second result states that any subsystem of a controllable system can be implemented by means of a singular feedback control law. These results yield pole placement and stabilization of controllable plants as a special case. These ideas are finally applied to the stabilization of a nonlinear system around an operating point.**

***Index Terms*— Behaviors, controllability, feedback, interconnection, invariant polynomials, linear systems, pole placement, regular interconnection, singular feedback, stabilization.**

I. Introduction

ONE OF THE MAIN features of the behavioral approach as a foundational framework for the theory of dynamical systems is that it does not take the input–output structure as the starting point for describing systems in interaction with their environment. Instead, a mathematical model is simply viewed as any relation among variables. In the dynamic case this relation constrains the time evolution which a set of variables can take. The collection of time trajectories which the model declares possible is called the *behavior* of the dynamical system. This basic definition proves to be a very convenient starting point for discussing dynamical systems in a variety of applications. In the present paper we will scrutinize this aspect in the context of control.

This behavior, hence a set of time functions, can be specified in many different ways. Often, as in Newton's second law or in Maxwell's equations, the behavior will be given as the solution set of a system of differential equations. Sometimes, as in Kepler's laws or in the theory of formal languages, the trajectories which the behavior declares feasible are described more directly, without the aid of behavioral equations. In many other examples the behavior will be specified through the intervention of auxiliary variables, which we will call *latent* variables, in order to distinguish them from the *manifest* variables, which are the variables whose time paths the model aims at describing. The usual state-space model forms an example of such a model structure.

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Thus it is to be expected that in the behavioral approach a variety of system representations will appear. The classical ones are the input–output representation, which suggest a cause/effect structure, and the input/state/output representation which, in addition to suggesting a cause/effect structure, also displays, through the state, the internal memory of the system. In control applications it has proven to be very convenient to adopt this input/state/output framework, to view the control and the other exogenous signals (for example, tracking signals or disturbances) as inputs, and the measurements and the to-be-controlled variables as outputs. In addition, it is often also easy to formulate the desired qualitative properties of the closed-loop response (as stability) in terms of input–output or state properties. Finally, the action of the controller and its cybernetic structure can sometimes be explained very effectively in a state-space context. Think of the separation theorem in LQG-control and of the elegant double-Riccati equation solution of the H_∞ problem. Nevertheless, models obtained from first principles will seldom be in input–output or in input/state/output form, and it is worth asking whether they form a reasonable starting point for the development of a theory which aims at treating physical models.

The behavioral point of view has received broad acceptance as an approach for modeling dynamical systems. It is now generally agreed upon that when modeling a dynamic component, it makes no sense to prejudice oneself (as one would be forced to do in a transfer function setting) as to which variables should be viewed as inputs and which variables should be viewed as outputs. There are a number of reasons for this. A pragmatic one is that ultimately this component will become part of an interconnected system. So, it will depend on the interconnection structure, which of the variables interconnecting this component with the rest of the system will act as inputs and which will act as outputs. A more philosophical, simpler, but perhaps more convincing reason is that when a physical system is not endowed with a natural signal flow graph, it is asking for difficulties to suggest that it has one (even if mathematically there would be nothing wrong with doing this). As an illustrative example, consider the port behavior of an RLC circuit. Assume that the current-to-voltage transfer function is biproper (proper with a proper inverse). Then it is possible, by any reasonable mathematical definition, to view the network both as being current-controlled or as being voltage-controlled. However, from a system theoretic point of view it is not logical to do either. An input–output, transfer function formulation has a tendency to suggest a signal flow structure which is not present in physical reality. We will not dwell here on the maneuvering which is sometimes needed in order to treat a differentiator in an input–output context. This is not to say, however, that there are no situations where the input–output structure is natural—*quite the contrary*. Whenever logic devices are involved, the input–output structure is often a *must*. Indeed, when in a typical physical device (say an electrical circuit) one variable (say the voltage at a port) is imposed, the other (say the current) will follow, but the situation can be turned around. When the second variable is imposed, the first will follow. However, the physics (or, better, the equivalent circuits acting as models) of logic devices will be such that this cannot be done. Imposing the output voltage of an operational amplifier will not lead to an input voltage that would correspond to that output when that input voltage was imposed.

The behavioral approach has, until now, met with much less acceptance in the context of control. We can offer a number of explanations for this fact. First, as already mentioned, there is something very natural in viewing controls as inputs and measured variables as outputs. When, subsequently, a controller is regarded as a feedback processor, one ends up with the feeling that the input–output structure is in fact an essential feature of control. Second, since it is possible to prove that every linear time-invariant system always admits a componentwise input–output partition, one gets the impression that the input–output framework can be adopted without second thoughts, that nothing is lost by taking it as the starting point.

The purpose of this paper is to present a framework for control which does not take *steering*, but which takes *interconnection* as the basic aim of controller design. The *steering* picture of a *feedback* sensor/actuator structure then emerges as an *important* special case. In the present paper we will treat general control problems. In subsequent papers we will study the LQ and H_∞ problem from this point of view. We will now illustrate by means of a detailed example a situation in which the classical signal flow graph approach in control is inadequate.

II. The Intelligent Control Paradigm

Present-day control theory centers around the signal flow graph shown in Fig. 1. The plant has four terminals (each supporting variables which are typically vector-valued). There are two input terminals, one for the control input and one for the other exogenous variables (as disturbances, set-points, reference signals, etc.), and there are two output terminals, one for the measurements and one for the to-be-controlled variables. By using feed-through terms in the plant equations this configuration accommodates, by incorporating these variables in the measurements, the possible availability to the controller of set-point settings, reference signals, or disturbance measurements for feedforward control, and, by incorporating the control input in the to-be-controlled outputs, the penalizing excessive control action in optimal control. The control inputs are generated by means of actuators, and the measurements are made available through sensors. Usually, the dynamics of the actuators and of the sensors are considered to be part of the plant.

In intelligent control, the controller is thought of as a microprocessor-type device which is driven by the sensor outputs and which produces the actuator inputs through a cleverly devised algorithm involving a combination of feedback, identification, and adaptation. The creation of such algorithms is considered to be the core of control theory. This picture is often completed by an A/D and a D/A converter interfacing the sensor with the microprocessor and the microprocessor with the actuators. Also, loops expressing model uncertainty often are incorporated in the above. Of course, many variations, refinements, and special cases of this structure are of interest, but the basic idea is that of a supervisor reacting in an intelligent way to observed events and measured signals.

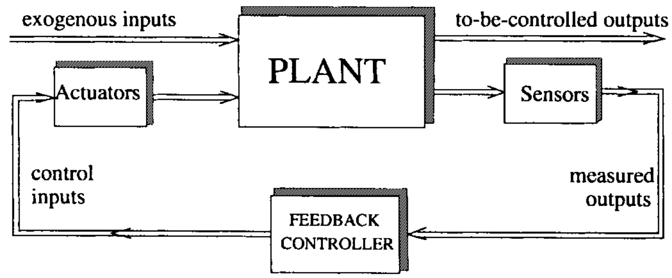

Fig. 1. Intelligent control.

The belief that the paradigm of Fig. 1 constitutes the essence of control has been prevalent ever since the beginning of the subject, from the Watt regulator (or at least its modern day interpretation), Black's feedback amplifier, and Wiener's cybernetics, to the ideas underlying modern adaptive and robust control. It is indeed a deep and very appealing paradigm which will undoubtedly gain relevance and impact as logic devices become ever more prevalent, reliable, and inexpensive. This paradigm has a number of features which are important for considerations which will follow. Some of these are as follows.

- There is an asymmetry between the plant and the controller; it remains apparent what part of the system is the plant and what part is the controller. This asymmetry disappears to some extent in the closed loop.
- The intelligent control paradigm tells us to be wary of errors and noise in the measurements. Thus it is considered ill-advised to differentiate measurements, presumably because this will lead to noise amplification.
- The plant and the controller are dynamical systems which can be interconnected at any moment in time. If for one reason or another the feedback controller temporarily fails to receive a sensor signal, then the control input can be set to a default value, and later on the controller can resume its action.

We will now present an example of a common controller in which none of these features are present.

III. An Example of a Common Controller

In this section we will analyze a very mundane and widespread automatic control mechanism, namely the traditional device which ensures the automatic closing of doors. There is nothing peculiar about this example. Devices based on similar

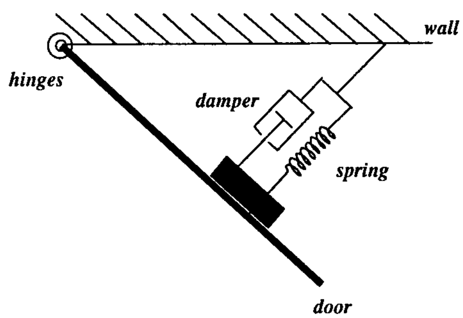

Fig. 2. A door-closing mechanism.

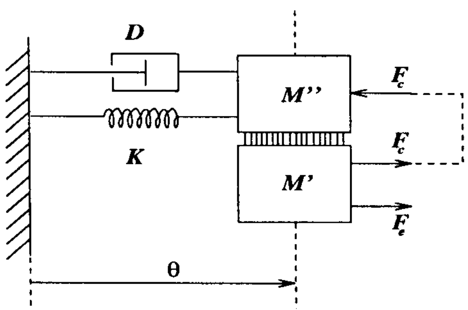

Fig. 3. A mass/spring/damper representation.

principles are used, for instance, for the suspension of cars, and the points which we will make through this example could also be made just as well through many temperature or pressure control devices. A typical automatic door-closing mechanism is schematically shown in Fig. 2.

Although there exists a large variety of such automatic-door-closing mechanisms, they invariably consist of a spring in order to force the closing of the door and a damper in order to make sure that it closes gently. In addition, these mechanisms often have considerable weight so that their mass cannot be neglected as compared to the mass of the door itself. These mechanisms can be modeled as a mass/spring/damper combination. In good approximation, the situation can be analyzed effectively as the mechanical system shown in Fig. 3.

We model the door as a free mass M' (neglecting friction in the hinges) on which two forces are acting. The first force, F_c, is the force exerted by the door-closing device, while the second force, F_e, is an exogenous force (exerted for example by a person pushing the door in order to open it). The equation of motion for the door becomes

$$M'\frac{d^2\theta}{dt^2} = F_c + F_e \tag{1}$$

where θ denotes the opening angle of the door and M' its mass. The door-closing mechanism, modeled as a mass/spring/damper combination, yields

$$M''\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = -F_c. \tag{2}$$

Here, M'' denotes the mass of the door-closing mechanism, D its damping coefficient, and K its spring constant. Combining (1) and (2) leads to

$$(M' + M'')\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = F_e. \tag{3}$$

In order to ensure proper functioning of the door-closing device, the designer can to some extent choose M'', D, and K (all of which must, for physical reasons, be positive). The desired response requirements are: small overshoot (to avoid banging of the door), fast settling time, and a not-too-low steady-state gain from F_e to θ (in order to avoid having to use an excessive force when opening the door). A good design will be achieved by choosing a light mechanism (M'' small) with a reasonably strong spring (K large), but not too strong so as to avoid having to exert excessive force in order to open the door, and the value of D is chosen so as to achieve slightly less than critical damping. In a sense, this is a perfect elementary example of a controller design. However, it does not fit many of the basic principles which are taught in control courses.

It is completely natural to view in this example the door as the plant and the door-closing mechanism as the controller. Then, if we insist on interpreting the situation in terms of control system configurations as Fig. 1, we will obtain the following equations:

$$\text{plant:}\quad M'\frac{d^2\theta}{dt^2} = u + v;\quad y = \theta;\quad z = \theta \tag{4}$$

with u the control input [$u = F_c$ in (1)], v the exogenous input [$v = F_e$ in (1)], y the measured output, and z the to-be-controlled output. The controller becomes

$$\text{controller:}\quad u = -M''\frac{d^2y}{dt^2} - D\frac{dy}{dt} - Ky. \tag{5}$$

This yields the closed-loop system, described by

$$\text{closed-loop system:}\quad (M' + M'')\frac{d^2z}{dt^2} + D\frac{dz}{dt} + Kz = v. \tag{6}$$

Observe that in the control law (5), the measurement y should be considered as the input and the control u should be considered as the output. Thus (5) suggests that we are using what would be called a PD^2 controller (a proportional twice differentiating controller), a singular controller which would be thought of as causing high noise amplification. Of course, no such noise amplification occurs in reality. Further, the plant is second order, the controller is second order, and the closed-loop system is also second order (thus unequal to the sum of the order of the plant and the controller). Hence, in order to connect the controller to the plant, we will have to "match" the initial states of the controller and the plant. Thus in order to interconnect the plant and the controller, preparation of these systems and their initial states is required. In attaching the door-closing mechanism to the door, we will indeed typically make sure that at the moment of attachment the initial values of θ and $d\theta/dt$ in (1) and (2) are zero for both the door and the door-closing mechanism.

We now come to our most important point concerning this example. Let us analyze the signal flow graph. In the plant [(1) and (4)] it is natural to view the forces F_c and F_e as inputs and θ as output. This input–output choice is logical both from the physical and from the cybernetic, control theoretic point of view. In the controller [(2) and (5)], on the other hand, the physical and the cybernetic points of view clash. From the cybernetic, control theoretic point of view, it is logical to regard the angle θ as input and the control force F_c as output. From the physical point of view, however, it is logical to regard (just as in the plant) the force F_c as input and θ as output. It is evident that as an interconnection of two mechanical systems, the door and the door-closing mechanism play completely symmetric roles. However, the cybernetic, control theoretic point of view obliges us to treat the situation as asymmetric, making the force the cause in one mechanical subsystem and the effect in another.

This simple but realistic example permits us to draw the following conclusions. Notwithstanding all its merits, the intelligent control paradigm of Fig. 1 gives an unnecessarily restrictive view of control. In many important practical control problems, the signal flow graph interpretation of Fig. 1 is untenable. The solution which we will propose to this dilemma is the following. We will keep the distinction between plant and controller with the understanding that this distinction is justified only from an *evolutionary* point of view, in the sense that it becomes evident only *after* we comprehend the genesis of the controlled system, after we understand the way in which the interconnected system has come into existence as a purposeful system. However, we will abandon the intelligent control signal flow graph as a paradigm for control. We will abandon the distinction between control inputs and measured outputs. Instead, we will put forward the *interconnection of a controller to a plant* as the central paradigm in control theory.

Other convincing examples of controllers in which the intelligent control paradigm is not suitable are car dampers and operational amplifiers. Also, we put question marks by the traditional feedback interpretation of this device. This is explained in [13]. It is perhaps somewhat ironic that we do not consider the *feedback* amplifier (since the logic device is in the forward loop) as a good example of *feedback*, as it is usually viewed (we are not the first to point out this anomaly; see, for instance, [5, pp. 145–163]).

However, we by no means claim that the intelligent control paradigm is without merits. To the contrary, it is an extremely useful and important way of thinking about many control problems. Claiming that the input–output framework is *not always* the suitable framework to approach a problem does not mean that one claims that it is *never* is.

IV. Control as the Interconnection

In this section we will describe mathematically how we can view control as the interconnection of a plant and a controller. We will do this in the context of the behavioral approach to dynamical systems (see [8]–[10]). Recall that a *dynamical system* Σ is defined as a triple, $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the *time axis*, $\mathbb{W}$ a set called the *signal space*, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior*. Thus $\mathbb{T}$ denotes the set of time instances relevant to the dynamical system under consideration. In the present paper, we will almost exclusively deal with continuous-time systems with $\mathbb{T} = \mathbb{R}$. The signal space denotes the set in which the time trajectories, which the system generates, take on their values. In lumped systems $\mathbb{W}$ will be (a subset of) $\mathbb{R}^q$, in distributed systems it is an infinite-dimensional function space, and in discrete-event systems (DES), it is a finite set. The prescription of the behavior $\mathcal{B}$ can occur in many different ways, from the solution set of a system of differential equations in continuous-time systems, to a prescription via grammars and substitution rules in DES. In the present paper we will discuss mainly systems with $\mathbb{W} = \mathbb{R}^q$.

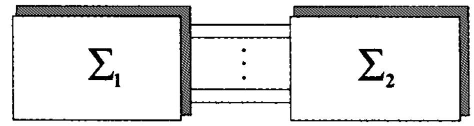

Fig. 4. Interconnection.

Let $\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathcal{B}_2)$ be two dynamical systems with the same time axis and the same signal space. The *interconnection of* Σ_1 *and* Σ_2, denoted as $\Sigma_1 \wedge \Sigma_2$, is defined as $\Sigma_1 \wedge \Sigma_2 := (\mathbb{T}, \mathbb{W}, \mathcal{B}_1 \cap \mathcal{B}_2)$; the behavior of $\Sigma_1 \wedge \Sigma_2$ consists simply of those trajectories w: $\mathbb{T} \rightarrow \mathbb{W}$ which are compatible with the laws of both Σ_1 (i.e., w belongs to $\mathcal{B}_1$) and Σ_2 (i.e., w belongs also to $\mathcal{B}_2$).

This definition stems from the mental picture shown in Fig. 4. In this picture, we view the signal space as a product space consisting of a Cartesian product of variables. We imagine that the components of this product space live on the terminals of Σ_1 and Σ_2. In the interconnected system, variables must be *acceptable* to both Σ_1 and Σ_2.

Two remarks are in order.

1) Of course, in most applications, systems are interconnected only through certain terminals and not along others. For example, the controller of Fig. 1 is connected to the plant only through the control input and measured output terminals. This situation can easily be incorporated in the definition of interconnection as follows. Assume $\Sigma_1 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathcal{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{W}_2 \times \mathbb{W}_3, \mathcal{B}_2)$ with their interconnection leading to $\Sigma_1 \wedge \Sigma_2 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \mathcal{B})$ with $\mathcal{B} = \{(w_1, w_2, w_3)$: $\mathbb{T} \rightarrow \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3 | (w_1, w_2) \in \mathcal{B}_1$ and $(w_2, w_3) \in \mathcal{B}_2\}$. This is illustrated in Fig. 5.

   By redefining Σ_1 to $\hat{\Sigma}_1 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \hat{\mathcal{B}}_1)$ with $\hat{\mathcal{B}}_1 = \mathcal{B}_1 \times \mathbb{W}_3^{\mathbb{T}}$, and $\hat{\Sigma}_2$ to $\hat{\Sigma}_2 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \hat{\mathcal{B}}_2)$ with $\hat{\mathcal{B}}_2 = \mathbb{W}_1^{\mathbb{T}} \times \mathcal{B}_2$, it is easily seen that this interconnection now becomes a special case of our general definition. Note that the definition of the behavior of $\hat{\Sigma}_1$ leaves the variables w_3 free in $\hat{\Sigma}_1$, while that of $\hat{\Sigma}_2$ leaves the variables w_1 free in $\hat{\Sigma}_2$.

2) In many interconnections, following the mental picture of Fig. 5, it is natural to suppress the interconnecting variables (w_2) after interconnection, yielding $\Sigma_1 \wedge \Sigma_2 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_3, \mathcal{B})$ with $\mathcal{B} = \{(w_1, w_3)$: $\mathbb{T} \rightarrow \mathbb{W}_1 \times \mathbb{W}_3 | \exists w_2$: $\mathbb{T} \rightarrow \mathbb{W}_2$ such that $(w_1, w_2) \in \mathcal{B}_1$ and $(w_2, w_3) \in \mathcal{B}_2\}$. This is illustrated in Fig. 6. This situation can be formalized using *manifest* and *latent* variables, one of the other central features of the behavioral approach.

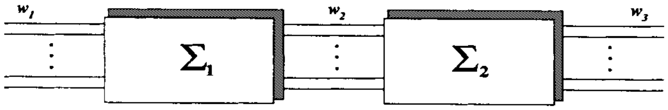

Fig. 5. Interconnection along certain terminals.

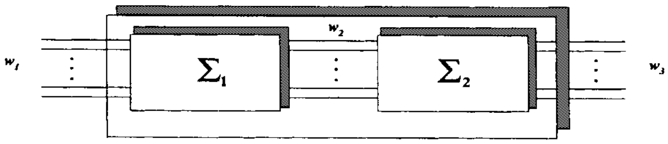

Fig. 6. Elimination of interconnecting variables.

The problem of control can now be described as follows. Assume that the *plant*, a dynamical system $\Sigma_p = (\mathbb{T}, \mathbb{W}, \mathcal{B}_p)$, is given. Let $\mathcal{C}$ be a family of dynamical systems, all with $\mathbb{T}$ as common time axis and $\mathbb{W}$ as common signal space. We will call $\mathcal{C}$ the set of *admissible controllers*. An element $\Sigma_c \in \mathcal{C}, \Sigma_c = (\mathbb{T}, \mathbb{W}, \mathcal{B}_c)$ is called an *admissible controller*. The interconnected system $\Sigma_p \wedge \Sigma_c$ will be called the *controlled system*. The controller Σ_c should be chosen so as to make sure that $\Sigma_p \wedge \Sigma_c$ has certain desirable properties. The problems of control theory are: *first*, to describe the set of admissible controllers; *second*, to describe what desirable properties the controlled system should have; and, *third*, to find an admissible controller Σ_c such that $\Sigma_p \wedge \Sigma_c$ has these desired properties.

It may be tempting to call *active control* what we have referred to as intelligent control, and *passive control* our idea of interconnection, implanting a device to act as a controller. However, we like to use the term passivity as a more specific property of devices which absorb energy. In fact, our approach to control leads to the question of what can be achieved using passive controllers, with passivity in the sense of dissipative systems. In other words, what can be achieved with controllers which function without an energy source? What systems can be stabilized using such controllers? We believe that such problems are of considerable practical importance.

In a sequence of papers, we will describe a number of concrete design philosophies following this point of view. They are familiar ones. The present one is inspired by stabilization and pole placement; the second will treat LQ control, and the third H_∞ control. However, the underlying philosophy, the problem formulations, and their solutions are sufficiently different from their classical counterparts that they merit a detailed coverage.

V. Linear Time-Invariant Differential Systems

For the sake of concreteness, we will now restrict our attention to a familiar class of dynamical systems, to systems described by constant coefficient linear differential equations, the analogues in a behavioral setting of the finite-dimensional linear systems, or, in an input–output setting, the systems described by rational transfer functions. In our earlier work we have discussed mainly the discrete-time case, whereas in the present paper we will be interested in the continuous-time case. We will therefore describe the general background in some detail.

Let ξ be an indeterminate, and denote by $\mathbb{R}^{\bullet \times q}[\xi]$ the set of real polynomial matrices with q rows and any (of course, finite) number of columns. Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$, written out explicitly, $R(\xi) = R_0 + R_1\xi + \cdots + R_N\xi^N$, and consider the system of differential equations

$$R_0 w + R_1 \frac{dw}{dt} + \cdots + R_N \frac{d^N w}{dt^N} = 0 \tag{7'}$$

or, in shorthand notation

$$R\left(\frac{d}{dt}\right)w = 0. \tag{7''}$$

Here w denotes the row-vector with components $w_1, w_2, \cdots, w_q$. Each of the w_k's is a mapping from $\mathbb{R}$ to $\mathbb{R}$, whence w: $\mathbb{R} \rightarrow \mathbb{R}^q$. Let $rowdim(R)$ denote the row dimension, i.e., the number of rows, of R. As is apparent, (7) consists of a system of $rowdim(R)$ scalar linear differential equations in q variables $w_1, w_2, \cdots, w_q$, with constant coefficients (the entries of the matrices $R_0, R_1, \cdots, R_N$). System (7) defines a continuous-time dynamical system with signal space $\mathbb{R}^q$ whose behavior consists of the functions col $(w_1, w_2, \cdots, w_q)$: $\mathbb{R} \rightarrow \mathbb{R}^q$ which satisfy (7). However, we need to spell out what it means for such a trajectory to be a solution to the set of differential equations (7).

We call w: $\mathbb{R} \rightarrow \mathbb{R}^q$ a *strong solution* of (7″) if w is N-times differentiable and if

$$R_0 w(t) + R_1 \frac{dw}{dt}(t) + \cdots + R_N \frac{d^N w}{dt^N}(t) = 0 \tag{8}$$

for all $t \in \mathbb{R}$. However, for many applications in control, this solution concept is too restrictive. For example, it would lead to difficulties when discussing the step-response. We will call w: $\mathbb{R} \rightarrow \mathbb{R}^q$ a *weak solution* of (7) if $w \in \mathcal{L}^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ and if for all functions $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ of compact support there hold

$$\left\langle R^T\left(-\frac{d}{dt}\right)f, w\right\rangle = 0$$

with

$$\langle f_1, f_2 \rangle := \int_{-\infty}^{+\infty} f_1^T(t) f_2(t)\, dt.$$

Note that w is a weak solution of (7) iff $R(d/dt)w$ is zero as a vector of distributions. We will define the behavior of (7) in terms of weak solutions. Thus (7) defines the dynamical system $\Sigma igma(R) := (\mathbb{R}, \mathbb{R}^q, \ker(R))$ with $\ker(R)$, the set of weak solutions of (7). It can be shown that $\ker(R) \cap C^\infty(\mathbb{R}, \mathbb{R}^q)$ is dense (in the topology of $\mathcal{L}^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$) in $\ker(R)$. In this sense every weak solution of (7) can actually be approximated by a classical (strong) one. Intuitively, for the purposes of this paper, one can therefore think of $\ker(R)$ as simply consisting of the collection of all solutions of (7).

Let us denote the family of dynamical systems obtained this way by $\mathcal{L}^q$ ($\mathcal{L}$ for *linear* and q for the number of variables). Thus each element of $\mathcal{L}^q$ is *parameterized* by a polynomial matrix $R \in \mathbb{R}^{\bullet \times q}[\xi]$. The notation used in the previous paragraph shows that the map $\Sigma igma$: $\mathbb{R}^{\bullet \times q}[\xi] \rightarrow \mathcal{L}^q$ associates with the "*parameter*" R, the dynamical system $\Sigma igma(R) = (\mathbb{R}, \mathbb{R}^q, \ker(R))$. It is easy to see that each element of $\mathcal{L}^q$ is a linear (its behavior being a linear subspace of $(\mathbb{R}^q)^{\mathbb{R}}$), time-invariant (its behavior being shift-invariant) dynamical system. We will refer to the elements of $\mathcal{L}^q$ as *linear differential systems* and to (7) as a *kernel representation* of the dynamical system $\Sigma igma(R)$.

The polynomial matrix R obviously defines $\Sigma(R)$, but the converse is not true. For example, it is easily seen that if U is a $rowdim(R) \times rowdim(R)$ unimodular polynomial matrix, then $\Sigma igma(R) = \Sigma igma(UR)$. We shall now see that under a simple additional condition, the converse also holds. We will call (7) (or briefly R) *minimal* if $R' \in \mathbb{R}^{\bullet \times q}[\xi]$ and $\Sigma igma(R') = \Sigma igma(R)$ imply $rowdim(R') \geq rowdim(R)$. Recall that $R \in \mathbb{R}^{\bullet \times q}[\xi]$ is of *full row rank* if $rank(R) = rowdim(R)$ (in $rank(R), R$ should be viewed as a matrix of polynomials, or, equivalently, as a matrix of rational functions). The following proposition shows in how far elements of $\mathcal{L}^q$ define the associated system of differential equations. This proposition is easily proven using the ideas used in the discrete-time case in [10].

Proposition 1:

1) $R \in \mathbb{R}^{\bullet \times q}[\xi]$ is minimal iff it is of full row rank.
2) If $R_1, R_2 \in \mathbb{R}^{\bullet \times q}[\xi]$ are both minimal or, by 1, both of full row rank, then $\Sigma igma(R_1) = \Sigma igma(R_2)$ iff there exists a unimodular polynomial matrix U such that $R_2 = UR_1$.
3) More generally, $\Sigma igma(R_1) = \Sigma igma(R_2)$ if and only if there exists $F_1, F_2 \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $R_1 = F_2 R_2$ and $R_2 = F_1 R_1$.

It is easily seen from 3) of Proposition 1 that $\Sigma igma(R_1) = \Sigma igma(R_2)$ implies $rank(R_1) = rank(R_2)$. Let $\Sigma \in \mathcal{L}^q$. Then by definition there exists a $R \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $\Sigma = \Sigma igma(R)$. It follows that $rank(R)$ depends only on Σ. Define now the map p: $\mathcal{L}^q \to \{0, 1, \cdots, q\}$ by $p(\Sigma) := rank(R)$ ($p(\Sigma)$ thus equals $rowdim(R)$ iff R is minimal). Define also m: $\mathcal{L}^q \to \{0, 1, \cdots, q\}$ by $m(\Sigma) := q - p(\Sigma)$. We shall later see that m and p are equal to the number of input and output variables.

One of the central notions in control theory is that of controllability. It ensures the very existence of reasonable controllers. This notion can be very nicely generalized to the behavioral setting. As can be expected, controllability will be of essential importance to us in the sequel.

We will call $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ *controllable* if for each $w_1, w_2 \in \mathcal{B}$ there exists a $\Delta \in [0, \infty)$ and a $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t) & \text{for } t < 0 \\ w_2(t - \Delta) & \text{for } t \geq \Delta. \end{cases}$$

We will call Σ *stabilizable* if for each $w \in \mathcal{B}$ there exists $w' \in \mathcal{B}$ such that $w(t) = w'(t)$ for $t < 0$, and $w'(t) \to 0$ as $t \to \infty$.

In our previous work [9], we have discussed the merits of this definition in much detail. It is a sweeping, but nevertheless natural, generalization of the classical definition of controllability. There is an elegant condition for the controllability of elements of $\mathcal{L}^q$ in terms of the coefficients of the defining differential equation (7).

Proposition 2: Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$. Then $\Sigma igma(R)$ is controllable iff the rank of the matrix $R(\lambda) \in \mathbb{C}^{rowdim(R) \times q}$ is the same for all $\lambda \in \mathbb{C}$, equivalently, iff $rank(R(\lambda)) = rank(R) = p(\Sigma igma(R))$ for all $\lambda \in \mathbb{C}$. It is stabilizable iff $rank(R(\lambda)) = rank(R) = p(\Sigma igma(R))$ for all $\lambda \in \mathbb{C}$ with $\Re e(\lambda) \leq 0$.

Note that if (7) is minimal, then consequently it is controllable if and only if the complex matrix $R(\lambda)$ is of full row rank for all $\lambda \in \mathbb{C}$.

At the other extreme of controllability are the autonomous systems. A system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ is said to be *autonomous* if $((w_1, w_2 \in \mathcal{B}) \wedge (w_1(t) = w_2(t) \text{ for } t < 0)) \Rightarrow (w_1 = w_2)$, in other words, if the past of a trajectory in $\mathcal{B}$ uniquely defines its future. There are many equivalent conditions for autonomy (see [10]).

Proposition 3: Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$. Then the following conditions are equivalent.

1) Σ is autonomous.
2) $\mathcal{B}$ is finite-dimensional.
3) $p(\Sigma) = q$, i.e., there exists $R \in \mathbb{R}^{q \times q}[\xi]$ with $\det(R) \neq 0$ such that $\Sigma igma(R) = \Sigma$.

Autonomous systems will be very important to us in the sequel. In particular, we will be interested in their characteristic polynomial and stability properties.

Assume that $\Sigma \in \mathcal{L}^q$ is autonomous and that $\Sigma = \Sigma igma(R)$ with $R \in \mathbb{R}^{q \times q}[\xi]$. Obviously, for any nonsingular diagonal matrix $\alpha \in \mathbb{R}^{q \times q}, \Sigma igma(R) = \Sigma igma(\alpha R)$. Therefore, we can always choose the kernel representation R of a given system in $\mathcal{L}^q$ such that $\det(R)$ is a monic polynomial. We will denote this polynomial by χ_Σ and call it the *characteristic polynomial* of Σ. It follows trivially from Proposition 1 that χ_Σ depends only on $\Sigma \in \mathcal{L}^q$ (and not on the matrix polynomial R which we have used to define it).

A polynomial $p \in \mathbb{R}[\xi]$ is called a *Hurwitz polynomial* if $p \neq 0$ and if it has all its roots in the open left-half of the complex plane. Similarly, we will call $R \in \mathbb{R}^{q \times q}[\xi]$ *Hurwitz* if $\det(R)$ is.

Assume that $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ is autonomous. We will call Σ *stable* if $w \in \mathcal{B}$ implies $\lim_{t \to \infty} w(t) = 0$. (Often this would be called *asymptotic stability*, but, in keeping with usage which has become customary, we will simply refer to this property as *stability*.) For the sake of completeness, let us state the following well-known result.

Proposition 4: Let $\Sigma \in \mathcal{L}^q$ be autonomous. Then Σ is stable iff χ_Σ is Hurwitz. Equivalently, let $R \in \mathbb{R}^{q \times q}[\xi]$ have $\det(R) \neq 0$. Then $\Sigma igma(R)$ is stable iff R is Hurwitz.

We need a couple of minor refinements, related to controllability, before embarking on control questions. Let $\Sigma \in \mathcal{L}^q$. Then, as we have just seen, Σ is controllable if and only if $\operatorname{rank}(R(\lambda))$ is constant for $\lambda \in \mathbb{C}$. The set $\Lambda(\Sigma) = \{\lambda \in \mathbb{C} | \operatorname{rank}(R(\lambda)) < \operatorname{rank}(R)\}$ is called the set of *uncontrollable exponents* of Σ. They play the role of the *uncontrollable modes* in state-space systems. More generally, assume that R is minimal. Then it can be factored as $R = FR'$ with $\Sigma igma(R') \in \mathcal{L}^q$ controllable, and $F \in \mathbb{R}^{p(\Sigma) \times p(\Sigma)}[\xi]$ having $\det(F) \neq 0$. Obviously, we can assume that $\det(F)$ is monic. It can be shown that $\det(F)$ depends on Σ only. We will call it the *characteristic polynomial of the uncontrollable part of* Σ and denote it as $\chi_{\Sigma^{uc}}$. This nomenclature can be justified as follows. Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$. Then there exists $\Sigma_1 = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1) \in \mathcal{L}^q$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_2) \in \mathcal{L}^q$ such that i) Σ_1 is controllable; ii) Σ_2 is autonomous; and iii) $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$. It can be shown that Σ_1 (called the *controllable part* of Σ) is uniquely defined by Σ. However, whereas Σ_2, the uncontrollable part, is not uniquely defined by Σ, its characteristic polynomial, χ_{Σ_2}, is. In terms of Σ, we have $\chi_{\Sigma_2} = \chi_{\Sigma^{uc}}$. It is clear from this that $\Sigma igma(R)$ is stabilizable iff the uncontrollable exponents of $\Sigma igma(R)$ have negative real parts, equivalently, iff $\chi_{\Sigma^{uc}}$, the characteristic polynomial of the uncontrollable part, is Hurwitz.

We include one final small notational element before we proceed. Above, we defined the characteristic polynomial for autonomous systems. If, however, $\Sigma \in \mathcal{L}^q$ is not autonomous, then we will define χ_Σ as the zero polynomial. We will consider the zero polynomial to be *monic* and *not* Hurwitz. This notation is consistent with our earlier one. Note that if we take a $R \in \mathbb{R}^{q \times q}[\xi]$ such that $\Sigma igma(R) = \Sigma$, and make sure that $\det(R)$ is monic, then $\chi_\Sigma = \det(R)$. Also, we will consider every polynomial to be a factor of the zero polynomial.

VI. Pole Placement and Stabilization in a Behavioral Framework

In this section we will study our first control problem, with control viewed as interconnection as explained in Section IV. The plant is a given dynamical system $\Sigma \in \mathcal{L}^q$. We will assume that the controller (and hence the controlled system) is also a linear differential system. Let $\Sigma_k = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_k) \in \mathcal{L}^q, k = 1, 2$. We will call Σ_2 a *subsystem* of Σ_1 (denoted $\Sigma_2 \leq \Sigma_1$) if $\mathcal{B}_2 \subseteq \mathcal{B}_1$. It follows from Proposition 1 that if $\Sigma_k = \Sigma igma(R_k)$, then $\Sigma_2 \leq \Sigma_1$ iff there exists $F_2 \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $R_1 = F_2 R_2$. This in particular implies that then χ_{Σ_2} is a factor of χ_{Σ_1}. Obviously, for any $\Sigma, \Sigma' \in \mathcal{L}^q, \Sigma \wedge \Sigma'$ will be a subsystem of Σ. Our first result is the analogue of the classical placement result.

Theorem 5: Let $\Sigma \in \mathcal{L}^q$ and assume that Σ is not autonomous. Then for any monic $r \in \mathbb{R}[\xi]$ there exists $\Sigma' \in \mathcal{L}^q$ such that $\chi_{\Sigma \wedge \Sigma'} = r$. If $\Sigma \in \mathcal{L}^q$ is autonomous, then there exists $\Sigma' \in \mathcal{L}^q$ such that $\chi_{\Sigma \wedge \Sigma'} = r$ if and only if r is a factor of χ_Σ.

Proof: Assume first that Σ is not autonomous. Let $R \in \mathbb{R}^{\bullet \times q}[\xi]$ be such that $\Sigma igma(R) = \Sigma$. By Proposition 1, we may as well assume that R is of full row rank. Now, there exist real unimodular polynomial matrices U and V such the URV is in Smith form, i.e., such that

$$URV = [\mathrm{diag}(d_1, d_2, \cdots, d_{p(\Sigma)}) \quad 0_{p(\Sigma) \times m(\Sigma)}]$$

with $0 \neq d_k \in \mathbb{R}[\xi]$ for $k = 1, 2, \cdots, p(\Sigma)$. (Of course we can make sure that d_{k+1} is also a factor of d_k, but we will not need this property.) Now define $C \in \mathbb{R}^{q \times q}[\xi]$ by $CV = \mathrm{diag}(1, 1, \cdots, 1, r)$. We will now show that $\Sigma' = \Sigma igma(C)$ achieves $\chi_{\Sigma \wedge \Sigma'} = r$.

Let $\mathcal{B}$ and $\mathcal{B}'$ be the behavior of Σ and Σ', respectively. Observe that $\mathcal{B}' \subseteq \mathcal{B}$. In fact, it is trivial to see that $\{((CV)(d/dt)w = 0) \Rightarrow ((URV)(d/dt)w = 0)\}$, from where the implication $\{(C(d/dt)w = 0) \Rightarrow ((UR)(d/dt)w = 0)\}$ follows, using the unimodularity of V. It follows that $\chi_{\Sigma \wedge \Sigma'} = \chi_{\Sigma'}$. Therefore it suffices to show that $\chi_{\Sigma'} = r$ which, fortunately, is true, since $r = \det(CV) = \det(V)\det(C)$. Whence $\det(C) = \alpha r$ for some $0 \neq \alpha \in \mathbb{R}$, as desired.

Now consider the case that Σ is autonomous. First observe that whatever be $\Sigma' \in \mathcal{L}^q, \chi_{\Sigma \wedge \Sigma'}$ will be a factor of χ_Σ. Thus it suffices to show that every such factor is achievable. Repeat the above proof with $CV = \mathrm{diag}(r_1, r_2, \cdots, r_q)$, where $r_k := gcd(d_k, r/r_{k-1})$ and $r_0 := 1$. It is easily seen that $r_1 r_2 \cdots r_q = r$. Further, since again $\mathcal{B}' \subseteq \mathcal{B}$, it follows that $x_{\Sigma \wedge \Sigma'} = \chi_\Sigma = r$, as desired. ■

The result just obtained guarantees pole placement (and hence stabilizability) for any $\Sigma \in \mathcal{L}^q$ which is not autonomous, i.e., as long as in system (1) describing Σ there are fewer equations than variables $p(\Sigma) < q$). (We shall later see that this means that at least one of the variables $w_1, w_2, \cdots, w_q$ is an input variable.) *Note that not even controllability or stabilizability of Σ is required for this to hold!* In particular, stabilizability thus holds by simple interconnection, regardless of the location of the uncontrollable exponents of Σ. It holds when Σ is not autonomous. This result goes against the grain. It invites protest. This result is due to the fact that the class of admissible controllers was chosen to be all of $\mathcal{L}^q$. In particular, by taking $\Sigma' = (\mathbb{R}, \mathbb{R}^q, 0)$ stability is trivially obtained. We will return later to the question of how such a control law could be implemented.

VII. Regular Interconnection

We will now introduce an important type of interconnection (which, as we shall see later, corresponds to singular feedback). Let $\Sigma, \Sigma' \in \mathcal{L}^q$. We will call $\Sigma \wedge \Sigma'$ a *regular interconnection* if $p(\Sigma \wedge \Sigma') = p(\Sigma) + p(\Sigma')$. There are a number of alternative equivalent ways of expressing this. In particular, if $\Sigma = \Sigma igma(R)$ and $\Sigma' = \Sigma igma(R')$, with R and R' of full row rank, then $\Sigma \wedge \Sigma'$ is a regular interconnection iff *col*$[R|R']$ is also a full row rank polynomial matrix.

It is trivial to see that any subsystem Σ'' of Σ can be realized through interconnection. Indeed if we take $\Sigma' = \Sigma''$, then obviously $\Sigma \wedge \Sigma' = \Sigma''$. However, this special interconnection is regular only in the trivial case $p(\Sigma) = 0$. The question thus arises when Σ'' can be achieved by regular interconnection. Actually, we shall now see that any subsystem of Σ can still be realized through regular interconnection provided that Σ is controllable!

Theorem 6: Assume that $\Sigma \in \mathcal{L}^q$ is controllable. Let $\Sigma'' \in \mathcal{L}^q$ be a subsystem of Σ. Then there exists a $\Sigma' \in \mathcal{L}^q$ such that $\Sigma \wedge \Sigma' = \Sigma''$ and such that this interconnection is regular.

Proof: Let $\Sigma = \Sigma igma(R)$ with R minimal. By controllability, its Smith form will yield $URV = [I_{p(\Sigma)} | O_{p(\Sigma) \times m(\Sigma)}]$. Let $\Sigma'' = \Sigma igma(R'')$ with R'' minimal. Then, since $\Sigma'' \leq \Sigma$, there will exist R'' with $\Sigma'' = \Sigma igma(R'')$ and $R''V$ of the form

$$R''V = \left[\begin{array}{c|c} I_{p(\Sigma)} & O_{p(\Sigma) \times m(\Sigma)} \\ \hline R''_{21} & R''_{22} \end{array}\right].$$

Note that we can always take $[R''_{21}|R''_{22}]$ of full row rank (in fact, with $R''_{21} = 0$). Now choose $\Sigma' = \Sigma igma([R''_{21}|R''_{22}]V^{-1})$. ∎

We shall soon see that a regular interconnection can be implemented by means of singular feedback. The important conclusion which may be drawn from the above theorem will be that singular feedback control problems for controllable systems amount to looking for a suitable subsystem.

One important variation of the above theorem worth stating is the following.

Theorem 7: Assume that $\Sigma \in \mathcal{L}^q$, and let $r \in \mathbb{R}[\xi]$ be monic. Then there exists $\Sigma' \in \mathcal{L}^q$ such that $\Sigma \wedge \Sigma'$ is i) a regular interconnection; and ii) $\chi_{\Sigma\wedge\Sigma'} = r$ if and only if χ_{Σ}^{uc} is a factor of r.

This theorem can be easily proven along the lines of our proof of Theorem 6. In particular, Theorem 7 implies that pole placement by means of regular feedback holds iff Σ is controllable, and there exists a Σ' such that $\Sigma \wedge \Sigma'$ is i) a regular interconnection; and ii) stable, iff Σ is stabilizable.

VIII. Implementation of Regular Interconnections by Singular Feedback

As we have argued before, we view interconnection as the basic idea of control. However, there remains the problem of controller implementation. In this section we will study this question for regular interconnections.

We have already encountered autonomous systems in $\mathcal{L}^q$, dynamical systems with a finite-dimensional behavior. If a system is not autonomous, then certain components of the signal vector are free. We will now formalize this.

Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$. Let the signal w be partitioned into two subvectors $w = \mathrm{col}(w_1, w_2)$. Assume that w_1 has q_1 components and that w_2 has q_2 components, with $q_1 + q_2 = q$. We will call w_1 *free* if for all $w_1 \in \mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{q_1})$ there exists $w_2 \in \mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{q_2})$ such that $\mathrm{col}(w_1, w_2) \in \mathcal{B}$. We will call it *maximally free* if no further free components are left in w_2, equivalently, it turns out, iff the set $\{w_2 \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{q_2}) | \mathrm{col}(0, w_2) \in \mathcal{B}\}$ is finite-dimensional. In addition to a component of w being free in $\mathcal{L}_1^{\mathrm{loc}}$, we are also interested in components which are free in C^∞. As we shall see, this constitutes a slightly different notion. We will call w_1, *C^∞-free* if for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^{q_1})$ there exists $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ such that $\mathrm{col}(w_1, w_2) \in \mathcal{B}$. The notion of *maximally C^∞-free* now follows.

Let us see how these notions translate into properties of a kernel representation. Write the minimal kernel representation $R(d/dt)w = 0$ in terms of w_1 and w_2, yielding

$$P\left(\frac{d}{dt}\right)w_2 = Q\left(\frac{d}{dt}\right)w_1. \tag{9}$$

Then w_1 is maximally C^∞-free iff P is square (i.e., $q_2 = p(\Sigma)$) and $\det(P) \neq 0$. It is maximally ($\mathcal{L}_1^{\mathrm{loc}}$-) free iff in addition the *transfer function*

$$G(\xi) := P^{-1}(\xi)Q(\xi) \tag{10}$$

is proper. Obviously, $G \in \mathbb{R}^{p(\Sigma)\times m(\Sigma)}(\xi)$. If $w = \mathrm{col}(w_1, w_2)$ is as in (9), then we will call this an *input–output partition* with input w_1 and output w_2. If in addition G is proper, we will speak of a *proper input–output partition*.

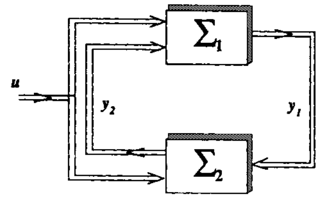

Fig. 7. A signal flow interpretation of $\Sigma_1 \wedge \Sigma_2$.

Now return to a given $\Sigma = \Sigma igma(R)$. Consider all $p(\Sigma) \times p(\Sigma)$ minors of R and assume that the $p(\Sigma) \times p(\Sigma)$ minor formed by the last $p(\Sigma)$ columns has maximal degree among all the $p(\Sigma) \times p(\Sigma)$ minors of R. This can always be achieved by permuting the columns of R. Note that this column permutation can be interpreted as a reordering of the components of w. Assuming now that this permutation has been done, partition w as $w = \mathrm{col}(w_1, w_2)$ with w_1, the first $m(\Sigma)$ and w_2 the last $p(\Sigma)$ elements of w. Now the maximal degree property of P implies that the corresponding transfer function (10) will be proper. The conclusion (of this reordering) is that every system in $\mathcal{L}^q$ is actually a proper input–output system. Note that the number of inputs in an input–output partition always equals $m(\Sigma)$, and that the number of outputs always equals $p(\Sigma)$, but that we may have a choice as to which components are actually taken to be inputs and which are taken to be outputs.

Let $\Sigma_1, \Sigma_2 \in \mathcal{L}^q$. We will now study the structure and the resulting implementation of the interconnection $\Sigma_1 \wedge \Sigma_2$ from an input–output point of view. Assume that Σ_1 and Σ_2 admit the following kernel representations:

$$\begin{aligned} \Sigma_1\colon \quad & P_1\left(\frac{d}{dt}\right)y_1 = Q_1\left(\frac{d}{dt}\right)y_2 + Q_1'\left(\frac{d}{dt}\right)u \\ \Sigma_2\colon \quad & P_2\left(\frac{d}{dt}\right)y_2 = Q_2\left(\frac{d}{dt}\right)y_1 + Q_2'\left(\frac{d}{dt}\right)u \end{aligned} \tag{11}$$

with P_1 and P_2 square and

$$\det(P_1) \neq 0, \quad \det(P_2) \neq 0, \quad \det\begin{bmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{bmatrix} \neq 0. \tag{12}$$

Let us first interpret these conditions. The condition $\det(P_1) \neq 0$ means that in $\Sigma_1, \mathrm{col}(u, y_2)$ serves as input and y_1 as output. The condition $\det(P_2) \neq 0$ means that in $\Sigma_2, \mathrm{col}(u, y_1)$ serves as input and y_2 as output. The condition

$$\det\begin{bmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{bmatrix}$$

implies two things: first, that $\Sigma_1 \wedge \Sigma_2$ is a regular interconnection, and second, that in $\Sigma_1 \wedge \Sigma_2, u$ serves as input, while $\mathrm{col}(y_1, y_2)$ serves as output. This interpretation can be illustrated nicely in a signal flow graph as shown in Fig. 7.

An interconnection $\Sigma_1 \wedge \Sigma_2$ in which the variables $w = \mathrm{col}(u, y_1, y_2)$ (perhaps after reordering of the components) admit the kernel representation (11) with the conditions (12) satisfied will be called a *feedback interconnection*. Note that in this interconnection we have not assumed that the transfer functions

$$P_1^{-1}(Q_1|Q_1'),\quad P_2^{-1}(Q_2|Q_2'),$$
$$\left[\begin{array}{c|c} P_1 & -Q_1 \\ \hline -Q_2 & P_2 \end{array}\right]^{-1}\left[\begin{array}{c} Q_1' \\ \hline Q_2' \end{array}\right]$$

are proper. If they are all proper, then we will call the interconnection $\Sigma_1 \wedge \Sigma_2$ a *regular feedback interconnection*. If a feedback interconnection is not regular, then it will be called *singular*.

We will now prove two results regarding interconnections. The first states that *every regular* interconnection can be viewed as an, in general, *singular*, feedback interconnection. The second states that every subsystem of a controllable system can be implemented as a (in general, singular) feedback interconnection with a proper input–output structure for the plant.

Theorem 8: Let $\Sigma_1, \Sigma_2 \in \mathcal{L}^q$, and assume that $\Sigma_1 \wedge \Sigma_2$ is a regular interconnection. Then the signal vector w admits a componentwise partition as $w = \text{col}(u, y_1, y_2)$ such that $\Sigma_1 \wedge \Sigma_2$ is a feedback interconnection. Moreover, this partition can be chosen such that in $\Sigma_1 \wedge \Sigma_2, \text{col}(u, \text{col}(y_1, y_2))$ is a proper input–output partition.

Proof: Write minimal kernel representation for Σ_1 and Σ_2

$$R_1\left(\frac{d}{dt}\right)w = 0$$
$$R_2\left(\frac{d}{dt}\right)w = 0.$$

Since $\Sigma_1 \wedge \Sigma_2$ is a regular interconnection

$$R := \left[\begin{array}{c} R_1 \\ \hline R_2 \end{array}\right]$$

will also be of full row rank. Now write $\Sigma_1 \wedge \Sigma_2$ in proper *input–output* form to yield

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$$

with $\det(P) \neq 0$ and $P^{-1}Q$ proper. Observe that the theorem follows if there exists an ordering of its columns such that P can be written as

$$P = \left[\begin{array}{c|c} P_1 & -Q_1 \\ \hline -Q_2 & P_2 \end{array}\right]$$

with

$$P_1 \in \mathbb{R}^{p(\Sigma_1)\times p(\Sigma_1)}[\xi],\quad \det(P_1) \neq 0$$
$$P_2 \in \mathbb{R}^{p(\Sigma_2)\times p(\Sigma_2)}[\xi],\quad \det(P_2) \neq 0.$$

Such an ordering exists, since $\det(P) \neq 0$, by Lagrange's formula, in which $\det(P)$ is written as the sum of the product of its $p(\Sigma_1)\times p(\Sigma_1)$ minors from its first $p(\Sigma_1)$ rows multiplied by the complementary $p(\Sigma_2) \times p(\Sigma_2)$ minors from its last $p(\Sigma_2)$ rows. ∎

The above theorem tells us that we can always choose the variables in a regular interconnection so as to achieve a feedback interconnection. The next theorem tells us that if we want to achieve a given subbehavior of a controllable system, then we can even start from a given input–output structure for the plant.

Theorem 9: Let $\Sigma, \Sigma' \in \mathcal{L}^q$ and assume that $\Sigma' \leq \Sigma$, i.e., Σ' is a subsystem of Σ. Assume, moreover, that Σ is controllable. Let $w = \text{col}(u, y_1, y_2)$ be a componentwise partition of w having the following properties.

1) $\text{col}(u, (y_1, y_2))$ is an input–output partition for Σ'.
2) $\text{col}(\text{col}(u, y_2), y_1)$ is an input–output partition for Σ.

(*Note that consequently both these input–output partitions may be chosen to be proper.)*

Then there exists a $\Sigma'' \in \mathcal{L}^q$ such that $\Sigma' = \Sigma \wedge \Sigma''$ and such that $\Sigma \wedge \Sigma''$ is a feedback interconnection relative to the partition $w = \text{col}(u, y_1, y_2)$.

Proof: Choose the partition $\text{col}(u, y_1, y_2)$ such that $\text{col}(u, \text{col}(y_1, y_2))$ is a proper input–output partition in Σ' and such that $\text{col}(\text{col}(u, y_2), y_1)$ is a proper input–output partition in Σ. Now write a minimal kernel representation for Σ

$$P_1\left(\frac{d}{dt}\right)y_1 = Q_1\left(\frac{d}{dt}\right)y_2 + Q_1'\left(\frac{d}{dt}\right)u \tag{13}$$

whence $\det(P_1) \neq 0$. Complete (13) so that, together with (14), they form a minimal kernel representation for Σ'

$$P_2\left(\frac{d}{dt}\right)y_2 = Q_2\left(\frac{d}{dt}\right)y_1 + Q_2'\left(\frac{d}{dt}\right)u. \tag{14}$$

That such a representation exists follows from the controllability of Σ and Theorem 6. Hence

$$\det\left(\left[\begin{array}{c|c} P_1 & -Q_1 \\ \hline -Q_2 & P_2 \end{array}\right]\right) \neq 0. \tag{15}$$

It is easy to see that (15) implies that there exists an $S \in \mathbb{R}^{(p(\Sigma')-p(\Sigma))\times p(\Sigma)}[\xi]$ such that $\det(P_2 + SQ_1) \neq 0$. Now consider the behavioral equation

$$(P_2 + SQ_1)\left(\frac{d}{dt}\right)y_2 = (Q_2 + SP_1)\left(\frac{d}{dt}\right)y_1 + (Q_2' - SQ_1')\left(\frac{d}{dt}\right)u \tag{16}$$

and define Σ'' as the system which has (16) as its kernel representation. It is now easily verified that Σ'' has the properties required in Theorem 9. ∎

Theorem 9 is, in our opinion, an important one in that it reduces the issue of the design of a feedback control law (provided that we allow it to be a singular feedback control law) to that of finding a suitable subsystem. Let us explain this in the case that the desired "closed-loop" system Σ' is autonomous in a bit more detail. We can start from any given input–output partition of the plant Σ. This partition could, for example, be imposed by the actuator/sensor structure of Σ, or it could ensure that this input–output partition is (strictly) proper. Theorem 9 states that the desired controlled system Σ' can be achieved by means of a feedback controller. In general, this feedback controller will, of course, need to be singular.

However, the example in Section III serves to illustrate that singular controllers need not be objectionable.

IX. Invariant Polynomial Assignment

The remainder of this paper is devoted to some refinements of our main results. These state: in a nonautonomous system every characteristic polynomial can be obtained by interconnection; for a controllable system this can be done by a regular interconnection and this is equivalent to a feedback interconnection. In the present section we consider the question which invariant polynomials can be achieved by interconnection, a sharpening of the characteristic polynomial assignability.

Let us first define the invariant polynomials of an element $\Sigma \in \mathcal{L}^q$. Introduce an equivalence relation on $\mathcal{L}^q$ by calling $\Sigma_1 = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1)$ equivalent to $\Sigma_2 = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_2)$ if there exists a unimodular $U \in \mathbb{R}^{q\times q}[\xi]$ such that $\mathcal{B}_2 = U(d/dt)\mathcal{B}_1$. Note that $U(d/dt)$ is a bijective differential operator, which gives this equivalence relation a very natural interpretation. In view of this we will call it *differential equivalence*. We will now construct a complete set of invariants for this equivalence relation, that is, a mapping f from $\mathcal{L}^q$ into a space F such that Σ_1 and Σ_2 are differentially equivalent if and only if $f(\Sigma_1) = f(\Sigma_2)$.

For F we will take the space of ordered q-tuples of monic real polynomials $(d_1, d_2, \cdots, d_q)$ such that d_{k+1} is a factor of d_k for $k = 1, 2, \cdots, q-1$. Now associate with $\Sigma \in \mathcal{L}^q$ the element of F formed by taking the diagonal elements of the Smith form of any $R \in \mathbb{R}^{q\times q}[\xi]$ such that $\Sigma igma(R) = \Sigma$. Let f: $\mathcal{L}^q \to F$ be the resulting map. This map is a complete invariant under differential equivalence. Let $f(\Sigma) = (d_1, d_2, \cdots, d_q)$. Then $d_1 = d_2 = \cdots = d_{m(\Sigma)} = 0; \Sigma$ is controllable iff $d_{m(\Sigma)+1} = \cdots = d_q = 1$. The polynomials $d_1, d_2, \cdots, d_q$ will be called the *invariant polynomials* of Σ. Now introduce a partial ordering on F by taking $((g_1', g_2', \cdots, g_q') = g' \le g'' = (g_1'', g_2'', \cdots, g_q'')) \;:\Leftrightarrow\; (g_k'$ is a factor of g_k'' for $k = 1, 2, \cdots, q)$.

Let $\Sigma, \Sigma' \in \mathcal{L}^q$ with $\Sigma' \le \Sigma$. Then it can be shown that $f(\Sigma) \le f(\Sigma')$. This leads to the following result.

Theorem 10: Let $\Sigma_1 \in \mathcal{L}^q$. Then for all $\Sigma_2 \in \mathcal{L}^q, f(\Sigma_1) \le f(\Sigma_1 \wedge \Sigma_2)$. Conversely, if $d \in F$ satisfies $d \ge f(\Sigma_1)$, there will exist a $\Sigma_2 \in \mathcal{L}^q$ such that $f(\Sigma_1 \wedge \Sigma_2) = d$.

We will not give the proof. Since χ_Σ equals the product of the elements of $f(\Sigma)$, this theorem is a generalization of Theorem 5. Specializing to the controllable case yields the following corollary.

Corollary 11: Let $\Sigma \in \mathcal{L}^q$ be controllable. Then $f(\Sigma) = (0, \cdots, 0, 1, \cdots, 1)$ with $m(\Sigma)$ zeros and $p(\Sigma)$ ones. Then for each set of real polynomials $d' = (d_1, \cdots, d_{m(\Sigma)}, 1, \cdots, 1) \in F$, there exists a $\Sigma' \in \mathcal{L}^q$ such that $\Sigma \wedge \Sigma'$ is a regular interconnection with $f(\Sigma \wedge \Sigma') = d'$.

Corollary 11 and Theorem 9 thus imply that for a controllable plant Σ every set of invariant polynomials $(d_1, d_2, \cdots, d_{m(\Sigma)}, 1, \cdots, 1)$ will hence be achievable by a (singular) feedback interconnection. Theorems 5 and 10 provide useful generalizations of the classical pole placement results. Further results on linear systems and interconnections from the behavioral point of view may be found in [2] and [3]. This last reference is particularly relevant for the purposes of the present paper. In it, stabilization from the behavioral point of view is related to the more classical input–output feedback definitions.

X. Regular Feedback

In this section we will discuss the "classical" notion of feedback in our setting. First, however, we introduce the dimension of the state space of an element of $\mathcal{L}^q$.

Let $E, F \in \mathbb{R}^{\bullet\times n}, G \in \mathbb{R}^{\bullet\times q}$, and assume that these matrices all have the same number of rows. Now consider the system of differential equations

$$E\frac{dx}{dt} + Fx + Gw = 0. \tag{17}$$

Let $\mathcal{B}_f$ denote all $(w, x) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n)$, which satisfy (17) in the distributional sense. Now define the external behavior of (17) as $\mathcal{B}_{\text{ext}} = \{w | \exists x$ such that $(w, x) \in \mathcal{B}_f\}$. It can be shown that systems described by equations such as (17) (first order in x, zeroth order in w) are state systems [10]. Further, for each element $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ there exist matrices E, F, G such that $\mathcal{B}_{\text{ext}} = \mathcal{B}$. Among all such representations there are some for which the dimension of the state space, n, is as small as possible. This minimal number will be denoted as $n(\Sigma)$. It is also equal to the minimal state-space dimension in an input/state/output representation of Σ.

It is possible to relate $n(\Sigma)$ to a minimal kernel representation $R(d/dt)w = 0$ of Σ. List all $p(\Sigma) \times p(\Sigma)$ minors of Σ. Since R has full row rank, at least one of these will be nonzero. The maximum of the degrees of all these minors is called the *McMillan degree* of R. It can be shown that it equals $n(\Sigma)$.

Let $\Sigma_1, \Sigma_2 \in \mathcal{L}^q$. Recall that we call the interconnection $\Sigma_1 \wedge \Sigma_2$ *regular* if $p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$. We will call it a *regular feedback interconnection* if in addition $n(\Sigma_1 \wedge \Sigma_2) = n(\Sigma_1) + n(\Sigma_2)$. We now state a theorem which explains this nomenclature.

Theorem 12: Let $\Sigma_1, \Sigma_2 \in \mathcal{L}^q$, and $\Sigma_1 \wedge \Sigma_2$ be a regular feedback interconnection. Then the signal vector w admits a componentwise partition as $w = \text{col}(u, y_1, y_2)$ such that:

1) in $\Sigma_1, (u, y_2)$ is input, y_1 is output, and the transfer function is proper;
2) in $\Sigma_2, (u, y_1)$ is input, y_2 is output, and the transfer function is proper;
3) in $\Sigma_1 \wedge \Sigma_2, u$ is input, (y_1, y_2) is output, and the transfer function is proper.

Proof: Follow the proof of Theorem 8, and observe that in this case it is possible to obtain (again by Lagrange's formula):

$$\begin{aligned} \textit{degree}\ (\det(P_1)) &= n(\Sigma_1) \\ \textit{degree}\ (\det(P_2)) &= n(\Sigma_2) \\ \textit{degree}\ (\det(P)) &= n(\Sigma_1 \wedge \Sigma_2). \end{aligned}$$

■

Regular feedback interconnections can be shown to be equivalent to a number of other statements in addition to the one of Theorem 12. In [3] regular feedback is shown to be equivalent to what is usually called *well-posedness*. Another equivalent condition is: let $\Sigma_i = (\mathbb{R}, \mathbb{R}^q, \mathcal{B}), i = 1, 2$. Then $\Sigma_1 \wedge \Sigma_2$ is a regular feedback connection iff for all $w_1 \in \mathcal{B}_1$ and $w_2 \in \mathcal{B}_2$, there exists $w \in \mathcal{B}_1 \cap \mathcal{B}_2$ such that $w_1 \wedge w \in \mathcal{B}_1$ and $w_2 \wedge w \in \mathcal{B}_2$ (recall that for time functions $\wedge$ denotes concatenation at $t = 0$). In words, whatever past trajectory has occurred in Σ_1 and Σ_2, it is possible to continue these trajectories in accordance simultaneously with the laws of Σ_1 and Σ_2.

This consideration allows us to interpret clearly the distinction between regular interconnection and regular feedback interconnection. The first requires preparing (the state of) both systems before interconnection; while in the second the control can start acting at any time. The second type of control action is the one which is usually pursued in control theory. However, our example in Section II indicates that regular interconnections which are not regular feedback interconnections have many applications in engineering practice. Moreover, Theorem 6 indicates that from a theoretical point of view they reduce the question of control design for a controllable system to that of finding a subsystem. In follow-up papers, we will exploit this in the context of LQ and H_∞ control.

The question of what can be achieved by regular feedback interconnection and which subsystems $\Sigma' \leq \Sigma$ of a given system $\Sigma \in \mathcal{L}^q$ can be implemented by means of a regular feedback interconnection remains a largely unexplored one. The problem promises to be unsolvable as it stands, since it has the pole placement problem by memoryless feedback as a special case. In the next section we will mention, without proof, some results which can be obtained on this problem for controllable systems.

XI. Remarks

1) The observability and controllability indexes of a system can be defined on the level of behaviors. However, it is much more insightful to do this for discrete-time than for continuous-time systems. Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}_d^q$, where $\mathcal{L}_d^q$ denotes the discrete time analogue of $\mathcal{L}^q$, i.e., the class of systems described by difference equations $R(\sigma)w = 0$, analogous to (8) but with σ the shift operator instead of differentiation. Define the *observability index* as the smallest L such that $w_1 \in \mathcal{B}, w_2 \in \mathcal{B}$, and $w_1(0) = w_2(0), w_1(1) = w_2(1), \cdots, w_1(L-1) = w_2(L-1)$ imply $w_1 \wedge w_2 \in \mathcal{B}$ ($\wedge$ denotes concatenation). Define the *controllability index* as the smallest L such that for all $w_1 \in \mathcal{B}, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t + L) = w_2(t)$ for $t \geq 0$. These indexes (in continuous—as well as in discrete-time systems) can be related to kernel and image representations, as follows. Consider the degrees of all the polynomial matrices R which induce a kernel representation of $\Sigma \in \mathcal{L}^q$. The observability index is the lowest of these degrees. Consider next the degrees of all the polynomial matrices M such that $w = M(d/dt)\ell$ induces an image representation [10] of $\Sigma \in \mathcal{L}^q$. The controllability index is the lowest of these degrees.

Let $\kappa(\Sigma)$ denote the controllability index of Σ. Then given any $\chi \in \mathbb{R}[\xi]$ of degree $n(\Sigma_1) + \kappa(\Sigma_1) - 1$, there exists a Σ_2 such that $\Sigma_1 \wedge \Sigma_2$ is a regular feedback interconnection with $\chi_{\Sigma_1 \wedge \Sigma_2} = \chi$, an analogous result with κ replaced by μ, the observability index.

2) Up to now we have considered the case in which all the variables w are available for interconnection. However, in applications there will be many situations where this will not be feasible, leading to the structure depicted in Fig. 8 in which the controller can impose only laws on the plant variables to the right.

This situation can be formalized in the context of systems with latent variables [10], leading to

$$R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell. \tag{18}$$

Since we assume (see Fig. 8) that only the variables w are available for interconnection, it is natural to view them as *manifest* variables, while viewing the variables ℓ which are not available for interconnection as *latent* variables. Control of (18) is very similar to the case treated earlier, provided we assume that (18) is an observable [10] latent variable system. Since observability is equivalent to the existence of a representation

$$R'\left(\frac{d}{dt}\right)w = 0 \quad R''\left(\frac{d}{dt}\right)w = \ell \tag{19}$$

which is equivalent to (18), it follows that any control law for (18), based on both w and ℓ, e.g.,

$$C_1\left(\frac{d}{dt}\right)w + C_2\left(\frac{d}{dt}\right)\ell = 0 \tag{20}$$

can actually be implemented (as far as the manifest behavior of the controlled system is concerned) by following control law:

$$(C_1 + C_2R'')\left(\frac{d}{dt}\right)w = 0. \tag{21}$$

The latter involves only the w variables. Hence, if we assume observability and if we do not worry about implementability by regular feedback, the theory changes little, in case only a limited number of variables are available for control interconnection. Note, however, that the usual additive disturbances in the measurements already obstruct observability.

3) Here are a few words about how state feedback fits in our framework. In [10] the notion of state (as a special type of latent variable) has been described in full detail. Let $\Sigma \in \mathcal{L}^q$ be described by $R(d/dt)w = 0$. Then there exists a $X \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $x = X(d/dt)w$ is a minimal state for Σ (see [4] for algorithms for constructing such an X). Hence, a control law acting on the state, say $Lw = K(d/dt)x$, will lead to a control law $C(d/dt)w = 0$ of a special form $C = L - KX$.

We describe one particularly important situation in a bit more detail. Let

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u \tag{22}$$

be a controllable input–output system with proper transfer function $G = P^{-1}Q$. Since this system is controllable, it admits an observable image representation [10]

$$u = D\left(\frac{d}{dt}\right)\ell, \quad y = N\left(\frac{d}{dt}\right)\ell. \tag{23}$$

Now assume that we try to implement by means of an interconnection a given subsystem Σ' of Σ. In this case a subsystem always comes down to adding to (23) a constraint on ℓ

$$S\left(\frac{d}{dt}\right)\ell = 0. \tag{24}$$

Now, it can be shown that (24) can be implemented [in the sense that (25) together with (23) will yield (24)] by a memoryless input-state law

$$Hu + Fx = 0 \tag{25}$$

if S (or VS with V unimodular) is such that SD^{-1} is proper. In fact, a suitable H will be given by $H = \lim_{\xi\to\infty} S(\xi)(D(\xi))^{-1}$. Hence, if this limit is square and nonsingular (24) can be implemented in the familiar

$$u = Fx \tag{26}$$

fashion.

4) Rather complete results concerning subsystem implementability by (regular) interconnection can be obtained when Σ is a single-input/single-output $(q = 2, m(\Sigma) = p(\Sigma) = 1)$ controllable system and when the subsystem $\Sigma' \leq \Sigma$ is autonomous $(q = p(\Sigma') = 2, m(\Sigma') = 0)$.
   a) There always exists a Σ'' such that $\Sigma' = \Sigma \wedge \Sigma''$ with $\Sigma \wedge \Sigma''$ a regular interconnection.
   b) If $n(\Sigma') \geq 2n(\Sigma) - 1$, then this interconnection can be taken to be a regular feedback interconnection. If $n(\Sigma') < n(\Sigma)$, then this interconnection cannot be taken to be a regular feedback interconnection.
   c) If $n(\Sigma') = n(\Sigma)$, then Σ'' can be taken to be a memoryless regular state feedback law $u = Fx$.
   d) If $n(\Sigma') \geq n(\Sigma)$, then Σ'' can be taken to be a dynamic state feedback law

$$H\left(\frac{d}{dt}\right)u = F\left(\frac{d}{dt}\right)x$$

   with $0 \neq H \in \mathbb{R}[\xi]$ and $H^{-1}F$ proper.

5) When an interconnection $\Sigma_1 \wedge \Sigma_2$ satisfies $p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$ (a regular interconnection), then we have seen that it can always be implemented by feedback structure with in general a nonproper transfer functions. Such structures are common in control engineering (e.g., the example in Section III, PD or PID control). Hence singular feedback interconnections are useful in practice, need not cause noise amplification, and cannot be dismissed for that reason. Much more attention should be paid to these situations in present day control theory. It is when in addition to $p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$, there holds $n(\Sigma_1 \wedge \Sigma_2) = n(\Sigma_1) + n(\Sigma_2)$, that we are in the realm of the usual intelligent control structure implementation.

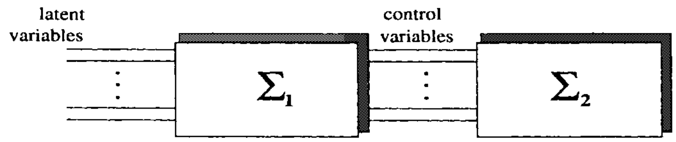

Fig. 8. Interconnection along control variables.

The issue of implementing a general interconnection which does not satisfy $p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$ as a control law remains unsettled: is it or is it not reasonable to allow such interconnections as control law? For example, it does not seem *a priori* unreasonable to allow $F = q = 0$ as a control law for a mechanical system. This could be achieved by pinning down the mass at a fixed position. Also, it does not seem *a priori* unreasonable to allow $V = 0, I = 0$ as the control law for an electrical circuit. In fact, there exist (active) one-ports whose external behavior is precisely $V = I = 0$ (such one-ports are called *nullators* (see [1], p. 75) for a realization of it). Terminating a passive (impedance or admittance) one-port with such a circuit will indeed result in an interconnection which is not regular.

6) One notoriously difficult problem in linear control theory is the question of generic eigenvalue assignment by memoryless output feedback. Consider a multivariable state-space system $dx/dt = Ax + Bu, y = Cx$, with m input, n state, and p output variables. Then it is easily seen that $n \leq m * p$ is a necessary condition for generic closed-loop pole assignability by memoryless linear output feedback. In [7] it is shown by analyzing the case $n = 4, m = p = 2$ that $n \leq m * p$ is not sufficient. Recently, Wang [6] has proven that $n < m * p$ is sufficient—*one mere extra degree of freedom suffices*! In [14] we have given a remarkable simple proof of this result. This proof is based on the idea of the present paper; by considering interconnection instead of feedback, we were able to get a much better handle on this theoretical problem.

XII. Stabilization of Nonlinear Systems

In this section we will apply the ideas of this paper to the stabilization of a nonlinear system around an equilibrium point.

Consider a nonlinear differential dynamical system described by

$$f\left(w, \frac{dw}{dt}, \cdots, \frac{d^L w}{dt^L}\right) = 0 \tag{27}$$

with f: $\mathbb{R}^q \times \mathbb{R}^q \times \cdots \times \mathbb{R}^q \to \mathbb{R}^g$. Equation (27) induces the nonlinear time-invariant dynamical system $\Sigma_{NL} := (\mathbb{R}, \mathbb{R}^q, \mathcal{B}_{NL})$ with

$$\mathcal{B}_{NL} := \Big\{ w \in C^L(\mathbb{R}, \mathbb{R}^q) | f \cdot \left(w(t), \frac{dw}{dt}(t), \cdots, \frac{d^L w}{dt^L}(t)\right) = 0 \text{ for all } t \in \mathbb{R} \Big\}. \tag{28}$$

Now assume that $a^* \in \mathbb{R}^q$ is an equilibrium point of (27), i.e.,

$$f(a^*, 0, \cdots, 0) = 0. \tag{29}$$

Linearizing (27) around this equilibrium yields

$$R_0 \Delta + R_1 \frac{d\Delta}{dt} + \cdots + R_L \frac{d^L \Delta}{dt^L} = 0 \tag{30}$$

where R_k denote the partial derivatives of f evaluated at $(a^*, 0, \cdots, 0)$. Specifically

$$R_k := \frac{\partial f}{\partial \sigma_k}(a^*, 0, \cdots, 0) \tag{31}$$

with f viewed as the map from $(\sigma_0, \sigma_1, \cdots, \sigma_k)$ to $f(\sigma_0, \sigma_1, \cdots, \sigma_k)$. Intuitively, (29) describes the behavior of $w(t) - w^* \cong \Delta(t)$. In [11] and [12] we have derived conditions under which (29) is indeed a linearization of (27) around the equilibrium a^*. We henceforth assume that this is the case.

Moreover, assume that (29) is minimal and stabilizable. Then there exists a $C_0, C_1, \cdots, C_L \in \mathbb{R}^{(q-g)\times q}$ such that the controller

$$C_0 \Delta + C_1 \frac{d\Delta}{dt} + \cdots + C_{L'} \frac{d^{L'} \Delta}{dt^{L'}} = 0 \tag{32}$$

stabilizes (30), i.e., such that $\begin{bmatrix} R \\ C \end{bmatrix}$ is Hurwitz.

Now it can be shown that the control law

$$C_0(w - a^*) + C_1 \frac{dw}{dt} + \cdots + C_{L'} \frac{d^{L'} w}{dt^{L'}} = 0 \tag{33}$$

will be such that a^* is a stable equilibrium of the nonlinear system described by (27) and (32). Note that the controller (32) has been derived without having to put (27) in input–output or in input/state/output form.

XIII. Example

We will now work out a very simple example in order to illustrate how the behavioral approach would proceed in modeling and stabilizing a very simple mechanical system around an equilibrium. Consider a rod of length L with a mass connected at one end. The problem is to stabilize this system as a vertical inverted pendulum in a particular position, as one would do when balancing a stick on one's hand. The relevant geometry is shown in Fig. 9.

Equations of motion for this system are

$$M \frac{d^2 \vec{q}_2}{dt^2} = \vec{F} - Mg\vec{1}_z \tag{33a}$$

$$\vec{F} = \alpha(\vec{q}_1 - \vec{q}_2) \tag{33b}$$

$$\|\vec{q}_1 - \vec{q}_2\| = L. \tag{33c}$$

These three equations describe the situation completely: (33a) are the equations of motion of the mass, with $\vec{F}$ the force exerted by the rod on the mass; (33b) tells us that the force $\vec{F}$ must act in the rod; (33c) guarantees that the rod is rigid. The positions $\vec{q}_2$ of the mass and $\vec{q}_1$ of the base of the rod remain a distance L apart.

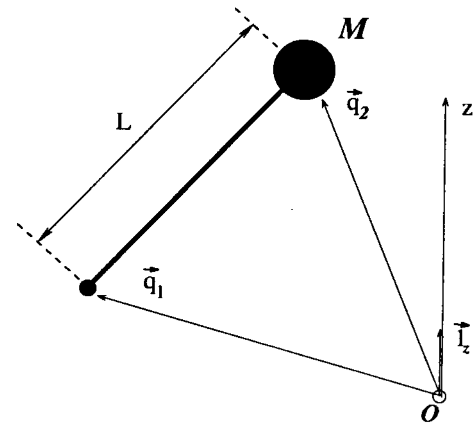

Fig. 9. An inverted pendulum.

Our purpose with (33) is to describe the relation between $\vec{q}_1$ and $\vec{q}_2$. For the case at hand these are our *manifest* variables. Note that it was essential to introduce the force $\vec{F}$ and the proportionality factor α. These are our *latent* variables. The constants L and M are to be viewed as system *parameters*, while the gravitational constant g is a *universal constant*.

Note that $\vec{q}_1^* = 0, \vec{q}_2^* = L\vec{1}_z$ is an equilibrium point, associated with $\vec{F}^* = Mg\vec{1}_z, \alpha^* = -(Mg/L)$. Linearizing around this equilibrium yields (in the obvious notation)

$$M \frac{d^2 \vec{\Delta}_{q_2}}{dt^2} = \vec{\Delta}_F \tag{34a}$$

$$\vec{\Delta}_F = -L\vec{1}_z \Delta_\alpha - \frac{Mg}{L}(\vec{\Delta}_{q_1} - \vec{\Delta}_{q_2}) \tag{34b}$$

$$\langle \vec{1}_z, \vec{\Delta}_{q_1} - \vec{\Delta}_{q_2} \rangle = 0. \tag{34c}$$

Next, eliminate $\vec{\Delta}_F$ and Δ_α from (34). This leads to

$$\frac{g}{L}\Delta_{q_2}^x - \frac{d^2 \Delta_{q_2}^x}{dt^2} = \frac{g}{L}\Delta_{q_1}^x \tag{35a}$$

$$\frac{g}{L}\Delta_{q_2}^y - \frac{d^2 \Delta_{q_2}^y}{dt^2} = \frac{g}{L}\Delta_{q_1}^y \tag{35b}$$

$$\Delta_{q_2}^z = \Delta_{q_1}^z \tag{35c}$$

as the relation between the $x, y,$ and z components of $\vec{\Delta}_{q_1}$ and $\vec{\Delta}_{q_2}$.

We should now stabilize this system around the equilibrium $\vec{\Delta}_{q_1} = \vec{0}, \vec{\Delta}_{q_2} = \vec{0}$.

1) Imposing $\vec{\Delta}_{q_1} = \vec{0}$ and $\vec{\Delta}_{q_2} = \vec{0}$ corresponds to nailing down the base of the rod and the mass in their desired positions. It is an example of an interconnection which is *not regular*. It is a harsh measure to arrive at stabilization.
2) Imposing the control law

$$\Delta_{q_1}^x = \alpha \Delta_{q_2}^x + \beta \frac{d\Delta_{q_2}^x}{dt} \tag{36a}$$

$$\Delta_{q_1}^y = \alpha \Delta_{q_2}^y + \beta \frac{d\Delta_{q_2}^y}{dt} \tag{36b}$$

$$\Delta_{q_1}^z = 0 \tag{36c}$$

with α, β chosen such that $(\alpha - 1)(g/L) + \beta(g/L)\xi + \xi^2$ is Hurwitz, leads to a regular interconnection corresponding to singular feedback (with $\vec{\Delta}_{q_1}$ as input and $\vec{\Delta}_{q_2}$ as output or memoryless state feedback).

3) Imposing the control law

$$\alpha\Delta_{q_1}^x + \frac{d\Delta_{q_1}^x}{dt} = \beta\Delta_{q_2}^x + \gamma\frac{d\Delta_{q_2}^x}{dt} \tag{37a}$$

$$\alpha\Delta_{q_1}^y + \frac{d\Delta_{q_1}^y}{dt} = \beta\Delta_{q_2}^y + \gamma\frac{d\Delta_{q_2}^y}{dt} \tag{37b}$$

$$\Delta_{q_1}^z = 0 \tag{37c}$$

with (α, β, γ) chosen such that $(\beta - \alpha)(g/L) + (\gamma - 1)(g/L)\xi + \alpha\xi^2 + \xi^3$ is Hurwitz, leads to a stabilizing regular feedback interconnection.

These control laws will lead to (locally) stabilizing control laws for the nonlinear system by replacing $\Delta_{q_1}^x$ by $q_1^x, \Delta_{q_2}^x$, by $q_2^x, \Delta_{q_1}^y$, by $q_1^y, \Delta_{q_2}^y$, by $q_2^y, \Delta_{q_1}^z$, by q_1^z, and $\Delta_{q_2}^z$ by $q_2^z + L$. Observe that these stabilizing control laws were arrived at without having to write the (non)linear system in state form, without having to examine what the inputs and outputs are. Also, they could be implemented in various ways, using physical (springs, etc.) devices or through sensor/actuator feedback connections.

XIV. Conclusions

In this paper, we have examined control from a behavioral point of view. Contrary to the classical picture which involves signal flow graphs processing inputs and outputs, we view control purely as imposing new additional laws on the system variables. We provided a physical example in order to convince the reader of the rationale of this view. Within this "interconnection" setting we studied the problems of stabilization, pole placement, and invariant factor assignment. Also various types of interconnections (regular, regular feedback) were introduced and related to classical signal flow graph feedback structures. The results were finally applied to the stabilization of nonlinear systems around an equilibrium. The moral of this paper can be captured in the maxim: "*It is vain to do with more what can be done with less*." This faded age-old wisdom, known as *Occam's Razor*, warns us not to introduce unnecessary structure, as inputs and outputs.

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