The Contact Problem for Linear Continuous-Time Dynamical Systems: A Geometric Approach

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***Abstract*—In this paper linear time-invariant dynamical systems described by a combination of differential equalities and static inequalities in state-space formulation are investigated. Of special interest is the contact problem: the effect of the boundary of the constraint set on the behavior of the system. This effect is studied by dividing the state-space in a number of disjunct subsets. It is shown that these subsets are invariant under linear state feedback. In our framework, a specific place is reserved for modeling the laws of collision, i.e., physical modeling, which are regarded as external factors. Our main results are a system theoretical framework in which we describe what happens upon contact and a definition of the constrained state-space system in terms of its restricted behavior. The results presented here can be considered as an extension for restricted linear systems of the classic positive invariance theory for linear systems.**

***Index Terms*—Constrained behaviors, constrained linear systems, contact problem, hybrid systems, inequality constraints.**

I. Introduction

The PURPOSE of this paper is to contribute to a better understanding of the influence of obstacles on the behavior of a physical system. Methods for physical modeling generally follow from certain principles, such as setting up the motion equations in the case of mechanical systems by use of the Lagrangian formalism. Since in many practical control problems geometric unilateral constraints enter the problem formulation, the resulting system of equations will in general contain both differential equalities as well as algebraic (in)equalities. For instance, during operations with a robotic arm, situations will occur where the manipulator is, or comes, in contact with its environment. In this setting the differential equations model the unconstrained behavior of the manipulator, whereas the algebraic equalities and inequalities model the environment. In a more general system theoretical framework the algebraic equations may model interconnections between subsystems [28] or general restrictions on the system imposed by the environment [26]. In this context we will be concerned with the preparation of the state to allow for a successful interconnection or tearing [28]. We also investigate what will happen if systems interconnect (or tear) while the state(s) of these systems have not been (or cannot be) prepared properly. And it is the latter that can be seen as the contact problem proper.

The contact problem is investigated from a system theoretical point of view: the effect of the presence of inequality constraints on the behavior of a linear continuous-time dynamical system. Inequality constraints can be found in, for instance, models for economic systems, thermal control systems, and biological systems. But perhaps the best known class of systems where inequalities arise is that of mechanical systems, where the contact problem can be discussed in terms of collisions and collision avoidance.

It is well known that control of constrained mechanical systems is much harder than control of unconstrained systems. One also has to ensure that impact forces remain within specified bounds and that bouncing is avoided as much as possible (see, e.g., [7] and [19]). Research into the mechanics of contact has a long history [3], [16]. When a trajectory makes contact with the boundary set of the region modeled by an inequality constraint, this is generally referred to as activation of the associated equality constraint. Activation and deactivation of a constraint, and the consequences of the addition and deletion of equations to a representation, have been studied for instance in [10], [13], and [18]. When a constraint becomes active, some derivative of (part of) the state can be discontinuous [18]. For example, the velocity of a bouncing rigid ball will change sign instantaneously when the ball touches the (rigid) ground. Depending on the physical properties of the modeled system, the new state (after collision) will in general depend on the state when the collision took place. A mapping that models the laws of collision is needed, which is regarded as external in our framework. The latter is not surprising because a general theoretical discussion on linear systems does not involve the notion of collisions, as apparent for mechanical systems. An analog can be found in [26], where certain mechanical properties, such as the notion of energy, are used to make a link between the framework of [25], [27], and "physical modeling." We will show that collisions can be brought into the studies of linear dynamical systems by using geometric terms, i.e., the use of particular structures of vector spaces [29].

To gain a clear understanding of the contact problem we make investigations into linear time-invariant systems in state representation. We will discuss inequality constraints that represent an arbitrary convex polyhedral set. The approach we take is to first investigate what can be deduced already

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from the mathematical models of the unconstrained dynamical system and the constraint set. It is only after this analysis that we look in more detail to the constrained dynamical system itself. This particular sequence of steps leads to important insights to the contact problem and to useful theoretical findings with respect to modeling and control of dynamical systems subject to unilateral constraints. Research in the field of positively invariant polyhedral sets (see for instance [4], [8], and the references therein), i.e., systems subject to constraints described by polyhedral sets, do not include a discussion on the contact problem itself. We will show that our approach generalizes, for restricted linear systems, the classic notion of positive invariance for linear systems.

The paper is organized a follows: in Section II the problem formulation is presented. A constrained mechanical system is discussed and the assumptions are stated. Since an arbitrary convex polyhedral set can be represented as the intersection of a finite system of closed affine half-spaces [24], it will prove fruitful to first consider the case where the state trajectory is restricted to be in one closed half-space. A first, and basic, subdivision of the state space, based on activation and deactivation of a boundary set, is given in Section III. In Section IV we complete our subdivision of the state space by examining the behavior of the unconstrained system on the boundary of the constraint set. It is shown that the subsets are invariant under linear state feedback. Some of the relations that hold between all the newly introduced subsets of the state space are presented. Algorithms that will compute all these subsets in a finite number of steps are derived in the Appendix. Sections V and VI are the core of the present paper: a detailed description of the restricted behavior of continuous-time constrained linear systems is given. A physical interpretation is added to the mathematical description, and the contact problem is discussed. The main results will be the allocation of the specific place that is reserved in our framework for modeling the laws of collision and a precise definition of the constrained state-space system in terms of its restricted behavior. In addition, the concepts of uncontrolled collisions and controlled collisions are introduced. It is shown that trajectories of the constrained system consist of concatenated trajectory pieces of the unconstrained system. In this framework, consistent initialization of constrained linear systems will be discussed. It is shown that dynamical systems subject to equality constraints are included in our analysis. In Section VII, relaxation of some of the assumptions is treated. Moreover, the results will be extended to cover continuous-time dynamical systems restricted by an arbitrary convex polyhedral set. In Section VIII, a comparison is made between our approach to constrained systems and constrained mechanical system models found in the literature. Finally, in Section IX the conclusions are stated. Throughout the paper simple examples illustrate the concepts that are introduced.

II. Motivation and Problem Formulation

The study of control/structure interaction in large spacecraft or complex robotic systems is facilitated by assuming an ideal, linear mathematical model for the dynamics (see, e.g., [2]). Such a model can, for instance, be obtained by (feedback) linearization of the nonlinear dynamics model. Consider the following multi-input/multi-output linear second-order differential equation as a model for a mechanical system:

$$M\ddot{y} + D\dot{y} + Ky = Lu \tag{1}$$

where $y \in \mathbb{R}^d$ is a generalized system coordinate vector, $u \in \mathbb{R}^m$ the generalized force vector, $M \in \mathbb{R}^{d\times d}$ the generalized positive definite inertia matrix, $D \in \mathbb{R}^{d\times d}$ the generalized structural damping matrix, $K \in \mathbb{R}^{d\times d}$ the generalized structural stiffness matrix, and $L \in \mathbb{R}^{d\times m}$ the actuator force distribution matrix.

The presence of an object in the environment implies a restriction of the behavior of the mechanical system. In many cases these restrictions can be represented, or approximated (locally), by a finite system of linear inequalities

$$Py \geq d \tag{2}$$

with $P \in \mathbb{R}^{p\times d}, d \in \mathbb{R}^p$. By convention, inequalities between vectors are componentwise. Restriction (2) determines a convex polyhedral set. Note that constraint (2) can also be used to model distances between subsystems in particular directions.

As a simple example consider a ball that is falling to the ground. Two principal observations can be made. First, the ball can start at any position on or above the ground and with any initial velocity. Second, due to the presence of gravity, the ball will inevitably come into contact with the boundary. Moreover, based on the conservation of momentum, the velocity of the (rigid) ball will change sign instantaneously when the ball touches the (rigid) ground.

More generally, it can be seen that if $Py(t) > d$ for $t \in [t_1, t_2]$, the behavior of the constrained system in that time-interval is described by (1). In that case the boundary set of the allowed region, i.e., the equality constraints $P_i y = d_i$, associated with the inequalities $P_i y \geq d_i$, are called passive [18]. (Here subscript i denotes the ith row of a matrix.) On the other hand, (part of) this boundary set is called active (at time t) when $P_i y(t) = d_i$ for some $i \in 1, \cdots, p$ [18]. In the latter case, (1) and (part of) (2) reduce to a differential/algebraic equation (DAE) on the time-interval $[t_1, t_2]$ and may be combined to yield a differential equation with fewer generalized variables than the one in (1) [13]. We will not follow this approach. For instantaneous collisions one has that a constraint can be active at a discrete-time point only. Another difference compared to DAE systems is that one cannot differentiate (2) to obtain an inequality constraint on the velocity level as the differential operator does not preserve sign [9]. On the other hand, for DAE's one also has $P_i\dot{y} = 0$, a fact that is referred to as a hidden constraint in constrained mechanical systems [5].

For (1), define the state $x := [y^T, \dot{y}^T]^T$. Then (1) and (2) can be written equivalently as

$$\begin{aligned} \dot{x} &= \hat{A}x + \hat{B}u \\ &:= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} u \\ \hat{C}x &:= [P \quad 0]x \geq d. \end{aligned} \tag{3}$$

Note that for constrained mechanical system (3) one has $\hat{C}\hat{B} = 0$, i.e., $\mathrm{im}(\hat{B}) \subseteq \ker(\hat{C})$.

Motivated by (3), we will investigate linear time-invariant dynamical systems

$$\Sigma\colon \dot{x} = Ax + Bu \tag{4}$$

subject to inequality constraints

$$Cx \geq d. \tag{5}$$

Here $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$, and $d \in \mathbb{R}^p$. Throughout, we will assume that polyhedral set (5) is nonempty, an assumption which is trivially satisfied if $d = 0$. It is remarked that no requirements, other than $C \neq 0$, will be put on constraint matrix C; we allow for redundant equations and implicit inequalities [9]. The combination of (4) and (5) gives rise to a constrained system $\Sigma^c(A, B, C, d)$, defined as

$$\Sigma^c\colon \begin{cases} \dot{x} = Ax + Bu \\ 0 \leq Cx - d. \end{cases} \tag{6}$$

To reduce notation we usually delete the arguments, for instance $\Sigma^c(A, B, C, d)$ will be denoted by Σ^c. In the remainder $\underline{x}$ will denote a trajectory of a dynamical system. Note that (4) is a linear system, whereas (6) is not. In [9] systems as in (6) are investigated in a convex conical setting. (A system Σ is called conical if $\{\underline{x}_1 \in \Sigma\} \Rightarrow \{\lambda\underline{x}_1 \in \Sigma, \forall\lambda \geq 0\}$, a system Σ is called convex if $\{\underline{x}_1, \underline{x}_2 \in \Sigma\} \Rightarrow \{\lambda\underline{x}_1 + (1-\lambda)\underline{x}_2 \in \Sigma, \forall\lambda\colon 0 \leq \lambda \leq 1\}$ [9].)

The following assumptions hold throughout this paper for system matrices A and B and constraint matrices C and d, unless stated otherwise.

1) $\mathrm{rank}[B\, AB \cdots A^{n-1}B] = n$.
2) $\mathrm{im}(B) \subseteq \ker(C)$.
3) $C \neq 0$.
4) $\{x \in \mathbb{R}^n | Cx \geq d\} \neq \emptyset$.

Assumption 1) is equivalent to controllability of the unconstrained system (4) [29]. Assumption 2) is motivated by representation (3) and is a natural one to make (see also [26] for a discussion on equality constraints in the case of Hamiltonian or gradient systems). Assumption 2) does cover mechanical systems subject to holonomic inequality constraints but is not limited to this case. Assumptions 3) and 4) are made to exclude the trivial cases $\Sigma^c = \Sigma$, or $\Sigma^c = \emptyset$ (depending on whether or not $0 \leq d$). Note that Assumption 4) is trivially satisfied for polyhedral cones. We will use the phrase "$\mathcal{X}_n(A, B, C, d)$ satisfies the assumptions" to indicate that in the state-space $\mathcal{X}$ with $\dim(\mathcal{X}) = n$, system matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ of the unconstrained system, and constraint matrices $C \in \mathbb{R}^{p\times n}$ and $d \in \mathbb{R}^p$, used to model a convex polyhedral set, satisfy the above assumptions. We also assume that the controls take their value in $\mathcal{U}$, the set of piecewise C^∞ functions. Relaxation of the assumptions is discussed in Section VII.

The main topic discussed in this paper is the contact problem. The questions we will address are as follows.

1) How do inequality constraints (5) interact with the behavior of an unconstrained system (4)?
2) How do collisions fit into a general theory on dynamical systems subject to unilateral constraints?

First we make investigations into the influence of the constraint on the unconstrained system Σ. Further discussion of the constrained system Σ^c itself is postponed until Section V. The second question, especially, will give rise to a specific place in our framework that is reserved for modeling the laws of collision, which are regarded as external factors. We will show that when the boundary of an inequality constraint becomes active, the state trajectory may become discontinuous with respect to time. It will also be shown that incorporating a physical interpretation into a system theoretical framework generalizes the notion of positively invariant (or controlled invariant) polyhedral sets.

We will show that collisions can be brought into the study of linear dynamical systems by using geometric terms, i.e., the use of particular structures of vector spaces. In particular it will be shown that the subsets can be computed using intermediate steps of the invariant subspace algorithm (ISA). For future reference we state this "$\mathcal{V}^*$-algorithm" [29]. Let $\mathcal{X}$ denote the state space. Let matrices A, B, and C be given and be of appropriate dimensions. Let $A^{-1}\mathcal{V} := \{x \in \mathbb{R}^n | Ax \in \mathcal{V}\}$. (Note that it is not required that A is invertible.) Define $\mathcal{V}^0 := \mathcal{X}$ and

$$\mathcal{V}^{k+1} = \ker(C) \cap A^{-1}(\mathcal{V}^k + \mathrm{im}(B)), \qquad k = 0, 1, \cdots. \tag{7}$$

This defines a nonincreasing sequence of subspaces of $\mathcal{X}$. Since $\dim(\mathcal{X})$ is finite, there exists a value of k such that $\mathcal{V}^{k+1} = \mathcal{V}^k$. This limit is denoted by $\mathcal{V}^*$. By construction one has $\mathcal{V}^* \subseteq \ker(C)$.

III. Contact and Release Sets

In this section we will start our subdivision of the state-space $\mathcal{X}$ by investigating the behavior of an unconstrained system in a convex polyhedral cone. Any convex polyhedral cone can be represented as the intersection of a finite system of closed half-spaces [24]

$$\{x \in \mathbb{R}^n | Cx \geq 0\} = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n | C_i x \geq 0\}$$

where C_i denotes the ith row of matrix $C \in \mathbb{R}^{p\times n}$. It will prove fruitful to first consider in detail the case of a single inequality constraint

$$Cx \geq 0, \quad \text{with } C \in \mathbb{R}^{1\times n}. \tag{8}$$

The constrained system $\Sigma^c(A, B, C)$ $(:= \Sigma^c(A, B, C, 0))$ with $C \in \mathbb{R}^{1\times n}$ will be denoted by Σ_1^c. (It is not until a full description is derived of the restricted behavior for Σ_1^c in Section V that we return to the case of multiple constraints, i.e., $C \in \mathbb{R}^{p\times n}(p \geq 1)$ in Section VI.)

A first, and basic, subdivision of the state-space $\mathcal{X}$ is based on inequality constraint (8) only. Define $\mathcal{X}_g$ (g for good) as the collection of states, where the inequality constraint is satisfied strictly and $\mathcal{X}_f$ (f for false) as the collection of states, where the inequality constraint is not satisfied. One has

$$\mathcal{X}_g = \{x \in \mathbb{R}^n | Cx > 0\} \tag{9}$$

$$\mathcal{X}_f = \{x \in \mathbb{R}^n | Cx < 0\} \tag{10}$$

$$\ker(C) = \{x \in \mathbb{R}^n | Cx = 0\}. \tag{11}$$

The set $\mathcal{X}_g$ represents the interior of the convex cone, whereas the boundary set models a hyperplane in $\mathbb{R}^n$. Note that $\mathcal{X}_g, \mathcal{X}_f$, and $\ker(C)$ are disjunct subsets whose union is again the state-space $\mathcal{X}$.

A further subdivision of the subspace $\ker(C)$ can be made based on the interaction of trajectories of unconstrained system Σ (4) with (8). Note that controllability (of the unconstrained system) implies that any state can be reached in finite time starting from an arbitrary initial state. Let $\mathcal{X}_{\rm con}$ (con for contact) denote the set of points where a trajectory of Σ that starts in $\mathcal{X}_g$ can come into contact with $\ker(C)$. Analogously, let $\mathcal{X}_{\rm rel}$ (rel for release) denote the set of points where a trajectory of Σ can leave the boundary set and remain (for some period of time) in $\mathcal{X}_g$. These sets are defined formally by the following definition.

Definition III.1 (Contact Set $\mathcal{X}_{\rm con}$ and Release Set $\mathcal{X}_{\rm rel}$):

1) $\mathcal{X}_{\rm con} := \{x \in \ker(C) | \exists \underline{x} \in \Sigma$ and $\exists t^* < 0$ such that $\underline{x}(0) = x$, and $\underline{x}(\tau) \in \mathcal{X}_g, \forall \tau\colon t^* < \tau < 0\}$.
2) $\mathcal{X}_{\rm rel} := \{x \in \ker(C) | \exists \underline{x} \in \Sigma$ and $\exists t^* > 0$ such that $\underline{x}(0) = x$, and $\underline{x}(\tau) \in \mathcal{X}_g, \forall \tau\colon 0 < \tau < t^*\}$.

Formally $\mathcal{X}_{\rm con} = \mathcal{X}_{\rm con}(A, B, C)$, but to shorten notation we will usually delete the arguments. The (finite-time) trajectory piece $\underline{x}(t)$ in the definition of $\mathcal{X}_{\rm con}$ is referred to as locally viable in [23], where the so-called target problem, i.e., how to reach a specific target subset, is discussed for differential inclusions.

It is intuitively clear that $\mathcal{X}_{\rm con}$ and $\mathcal{X}_{\rm rel}$, defined for system $\Sigma(A, B, C)$, switch roles for the time-reversed system $\Sigma(-A, -B, C)$. This is proven in the Appendix.

Finally, the set of points where a control exists such that the state remains in $\ker(C)$ is given by $\mathcal{V}^*$ [29].

Example III.2 (A Single Train Moving Along a Track): Let the position of (a point on) a train be denoted by y. Consider the single-input/single-output (SISO) representation $\ddot{y} + \dot{y} + y = u$, obtained from (1) by setting $M = D = K = L = 1$, subject to the inequality constraint $y \geq 0$, obtained from (2) by setting $P = 1, d = 0$. Define $x := [y^T, \dot{y}^T]^T$. The system matrices are

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The constraint matrix C reads: $C = [1\ 0]$. It can be verified that (A, B) is controllable and that $CB = 0$. Inspection shows that

$$\mathcal{X}_{\rm con} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 \leq 0\}$$
$$\mathcal{X}_{\rm rel} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 \geq 0\}$$

and

$$\mathcal{V}^* = \{0\}.$$

It follows that

$$\mathcal{X}_{\rm con} \cup \mathcal{X}_{\rm rel} = \ker(C), \quad \text{and} \quad \mathcal{X}_{\rm con} \cap \mathcal{X}_{\rm rel} = \mathcal{V}^*. \qquad \square$$

Even on this basic level some interesting properties arise. It can be seen that on $\mathcal{X}_{\rm rel} \backslash \mathcal{V}^*$ no special measures need to be taken as the trajectory will leave the constraint and will go to $\mathcal{X}_g$. On $\mathcal{X}_{\rm con} \backslash \mathcal{V}^*$, however, a collision can take place as there is no control such that the trajectory will remain on $\ker(C)$. In general, the situation will be more complicated than the one given in the example above. We will locate exactly, in subsequent sections, the subset of $\ker(C)$ where collisions will take place.

IV. Subdivision of the Boundary Set

If a trajectory $\underline{x} \in \Sigma$ enters the boundary of the constraint set, i.e., if at some time t one has $C\underline{x}(t) = 0$, it depends on the characteristics of the state (and its derivatives) at this contact point whether or not a collision takes place. Of particular interest are the components of the state (and its derivatives) that are not in the boundary set. A measure of this is the value of $C\underline{x}^{(i)}(t)$, where $\underline{x}^{(i)}(t)$ denotes the ith derivative of the state vector at time t, which can be obtained from (4)

$$\underline{x}^{(i)}(t) = A^i \underline{x}(t) + A^{i-1} B u(t) + \cdots + B u^{(i-1)}(t).$$

We will be concerned mainly with a further subdivision of $\ker(C)$. To shorten notation some definitions are given that will enable us to present alternative representations of the subsets defined in the previous section. Let $\underline{u} \in \mathcal{U}^{\mathbb{N}}$, i.e., $\underline{u}$ is a countable dimensional vector whose elements take their values in $\mathcal{U}$.

Definition IV.1: Let matrices A, B, and C be given. Define the following.

1) h_i: $\ker(C) \times \mathcal{U}^{\mathbb{N}} \mapsto \mathbb{R}$ as $h_i(x, \underline{u}) := CA^i x + \Sigma_{j=1}^{i} CA^{j-1} B \underline{u}_{i-j}$.
2) r: $\ker(C) \times \mathcal{U}^{\mathbb{N}} \mapsto \mathbb{N} \cup \{\infty\}$ as $r(x, \underline{u}) := \min\{i \in \mathbb{N} | h_i(x, \underline{u}) \neq 0\}$ with $r(x, \underline{u}) := \infty$ if $h_i(x, \underline{u}) = 0, \forall i \in \mathbb{N}$.
3) r_C: $\ker(C) \mapsto \mathbb{N}$ as $r_C(x) := \min\{i \in \mathbb{N} | h_i(x, \underline{u}) \neq 0, \underline{u} \in \mathcal{U}^{\mathbb{N}}\}$.
4) $r_0 := \min\{i \in \mathbb{N} | CA^{i-1} B \neq 0\}$.

Both r_0 and $r_C(x)$ are finite since the pair (A, B) is controllable [29]. It is easy to see that $\forall x \in \ker(C)$ and $\forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$, one has $r_C(x) \leq r(x, \underline{u})$. Observe that r_0 is the smallest integer $i \in \mathbb{N}$ for which $h_i(x, \underline{u})$ depends on the control. In an input/output setting the integer r_0 can be used to derive, for instance, an input/output decoupling control law [12], and is also known as the order of the infinite zero in case $p = 1$, and the least of the order at infinity in the general case [11]. It is remarked that the integer r_0 has its counterpart in nonlinear systems theory, where it is referred to as a characteristic number, and represents "the inherent number of integrations between inputs and an output $y := h_0(x, u)$" [22]. In such a nonlinear context the map $h_i(x, \underline{u})$ denotes the Lie-derivative of the one-form $x^{(i)}(t)$ along vectorfields built up from Ax and B. From the assumptions it is easy to see that $r_0 \geq 2$ and $r_C(x) \geq 1$. The following corollary is straightforward from the definitions and the assumption $CB = 0$.

Corollary IV.2 (Some Useful Properties): Let $x \in \ker(C)$. Then:

1) $r_C(x) = \min\{r_0, \inf\{i \in \mathbb{N} | CA^i x \neq 0\}\}$;
2) $\forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $1 \leq i < r_C(x) \Rightarrow h_i(x, \underline{u}) = 0$;
3) $r_C(x) < r_0 \Rightarrow \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $h_i(x, \underline{u}) = h_i(x), i < r_0$;
4) $\exists \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $i \geq r_0 \Rightarrow h_i(x, \underline{u}) = 0$.

It can be seen from Corollary IV.2-3) that for $i \leq r_0$, one has that $h_i(x, \underline{u})$ is independent of the control. The integers r_0 and $r_C(x)$ also have another important property: they are invariant under linear state feedback.

Lemma IV.3: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1 \times n}$. Then the integers r_0 and $r_C(x)$ are invariant under the linear state feedback $u = Fx + v$, with v the new control.

Proof: Omitted. □

It is remarked that Lemma IV.3 also holds if $CB \neq 0$, i.e., if $r_0 = 1$. Next we give alternative representations of the subsets $\mathcal{X}_{\text{con}}, \mathcal{X}_{\text{rel}}$, and $\mathcal{V}^*$.

Lemma IV.4 (The Sets $\mathcal{X}_{\text{con}}, \mathcal{X}_{\text{rel}}$, and $\mathcal{V}^*$):

1) $\mathcal{X}_{\text{con}} = \{x \in \ker(C) | \exists \underline{u} \in \mathcal{U}^{\mathbb{N}}$ such that $\{r(x, \underline{u}) < \infty$ and even, and $h_{r(x,\underline{u})}(x, \underline{u}) > 0\}$, or $\{r(x, \underline{u}) < \infty$ and odd, and $h_{r(x,\underline{u})}(x, \underline{u}) < 0\}\}$.
2) $\mathcal{X}_{\text{rel}} = \{x \in \ker(C) | \exists \underline{u} \in \mathcal{U}^{\mathbb{N}}$ such that $r(x, \underline{u}) < \infty$ and $h_{r(x,\underline{u})}(x, \underline{u}) > 0\}$.
3) $\mathcal{V}^* = \{x \in \ker(C) | \exists \underline{u} \in \mathcal{U}^{\mathbb{N}}$ such that $r(x, \underline{u}) = \infty\}$.

Proof: Let $\underline{x} \in \Sigma$ and let $\underline{x}(0) = x$ with $x \in \ker(C)$. If $h_i(x, \underline{u}) = 0$ for all i, then $r(x, \underline{u}) > n$. It follows that $x \in \mathcal{V}^*$. This gives 3). Now suppose that $\exists i$ such that $h_i(x, \underline{u}) \neq 0$. From a Taylor-series expansion it follows that it suffices to look at the first derivative that is not equal to zero. From the definition of the sets $\mathcal{X}_{\text{con}}$ and $\mathcal{X}_{\text{rel}}$ the statements in 1) and 2) now follow. □

It is important to note that Lemma IV.4 does not say that, for instance, $\mathcal{X}_{\text{con}}$ and $\mathcal{X}_{\text{rel}}$ are disjunct. This is discussed in more detail in the remainder.

It is clear that the integers $r_0, r(x, \underline{u})$, and $r_C(x)$ play an important role. The decision of whether or not a point $x \in \ker(C)$ belongs, for instance, $\mathcal{X}_{\text{rel}}$, can be based on the value of $r_C(x)$. It is easy to see that if $\exists \underline{v}$ such that $h_{r(x,\underline{v})}(x, \underline{v}) > 0$ for $r_0 < r(x, \underline{v}) < \infty$, then there is also a control $\underline{u}$ such that $h_{r_0}(x, \underline{u}) > 0$. If $i \geq r_0$, it follows from Corollary IV.2-4) that it is basically a controller design problem to keep the derivatives of the state along the manifold C equal to zero (or not). This motivates the definitions in the remainder of this section.

Let $\mathcal{V}_g$ denote all $x \in \ker(C)$ for which all trajectories of Σ passing through x do so coming from $\mathcal{X}_g$ and going to $\mathcal{X}_g$. Similarly, let $\mathcal{V}_f$ denote all $x \in \ker(C)$ for which all trajectories of Σ passing through x do so coming from $\mathcal{X}_f$ and going to $\mathcal{X}_f$. Let $\mathcal{V}_c$ denote the collection of states that belong to $\ker(C)$ of which the smallest $i \in \mathbb{N}$ for which h_i can be unequal to zero depends on the control. These sets are defined formally by the following definition.

Definition IV.5 (The Sets $\mathcal{V}_g, \mathcal{V}_f$, and $\mathcal{V}_c$ (c for Control):

1) $\mathcal{V}_g := \{x \in \ker(C) | \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $r_C(x)$ is even, and $h_{r_C(x)}(x, \underline{u}) > 0\}$.
2) $\mathcal{V}_f := \{x \in \ker(C) | \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $r_C(x)$ is even, and $h_{r_C(x)}(x, \underline{u}) < 0\}$.
3) $\mathcal{V}_c := \{x \in \ker(C) | r_C(x) = r_0\}$.

Based on our motivation of mechanical systems, we make a further subdivision in contact (or release) with "velocity" equal to zero and "velocity" unequal to zero.

Definition IV.6 (The Sets $\mathcal{X}_{\text{con},v}, \mathcal{X}_{\text{rel},v}$ (v for Velocity), $\mathcal{X}_{\text{con},h}$, and $\mathcal{X}_{\text{rel},h}$ (h for Higher Derivatives)):

1) $\mathcal{X}_{\text{con},v} := \{x \in \mathcal{X}_{\text{con}} | r_C(x) = 1\}$.
2) $\mathcal{X}_{\text{rel},v} := \{x \in \mathcal{X}_{\text{rel}} | r_C(x) = 1\}$.
3) $\mathcal{X}_{\text{con},h} := \{x \in \mathcal{X}_{\text{con}} | \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $1 < r_C(x), r_C(x)$ is odd and $h_{r_C(x)}(x, \underline{u}) < 0\}$.
4) $\mathcal{X}_{\text{rel},h} := \{x \in \mathcal{X}_{\text{rel}} | \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$: $1 < r_C(x), r_C(x)$ is odd and $h_{r_C(x)}(x, \underline{u}) > 0\}$.

Finally, we prove that all subsets defined so far are invariant under linear state feedback (which is not to be confused with the notion of controlled invariance). For instance, the set where contact can be made with "velocity" component unequal to zero does not change if we apply linear state feedback.

Proposition IV.7: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1 \times n}$. Then the subsets $\mathcal{V}_c, \mathcal{V}_g, \mathcal{V}_f, \mathcal{X}_{\text{con},v}, \mathcal{X}_{\text{con},h}, \mathcal{X}_{\text{rel},v}$, and $\mathcal{X}_{\text{rel},h}$ are invariant under the linear state feedback $u = Fx + v$, with v the new control, for the system Σ.

Proof: For the subsets $\mathcal{V}_f, \mathcal{V}_g, \mathcal{X}_{\text{con},v}, \mathcal{X}_{\text{con},h}, \mathcal{X}_{\text{rel},v}$, and $\mathcal{X}_{\text{rel},h}$ there holds: $1 \leq r_C(x) < r_0$. From Lemma IV.3 follows that these integers are invariant. It is easy to show that the sign of h_i is preserved. Finally, for $\mathcal{V}_c$ one has $r_C(x) = r_0$ (without conditions on the sign of h_i). □

It will be important for a description of the restricted behavior to have available some of the relations that exist between the subsets defined so far. In the Appendix, Lemma A.1, it is shown that $\mathcal{V}_c = \mathcal{V}^*$. The following result will be important.

Theorem IV.8 (Relations Between Subsets of the State Space): Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1 \times n}$. Then we have the following.

1) $\ker(C) = \mathcal{V}_c \cup \mathcal{V}_f \cup \mathcal{V}_g \cup \mathcal{X}_{\text{con},v} \cup \mathcal{X}_{\text{con},h} \cup \mathcal{X}_{\text{rel},v} \cup \mathcal{X}_{\text{rel},h}$.
2) The subsets $\mathcal{X}_g, \mathcal{X}_f, \mathcal{V}_c, \mathcal{V}_f, \mathcal{V}_g, \mathcal{X}_{\text{con},v}, \mathcal{X}_{\text{con},h}, \mathcal{X}_{\text{rel},v}$, and $\mathcal{X}_{\text{rel},h}$ are two by two disjunct.
3) $\mathcal{X}_{\text{con}} = \mathcal{V}_c \cup \mathcal{V}_g \cup \mathcal{X}_{\text{con},v} \cup \mathcal{X}_{\text{con},h}$.
4) $\mathcal{X}_{\text{rel}} = \mathcal{V}_c \cup \mathcal{V}_g \cup \mathcal{X}_{\text{rel},v} \cup \mathcal{X}_{\text{rel},h}$.

Proof: The first two statements are straightforward from the definitions and Lemma IV.4. The last two statements then follow from Lemma A.1 and Lemma A.2 (see the Appendix) by straightforward argumentation. □

Theorem IV.8 together with Proposition IV.7 yields that we can make a complete subdivision of $\ker(C)$ in disjunct subsets that are invariant under linear state feedback. In the Appendix, algorithms are derived that calculate these subsets in a finite number of steps. The following result is immediate.

Corollary IV.9: The subsets $\mathcal{X}_{\text{con}}$ and $\mathcal{X}_{\text{rel}}$ are invariant under the linear state feedback $u = Fx + v$, with v the new control.

Next, as a summary, an intuitive explanation is given of all subsets that have been defined so far. Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and take $C \in \mathbb{R}^{1 \times n}$. We can make the following statements relating $x \in \mathcal{X}$ with a trajectory of the unconstrained system Σ.

1) $x \in \mathcal{X}_g \Leftrightarrow x$ satisfies the inequality constraint strictly.
2) $x \in \ker(C) \Leftrightarrow x$ belongs to the boundary set.

3) $x \in \mathcal{X}_f \Leftrightarrow x$ does not satisfy the inequality constraint.
4) $x \in \mathcal{X}_{\mathrm{con},v} \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go transversally through the boundary set from $\mathcal{X}_g$ to $\mathcal{X}_f$.
5) $x \in \mathcal{X}_{\mathrm{con},h} \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go tangentially "through" the boundary set and go from $\mathcal{X}_g$ to $\mathcal{X}_f$.
6) $x \in \mathcal{V}_g \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go tangentially "through" the boundary set and go from $\mathcal{X}_g$ to $\mathcal{X}_g$.
7) $x \in \mathcal{V}_c \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go tangentially "through" the boundary set, and there exists at least one trajectory that remains in $\ker(C)$.
8) $x \in \mathcal{V}_f \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go tangentially "through" the boundary set from $\mathcal{X}_f$ to $\mathcal{X}_f$.
9) $x \in \mathcal{X}_{\mathrm{rel},h} \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go tangentially "through" the boundary set from $\mathcal{X}_f$ to $\mathcal{X}_g$.
10) $x \in \mathcal{X}_{\mathrm{rel},v} \Leftrightarrow$ all trajectories $\underline{x} \in \Sigma$ with $\underline{x}(\underline{t}) = x$ for some $\underline{t}$ go transversally through the boundary set from $\mathcal{X}_f$ to $\mathcal{X}_g$.

Until now we have investigated how trajectories of an unconstrained dynamical system interact with a boundary set. If the boundary set is looked upon as a mathematical constraint rather than a hard environment constraint, it follows that in $\mathcal{X}_{\mathrm{con}} \backslash \mathcal{V}_c$ application of a smooth control cannot prevent a trajectory of system (4) to enter $\mathcal{X}_f$. It is clear that this finding has consequences for (feedback) controller design if one of the objectives is smooth contact with the boundary set. Moreover, the control should be chosen appropriately in $\mathcal{V}^*$, i.e., one should exclude controls that will drive the system into $\mathcal{X}_f$. This leads to the concepts of (locally) applicable and forbidden controls which will prove useful when the collision maps are discussed.

Definition IV.10: Consider (4). Let $x \in \mathcal{V}_c$. The set of (locally) applicable controls is defined as $\mathcal{U}_g(x) = \{u \in \mathcal{U} | \exists \underline{x} \in \Sigma, \exists t^* > 0$ such that $\underline{x}(0) = x$, and $\underline{x}(\tau) \in \mathcal{X}_g \cup \ker(C), \forall \tau\colon 0 < \tau < t^*\}$. The set of (locally) applicable boundary controls is defined as $\mathcal{U}_b(x) = \{u \in \mathcal{U} | \exists \underline{x} \in \Sigma, \exists t^* > 0$ such that $\underline{x}(0) = x$, and $\underline{x}(\tau) \in \ker(C), \forall \tau\colon 0 < \tau < t^*\}$. The set of (locally) forbidden controls is defined as $\mathcal{U}_f(x) = \{u \in \mathcal{U} | \exists \underline{x} \in \Sigma, \exists t^* > 0$ such that $\underline{x}(0) = x$, and $\underline{x}(\tau) \in \mathcal{X}_f, \forall \tau\colon 0 < \tau < t^*\}$.

Definition IV.10 can be seen as a further subdivision of the boundary set. However, as far as the points $x \in \mathcal{V}_c$ are concerned, these sets are not disjunct. The set of forbidden controls will prove useful when we introduce the collision maps in our framework. From the definitions it follows that for (4) the set of forbidden controls is given by

$$\mathcal{U}_f(x) = \bigcup_i \mathcal{U}_f^i(x), \qquad i \geq r_0$$

where

$$\mathcal{U}_f^i(x) := \{u \in \mathcal{U} | h_i(x, \underline{u}) < 0\}.$$

The set $\mathcal{U}_b(x)$ is also important in the case of bilaterally constrained systems, where (6) reduces to a DAE. For the set $\mathcal{U}_b(x)$ an explicit expression can be derived. In fact, it is easy to see that for (4)

$$\mathcal{U}_b(x) = \{u \in \mathcal{U} | CA^{r_0}x + CA^{r_0 - 1}Bu = 0\}.$$

The above sets, and some implications of our findings for controller synthesis, will be discussed in Section VIII. (A complete treatment of the latter topic is outside the scope of the present paper.)

V. Restricted Behaviors: The Single Constraint Case

In this section we will define what we mean by a constrained linear system Σ_1^c, i.e., how an inequality constraint affects the behavior of a dynamical system. Recall from Section II that the constrained behavior (with $d = 0$) is given by

$$\Sigma^c\colon \begin{cases} \dot{x} = Ax + Bu \\ 0 \leq Cx. \end{cases} \tag{12}$$

A more detailed description of Σ^c is based on the subsets defined in the previous sections.

For constrained mechanical systems one has in general $r_0 = 2$, i.e., $\mathcal{X}_{\mathrm{con},v} \neq \emptyset$. This can be seen from (3). If each (sub)system is actuated, then in general $\hat{C}\hat{A}\hat{B} = PM^{-1}L \neq 0$. Now, if contact is made in $\mathcal{X}_{\mathrm{con},v}$ a problem arises since all trajectories will proceed to $\mathcal{X}_f$. Consequently, for mechanical systems subject to unilateral constraints the contact problem arises: collisions do happen and a discussion on this subject should thus be an integral part of a general theory on constrained mechanical systems. Now in mechanics, a collision will not change the position but will affect the velocity component. Note that this change will in general depend in a unique way on the state at the moment of collision. We make the following assumption.

Assumption V.1: Collisions are instantaneous.

For a description of Σ^c we need two more notions: (in)consistent initial conditions and a mapping that can be used to model the laws of collision. First we define the set of (in)consistent initial conditions.

Definition V.2 (Initial Value Sets): The set of consistent initial conditions, $\mathcal{I}^c$, for Σ^c is defined as $\mathcal{I}^c := \{x \in \mathcal{X} | \exists \underline{x} \in \Sigma^c$ with $\underline{x}(0) = x\}$. The set of inconsistent initial conditions, $\mathcal{N}^c$, for Σ^c is defined as $\mathcal{N}^c := \{x \in \mathcal{X} | \nexists \underline{x} \in \Sigma^c$ with $\underline{x}(0) = x\}$.

Clearly $\mathcal{I}^c \cap \mathcal{N}^c = \emptyset$, and $\mathcal{I}^c \cup \mathcal{N}^c = \mathcal{X}$.

In the remainder of this section we again concentrate on the single constraint case: $C \in \mathbb{R}^{1 \times n}$.

In our framework contact with the boundary set of a single inequality constraint is modeled by a map T. A general expression for the map T will involve entering the physical nature of the system and the constraint, and we will not pursue such a general expression in the present paper. The map T will be used to provide a continuation of trajectories by mapping the set of points where contact is made to the set where release can take place. This will make (5) an invariant set for (4). We will focus on the local behavior upon contact. For the global behavior one also needs to consider an infinite number of collisions in a finite period of time, which is a modeling topic in itself (see [1] for a related discussion). For mechanical systems, however, some general remarks are made in Section VIII.

Recall that if contact is made in $\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c$, then whatever (smooth) control is used all trajectories will go to $\mathcal{X}_f$ unless some special measures are taken. In $\mathcal{V}_c$ it depends also on the control whether a trajectory will proceed in $\mathcal{X}_g$ or will remain in $\ker(C)$. This motivates the following definition.

Definition V.3: Let $x \in \ker(C)$. If $x \in (\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c) \cup \mathcal{V}_f$ we will call contact at x an uncontrolled collision. Likewise, if $x \in \mathcal{X}_{\text{rel}} \backslash \mathcal{V}_c$ we will call release at x an uncontrolled release. If $x \in \mathcal{V}_c$, then we will say that contact, release, at x is a controlled collision, controlled release, respectively.

From a system theoretical point of view, in the case of elastic collisions, the map T can be decomposed into a number of different maps by making use of the symmetry with respect to the integer $r_C(x)$. For this observe that for the sets $\mathcal{X}_{\text{con},v}$ and $\mathcal{X}_{\text{rel},v}$ the value of $r_C(x)$ is the same, as it is for the sets $\mathcal{X}^i_{\text{con},h}$ and $\mathcal{X}^i_{\text{rel},h}$, and for the sets $\mathcal{V}^i_g$ and $\mathcal{V}^i_f$ (c.f., Proposition A.4). For the set $\mathcal{V}_c$ we have $r_0 = r_C(x)$, and once contact is made we can instantly choose a control such that we remain in $\mathcal{V}^*$. Since all subsets are disjunct, and following the line of reasoning above, we introduce the collision maps, making use of the result in Theorem IV.8 (and the notation in Algorithm A.8).

Definition V.4 (Elastic Collisions): In the case of uncontrolled elastic collisions, the collision map T: $\mathcal{X}_{\text{con}} \cup \mathcal{V}_f \mapsto \mathcal{X}_{\text{rel}}$ can be decomposed on its (co)domain as follows.

1) T_v: $\mathcal{X}_{\text{con},v} \mapsto \mathcal{X}_{\text{rel},v}$, for collisions with "velocity" unequal to zero.
2) T^i_h: $\mathcal{X}^i_{\text{con},h} \mapsto \mathcal{X}^i_{\text{rel},h}$, for collisions with a higher odd derivative unequal to zero.
3) $T^i_{f,g}$: $\mathcal{V}^i_f \mapsto \mathcal{V}^i_g$, for collisions with a higher even derivative unequal to zero.
4) T^i_g: $\mathcal{V}^i_g \mapsto \mathcal{V}^i_g$, for collisions with a higher even derivative unequal to zero.

In the case of controlled elastic collisions, the collision map T can be modeled as T_c: $\mathcal{V}_c \times \mathcal{U} \mapsto \mathcal{V}_c \times (\mathcal{U} \backslash \mathcal{U}_f)$, for collisions in $\mathcal{V}_c$.

Note that the map $T^i_{f,g}$ is only needed during initialization since for $t > t_0$ no trajectory of Σ^c_1 will ever enter $\mathcal{V}_f$ again. The map T^i_g is defined for completeness reasons. Also, in $\mathcal{V}_c$, a major difference with respect to classical contact theory occurs: control enters the formulation. Whereas for uncontrolled collisions only the state needs to be adapted, for controlled collisions the control also needs to be adapted. Note that for uncontrolled collisions the map T will introduce a jump in the state-variables, from which an impulsive input can be calculated [15]. Another difference is that for uncontrolled collisions the collision map acts at discrete points in time, whereas controlled collisions may need continuous application of the map T_c. This difference is further detailed in Section VIII for constrained mechanical systems.

In the case of inelastic collisions (in mechanical systems), part of the velocity component of the state is set to zero. This can be captured in our framework by mapping the state x into $\hat{x}$, with $\hat{x}$ in the first nonempty subset in $\mathcal{X}_{\text{rel}}$, where $r_C(x) < r_C(\hat{x})$, leaving the possible higher-order discontinuities intact. This is possible since we can always map x into the set $\mathcal{V}^*$ (for which $r_C(\hat{x}) = r_0$). Again, if originally $r_C(x) = r_0$, then control enters the formulation.

Definition V.5 (Inelastic Collisions): Let $x \in \ker(C)$ denote the contact point. Define $r_T := \min\{r_C(\hat{x}) | \exists \hat{x} \in \mathcal{X}_{\text{rel}}$ such that $r_C(x) < r_C(\hat{x}) \le r_0\}$ with $r_T = r_0$ if $r_C(x) = r_0$. Then inelastic collisions can be modeled by a map as follows.

1) T_u: $(\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c) \cup \mathcal{V}_f \mapsto \mathcal{X}_{\text{rel}} \cap \{x \in \ker(C) | r_C(x) = r_T\}$ if $r_T < r_0$.
2) T_p: $(\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c) \cup \mathcal{V}_f \mapsto \mathcal{V}_c$ if $r_T = r_0$.
3) T_c: $\mathcal{V}_c \times \mathcal{U} \mapsto \mathcal{V}_c \times (\mathcal{U} \backslash \mathcal{U}_f)$ if $r_T = r_0$.

It can be seen that inelastic collisions differ in a number of ways from elastic collisions, especially when control enters the formulation. Notably, if $x \in (\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c) \cup \mathcal{V}_f$ is mapped into $\mathcal{V}_c$, (so called "plastic" collisions), then at the same time instance one must prevent the solution from entering into $\mathcal{X}_f$ by adapting the control (if necessary). This can be done by applying the map T_c immediately after the map T_p. Of course, one could also combine the latter two maps in one controlled collision map, but the present definitions will allow us to unify elastic and inelastic collisions when discussing the behavior of constrained dynamical systems below. Note also that the map T_c in Definition V.5 is similar to the map T_c in Definition V.4.

Example V.6 (Example III.2 Revisited): Consider again the system in Example III.2. From $CB = 0$ and $CAB = 1$, it follows that $r_0 = 2$. From Algorithm A.8 it follows that

$$\mathcal{X}_{\text{con},v} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 < 0\}$$
$$\mathcal{X}_{\text{rel},v} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 > 0\}, \quad \mathcal{V}^* = \{0\}.$$

The sets $\mathcal{V}_g, \mathcal{V}_f, \mathcal{X}_{\text{con},h}$, and $\mathcal{X}_{\text{rel},h}$ are empty. From Theorem IV.8 it now follows that

$$\mathcal{X}_{\text{con}} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 \le 0\}$$

and

$$\mathcal{X}_{\text{rel}} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 \ge 0\}$$

c.f., Example III.2. In the case of elastic collisions the intuitive "change of the sign of the velocity component" follows by defining $T(0, x_2) = (0, -\delta x_2), x_2 \ne 0$. Here $0 < \delta \le 1$ is the elasticity parameter. The inelastic collisions can correspond to $\delta = 0$. □

Due to the introduction of the map T in to our framework, it can be seen that any $x \in \{x \in \mathcal{X} | Cx \ge 0\}$ can be chosen as initial condition: in order to cross the boundary set, contact must be made in $\mathcal{X}_{\text{con}} \cup \mathcal{V}_f$. By choice of the collision map, the trajectory will proceed from $\mathcal{X}_{\text{rel}}$, where the new state acts as an initial condition for the system. Thus, in our framework, trajectories of the constrained system consist of concatenated path pieces of the unconstrained system.

Lemma V.7: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1 \times n}$. The (in)consistent initial value sets for (12) are given by

$$\mathcal{I}^c = \mathcal{X}_g \cup \mathcal{X}_{\text{con}} \cup \mathcal{X}_{\text{rel}} \cup \mathcal{V}_f, \quad \text{and} \quad \mathcal{N}^c = \mathcal{X}_f.$$

Proof: The proof is straightforward from the definitions of the subsets and the collision map T. □

It is remarked that if one requires that all trajectories are smooth at t_0, the condition $\lim_{t\downarrow t_0} \underline{x}(t) = x(t_0)$ must be added to our definitions of (in)consistent initial conditions. In that case the set $\mathcal{X}_{\text{con},v} \cup \mathcal{X}_{\text{con},h} \cup \mathcal{V}_f$ cannot be part of the set $\mathcal{I}^c$ (and should be added to $\mathcal{N}^c$).

Example V.8 (Example III.2 Revisited): Consider again the mechanical system in Example III.2. Use of the map T: $(0, x_2) \mapsto (0, -\delta x_2)$, for $x_2 \neq 0, 0 \leq \delta \leq 1$ shows that for the constrained system Σ_1^c: $\mathcal{I}^c = \mathcal{X}_g \cup \ker(C)$ and $\mathcal{N}^c = \mathcal{X}_f$. □

It is of interest to note that in the classic positive invariance theory [4], [8], the set $\{x \in \mathcal{X} | [P \;\; 0]x \geq 0\}$ is not a positively invariant set for the system in (3). This can be seen using a result from [8], provided that there are no redundant constraints. In that case, classical positive invariance holds if and only if there exists an essentially nonnegative matrix $H \in \mathbb{R}^{p\times p}$ such that for the system with linear feedback $u = Fx$, there holds $\hat{C}(\hat{A} + \hat{B}F) = H\hat{C}$ [8], with $\hat{C} = [P\; 0]$. In the present case this reduces to finding matrix H such that $\hat{C}\hat{A} = H\hat{C}$, since $\hat{C}\hat{B} = 0$. From $\hat{C}\hat{A} = [0 \;\; P]$ and $H\hat{C} = [HP \;\; 0]$ it can be seen that matrix H does not exist unless $P = 0$. Moreover, if $\mathcal{X}_{\text{con},v} \neq \emptyset$, then (classical) positive invariance will never hold. We conclude that our approach is an extension of the classic positive invariance theory for linear systems.

We are now ready to present a more detailed description of a constrained linear system. For Σ_1^c it is clear that the initial conditions must be in the set $\mathcal{I}^c$. If a trajectory $\underline{x}$ at time t belongs to $\mathcal{X}_g$, then the boundary of the constraint set does not influence the trajectory. Hence the state satisfies the unconstrained system equations. Furthermore, if contact is made, collision can take place. This is modeled by the uncontrolled and controlled collision maps from Definitions V.4 and V.5. In fact, we have now proven the following theorem.

Theorem V.9: The constrained system Σ_1^c in (12) can alternatively be presented as $\Sigma_1^c = \{\underline{x}\colon \mathbb{R} \mapsto \mathcal{X} | \exists u$ piecewise C^∞ such that $\forall t_0 \in \mathbb{R}$:

1) $\underline{x}(t_0) \in \mathcal{I}^c = \mathcal{X}_g \cup \ker(C)$;
2) $\underline{x}(t) \in \mathcal{X}_g \cup \mathcal{X}_{\text{rel}}, t \geq t_0 \Rightarrow \dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$;
3) $\underline{x}(t) \in (\mathcal{X}_{\text{con}} \backslash \mathcal{V}_c) \cup \mathcal{V}_f, t \geq t_0 \Rightarrow \lim_{t^*\downarrow t} \underline{x}(t^*) = T(\underline{x}(t))$;
4) $\underline{x}(t) \in \mathcal{V}_c, t \geq t_0 \Rightarrow \lim_{t^*\downarrow t}(\underline{x}, \underline{u})(t^*) = T(\underline{x}(t), \underline{u}(t))$, with $\underline{u}_i(t) := u^{(i)}(t)\}$.

Here T is defined as in Definitions V.4 and V.5.

Proof: Omitted. □

System Σ_1^c is a complex hybrid system, where features of continuous dynamical systems are combined with characteristics of finite automata [6]. Controller synthesis for such systems is a nontrivial task [6]. For nonlinear constrained mechanical systems, some of these difficulties are described in [13] and [18].

Finally, we state the following result.

Corollary V.10: $\Sigma_1^c(A, B, C) = \Sigma_1^c(A + BF, B, C)$, i.e., the constrained system Σ_1^c in Theorem V.9 is invariant under linear feedback $u = Fx + v$, with v the new control.

Example V.11 (Example III.2 Extended): Suppose that we want to model two trains riding on the same track, where the second train is initially to the right of the first train. Denote $y = [y_1^T, y_2^T]^T$, with y_1, y_2, the position of the first train and the second train, respectively. The position constraint reads: $y_2 - y_1 \geq 0$. If we further assume that $m_i = d_i = k_i = l_i = 1$, it follows that $r_0 = 2$. From Algorithm A.8 we obtain that

$$\mathcal{X}_{\text{con},v} = \{y_2 - y_1 = 0, \dot{y}_2 - \dot{y}_1 < 0\}.$$

So, collisions occur when the trains make contact, and the second train is moving faster to the left, or slower to the right, than the first train. Assuming rigid trains, the elastic collision map T_v, based on conservation of momentum, can read

$$T_v(y_1, y_2, \dot{y}_1, \dot{y}_2) = (y_1, y_2, \dot{y}_2, \dot{y}_1). \qquad \square$$

VI. Restricted Behaviors: The Multiple Constraints Case

In this section we will extend the results of Section V to the multiple constraints case, i.e.,

$$\Sigma^c\colon \begin{cases} \dot{x} = Ax + Bu \\ 0 \leq Cx \end{cases} \tag{13}$$

with $C \in \mathbb{R}^{p\times n}, p \geq 1$. For later reference we denote

$$\mathcal{C} := \{x \in \mathbb{R}^n | Cx \geq 0\}, \quad C \in \mathbb{R}^{p\times n}, \qquad p \geq 1.$$

To make full use of the results of the previous sections we make the following observations. Note that $\mathcal{C} = \cap_{i=1}^p \{x \in \mathbb{R}^n | C_i x \geq 0\}$. (In this section the subscript i is used to denote subsets and matrices of subsystem i.) Next, define the constrained systems $\Sigma_i^c, i = 1, \cdots, p$, by

$$\Sigma_i^c\colon \begin{cases} \dot{x} = Ax + Bu \\ 0 \leq C_i x. \end{cases} \tag{14}$$

It follows that $\Sigma^c = \cap_{i=1}^p \Sigma_i^c$. For system Σ^c contact or release with the boundary set is to be understood as contact or release with at least one boundary set. For the polyhedral cone $\mathcal{C}$ the boundary set $\mathcal{C}_b$, and the regions $\mathcal{X}_g$ and $\mathcal{X}_f$, are given by

$$\mathcal{C}_b = \mathcal{C} \cap \left\{ \bigcup_{i=1}^p \{x \in \mathbb{R}^n | C_i x = 0\} \right\} \tag{15}$$

$$\mathcal{X}_g = \bigcap_{i=1}^p \{x \in \mathbb{R}^n | C_i x > 0\} \tag{16}$$

$$\mathcal{X}_f = \bigcup_{i=1}^p \{x \in \mathbb{R}^n | C_i x < 0\}. \tag{17}$$

It can be seen that

$$\mathcal{C}_b = \mathcal{C} \cap \left(\bigcup_{i=1}^p (\ker(C_i)) \right), \quad \mathcal{X}_g = \bigcap_{i=1}^p \mathcal{X}_{g,i}$$

and

$$\mathcal{X}_f = \bigcup_{i=1}^p \mathcal{X}_{f,i}.$$

For each subsystem $\Sigma_i^c, (i = 1 \cdots p)$, the subsets of interest can be computed from Algorithm A.8. The (in)consistent initial conditions sets are given by

$$\mathcal{I}^c = \bigcap_{i=1}^{p} \mathcal{I}_i^c = \mathcal{C}, \quad \text{and} \quad \mathcal{N}^c = \bigcup_{i=1}^{p} \mathcal{N}_i^c = \mathcal{X}_f.$$

Taking the sets $\mathcal{I}^c$ and $\mathcal{N}^c$ into account, the following is obtained for the multiple constraints case:

$$\mathcal{V}_c = \left(\bigcup_{i=1}^{p} \mathcal{V}_{c,i}\right) \cap \mathcal{C}_b \tag{18}$$

$$\mathcal{V}^* = \bigcap_{i=1}^{p} \mathcal{V}_{c,i} \tag{19}$$

$$\mathcal{V}_g = \left(\bigcup_{i=1}^{p} \mathcal{V}_{g,i}\right) \cap \mathcal{C}_b \tag{20}$$

$$\mathcal{V}_f = \left(\bigcup_{i=1}^{p} \mathcal{V}_{f,i}\right) \cap \mathcal{C}_b \tag{21}$$

$$\mathcal{X}_{\mathrm{con},v} = \left(\bigcup_{i=1}^{p} \mathcal{X}_{\mathrm{con},v,i}\right) \cap \mathcal{C}_b \tag{22}$$

$$\mathcal{X}_{\mathrm{rel},v} = \left(\bigcup_{i=1}^{p} \mathcal{X}_{\mathrm{rel},v,i}\right) \cap \mathcal{C}_b \tag{23}$$

$$\mathcal{X}_{\mathrm{rel},h} = \left(\bigcup_{i=1}^{p} \mathcal{X}_{\mathrm{rel},h,i}\right) \cap \mathcal{C}_b \tag{24}$$

$$\mathcal{X}_{\mathrm{con},h} = \left(\bigcup_{i=1}^{p} \mathcal{X}_{\mathrm{con},h,i}\right) \cap \mathcal{C}_b. \tag{25}$$

There are a number of things that make multiple constraints notably different in character than single constraints. First, the subsets in (18)–(25) need not be disjunct as on the intersection of the constraints, for each individual boundary set different characteristics can hold. Nevertheless, all subsets can still be computed. The multiple number of combinations of subsets that is possible also reveals that for dynamical systems subject to multiple constraints the organization of all the subsets is a problem in itself, which we will not tackle here.

Secondly, in contrast to the single constraint case, it is now possible that, for instance, $\mathcal{V}_{c,i}$ is not entirely in $\mathcal{C}$, or that $\mathcal{X}_{\mathrm{con},v,i} \in \mathcal{C}$, but $\mathcal{X}_{\mathrm{rel},v,i} \notin \mathcal{C}$. This is shown in the following example.

Example VI.1: Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is readily verified that $\mathcal{X}_n(A, B, C)$ satisfies the assumptions. From Algorithm A.8 it follows that

$$\begin{aligned} \mathcal{X}_{\mathrm{con},v,1} &= \{x \in \mathbb{R}^3 | x_1 = 0, -x_2 < 0\} \\ \mathcal{X}_{\mathrm{rel},v,1} &= \{x \in \mathbb{R}^3 | x_1 = 0, x_2 < 0\} \\ \mathcal{X}_{\mathrm{con},v,2} &= \{x \in \mathbb{R}^3 | x_2 = 0, x_3 < 0\} \end{aligned}$$

and

$$\mathcal{X}_{\mathrm{rel},v,2} = \{x \in \mathbb{R}^3 | x_2 = 0, x_3 > 0\}.$$

From (22) and (23) it follows that $\mathcal{X}_{\mathrm{con},v,1} \in \mathcal{X}_{\mathrm{con}}$ and $\mathcal{X}_{\mathrm{rel},v,1} \notin \mathcal{X}_{\mathrm{rel}}$. □

Two solutions can be proposed for this loss of symmetry. The first is to treat the loss of symmetry as a control problem: the system should be controlled such that contact is not made in $\mathcal{X}_{\mathrm{con},v,1}$. In light of the research in the field of positive invariance, this seems to be a difficult problem, and we will not discuss it in the present paper. Another solution is to view this as an inelastic collision: simply map the contact point in the first subset (with higher value of $r_C(x)$) such that release can take place. Such a subset always exists since $\mathcal{V}^*$ is nonempty. Fortunately, symmetry is preserved in the case of mechanical systems (with $r_0 = 2$), subject to position constraints, as in (2). This is immediate from the observation that the contact and release sets yield inequality constraints on the velocities, which can never be in conflict with the position constraints. (They can be empty if the original position constraints are in fact implicit equalities [9].) Obviously, Example VI.1 does not deal with a mechanical system.

The third difficulty with multiple constraints is that for all intersections of boundary sets, new collision maps may need to be defined even if the collision maps for the single boundary sets $\{x \in \mathcal{X} | C_i x = 0\} \backslash \{\cup_{j=1, j\neq i}^{p} \{x \in \mathcal{X} | C_j x = 0\}\}$ have already been specified. However, this requires specific knowledge of the application, which can be seen as follows. Suppose that there are two different inequality constraints, numbered 1 and 2, respectively, leading to two boundary sets. On the intersection of the boundary sets the combinations of all subsets introduce subsets with new characteristics. For instance, there may be a subset where for one constraint we enter the boundary set in $\mathcal{X}_{\mathrm{con},v,1}$, whereas for the other constraint we enter the boundary set in $\mathcal{X}^3_{\mathrm{con},h,2}$. Based on Section V, the obvious choice of the collision map T would be to map $(\mathcal{X}_{\mathrm{con},v,1} \cap \mathcal{X}^3_{\mathrm{con},h,2})$ onto $(\mathcal{X}_{\mathrm{rel},v,1} \cap \mathcal{X}^3_{\mathrm{rel},h,2})$. It is unclear, however, how the original collision maps $T_{v,1}$ and $T^3_{h,2}$ should be combined. And compositions of $T_{v,1}$ and $T^3_{h,2}$ also may not be a correct expression as the map $T^3_{h,2}$ may result in a (intermediate) state that violates the first inequality constraint. A further complication arises if the boundary sets have different elasticity properties. Clearly, the physics of the problem should not only specify (expressly) the collision maps on the boundary set of each individual constraint, but also on the intersection of the boundary sets. For the present paper it suffices to remark that we have identified the places where collisions can occur and where the collision maps should be defined to deal with these collisions. Note that the results presented here can also be used as a starting point for controller design such that simultaneously a state trajectory remains on one constraint and makes contact with another constraint in a smooth manner, i.e., contact with velocity components equal to zero. For instance, if $\mathcal{V}_{c,1} \subseteq \mathcal{V}_c$, such a design is possible if and only if $\mathcal{V}_{c,1} \cap (\mathcal{X}_{\mathrm{con},2} \backslash \mathcal{X}_{\mathrm{con},v,2}) \neq \emptyset$.

It is clear that with the above definitions the analog of Theorem IV.8-3) and -4) still holds, with the obvious redefinition of the sets $\mathcal{X}_{\mathrm{con}}$ and $\mathcal{X}_{\mathrm{rel}}$, keeping in mind that contact and release is to be understood as contact or release with at least one boundary set.

Based on the results of Section V and the discussion above, we can now give the following result, with the obvious redefinition of the set $\mathcal{U}_f$.

Theorem VI.2: The constrained system Σ^c given in (13) can alternatively be presented as $\Sigma^c = \{\underline{x}\colon \mathbb{R} \mapsto \mathcal{X} | \exists u$ piecewise C^∞ such that $\forall t_0 \in \mathbb{R}$:

1) $\underline{x}(t_0) \in \mathcal{I}^c$;
2) $\underline{x}(t) \in \mathcal{X}_g \cup \mathcal{X}_{\rm rel}, t \geq t_0 \Rightarrow \dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$;
3) $\underline{x}(t) \in (\mathcal{X}_{\rm con} \backslash \mathcal{V}_c) \cup \mathcal{V}_f, t \geq t_0 \Rightarrow \lim_{t^* \downarrow t} \; \underline{x}(t^*) = T(\underline{x}(t))$;
4) $\underline{x}(t) \in \mathcal{V}_c, t \geq t_0 \Rightarrow \lim_{t^* \downarrow t} (\underline{x}, \underline{u})(t^*) = T(\underline{x}(t), \underline{u}(t))$, with $\underline{u}_i(t) := u^{(i)}(t)\}$.

Proof: The proof follows from Theorem V.9 and the definitions in this section. □

Again, system Σ^c is a complex hybrid system, where the collision maps are now much more complicated, especially on intersections of boundary sets. As an example we will consider the case where the inequalities model a linear subspace, showing that systems subject to equality constraints can also be treated with the theory presented here.

Example VI.3 (Example III.2 Extended): Consider again the system in Example III.2, but now subject to the constraint pair $y \geq 0, -y \geq 0$. For system Σ_1^c we have (from Example V.6)

$$\mathcal{X}_{{\rm con},v,1} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 < 0\}$$

and

$$\mathcal{X}_{{\rm rel},v,1} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 > 0\}.$$

From Algorithm A.8 it follows for system Σ_2^c that (with constraint $-y \geq 0$):

$$\mathcal{X}_{{\rm con},v,2} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 > 0\}$$

and

$$\mathcal{X}_{{\rm rel},v,2} = \{x \in \mathbb{R}^2 | x_1 = 0 \wedge x_2 < 0\}.$$

Clearly, $\mathcal{C}_b = \{y = 0\}, \mathcal{N}^c = \{y < 0\} \cup \{y > 0\}, \mathcal{I}^c = \{0\}$. Since for Σ_1^c and Σ_2^c the boundary set is the same, it is obvious that also the intersection of the contact sets becomes important for trajectories of $\Sigma_1^c \cap \Sigma_2^c$. From $\mathcal{X}_{{\rm con},v,1} \cap \mathcal{X}_{{\rm con},v,2} = \emptyset$ and $\mathcal{X}_{{\rm rel},v,1} \cap \mathcal{X}_{{\rm rel},v,2} = \emptyset$ it follows that a trajectory cannot leave $\mathcal{C}_b$, nor make contact with it. Consequently, $y = 0$ is the only solution, as expected. □

Corollary VI.4: $\Sigma^c(A, B, C) = \Sigma^c(A + BF, B, C)$, i.e., the constrained dynamical system Σ^c in Theorem VI.2 is invariant under linear feedback $u = Fx + v$, with v the new control.

VII. GENERALIZATIONS

In this section we will first discuss the relaxation of one or more of the assumptions. Secondly, we will extend the results to cover arbitrary convex polyhedral sets.

First consider the case where the assumption ${\rm im}(B) \subseteq \ker(C)$ does not hold, i.e., $CB \neq 0$. The following result is valid independent of the controllability of (A, B) and states that all collisions are controlled collisions.

Lemma VII.1: Let $C \in \mathbb{R}^{1 \times n}, C \neq 0$ and $d = 0$. If ${\rm im}(B) \not\subseteq \ker(C)$, then $\ker(C) = \mathcal{V}^* = \mathcal{V}_c = \mathcal{X}_{{\rm con},v} = \mathcal{X}_{{\rm rel},v}$, and $\mathcal{X}_{{\rm con},h} = \mathcal{X}_{{\rm rel},h} = \mathcal{V}_f = \mathcal{V}_g = \emptyset$.

Proof: From $\dim(\ker(C)) = n - 1$ and ${\rm im}(B) \not\subseteq \ker(C)$ follows $\ker(C) + {\rm im}(B) = \mathcal{X}$. ISA now gives $\mathcal{V}^* = \ker(C)$. Moreover, since $CB \neq 0$ one has $r_0 = 1$. And from $h_1(x, \underline{u}) = CAx + CB\underline{u}_0$ it follows that, depending on value of $\underline{u}_0, h_1(x, \underline{u}) < 0, h_1(x, \underline{u}) = 0$ or $h_1(x, \underline{u}) > 0$. □

Next suppose that (A, B) is not controllable, and $CB = 0$. Note that in case $B = 0$, i.e., in the case of an autonomous system, obviously (A, B) is not controllable. For simplicity assume that we are dealing with a single inequality constraint. Two cases are distinguished: $r_0 < \infty$ and $r_0 = \infty$. In case $r_0 < \infty$, the analysis of the previous sections still holds. The interesting case is when $r_0 = \infty$. In that case $r_{\min} = r_1$ (see the Appendix, Definition A.5). It is straightforward to show that now $\mathcal{V}^* = \mathcal{V}^{r_1}$ and $\mathcal{V}_c = \emptyset$. So, for instance, step 10) in Algorithm A.8 should be adapted to $\mathcal{V}^{r_1} = \mathcal{V}^*$. Furthermore, from $r_0 = \infty$ it follows that $CA^{j-1}B = 0$ $(j \geq 1)$. It follows that, in the notation of [29], $\langle A | {\rm im}(B) \rangle \subseteq \ker(C)$. Consequently, the state trajectories of the controllable part of the system are entirely in $\ker(C)$. It depends on the characteristics of the uncontrollable part of the system whether collisions can happen.

Example VII.2: Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = [1 \quad 0 \quad 0].$$

It follows that

$$\langle A | {\rm im}(B) \rangle = \mathcal{V}^* = \{x \in \mathbb{R}^n | x_1 = 0 = x_2\}.$$

Algorithm A.8 yields

$$\mathcal{X}_{{\rm con},v} = \{x \in \mathbb{R}^n | x_1 = 0, x_2 < 0\}.$$

These collisions are uncontrolled. On the other hand if $A = I$, then $\mathcal{X}_{{\rm con},v} = \emptyset$ since for the uncontrolled system we have $\underline{x}_1(t) = e^t \underline{x}_1(t_0)$, which never comes in contact with the boundary $x_1 = 0$ if $\underline{x}_1(t_0) \neq 0$. □

We have now set the stage to extend our results to linear systems, where the state trajectory is constrained to an arbitrary convex polyhedral set. It is well known that any polyhedral set in $\mathbb{R}^n$ can be written as a convex polyhedral cone in $\mathbb{R}^{n+1}$ by introducing an auxiliary variable [24]. The original polyhedral set $\mathcal{C} := \{x \in \mathbb{R}^n | Cx \geq d\}$ can then be obtained by projection onto the original space $\mathbb{R}^n$ [24]. Let α denote the auxiliary variable. Now define the extended polyhedral cone

$$\mathcal{C}^+ := \{(x, \alpha) \in \mathbb{R}^{n+1} | Cx\alpha - d\alpha \geq 0, \alpha \geq 0\}.$$

Clearly, taking $\alpha = \alpha_0 > 0$ constant, by means of the projection map Π_x, i.e., $\Pi_x\colon (x, \alpha) \mapsto x$, we obtain the original polyhedral set $\mathcal{C}$ again.

This idea for static cones can be extended as follows to the dynamical system $\Sigma^c(A, B, C, d)$ given by

$$\Sigma^c\colon \begin{cases} \dot{x} = Ax + Bu \\ d \leq Cx. \end{cases} \tag{26}$$

To avoid some technicalities we assume without loss of generality that there are no redundant inequalities [9]. It can be seen that contact and release take place on the boundary set

$$\mathcal{C}_b := \mathcal{C} \cap \left\{ \bigcup_{i=1}^{p} \{x \in \mathcal{X} | C_i x = d_i\} \right\}.$$

For (26) the integer $r_b(x) := \min\{r(x, \underline{u}) | \underline{u} \in \mathcal{U}^{\mathbb{N}}\}$ is of interest. (Compare with $r_C(x)$ from Definition IV.1.) Define as new state variable $z := [(x\alpha)^T, \alpha^T]^T$. Note that auxiliary variable α is used as a scaling variable and is kept to its initial value α_0 by taking $\dot{\alpha} = 0$. In that case the original state x can be obtained from the new state since $x = z_1/\alpha_0$. Clearly, there is a one-to-one correspondence between the old state-variable x and the new state-variable z_1. For the dynamics of Σ^{c+}, the extended system, we obtain

$$\dot{z_1} = \frac{d(x\alpha)}{dt} = \dot{x}\alpha + \dot{\alpha}x.$$

Substitution of this equation into (26), using $\dot{\alpha} = 0$, yields

$$\dot{z_1} = (Ax + Bu)\alpha = Az_1 + Buz_2.$$

We obtain for the extended system

$$\Sigma^{c+}\colon \begin{cases} \dot{z}_1 = Az_1 + Buz_2 \\ \dot{z}_2 = 0 \\ 0 \le Cz_1 - dz_2 \\ 0 < z_2. \end{cases} \tag{27}$$

Observe that in (27) the dynamics are (in part) represented by a nonlinear differential equation. Also note that the restricting cone is not closed. In order to put (27) in the standard form discussed in Section VI, we first observe that $z_2 > 0$ is trivially satisfied if we take as initial condition

$$z(t_0) := [(\alpha_0 x(t_0))^T, \alpha_0^T]^T$$

with $\alpha_0 > 0$. This follows from $\dot{z}_2 = 0$. Next, and again using the fact $\alpha > 0$ is constant, we make a change of the basis of $\mathcal{U}$, i.e., $v := \alpha u$, with v the new control. Since $\alpha \neq 0$, we can always recover the original input u. Substitution of $v = \alpha u$ into (27) gives

$$\Sigma^{c+}\colon \begin{cases} \dot{z}_1 = Az_1 + Bv \\ \dot{z}_2 = 0 \\ 0 \le Cz_1 - dz_2 \end{cases} \tag{28}$$

with

$$z(t_0) := [(\alpha_0 x(t_0))^T, \alpha_0^T]^T, \qquad \alpha_0 > 0.$$

As a last step, define

$$\overline{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \text{and} \quad \overline{C} := [C \quad -d].$$

The resulting system equations now read

$$\Sigma^{c+}\colon \begin{cases} \dot{z} = \overline{A}z + \overline{B}v \\ 0 \le \overline{C}z. \end{cases} \tag{29}$$

Clearly, system (29) is a linear system restricted by a polyhedral cone and fits the framework of Section VI.

With respect to the assumptions made in Section II we remark the following. First observe that

$$\overline{C}\,\overline{B} = [C \quad -d] \begin{bmatrix} B \\ 0 \end{bmatrix} = CB.$$

Hence $\{\mathrm{im}(B) \subseteq \ker(C)\} \Leftrightarrow \{\mathrm{im}(\overline{B}) \subseteq \ker(\overline{C})\}$. Also $\{C \neq 0\} \Leftrightarrow \{\overline{C} \neq 0\}$ and $\{\mathcal{C} \neq \emptyset\} \Leftrightarrow \{\mathcal{C}^+ \neq \emptyset\}$. However, controllability of the pair (A, B) is not preserved by introduction of the auxiliary variable α: the pair $(\overline{A}, \overline{B})$ is obviously not controllable. It can be seen, however, that $\langle \overline{A} | \mathrm{im}(\overline{B}) \rangle = \mathcal{X} \cup \{\alpha_0\}$, and $\Pi_x \langle \overline{A} | \mathrm{im}(\overline{B}) \rangle = \mathcal{X}$. We state the following result.

Proposition VII.3: Let $C \in \mathbb{R}^{1 \times n}, d \in \mathbb{R}$, and $\alpha > 0$. Consider systems Σ^c and Σ^{c+}. Then the following relations hold.

1) $r_0(A, B, C, d) = r_0(\overline{A}, \overline{B}, \overline{C})$.
2) $\{[(\alpha x)^T, \alpha^T]^T \in \mathbb{R}^{n+1} | r_{\overline{C}}(\alpha x, \alpha) = \ell\} = \{x \in \mathbb{R}^n | r_b(x) = \ell\}$.

Proof: Straightforward computation gives

$$\overline{A}^j = \begin{bmatrix} A^j & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overline{C}\,\overline{A}^j = [C \quad -d] \begin{bmatrix} A^j & 0 \\ 0 & 0 \end{bmatrix} = [CA^j \quad 0].$$

The result in 1) now follows from $\overline{C}\,\overline{A}^j \overline{B} = CA^j B$. From the definitions it follows that

$$\begin{aligned} h_i(z, \underline{v}) &= h_i([(\alpha x)^T, \alpha^T]^T, \underline{v}) \\ &= \overline{C}\,\overline{A}^i \begin{bmatrix} x\alpha \\ \alpha \end{bmatrix} + \sum_{j=1}^{i} \overline{C}\,\overline{A}^{j-1} \overline{B}\underline{v} \\ &= CA^i x\alpha + \sum_{j=1}^{i} CA^{j-1} B \underline{v}_{i-j} \\ &= \alpha \left(CA^i x + \sum_{j=1}^{i} CA^{j-1} B \underline{u}_{i-j} \right) \\ &= \alpha h_i(x, \underline{u}). \end{aligned}$$

The result in 2) now follows from $\alpha > 0$. □

Clearly, for Σ^{c+} we can use the results presented in Sections III–VI, with the notable exception that we no longer have that $\mathcal{V}^* = \mathcal{V}_c$ for the constrained system Σ^c if $d \neq 0$.

VIII. Constrained Mechanical Systems

In this section our approach to constrained dynamical systems is applied and compared to constrained manipulator models. A primary goal of controller design for constrained manipulators is to ensure that impact forces remain within specified bounds and that bouncing of the manipulator is avoided as much as possible. In general, for constrained manipulator systems there are two approaches to the contact problem: in the first approach the surface is assumed to be stiff, but deformable. In the second approach the surface is assumed to be rigid. We will only discuss the latter approach for an ideal linear model of a constrained robotic manipulator: a series of connected links with a so-called end-effector on the end of the last link. Numerous publications exist which address constrained manipulators, and we refer to [7], [13], [14], [17], [19], [21], and [30] (and the references therein) for details.

As in our approach, three phases are recognized in constrained motion: a free motion phase, a collision phase, and a constrained motion phase. More often than not, the dynamics model of a robotic manipulator is in terms of generalized joint-coordinates. Using the kinematic relations it can be transformed into a model in so-called task-space coordinates [21]. Assuming that contact is frictionless, the following model is taken from [7], adapted to a linear setting and to our notation:

$$M\ddot{y} + D\dot{y} + Ky = Lv + P^T f \tag{30}$$

$$f = \begin{cases} 0 & \text{during free motion,} \\ \Gamma & \text{during collision,} \\ \lambda & \text{during constrained motion} \end{cases} \tag{31}$$

subject to (2). Here y represents the end-effector position, $P^T f$ represents the contact force matrix, Γ denotes the impulsive force due to collisions, and λ denotes the Lagrange multiplier which has the dimension of the force for a holonomic equality constraint $Py = p$ [7]. We will assume that $d \leq 0$, which gives that the constraint set is nonempty.

Model (30) is based on mechanical systems subject to equality constraints. Clearly, (30) now contains two inputs, i.e., v and f, of which v is the control input and f is an input which is not available for control purposes. In the literature on constrained mechanical systems matrix L is usually chosen to be the identity. We will assume that each degree of freedom can be actuated and that L is invertible (to avoid some singularities). As in Section II a state-space model can be obtained by defining $x = [y^T, \dot{y}^T]^T$

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} v + \begin{bmatrix} 0 \\ M^{-1}P^T \end{bmatrix} f \tag{32}$$

in conjunction with (31) and (5).

In our approach to restricted systems, c.f., Theorem VI.2, in the free motion phase the original unconstrained dynamics model is recovered. This is also true in model (30), (31), and (2) since f is set to zero in that case.

Next we discuss the collision phase. In (31), Γ models an impulsive force due to collisions. In the case of a robotic manipulator, this impulsive force is distributed throughout all joints of the system. In general it is difficult to obtain an explicit expression of this force, and in [7] an estimate is given based on the momentum change in the robotic system due to the collision. This means that knowledge of the collision map T, introduced in Section V, is a prerequisite to obtaining an expression for Γ, since the velocity characteristics on the end-effector level immediately after the collisions are necessary. After estimating Γ, the jump discontinuity in the joint angular velocity components can be calculated. The importance of (the estimate of) Γ is a consequence of the fact that a small impulsive force leads (in general) to small changes in the velocities. The estimated value of Γ can be used in the path planning stage: a feedforward control can be designed such that impulse forces are small.

With the aid of the homogenization procedure from Section VII, it can be shown that from (3) and Definition IV.1 that $r_0 = 2$. From Algorithm A.8 it follows that for (3), with $\hat{C} \in \mathbb{R}^{1\times n}, d \in \mathbb{R}$, the uncontrolled collision and release sets are

$$\mathcal{X}_{\text{con},v} = \{Py = d \wedge P\dot{y} < 0\} \tag{33}$$

$$\mathcal{X}_{\text{rel},v} = \{Py = d \wedge P\dot{y} > 0\}. \tag{34}$$

Consequently, for (3), it follows that the collision map T (introduced in Section V) does not need to affect the positions, but causes a jump discontinuity in (part of) the velocity components of the state. This is completely in line with the results found in, for instance, [7], [13], and [30] (when adapted to the linear case). Since the system has a nonzero mass, the discontinuity in the state is caused by an impulsive force. Our approach to constrained dynamical systems covers a result obtained in [7] for constrained mechanical systems. (The original proof for the nonlinear case can be found in [7, Th. 1].)

Proposition VIII.1 [7]: Let $\dot{y}_n$ denote the velocity component normal to the boundary set of (2) at the moment of collision. Then there is no elastic collision $(\Gamma = 0)$ if and only if $\dot{y}_n = 0$.

Proof: From Theorems IV.8 and V.9 and (33), it follows that there are no collisions if and only if $Py = d$ and $P\dot{y} = 0$. Hence, $\dot{y} \in \ker(P)$, i.e., $\dot{y}_n = 0$. The collision map T of Section V can be taken as the identity on $\mathcal{X}$, meaning that there is no discontinuity in the state. Consequently, $\Gamma = 0$. □

Finally, we discuss the constrained motion phase. System (3) reduces to a singular system (or descriptor system) [20]. If the motion is assumed to proceed in $\mathcal{V}^*$, i.e., if one assumes that the trajectory of (3) remains in $\ker(C)$, we effectively have a bilateral constraint. An expression for the collision map T_c can be derived with the aid of the class of applicable controls $\mathcal{U}_b$ in Definition IV.10. Since $\hat{C}\hat{A}^{r_0-1}\hat{B}$ has full-row rank, an expression for a right-inverse can be found. To arrive at (30) we take $u := v + w$ and solve

$$\hat{C}\hat{A}^{r_0}x + \hat{C}\hat{A}^{r_0-1}\hat{B}(v+w) = 0$$

for the vector w. This leads to $w = L^{-1}P^T\lambda$, where

$$\lambda = -(PM^{-1}P^T)^{-1}(PM^{-1}(Lv - D\dot{y} - Ky)).$$

If we now define the collision map $T_c(x, u) = (x, v + L^{-1}P^T\lambda)$, c.f., Definition V.4, then we arrive at (30). Note that the state needs no adaptation. The expression for λ equals the one in, for instance, [10], [17], and [18] (for the linear case). Upon making contact, whatever the control v, this control is corrected by means of λ such that the control u defined by $u := v + L^{-1}P^T\lambda$ is not forbidden, i.e., $u \notin \mathcal{U}_f$ (c.f., Definition IV.10). This is part of the reason why model (30) is widely used. On the other hand, if u is given, then (v, λ) is not uniquely determined. This also follows from practical experience; it is possible to vary the force exerted on a table while maintaining the position. This means that there are infinitely many combinations (v, λ) that give

$u = v + L^{-1}P^T\lambda$. Note that by introduction of the auxiliary variable λ, recognition of deactivation of the boundary set is a problem in itself, i.e., the sign of λ is important. For a detailed discussion on whether or not a boundary set becomes active or passive in the case of constrained mechanical systems, we refer to [13] and [18].

It is concluded that our approach to restricted systems fits the general approach taken in the literature on constrained mechanical systems. Moreover, our approach makes it possible to compute explicitly the regions where uncontrolled collisions occur and where smooth controlled contact is possible. Our study indicates that control effort may be aimed at making contact with small velocities and at effective avoidance of multiple bounces. It is remarked that (30) also provides clues for extending the theory presented in this paper to the case where, after impact, the motion remains (or is to remain) on the boundary of the constraint set [14], [19]. During this constrained motion phase, system Σ^c (6) becomes a descriptor system in differential algebraic form. If part of the input variables are to be interpreted as contact forces as in (30), it is practical to maintain all input variables in the system representation. An auxiliary variable can be introduced which plays the role of Lagrange multiplier [10], [17]. A promising approach is to combine our framework to model the contact problem, with the framework advocated in [26] to cover subspace restrictions.

IX. Conclusions

In this paper we have studied the contact problem: the effect (of the boundary) of inequality constraints on the behavior of linear continuous-time dynamical systems. A number of (two-by-two disjunct) subsets of the state space have been introduced. It was shown that these subsets are invariant under linear state feedback. Algorithms have been derived that calculate these subsets in a finite number of steps. Our main results are a system theoretical framework in which we described exactly what happens upon contact, identified the specific places for modeling the laws of collision (which are regarded as external factors), and presented a precise definition of the constrained state-space system in terms of the restricted behavior. In this framework, the consistent initialization of a constrained linear system has been discussed. The results presented here can be considered an extension for restricted linear systems of the classic positive invariance theory for linear systems.

In practice there are usually restrictions on the control. By linear state feedback these constraints are transformed into state constraints. Clearly, by choice of the control designer, contact with the virtual boundary can lead to saturation of the controls. One of the features of classical positive invariance theory is that it usually leads, under restricting conditions, to controls that do not saturate. It is of interest to combine the theory presented here for systems with state restrictions with the (classical) positive invariance theory for controller restrictions (under linear state feedback). Such a theory can lead to nonsaturating controller design, where realistic collision maps are included for the state restrictions.

It is also of interest to apply the approach taken in the present paper to cover other (more general) system representations. If the restricting constraints contain a mixture of equalities and inequalities, it is natural to investigate the contact problem in a descriptor representation: the equalities, i.e., the dynamics and the equality constraints represent the descriptor system, and the inequality constraints model behavioral restrictions. The equalities can also model contact with a boundary for a longer period of time. In that case, combining our framework with the results reported in [10] and [26] may allow for easier resolution of constrained motion problems in a system theoretical setting.

Finally, we remark that our approach shows promise for extension to nonlinear systems. Indeed, (some of) the integers introduced in the present paper already have their counterpart for nonlinear systems. Also, for controlled collisions the role of the largest controlled invariant set is taken over by the maximal controlled invariant distribution. This is currently under investigation.

Appendix
Algorithms

In this Appendix first some relations between subsets of the boundary set are presented. Next, algorithms are derived that will, in principle, produce all the subsets in a finite number of steps.

Lemma A.1: Let $\mathcal{X}_n(A,B,C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Then $\mathcal{V}_c = \mathcal{V}^*$.

Proof: ($\mathcal{V}_c \subseteq \mathcal{V}^*$): Suppose $x \in \mathcal{V}_c$. Then $r_0 = r_C(x)$. From Corollary IV.2-2) and -4) it follows that $\exists \underline{u}$ such that $h_i(x,\underline{u}) = 0, \forall i$. This gives $r(x,\underline{u}) = \infty$. From Lemma IV.4 it follows that $x \in \mathcal{V}^*$. ($\mathcal{V}^* \subseteq \mathcal{V}_c$): Let $x \in \mathcal{V}^*$. Now suppose that $x \notin \mathcal{V}_c$. By definition $r_C(x) \neq r_0$. From Corollary IV.2-1) it follows $r_C(x) < r_0$. Moreover, from Corollary IV.2-3) it follows that $h_{r_C(x)}(x,\underline{u}) \neq 0, \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$. This contradicts that $\exists \underline{u} \in \mathcal{U}^{\mathbb{N}}$ such that $r(x,\underline{u}) = \infty$, as follows from $x \in \mathcal{V}^*$.

Lemma A.2: Let $\mathcal{X}_n(A,B,C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Then we have the following.

1) $\mathcal{X}_{\text{con}} \cap \mathcal{X}_{\text{rel}} = \mathcal{V}^* \cup \mathcal{V}_g$.
2) $\mathcal{X}_{\text{con},v} \cap \mathcal{X}_{\text{rel}} = \emptyset$, and $\mathcal{X}_{\text{rel},v} \cap \mathcal{X}_{\text{con}} = \emptyset$.
3) $\mathcal{X}_{\text{con}} \cup \mathcal{X}_{\text{rel}} = \ker(C) \backslash \mathcal{V}_f$.
4) $\mathcal{V}_c \cap \mathcal{V}_g = \emptyset$.

Proof ($1, \subseteq$): Suppose that $x \in \mathcal{X}_{\text{con}} \cap \mathcal{X}_{\text{rel}}$. If $r_C(x) = r_0$, then $x \in \mathcal{V}_c$. From Lemma A.1 it follows that $x \in \mathcal{V}^*$. If on the other hand, $r_C(x) \neq r_0$, then $r_C(x) < r_0$, c.f., Corollary IV.2-1). From $x \in \mathcal{X}_{\text{rel}}$, Lemma IV.4-2), and Corollary IV.2-3) it follows that $h_{r_C(x)}(x,\underline{u}) > 0, \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$. From $x \in \mathcal{X}_{\text{con}}$ it follows from Lemma IV.4-1) that $h_{r_C(x)}(x,\underline{u})$ is even. From Definition IV.5 it now follows that $x \in \mathcal{V}_g$. ($1, \supseteq$): from Definition IV.5 and Lemma IV.4 it follows that $\mathcal{V}_g \subseteq \mathcal{X}_{\text{con}} \cap \mathcal{X}_{\text{rel}}$. Suppose that $x \in \mathcal{V}_c$. Then $r_C(x) = r_0$, c.f., Definition IV.5. From Corollary IV.2-2) and Definition IV.1 it follows that $h_i(x,\underline{u}) = 0\ \forall i < r_0, h_{r_0}(x,\underline{u}) = CA^{r_0}x + CA^{r_0-1}B\underline{u}_0$, with $CA^{r_0-1}B \neq 0$. By appropriate choice of $\underline{u}_0$ one can make $h_{r_0}(x,\underline{u})$ positive or negative, taking into account the parity of r_0. From Lemma IV.4 it now follows that $x \in \mathcal{X}_{\text{con}} \cap \mathcal{X}_{\text{rel}}$.

2): We only prove the first equality. The second equality can be proven analogously. Suppose $x \in \mathcal{X}_{\text{con},v}$. Then $r_C(x) = 1$, c.f., Definition IV.6. It follows from Lemma IV.4 and $CB = 0$ that $h_1(x, \underline{u}) < 0, \forall \underline{u} \in \mathcal{U}^{\mathbb{N}}$. Lemma IV.4-2) now gives that $x \notin \mathcal{X}_{\text{rel}}$.

3): Let $x \in \ker(C)$. First assume that $x \in \mathcal{V}_f$. Definition IV.5-2) gives that $\forall \underline{u} \in \mathcal{U}^{\mathbb{N}}, r_C(x)$ is even and $h_{r_C(x)}(x, \underline{u}) < 0$. From Lemma IV.4 it now follows that $x \notin \mathcal{X}_{\text{con}} \cup \mathcal{X}_{\text{rel}}$. Next assume that $x \in \ker(C) \backslash \mathcal{V}_f$. If $r_C(x)$ is odd and $h_{r_C(x)}(x, \underline{u}) < 0$, then it follows from Lemma IV.4 that $x \in \mathcal{X}_{\text{con}}$. If $h_{r_C(x)}(x, \underline{u}) > 0$, then it follows from Lemma IV.4 that $x \in \mathcal{X}_{\text{rel}}$. This leads to $x \in \mathcal{X}_{\text{con}} \cup \mathcal{X}_{\text{rel}}$.

4): This follows from the definitions. □

Note that the result in Lemma A.2-1) shows that Example III.2 is indeed nongeneral. That example, however, does show that some of the subsets can be empty. For the sets $\mathcal{X}_g, \mathcal{X}_f, \ker(C)$ expressions in terms of constraint matrix C are given by (9)–(11), respectively. Moreover, from $\mathcal{V}_c = \mathcal{V}^*$ it follows that ISA (7) yields the set $\mathcal{V}_c$ in a finite number of steps. Therefore, in the remainder we will concentrate on the subsets $\mathcal{V}_g, \mathcal{V}_f, \mathcal{X}_{\text{con},v}, \mathcal{X}_{\text{con},h}, \mathcal{X}_{\text{rel},v}$, and $\mathcal{X}_{\text{rel},h}$. Recall from Theorem IV.8 that these sets are two-by-two disjunct. For the sets $\mathcal{X}_{\text{con},v}$ and $\mathcal{X}_{\text{rel},v}$ there holds $r_C(x) = 1$, by definition. From Definition IV.1 it now follows that $h_1(x, \underline{u}) = CAx$. This yields by Lemma IV.4 the following proposition.

Proposition A.3: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Then

$$\mathcal{X}_{\text{con},v} = \{x \in \mathcal{X} | Cx = 0 \wedge CAx < 0\}$$

and

$$\mathcal{X}_{\text{rel},v} = \{x \in \mathcal{X} | Cx = 0 \wedge CAx > 0\}.$$

Proof: Omitted. □

For the subsets $\mathcal{V}_g, \mathcal{V}_f, \mathcal{X}_{\text{con},h}$, and $\mathcal{X}_{\text{rel},h}$ the derivation of alternative representations is based on the observation that either $r_C(x)$ must be odd or $r_C(x)$ must be even.

Propostion A.4: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Then we have the following.

1) $\mathcal{V}_g = \cup_{1\leq i<(1/2)r_0} \mathcal{V}_g^i; \mathcal{V}_g^i = \{x \in \ker(C) | CA^{2i}x > 0\} \cap \{\cap_{0\leq j<2i} A^{-j} \ker(C)\}, 1 \leq i < \frac{1}{2}r_0$.
2) $\mathcal{V}_f = \cup_{1\leq i<(1/2)r_0} \mathcal{V}_f^i; V_f^i = \{x \in \ker(C) | CA^{2i}x < 0\} \cap \{\cap_{0\leq j<2i} A^{-j} \ker(C)\}, 1 \leq i < \frac{1}{2}r_0$.
3) $\mathcal{X}_{\text{con},h} = \cup_{1\leq i<(1/2)(r_0-1)} \mathcal{X}_{\text{con},h}^i; \mathcal{X}_{\text{con},h}^i = \{x \in \ker(C) | CA^{2i+1}x < 0\} \cap \{\cap_{0\leq j\leq 2i} A^{-j} \ker(C)\}, 1 \leq i < \frac{1}{2}(r_0 - 1)$.
4) $\mathcal{X}_{\text{rel},h} = \cup_{1\leq i<(1/2)(r_0-1)} \mathcal{X}_{\text{rel},h}^i; \mathcal{X}_{\text{rel},h}^i = \{x \in \ker(C) | CA^{2i+1}x > 0\} \cap \{\cap_{0\leq j\leq 2i} A^{-j} \ker(C)\}, 1 \leq i < \frac{1}{2}(r_0 - 1)$.

Proof: We will prove only 1). The other statements are proven analogously. From Definition IV.5 it follows that $\{x \in \mathcal{V}_g\} \Leftrightarrow \{r_C(x) = 2i$ and $h_{r_C(x)}(x, \underline{u}) > 0, 1 \leq i < \frac{1}{2}r_0\}$. The result now follows from Corollary IV.2-2) and -3), and the definition of the map $h_i(x, \underline{u})$.

In the remainder of this section we show that the calculation of the subspaces can be based on the ISA (7). The integer r_0 is well defined as a consequence of the controllability of the pair (A, B). Obviously, observability of the pair (C, A) has an influence on the set $\mathcal{V}^*$. For this, it is useful to define two more integers.

Definition A.5: Define the integers r_1 and $r_{\min}$ as $r_1 := \min\{i \in \mathbb{N} | \cap_{0\leq j<i} A^{-j} \ker(C) \subseteq A^{-i} \ker(C)\}$, and $r_{\min} := \min(r_0, r_1)$.

Note that the set $N := \cap_{0\leq j<r_1} A^{-j} \ker(C)$ is the unobservable subspace, and if $N = \{0\}, (C, A)$ is observable and integer r_1 equals the observability index [29].

Lemma A.6: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Let $\mathcal{V}^i$ denote the subspaces obtained in step i of ISA (7). Then:

1) $\text{im}(B) \not\subseteq A^{-(r_0-1)} \ker(C)$;
2) $\text{im}(B) \subseteq A^{-(i-1)} \ker(C), \forall i$: $1 \leq i < r_0$;
3) $\mathcal{V}^i = \cap_{0\leq j<i} A^{-j} \ker(C), \forall i$: $1 \leq i \leq r_0$;
4) $r_{\min} = r_0$, and $\mathcal{V}^{r_0} = \mathcal{V}^*$.

Conditions 1) and 2) follow immediately from the assumptions and the definitions. From ISA (7) it follows that for $i = 1$ the equality 3) holds since $\mathcal{V}^1 = \ker(C)$. Now assume that $i = n < r_0$. Then, from (7) $\mathcal{V}^{n+1} = \ker(C) \cap (A^{-1}(\cap_{0\leq j<n} A^{-j} \ker(C)) + \text{im}(B))$. Using 2) now gives $\mathcal{V}^{n+1} = \ker(C) \cap (A^{-1} \cap_{0\leq j<n} A^{-j} \ker(C)) = \cap_{0\leq j<n+1} A^{-j} \ker(C)$. As for 4): from 1), 2), and Definition A.5 it follows that $r_0 \leq r_1$. The second statement now follows from 1), 2), and ISA.

Combining ISA (7) and Lemma A.6 now gives the following corollary.

Corollary A.7: Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Then the subspaces $\mathcal{V}^i$, calculated with ISA, have the following characteristics:

$$\begin{aligned} &\mathcal{V}^0 = \mathcal{X}; \quad \mathcal{V}^1 = \ker(C) = \mathcal{V}^0 \backslash (\mathcal{X}_g \cup \mathcal{X}_f) \\ &\mathcal{V}^2 = \mathcal{V}^1 \backslash (\mathcal{X}_{\text{con},v} \cup \mathcal{X}_{\text{rel},v}); \quad \mathcal{V}^3 = \mathcal{V}^2 \backslash (\mathcal{V}_g^1 \cup \mathcal{V}_f^1) \\ &\mathcal{V}^4 = \mathcal{V}^3 \backslash (\mathcal{X}_{\text{con},h}^1 \cup \mathcal{X}_{\text{rel},h}^1); \cdots; \mathcal{V}^{r_0} = \mathcal{V}_c = \mathcal{V}^*. \end{aligned}$$

Finally, combining Corollary A.7 with Propositions A.3 and A.4 and Lemma A.6 yields the following algorithm, which produces all the required subsets in a finite numbers of steps. Since $\mathcal{V}^* = \mathcal{V}^{r_0}$ the algorithm can be terminated at most in r_0 steps.

Algorithm A.8 (Computation of the Subsets of the State-Space): Let $\mathcal{X}_n(A, B, C)$ satisfy the assumptions and let $C \in \mathbb{R}^{1\times n}$. Let $\mathcal{V}^i$ denote the subspaces obtained in step i of ISA (7). Let $\mathcal{V}^0 = \mathcal{X}$. Then:

1) $\mathcal{V}^i = \{x \in \mathcal{V}^{i-1} | CA^{i-1}x = 0\}, 1 \leq i \leq r_0$;
2) $\mathcal{X}_g = \{x \in \mathcal{V}^0 | Cx > 0\}$;
3) $\mathcal{X}_f = \{x \in \mathcal{V}^0 | Cx < 0\}$;
4) $\mathcal{X}_{\text{con},v} = \{x \in \mathcal{V}^1 | CAx < 0\}$;
5) $\mathcal{X}_{\text{rel},v} = \{x \in \mathcal{V}^1 | CAx > 0\}$;
6) $\mathcal{V}_g = \cup_{1\leq i<(1/2)r_0} \mathcal{V}_g^i$ with $\mathcal{V}_g^i = \{x \in \mathcal{V}^{2i} | CA^{2i}x > 0\}, 1 \leq i < \frac{1}{2}r_0$;
7) $\mathcal{V}_f = \cup_{1\leq i<(1/2)r_0} \mathcal{V}_f^i$ with $\mathcal{V}_f^i = \{x \in \mathcal{V}^{2i} | CA^{2i}x < 0\}, 1 \leq i < \frac{1}{2}r_0$;
8) $\mathcal{X}_{\text{con},h} = \cup_{1\leq i<(1/2)(r_0-1)} \mathcal{X}_{\text{con},h}^i$ with $\mathcal{X}_{\text{con},h}^i = \{x \in \mathcal{V}^{2i+1} | CA^{2i+1}x < 0\}, 1 \leq i < \frac{1}{2}(r_0 - 1)$;
9) $\mathcal{X}_{\text{rel},h} = \cup_{1\leq i<(1/2)(r_0-1)} \mathcal{X}_{\text{rel},h}^i$ with $\mathcal{X}_{\text{rel},h}^i = \{x \in \mathcal{V}^{2i+1} | CA^{2i+1}x > 0\}, 1 \leq i < \frac{1}{2}(r_0 - 1)$;
10) $\mathcal{V}_c = \mathcal{V}^{r_0} (= \mathcal{V}^*)$.

Proof: Omitted. □

As remarked in Section III, it is intuitively clear that the contact sets and the release sets switch roles for the time-reversed system, i.e., the system with system matrices $-A$ and $-B$. Note that $\mathcal{X}_n(A,B,C)$ satisfies the assumptions stated in Section II if and only if $\mathcal{X}_n(-A,-B,C)$ satisfies these assumptions.

Proposition A.9: Let $\mathcal{X}_n(A,B,C)$ satisfy the assumptions. Then the following relations hold between subsets of system $\Sigma(A,B,C)$ and the time-reversed system $\Sigma(-A,-B,C)$:

$$\begin{aligned}\mathcal{X}_g(A,B,C) &= \mathcal{X}_g(-A,-B,C)\\ \mathcal{X}_{\mathrm{con},v}(A,B,C) &= \mathcal{X}_{\mathrm{rel},v}(-A,-B,C)\\ \mathcal{X}_{\mathrm{rel},v}(A,B,C) &= \mathcal{X}_{\mathrm{con},v}(-A,-B,C)\\ \mathcal{V}_g(A,B,C) &= \mathcal{V}_g(-A,-B,C)\\ \mathcal{V}_f(A,B,C) &= \mathcal{V}_f(-A,-B,C)\\ \mathcal{X}_{\mathrm{con},h}(A,B,C) &= \mathcal{X}_{\mathrm{rel},h}(-A,-B,C)\\ \mathcal{X}_{\mathrm{rel},h}(A,B,C) &= \mathcal{X}_{\mathrm{con},h}(-A,-B,C)\\ \mathcal{V}_c(A,B,C) &= \mathcal{V}_c(-A,-B,C).\end{aligned}$$

Proof: From Definition IV.1 it follows that the sign of B is not important. The result now follows from Algorithm A.8.

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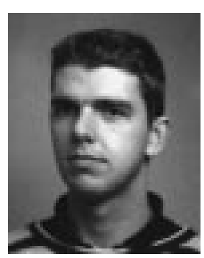

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