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Mathematics of Control, Signals, and Systems

Interconnections and Symmetries of Linear Differential Systems*

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Abstract. In this paper we study the interplay between control problems and symmetries in the context of linear systems. In particular, we establish sufficient conditions under which it is possible to control a symmetric system in order to make it achieve control objectives, without "breaking" its symmetry.

Key words. Feedback, Interconnection, Linear system, Reductive group, Symmetry.

1. Introduction

A central issue in control theory is to study the modifications which can be induced on an input/output system by means of a (feedback) controller. Typical problems in this context are stabilizability, pole placement, disturbance decoupling, robustness, and many others.

In this paper we want to study these problems in the context of linear systems with symmetries. The main question which we want to address is if it is possible to control a system which has a certain symmetry, in order to achieve control objectives, such as stability, without "symmetry breaking." In order to clarify the type of problem we are interested in, we present a couple of simple illustrative examples. We are a bit informal, precise definitions and notation being given in further sections.

Consider, first, q identical particles $P_1, \ldots, P_q$ in $\mathbf{R}^3$ which are under the simultaneous influence of a fixed potential field, potential interactions, and external control forces $F_1, \ldots, F_q$. Denote the position of the particle P_i by s_i. Assume that the system has an unstable symmetric equilibrium for $s_i = s^*$ for $i = 1, \ldots, q$ with $F_i = 0$ for $i = 1, \ldots, q$ and that our goal is to stabilize asymptotically the system around this point. Linearization around the equilibrium point yields an input/output linear system of the following type:

$$R\left(\frac{d}{dt}\right)x = F, \tag{1}$$

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where $x = {}^t(x_1, \ldots, x_q)$ with $x_i = s_i - s^* \in \mathbf{R}^3$, $F = {}^t(F_1, \ldots, F_q)$, and $R(d/dt)$ is the constant coefficient matrix differential operator induced by the $3N \times 3N$ polynomial matrix

$$R = \begin{bmatrix} R_1 & R_2 & \cdots & R_2 & R_2 \\ R_2 & R_1 & \cdots & R_2 & R_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_2 & R_2 & \cdots & R_2 & R_1 \end{bmatrix}, \tag{2}$$

where R_1 and R_2 are 3×3 polynomial matrices. The system has a symmetry which is also evidenced by motion equations: if (F, x) is an input/output pair of functions solving (1) and if $\sigma \in S_q$ (the group of permutations of the set $\{1, \ldots, q\}$), then (F^σ, x^σ), where $F^\sigma = {}^t(F_{\sigma(1)}, \ldots, F_{\sigma(q)})$ and $x^\sigma = {}^t(x_{\sigma(1)}, \ldots, x_{\sigma(q)})$, is also an input/output pair solving (1). It can easily be checked (using the behavioral theory or classically, passing through state space realizations) that the system (2) is asymptotically stabilizable by dynamic feedback. In other words, two $3N \times 3N$ polynomial matrices A and B with A invertible over the field of rational functions $\mathbf{R}(z)$ and with $A^{-1}B$ (the transfer function) a proper rational matrix exist, such that the linear x-input/F-output linear system

$$A\left(\frac{d}{dt}\right)F = B\left(\frac{d}{dt}\right)x \tag{3}$$

is an asymptotically stabilizing controller for (1), namely, the closed-loop system

$$\begin{aligned} R\left(\frac{d}{dt}\right)x &= F, \\ A\left(\frac{d}{dt}\right)F &= B\left(\frac{d}{dt}\right)x \end{aligned} \tag{4}$$

is asymptotically stable: if (F, x) solves (4), then $\lim_{t \to +\infty} (F, x)(t) = 0$. The stabilization problem becomes less obvious if we impose additional requirements on the feedback controller. One possible request which is of evident importance, is that, in the controller, each F_i only depends on the corresponding x_i and that such dependence does not depend on i. Namely, we require both A and B to be of block diagonal form with the same 3×3 repeated blocks on the diagonals. Stabilization under this requirement is a typical problem considered in decentralized control and it turns out to be a quite difficult one which will not always be solvable. A reasonable, less restrictive request on the controller could be that it has to have the same symmetry as the system which it has to control. Notice that the decentralized controller is indeed a symmetric system of a very special type. A consequence of our main result, Corollary 11, illustrated in Example 3 of Section 5, shows that this is possible, that such a symmetric stabilizing feedback controller does indeed exist. The first consequence of using symmetric controllers is that the symmetry is also preserved at the level of the closed-loop system (4), which is a fact of independent interest. Moreover, it follows from the representation results in [FW2] that such a symmetric controller admits "symmetric" equations in the sense that A and B can be choosen to have the same structure as R. Namely,

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 & A_2 \\ A_2 & A_1 & \cdots & A_2 & A_2 \\ \vdots & \vdots & & \vdots & \vdots \\ A_2 & A_2 & \cdots & A_2 & A_1 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 & \cdots & B_2 & B_2 \\ B_2 & B_1 & \cdots & B_2 & B_2 \\ \vdots & \vdots & & \vdots & \vdots \\ B_2 & B_2 & \cdots & B_2 & B_1 \end{bmatrix} \tag{5}$$

for suitable 3×3 polynomial matrices A_1, A_2, B_1, B_2. This shows that the controller has a nice internal structure reflecting the symmetry of the problem. Other symmetries could arise in this problem if the fixed potential field has some geometric symmetry. For instance, if the potential has radial symmetry, then there is an obvious $SO(3)$-symmetry in the physical system. This can be put in evidence by the fact that in this case the polynomial matrices R_1 and R_2 are of the type $R_1 = r_1 Id$ and $R_2 = r_2 Id$ where $r_1, r_2 \in \mathbf{R}[z]$. This symmetry can also be incorporated in the stabilizing controller which will have the matrices A_1, A_2, B_1, B_2 diagonal with the same structure as R_1 and R_2. This example is discussed again in Example 3 of Section 5 where other control problems are also considered.

Another example comes from the problem of designing active dampers to stabilize vibration tables or drilling towers which have a certain rotational symmetry with the location of the dampers respecting such symmetry. Of course this is, in principle, a distributed parameter control problem. However, a suitable finite-dimensional simplification is the linear model described by the following equations:

$$R\left(\frac{d}{dt}\right)x = F, \tag{6}$$

where $x = {}^t(x_1, \ldots, x_q)$ with $x_i \in \mathbf{R}$, $F = {}^t(F_1, \ldots, F_q)$ with $F_i \in \mathbf{R}$, and

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_{q-1} & r_q \\ r_q & r_1 & \cdots & r_{q-2} & r_{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ r_2 & r_3 & \cdots & r_q & r_1 \end{bmatrix}, \tag{7}$$

where $r_i \in \mathbf{R}[z]$ for all $i = 1, \ldots, q$. We can think of this as the model of a simplified version of the original system where we have concentrated all the mass in the points $P_1, \ldots, P_q$ where the dampers are located. x_i represents the displacement from equilibrium of the point P_i and F_i is the control force acting on P_i. It is clear from the equations that we are in a less symmetric situation than in previous example: in this case we do not have invariance of the solution set of (6) by the complete permutation group S_q as in the previous example but only with respect to the cyclic permutations. They form the subgroup $\mathbf{Z}_N$. However, using similar arguments based on Corollary 11, we can again show that a stabilizing symmetric feedback controller of type

$$A\left(\frac{d}{dt}\right)F = B\left(\frac{d}{dt}\right)x \tag{8}$$

with

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{q-1} & a_q \\ a_q & a_1 & \cdots & a_{q-2} & a_{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_q & a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_{q-1} & b_q \\ b_q & b_1 & \cdots & b_{q-2} & b_{q-1} \\ \vdots & \vdots & & \vdots & \vdots \\ b_2 & b_3 & \cdots & b_q & b_1 \end{bmatrix} \tag{9}$$

exists. It can be shown that the transfer function from x to F, $A^{-1}B$, will have the same structure as A and B. This can be interpreted as follows. The output response F_i of the ith damper to the input displacements trajectory $(x_1, \dots, x_q)$ is equal to the output response F_j of the jth damper to the rotated input displacements trajectory $(x_{1+i-j}, \dots, x_{q+i-j})$ (where addition is modulo q). This is a nice symmetric solution of the problem which again reflects the special structure of the plant.

In these two examples the symmetry considered was of a static type, namely, the action of the group was only at the level of the input and output spaces, and time was not involved in the symmetry. Control problems including stabilization for this type of symmetries were also studied in [HLM], [HM1], [HM2], and [M]. However, in these papers the authors consider only linear systems in input/state/output form with the symmetry also in the state variables. The theory which we have developed in this paper is, on the other hand, based on the behavioral approach to systems and, as a consequence, it is completely intrinsic in the sense that all the results proven only depend on the intrinsic properties of a linear system and not on its possible representations (input/output, input/state/output, etc.). A related advantage of using the behavioral approach is that we can also consider systems with no *a priori* distinction between input and output variables: this is quite useful in studying interconnections of electrical networks with symmetries. Moreover, our theory also applies to symmetries which are not of a static type and to a wide range of different control problems. The paradigm underlying Corollary 11 is the following: If, given a plant with a symmetry, a certain feedback control problem can be solved and the control problem is "symmetric" in a suitable sense, then the control problem can be solved by means of a controller which has the same symmetry as the plant. The affirmative answer in the previous examples are exactly due to the fact that asymptotic stabilization is "symmetric" with respect to static symmetries. An important example of nonstatic symmetry is time-reversibility: the many-particles system in the first example has this property. Indeed, since there are no dissipative forces in the equations of motion, only even-order derivatives appear so that the input/output solution set of (1) is closed under reversing the arrow of time. However, it is clear that time-reversibility cannot be mantained in the closed-loop system if we want asymptotic stability. The symmetry has necessarily to be "broken" if we want to achieve this control objective. In the language used previously we could say that asymptotic stability is not "symmetric" with respect to time-reversibility. Of course, there are instead control problems other than stabilization which are "symmetric" with respect to time-reversibility: some are considered in Examples 3 and 4 of Section 5.

A related problem which we also consider in this paper is the one of interconnecting a system with a feedback controller in order to make the closed-loop system symmetric with respect to a given symmetry. Such a symmetrization problem actually encompasses many classical problems in control theory like disturbance decoupling and noninteracting control.

In order to discuss systems and symmetries we make use of the behavioral approach to systems theory [W1]–[W3]. This permits us to introduce the concept of system interconnection [W4] at a general level such that it encompasses the case of feedback and to treat a quite general class of symmetries. In this paper we consider linear differential behaviors, namely, systems which are described by a set of linear constant coefficient differential equations. All the results we need on these systems are collected in Section 2 where all the relevant notation is also introduced. In Section 3 we introduce the concept of symmetry as an action of a group G on the C^∞-trajectories which preserves differential behaviors. We then recall some results proven in [FW3]. To each symmetry we can associate, in a canonical way, a dual action on the space of equations. This permits us to shift our investigations to a more algebraic level. Most of our results are established for symmetries induced by linearly reductive matrix groups. In Section 4 we introduce the concept of feedback and regular feedback interconnection of differential behaviors and we establish a few results, of independent interest, which characterize the subspaces of a given differential behavior which can be obtained through feedback and regular feedback. In Section 5 we discuss the problem of symmetric feedback: our main results are Theorem 10 and Corollary 11 which give sufficient conditions under which certain control objectives can be achieved, without "breaking" the original symmetry of the plant. These results are then applied to some examples such as symmetric stabilization and symmetric decoupling. Finally, in Section 6 we briefly consider the problem of making a system symmetric by feedback.

As a final remark, we notice that symmetric control problems such as the one considered in this paper are related to control problems for systems defined over rings [BBV], [BSSV], [S].

2. Differential Behaviors. Preliminary Facts

Let k be equal to $\mathbf{R}$ or $\mathbf{C}$ and denote by k^* the multiplicative group $k\backslash\{0\}$. Throughout this paper W and E always denote finite-dimensional vector spaces over k. Denote by C_W^∞ the k-vector space of infinitely differentiable functions from $\mathbf{R}$ to W equipped with the canonical Frechet topology of uniform convergence on compact subsets of $\mathbf{R}$. Denote $R = k[z]$ and $E[z] = R \otimes_k E$, the R-module of polynomials with coefficients in E.

Following [W3] we define a *linear differential behavior* over W (called the *signal space*), a subspace $\mathscr{B}$ of C_W^∞ which is the kernel of a linear contant coefficient differential operator. Namely, a vector space E exists and $D = \sum_{i=0}^n D_i z^i \in \mathrm{Hom}_k(W, E)[z]$ ($D_i \in \mathrm{Hom}_k(W, E)$), such that

$$\mathscr{B} = \left\{ w \in C_W^\infty | D\left(\frac{d}{dt}\right) w := \sum_{i=0}^n D_i \frac{d^i w}{dt^i} = 0 \right\}. \tag{10}$$

D is called a *polynomial matrix representation* of $\mathscr{B}$ and E is the *equating space* of

D. The class of linear differential behaviors over W is denoted by $\mathscr{D}[W]$. In the examples presented in Section 1, (1), (3), (6), and (8) were exactly polynomial matrix representations of linear differential behaviors. In all those cases we have $w = (F, x)$: the signal space is the product of the input space and the output space. As we have already mentioned, one of the advantages of the behavioral approach to systems is that we do not need to assume such a decomposition in the signal space.

A key concept in systems theory is the one of controllability. $\mathscr{B} \in \mathscr{D}[W]$ is said to be *controllable* if for all $w_1, w_2 \in \mathscr{B}$ there are $t_0 \geq 0$ and $w \in \mathscr{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - t_0)$ for $t \geq t_0$. At the opposite extreme of the controllable behaviors there are the autonomous ones: $\mathscr{B} \in \mathscr{D}[W]$ is said to be *autonomous* if $w_1, w_2 \in \mathscr{B}$ and $w_1(t) = w_2(t)$ for $t < 0$ implies $w_1 = w_2$. If $\mathscr{B} \in \mathscr{D}[W]$, then a largest controllable linear differential behavior contained in $\mathscr{B}$ exists. It is denoted by $\mathscr{B}_c$ and is called the *controllable subbehavior* of $\mathscr{B}$. For the relationship of these notions with the classical ones see [W1]–[W3].

Let $\mathscr{B} \in \mathscr{D}[W]$. Consider the annihilators of $\mathscr{B}$, defined by

$$\mathscr{B}^{\perp} := \left\{ p \in W^*[z] | p\left(\frac{d}{dt}\right) w = 0, \forall w \in \mathscr{B} \right\}, \tag{11}$$

where $W^* := \operatorname{Hom}_k(W, k)$. Clearly, $\mathscr{B}^{\perp}$ is an R-submodule of the R-free module $W^*[z]$. On the other hand, if M is an R-submodule of $W^*[z]$, we can consider

$$^{\perp}M := \left\{ w \in C_W^{\infty} | p\left(\frac{d}{dt}\right) w = 0, \forall p \in M \right\}. \tag{12}$$

Since M is finitely generated, it follows that $^{\perp}M \in \mathscr{D}[W]$. The following theorem gathers some relevant results on differential behaviors.

Theorem 1. *Let $\mathscr{B} \in \mathscr{D}[W]$ and let M be an R-submodule of $W^*[z]$. Then*

(1)

$$^{\perp}(\mathscr{B}^{\perp}) = \mathscr{B}, \qquad (^{\perp}M)^{\perp} = M. \tag{13}$$

This yields a bijection between $\mathscr{D}[W]$ and the class of all R-submodules of $W^*[z]$.

(2) *Let $\{\mathscr{B}_i | i \in I\}$ be a family of elements in $\mathscr{D}[W]$. Then*

$$\bigcap_{i \in I} \mathscr{B}_i \in \mathscr{D}[W] \quad \text{and} \quad \left(\bigcap_{i \in I} \mathscr{B}_i\right)^{\perp} = \sum_{i \in I} \mathscr{B}_i^{\perp}. \tag{14}$$

Moreover, if I is finite, then

$$\sum_{i \in I} \mathscr{B}_i \in \mathscr{D}[W] \quad \text{and} \quad \left(\sum_{i \in I} \mathscr{B}_i\right)^{\perp} = \bigcap_{i \in I} \mathscr{B}_i^{\perp}. \tag{15}$$

(3) *$\mathscr{B}$ is controllable if and only if $W^*[z]/\mathscr{B}^{\perp}$ is free.*

(4) *$\mathscr{B}$ is autonomous if and only if $W^*[z]/\mathscr{B}^{\perp}$ is torsion.*

(5) *If $W^*[z]/\mathscr{B}^{\perp} = T \oplus F$ where T is the torsion of $W^*[z]/\mathscr{B}^{\perp}$ and F is an R-free complement, then $\mathscr{B}_c = {}^{\perp}(\pi^{-1}(T))$ where π: $W^*[z] \to W^*[z]/\mathscr{B}^{\perp}$ is the quotient projection. Moreover, $X = {}^{\perp}(\pi^{-1}(F))$ is autonomous and $\mathscr{B} = \mathscr{B}_c \oplus X$.*

Proof. (1) is contained in [W3].

(2) The fact that $\bigcap_{i \in I} \mathscr{B}_i \in \mathscr{D}[W]$ follows from the fact that $W^*[z]$ is noetherian. That $\mathscr{D}[W]$ is closed under finite summation is standard: see [W3]. The rest are straightforward verifications.

(3) and (4) are also standard (see [W3]).

(5) It easily follows from (3) and (4) that ${}^{\perp}(\pi^{-1}(T))$ is controllable and ${}^{\perp}(\pi^{-1}(F))$ is autonomous. Moreover, it is clear that ${}^{\perp}(\pi^{-1}(T)) \oplus {}^{\perp}(\pi^{-1}(F)) = \mathscr{B}$. This implies that $\mathscr{B}_c = {}^{\perp}(\pi^{-1}(T))$. ■

If $\mathscr{B} \in \mathscr{D}[W]$, denote the rank of the free module $\mathscr{B}^{\perp}$ by $p(\mathscr{B})$. Clearly, $p(\mathscr{B}) \leq \dim_k W$ and we can find $D \in \mathrm{Hom}_k(W, k^{p(\mathscr{B})})[z]$ such that $\ker D(d/dt) = \mathscr{B}$. This simply shows that $\mathscr{B}$ can be described by $p(\mathscr{B})$ differential equations and no less. Any polynomial matrix representation of $\mathscr{B}$ with equating space of dimension $p(\mathscr{B})$ is therefore called *minimal*. An example of minimal representation is given by (1) in Section 1. If $\mathscr{B} \in \mathscr{D}[W]$, consider

$$\mathscr{B}_i^{\perp} := \{p \in \mathscr{B}^{\perp} | \deg(p) \leq i\}, \tag{16}$$

where $\deg(p)$ denotes the degree of the polynomial $p \in W^*[z]$. Clearly, the $\mathscr{B}_i^{\perp}$'s are k-vector spaces and it holds that

$$\mathscr{B}_i^{\perp} + z\mathscr{B}_i^{\perp} \subseteq \mathscr{B}_{i+1}^{\perp}. \tag{17}$$

It is a standard fact (see [W1] for more details) that equality holds in (17) except for at most finitely many indices. Define

$$\begin{aligned} \gamma_0(\mathscr{B}) &= \dim_k \mathscr{B}_0^{\perp}, \\ \gamma_i(\mathscr{B}) &= \dim_k \mathscr{B}_i^{\perp} - \dim_k(\mathscr{B}_{i-1}^{\perp} + z\mathscr{B}_{i-1}^{\perp}), \qquad i \geq 1. \end{aligned} \tag{18}$$

It is well known that $p(\mathscr{B}) = \sum_i \gamma_i(\mathscr{B})$. Define, moreover, $n(\mathscr{B}) = \sum_i i\gamma_i(\mathscr{B})$. The two integers $p(\mathscr{B})$ and $n(\mathscr{B})$ we have introduced have an important system-theoretic interpretation: $p(\mathscr{B})$ is the number of output variables in any input/output representation of $\mathscr{B}$, while $n(\mathscr{B})$ is the McMillan degree of $\mathscr{B}$, namely, the dimension of the state space in any minimal state-space representation of $\mathscr{B}$. See [W1]–[W3] for precise statements and details.

Remark. The choice of working in C^{∞} is mostly done for simplicity. More general settings can indeed be chosen, see [W3] and [F]. In particular, all the results we present in this paper are still true if we replace C^{∞} with the space of distributions $\mathscr{D}'$.

3. Symmetries of Differential Behaviors

Denote the group of all the topological vector space isomorphism of C_W^{∞} by $GL_k(C_W^{\infty})$. Let G be a group and let $T\colon G \to GL_k(C_W^{\infty})$ be a representation of G. Clearly, T induces an action of G on the class of all (closed) subspaces of C_W^{∞}. We say that (G, T) is a *symmetry on* $\mathscr{D}[W]$ if the class $\mathscr{D}[W]$ is invariant under the action of G (i.e., if $\mathscr{B} \in \mathscr{D}[W]$, then $T_g\mathscr{B} := \{T_g w | w \in \mathscr{B}\} \in \mathscr{D}[W]$ for all $g \in G$). $\mathscr{B} \in \mathscr{D}[W]$ is called *symmetric* if it is fixed by the action of G (i.e., $T_g w \in \mathscr{B}$ for every $w \in \mathscr{B}$ and for all $g \in G$). The subclass of symmetric behaviors in $\mathscr{D}[W]$ is denoted by $\mathscr{D}[W]^G$.

We now need to introduce some notation and to recall some results established in [FW3]. Aside from the differential operators there are other relevant operators on C^∞. If $x \in \mathbf{R}$, denote by σ^x the *shift* operator given by $(\sigma^x w)(t) := w(t + x)$. If $\xi \in k$, denote by $\mathscr{M}_\xi$ the *multiplicative* operator given by $(\mathscr{M}_\xi w)(t) := e^{\xi t} w(t)$. If $\eta \in \mathbf{R}^*$, denote by $\mathscr{S}_\eta$ the *scaling* operator given by $(\mathscr{S}_\eta w)(t) := w(\eta t)$. All these operators can be trivially extended to any space of type C_W^∞ and they are denoted in the same way. The following was proven in [FW3].

Theorem 2. *Let G be a group and let $T: G \to GL_k(C_W^\infty)$ be a homomorphism. The following conditions are equivalent:*

(1) *(G, T) is a symmetry in $\mathscr{D}[W]$.*
(2) *There are (unique) maps $x: G \to \mathbf{R}$, $\xi: G \to k$, $\eta: G \to \mathbf{R}^*$, and $R: G \to GL_R(W[z])$ such that*

$$T_g = \sigma^{x_g} \circ R_g\left(\frac{d}{dt}\right) \circ M_{\xi_g} \circ S_{\eta_g}. \tag{19}$$

Consider now $\tau: G \to GA(1, k)$ given by $\tau_g w = \eta_g w + \xi_g$ where $w \in k$ and $g \in G$. From the fact that T is a homomorphism, it follows that τ is also a homomorphism and it naturally induces a G-action on R defined by the k-algebra automorphisms

$$g \cdot p = p \circ \tau_{g^{-1}}, \qquad p \in R, \quad g \in G. \tag{20}$$

The action (20) of G on R can be extended to any tensor product of type $R \otimes_k W$. Consider now $R^*: G \to GL_R(W^*[z])$ (if $R_g = \sum_i R_{i,g} z^i$, then $R_g^* = \sum_i R_{i,g}^* z^i$). Define $U: G \to GL_k(W^*[z])$ by

$$U_g p := (\det R_{g^{-1}}^*)(g \cdot (R_g^* p)), \qquad g \in G, \quad p \in W^*[z]. \tag{21}$$

A straightforward verification shows that U is a k-representation. Moreover,

$$(T_g \mathscr{B})^\perp = U_g(\mathscr{B}^\perp), \qquad \forall g \in G, \tag{22}$$

which, in particular, shows that $\mathscr{B} \in \mathscr{D}[W]^G$ if and only if $\mathscr{B}^\perp$ is a G-invariant R-submodule of $W^*[z]$. U is called the *dual action* associated with the symmetry (G, T). For the sake of simplicity of notation from now on all the actions of G are denoted by left multiplication by g (e.g., $T_g w = g \cdot w$, $U_g p = g \cdot p$).

Example 1. A symmetry (G, T) on $\mathscr{D}[W]$ is called *time-invariant* if $T_g \circ \sigma^x = \sigma^x \circ T_g$ for all $g \in G$. It is easy to see that (G, T) is time-invariant if and only if U_g is R-linear for all $g \in G$. A particular class of time-invariant symmetries are the static ones: a symmetry (G, T) is called *static* if a representation ρ of G on W exists such that $(T_g w)(t) = \rho_g(w(t))$ for all $t \in \mathbf{R}$ and $g \in G$. In what follows we identify T and ρ. In this case it is easy to see that the dual action U is given by ρ^*. Static symmetries have been studied in much detail in [F] and [FW2].

Example 2. An important example of nonstatic (actually not even time-invariant) symmetry is *time-reversibility*. In this case $G = \mathbf{Z}_2 = (\{-1, +1\}, \cdot)$, $(T_{-1}w)(t) = w(-t)$. Differential behaviors which are symmetric with respect to this symmetry are called *time-reversible*. In this case the dual action on $W^*[z]$ is the following involution: for $p(z) \in W^*[z]$ and $g \in G$, define $(gp)(z) := p(gz)$. See [FW1] for more details about time-reversibility and its generalizations.

From now on we assume that G is a matrix group, namely, that it is a subgroup of $GL(n, \mathbf{C})$ for some $n \in \mathbf{N}$. Consider the Zariski topology in $GL(n, \mathbf{C})$. We do not assume that G is closed, since this permits us to consider real groups also. It is a classical result that every finite group and, more generally, every affine algebraic group is (isomorphic to) a closed matrix group. If X is an affine algebraic set, a map $\varphi: G \to X$ is called algebraic if it is the restriction of an algebraic map from $\overline{G}$ to X. If V is a finite-dimensional k-vector space, a representation $\rho: G \to GL_k(V)$ is said to be *rational* if it is algebraic (think of $GL_k(V)$ as a subset of $GL_{\mathbf{C}}(V \otimes_k \mathbf{C})$). In other words, ρ is rational if, with a fixed basis of V, the entries of the matrix which represents ρ are all polynomial functions of G. Notice that if G is finite, the condition of rationality is always verified. If V is a (possibly infinite-dimensional) k-vector space, a representation $\rho: G \to GL_k(V)$ is called rational if a sequence V_n of G-invariant k-subspaces of V exists such that $\bigcup_n V_n = V$ and the subrepresentations $\rho: G \to GL_k(V_n)$ are all rational. We also say that V is a G-module. A matrix group G is called *linearly reductive* if all its rational finite-dimensional complex representations are completely reducible (i.e., they can be written as a direct sum of irreducible representations). We refer the reader to [N] for a detailed analysis of linearly reductive matrix groups. We just remind the reader that a lot of groups have this property: all finite groups and, more generally, every compact topological group; further, every semisimple connected affine algebraic group, like the classical groups SL_n, O_n, and SP_n.

Consider now an action of the group G on the polynomial algebra $R = k[z]$ given by k-algebra automorphisms, which is rational. Let M be an R-module, equipped with a G-action which makes it into a rational G-module and such that

$$g \cdot (rm) = (g \cdot r)(g \cdot m). \tag{23}$$

M is said to be an *$(R - G)$-module* and the action of G on M is called a *quasi-linear representation* of G. If M, N are $(R - G)$-modules and M is R-finitely generated, we can make $\mathrm{Hom}_R(M, N)$ into an $(R - G)$-module by defining $(g \cdot f)(m) := g \cdot (f(g^{-1} \cdot m))$ where $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$. We refer the reader to [BH] for all the results on $(R - G)$-modules. We will need the following.

Proposition 3 (Splitting Lemma). *Let G be a lineary reductive matrix group and let*

$$0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0 \tag{24}$$

be an exact sequence of $(R - G)$-modules (i and p are G-equivariant). If (24) splits as a sequence of R-modules, it also splits as sequence of $(R - G)$-modules (i.e., the splitting map can be chosen to be G-equivariant).

Proof. In the case $k = \mathbf{C}$ it is proven in [BH]. The case $k = \mathbf{R}$ is analogous. ■

Let G be a matrix group, and let (G, T) be a symmetry on $\mathscr{D}[W]$. Consider the associated maps x, ξ, η, and R as in Theorem 2. (G, T) is said to be a *rational symmetry* if:

(i) x, ξ, η, and $R(\lambda)$ (for all $\lambda \in \mathbf{C}$) are algebraic maps.
(ii) $\sup\{\deg(R_g) | g \in G\} < +\infty$.

If (G, T) is rational, the associated homomorphism $\tau\colon G \to GA(1, k)$ is algebraic, hence the action of G on R given in (20) turns R into a G-module. Moreover, it is immediate to check that the dual action U in (21) is quasi-linear and turns $W^*[z]$ into an $(R - G)$-module. Notice that if G is finite, conditions (i) and (ii) are always automatically verified. Further, a static symmetry (G, ρ) (with G possibly not finite) is rational if and only if $\rho\colon G \to GL_k(W)$ is a rational representation.

A rational symmetry (G, T) is called *degree-preserving* if $\deg(U_g p) = \deg(p)$ for all $g \in G$, $p \in W^*[z]$. Notice that this is equivalent to having R constant in Theorem 2. (G, T) is called *finite* if $\tau(G)$ is a finite group (hence $\{0\}$ or isomorphic to $\mathbf{Z}_2$). Notice that if G is finite (or compact), this is always verified. It is clear that static symmetries and time-reversibility are degree-preserving and finite.

We close the section with a couple of simple results on symmetric differential behaviors which will be needed in what follows.

Proposition 4. *Let G be a linearly reductive group and let (G, T) be a rational symmetry on $\mathscr{D}[W]$. Let $\mathscr{B} \in \mathscr{D}[W]$. The following conditions are equivalent:*

(1) $\mathscr{B} \in \mathscr{D}[W]^G$.
(2) *$\mathscr{B}_c \in \mathscr{D}[W]^G$ and $X \in \mathscr{D}[W]^G$ autonomous exists such that $\mathscr{B} = \mathscr{B}_c \oplus X$.*

Proof. (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2) Consider the canonical exact sequence of $(R - G)$-modules

$$0 \to \mathscr{B}^\perp \to W^*[z] \xrightarrow{\pi} W^*[z]/\mathscr{B}^\perp \to 0. \tag{25}$$

Let T be the torsion of $W^*[z]/\mathscr{B}^\perp$. It is easy to see that T is G-invariant. It follows from Proposition 3 that a G-invariant free submodule of $W^*[z]/\mathscr{B}^\perp$ exists such that $W^*[z]/\mathscr{B}^\perp = T \oplus F$. The result now follows from (5) of Theorem 1. ■

Let (G, T) be a symmetry on $\mathscr{D}[W]$ and let $\mathscr{B} \in \mathscr{D}[W]$. There is a largest G-invariant subspace contained in $\mathscr{B}$:

$$\mathscr{B}_G := \bigcap_{g \in G} g \cdot \mathscr{B}; \tag{26}$$

and a smallest G-invariant subspace containing $\mathscr{B}$:

$$\mathscr{B}^G := \sum_{g \in G} g \cdot \mathscr{B}. \tag{27}$$

Clearly, $\mathscr{B}_G = \mathscr{B} = \mathscr{B}^G$ if and only if $\mathscr{B} \in \mathscr{D}[W]^G$. Are $\mathscr{B}_G$ and $\mathscr{B}^G$ also differential behaviors? It follows from (2) of Theorem 1 that the answer is in the affirmative for

$\mathscr{B}_G$. The question for $\mathscr{B}^G$ is more delicate and in general the answer is negative. However, there is a positive result for an important class of symmetries.

Proposition 5. *Let (G, T) be a finite rational symmetry and let $\mathscr{B} \in \mathscr{D}[W]$. Then $\mathscr{B}^G \in \mathscr{D}[W]$.*

Proof. Let $X \in \mathscr{D}[W]$ be such that $\mathscr{B} = \mathscr{B}_c \oplus X$. Clearly, $\mathscr{B}^G = (\mathscr{B}_c)^G + X^G$. Because of (2) of Theorem 1, it suffices to prove that $(\mathscr{B}_c)^G$ an X^G are differential behaviors. First, let us consider $(\mathscr{B}_c)^G = \sum_g g \cdot \mathscr{B}_c$. We claim that a finite set $\{g_1, \ldots, g_n\} \subseteq G$ exists such that $(\mathscr{B}_c)^G = \sum_{i=1}^n g_i \cdot \mathscr{B}_c$. In order to see this, assume the contrary. Then a sequence $g_1, g_2, \ldots, g_n, \ldots$ of elements of G exists such that

$$g_1 \cdot \mathscr{B}_c \neq g_1 \cdot \mathscr{B}_c + g_2 \cdot \mathscr{B}_c \neq \cdots \neq \sum_{i=1}^{n} g_i \cdot \mathscr{B}_c \neq \cdots. \tag{28}$$

Now consider the annihilators:

$$g_1 \cdot \mathscr{B}_c^\perp \neq g_1 \cdot \mathscr{B}_c^\perp \cap g_2 \cdot \mathscr{B}_c^\perp \neq \cdots \neq \bigcap_{i=1}^{n} g_i \cdot \mathscr{B}_c^\perp \neq \cdots. \tag{29}$$

Since, by (3) of Theorem 1, they are all direct addends in $W^*[z]$ it follows that the rank is strictly decreasing. This is a contradiction.

Let us now prove that $X^G \in \mathscr{D}[W]$. Consider the complexification $Y = X \otimes_k \mathbf{C}$. Clearly, $Y \in \mathscr{D}[W \otimes_k \mathbf{C}]$. Further, (G, T) is also a symmetry on $\mathscr{D}[W \otimes_k \mathbf{C}]$ and $Y^G = X^G \otimes_k \mathbf{C}$. If Γ is a finite subset of $\mathbf{C}$ and $n \in N$, consider

$$Y_{\Gamma,n} := \{t \mapsto p(t)e^{\lambda t} | p \in W \otimes_k \mathbf{C}[z] \deg(p) \leq n, \lambda \in \Gamma\}. \tag{30}$$

It is well known that Γ and n can be chosen such that $Y \subseteq Y_{\Gamma,n}$. Since (G, T) is finite, we have that $\Gamma^G := \{\eta_g \lambda + \xi_g | g \in G, \lambda \in \Gamma\}$ is also finite. It easily follows from Theorem 2 that

$$(Y_{\Gamma,n})^G \subseteq Y_{\Gamma^G,n}. \tag{31}$$

This implies that Y^G, and therefore also X^G, is finite-dimensional. On the other hand, X^G is shift-invariant and hence (see [F] for instance) is a differential behavior. ■

Remark. It follows from the proof of the previous proposition that if (G, T) is a finite rational symmetry, $\mathscr{B}^G$ can be represented as sum of only a finite number of terms of the form $g \cdot \mathscr{B}$.

4. Interconnections

Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{D}[W]$. Define the *interconnection* of $\mathscr{B}_1$ and $\mathscr{B}_2$ simply as $\mathscr{B}_1 \cap \mathscr{B}_2$. The interconnection is said to be *feedback* if

$$p(\mathscr{B}_1 \cap \mathscr{B}_2) = p(\mathscr{B}_1) + p(\mathscr{B}_2), \tag{32}$$

and *regular feedback* if it is feedback and in addition

$$n(\mathscr{B}_1 \cap \mathscr{B}_2) = n(\mathscr{B}_1) + n(\mathscr{B}_2). \tag{33}$$

Feedback and regular-feedback interconnections are denoted by means of special symbols, respectively, $\mathscr{B}_1 \cap_{\mathrm{f}} \mathscr{B}_2$ and $\mathscr{B}_1 \cap_{\mathrm{rf}} \mathscr{B}_2$. It can be shown that this notion of regular feedback corresponds, for input/state/output systems, to the usual classical notion of (nonsingular) feedback. On the other hand, feedback interconnections which are not regular classically appear when we do a feedback interconnection of two input/state/output systems which both have feedthrough terms. For more details regarding these definitions and their relation with the classical ones in systems theory, see [W3] and [W4].

Remark. It follows from (2) of Theorem 1 that the interconnection between $\mathscr{B}_1$ and $\mathscr{B}_2$ is feedback if and only if

$$\mathscr{B}_1^\perp \cap \mathscr{B}_2^\perp = \{0\}. \tag{34}$$

Moreover, the interconnection is regular feedback if and only if (see [W1])

$$(\mathscr{B}_1^\perp)_i \cap (\mathscr{B}_2^\perp)_i = \{0\}, \qquad (\mathscr{B}_1^\perp)_i \oplus (\mathscr{B}_2^\perp)_i = (\mathscr{B}_1^\perp + \mathscr{B}_2^\perp)_i, \qquad \forall i. \tag{35}$$

Let $\mathscr{B}', \mathscr{B} \in \mathscr{D}[W]$ with $\mathscr{B}' \subseteq \mathscr{B}$. $\mathscr{B}'$ is said to be a *feedback subbehavior* (resp. a *regular-feedback subbehavior*) of $\mathscr{B}$ if $\tilde{\mathscr{B}} \in \mathscr{D}[W]$ exists such that $\mathscr{B}' = \mathscr{B} \cap_{\mathrm{f}} \tilde{\mathscr{B}}$ (resp. $\mathscr{B}' = \mathscr{B} \cap_{\mathrm{rf}} \tilde{\mathscr{B}}$). If $\mathscr{B} \in \mathscr{D}[W]$, we denote by $\mathscr{F}(\mathscr{B})$ (resp. $\mathscr{RF}(\mathscr{B})$) the set of all the feedback (resp. regular-feedback) subbehaviors of $\mathscr{B}$.

The next result is a characterization of feedback subbehaviors.

Theorem 6. *Let $\mathscr{B}, \mathscr{B}' \in \mathscr{D}[W]$ and $\mathscr{B}' \subseteq \mathscr{B}$. The following conditions are equivalent*:

(1) $\mathscr{B}' \in \mathscr{F}(\mathscr{B})$.
(2) $\mathscr{B}'^\perp/\mathscr{B}^\perp$ *is free.*
(3) *If $X \in \mathscr{D}[W]$ is such that $\mathscr{B}' = \mathscr{B}'_{\mathrm{c}} \oplus X$, then $\mathscr{B} = \mathscr{B}_{\mathrm{c}} + X$.*

In particular, it follows from (3) *that if $\mathscr{B}$ is controllable, then $\mathscr{B}' \subseteq \mathscr{B}$ implies $\mathscr{B}' \in \mathscr{F}(\mathscr{B})$.*

Proof. (1)$\Leftrightarrow$(2) follows from the previous remark and from (2) of Theorem 1.

(2)$\Rightarrow$(3) Clearly, $\mathscr{B}_{\mathrm{c}} + X \subseteq \mathscr{B}$, so we only have to prove that $\mathscr{B} \subseteq \mathscr{B}_{\mathrm{c}} + X$ which is equivalent to $\mathscr{B}_{\mathrm{c}}^\perp \cap X^\perp \subseteq \mathscr{B}^\perp$. Let M be a submodule of $W^*[z]$ such that $\mathscr{B}'^\perp = \mathscr{B}^\perp \oplus M$. This yields the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] & W^*[z]/\mathscr{B}^\perp \arrow[r, "p"] & W^*[z]/\mathscr{B}'^\perp \arrow[r] & 0 \\ & & W^*[z] \arrow[u, "\pi"] \arrow[ur, "\pi'"'] & & \end{tikzcd} \tag{36}$$

where p, π, and π' are the canonical projections and where the top row is exact. Let $r \in \mathscr{B}_{\mathrm{c}}^\perp \cap X^\perp$. Since $\mathscr{B}'_{\mathrm{c}} \subseteq \mathscr{B}_{\mathrm{c}}$, it follows that $r \in \mathscr{B}'^\perp$. Hence $p(\pi(r)) = \pi'(r) = 0$ which yields $\pi(r) \in \ker p = i(M)$. On the other hand, $r \in \mathscr{B}_{\mathrm{c}}^\perp$, hence $\pi(r) \in \mathscr{B}_{\mathrm{c}}^\perp/\mathscr{B}^\perp =$

$\mathscr{B}_c^\perp/\mathscr{B}_c^\perp \cap X^\perp \hookrightarrow W^*[z]/X^\perp$. Since $i(M)$ is free and $W^*[z]/X^\perp$ is torsion, this implies that $\pi(r) = 0$. This yields $r \in \mathscr{B}^\perp$.

(3) $\Rightarrow$ (2) We have

$$\mathscr{B}'^\perp/\mathscr{B}^\perp \hookrightarrow \mathscr{B}_c'^\perp/\mathscr{B}_c^\perp \hookrightarrow W^*[z]/\mathscr{B}_c^\perp. \tag{37}$$

It follows from (3) of Theorem 1 that the last module in (37) is free, hence the first one also is. ■

Corollary 7. *Let $\mathscr{B}, \mathscr{B}', \mathscr{B}'' \in \mathscr{D}[W]$ and suppose that $\mathscr{B}'' \subseteq \mathscr{B}' \subseteq \mathscr{B}$. Then*

$$\mathscr{B}'' \in \mathscr{F}(\mathscr{B}) \quad \Rightarrow \quad \mathscr{B}' \in \mathscr{F}(\mathscr{B}). \tag{38}$$

Proof. It immediately follows from condition (2) of Theorem 6. ■

We would like to establish a result analogous to Theorem 6 for regular-feedback subbehaviors. The problem is much more complicated and of central importance in linear systems theory. We give here a necessary and sufficient condition at the level of annihilators.

Theorem 8. *Let $\mathscr{B}, \mathscr{B}' \in \mathscr{D}[W]$ and $\mathscr{B}' \subseteq \mathscr{B}$. The following conditions are equivalent*:

(1) $\mathscr{B}' \in \mathscr{RF}(\mathscr{B})$.
(2) $(\mathscr{B}_i'^\perp + z\mathscr{B}_i'^\perp) \cap \mathscr{B}_{i+1}^\perp = \mathscr{B}_i^\perp + z\mathscr{B}_i^\perp$, $\forall i$.

Proof. (1) $\Rightarrow$ (2) Let $\tilde{\mathscr{B}} \in \mathscr{D}[W]$ such that $\mathscr{B}' = \mathscr{B} \cap_{\mathrm{rf}} \tilde{\mathscr{B}}$. Then, by the previous remark,

$$\mathscr{B}_i^\perp \oplus \tilde{\mathscr{B}}_i^\perp = \mathscr{B}_i'^\perp, \qquad \forall i. \tag{39}$$

Notice that in (2) "$\supseteq$" is always verified, so we only have to check "$\subseteq$." Let $p \in (\mathscr{B}_i'^\perp + z\mathscr{B}_i'^\perp) \cap \mathscr{B}_{i+1}^\perp$. Then $p = p_1 + zp_2$ with $p_1, p_2 \in \mathscr{B}_i'^\perp$. It follows from (39) that $p_1 = q_1 + \tilde{q}_1$ and $p_2 = q_2 + \tilde{q}_2$ with $q_1, q_2 \in \mathscr{B}_i^\perp$ and $\tilde{q}_1, \tilde{q}_2 \in \tilde{\mathscr{B}}_i^\perp$. Hence $p = (q_1 + zq_2) + (\tilde{q}_1 + z\tilde{q}_2)$. Since $p \in \mathscr{B}_{i+1}^\perp$, it follows from (39) that $\tilde{q}_1 + z\tilde{q}_2 = 0$. This implies that $p \in \mathscr{B}_i^\perp + z\mathscr{B}_i^\perp$.

(2) $\Rightarrow$ (1) Let us prove, by induction, that k-subspaces $M^i \subseteq \mathscr{B}_i'^\perp$ exist such that:

(i) $\mathscr{B}_i^\perp \oplus M^i = \mathscr{B}_i'^\perp$.
(ii) $M^i + zM^i \subseteq M^{i+1}$.

Choose M^0 such that $\mathscr{B}_0^\perp \oplus M^0 = \mathscr{B}_0'^\perp$. Suppose that we have obtained $M^0, \ldots, M^j$ which verify (i) and (ii) (for $i < j$). Let us first prove then that

$$(M^j + zM^j) \cap \mathscr{B}_{j+1}^\perp = \{0\}. \tag{40}$$

It follows from (2) of Theorem 8 that (40) is equivalent to

$$(M^j + zM^j) \cap (\mathscr{B}_j^\perp + z\mathscr{B}_j^\perp) = \{0\}. \tag{41}$$

Let $p \in (M^j + zM^j) \cap (\mathscr{B}_j^\perp + z\mathscr{B}_j^\perp)$. Then $p = m + zn = \alpha + z\beta$ with $m, n \in M^j$ and

$\alpha, \beta \in \mathscr{B}_j^\perp$. Therefore,

$$z(n - \beta) = \alpha - m \in \mathscr{B}_j'^\perp. \tag{42}$$

If $j = 0$, this implies $n = \beta \in M^0 \cap \mathscr{B}_0^\perp = (0)$. Similarly, $m = \alpha = 0$. Hence $p = 0$. If $j > 0$, (42) yields $n - \beta \in \mathscr{B}_{j-1}'^\perp$. From conditions (i) and (ii) it now follows that $n \in M^{j-1}$ and $\beta \in \mathscr{B}_{j-1}^\perp$. Hence $p \in M^j \cap \mathscr{B}_j^\perp = (0)$. This proves (41), which yields (40). Let us now consider any complementary k-subspace of $\mathscr{B}_{j+1}^\perp \oplus (M^j + zM^j)$ inside $\mathscr{B}_{j+1}'^\perp$ and define $M^{j+1} = N \oplus (M^j + zM^j)$. Clearly, the sequence $M^0, \ldots, M^{j+1}$ satisfies (i) and (ii) and thus, by induction, we obtain the sequence M^i.

Denote by M the R-submodule of $\mathscr{B}'^\perp$ generated by $\bigcup_i M^i$. We claim that

$$M_j = M^j, \qquad \forall j. \tag{43}$$

In (43) "$\supseteq$" is clear from the definition of M. "$\subseteq$" is proven as follows: let $m \in M_j$. Then $m = \sum_{i=0}^{l} \lambda_i m_i$ with $\lambda_i \in R$ and $m_i \in M^{k_i}$. Let $s = \max\{k_i + \deg(\lambda_i) | i = 0, \ldots, l\}$. A repeated application of (ii) shows that $m \in M^s$. On the other hand, $m \in \mathscr{B}_j'^\perp = \mathscr{B}_j^\perp \oplus M^j$. Again, the properties of the sequence M^i imply that $m \in M^j$. This proves (43).

It now immediately follows from the previous remark that $\mathscr{B} \cap_{\mathrm{rf}} {}^\perp M = \mathscr{B}'$. ∎

Remark. If $\mathscr{B}_{i+1}^\perp = \mathscr{B}_i^\perp + z\mathscr{B}_i^\perp$ for $i > h \in \mathbf{N}$, condition (2) in Theorem 8 is automatically verified for $i > h$. This shows that (2) in Theorem 8 really consists of only a finite number of conditions.

Corollary 9. *Let $\mathscr{B}, \mathscr{B}', \mathscr{B}'' \in \mathscr{D}[W]$ and assume that $\mathscr{B}'' \subseteq \mathscr{B}' \subseteq \mathscr{B}$. Then*

$$\mathscr{B}'' \in \mathscr{RF}(\mathscr{B}) \quad \Rightarrow \quad \mathscr{B}' \in \mathscr{RF}(\mathscr{B}). \tag{44}$$

Proof. Follows immediately from condition (2) of Theorem 8. ∎

5. Symmetric Interconnections

In this section assume that G is a linearly reductive matrix group and let (G, T) be a rational symmetry on $\mathscr{D}[W]$. The following important result says that all the (regular) feedback symmetric subbehaviors of a symmetric behavior $\mathscr{B}$ can be obtained by (regular) feedback interconnection of $\mathscr{B}$ with some other symmetric behavior.

Theorem 10. *Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{D}[W]$ and assume that $\mathscr{B}_1, \mathscr{B}_1 \cap \mathscr{B}_2 \in \mathscr{D}[W]^G$. Then $\tilde{\mathscr{B}}_2 \in \mathscr{D}[W]^G$ exists such that*

$$\mathscr{B}_1 \cap \mathscr{B}_2 = \mathscr{B}_1 \cap \tilde{\mathscr{B}}_2. \tag{45}$$

Moreover, if $\mathscr{B}_1 \cap \mathscr{B}_2$ is feedback, then $\tilde{\mathscr{B}}_2$ can be chosen such that $\mathscr{B}_1 \cap \tilde{\mathscr{B}}_2$ is also feedback. If $\mathscr{B}_1 \cap \mathscr{B}_2$ is regular feedback and (G, T) is degree-preserving, then $\tilde{\mathscr{B}}_2$ can be chosen such that $\mathscr{B}_1 \cap \tilde{\mathscr{B}}_2$ is regular feedback.

Proof. Of course, $\tilde{\mathscr{B}}_2 := \mathscr{B}_1 \cap \mathscr{B}_2$ satisfies (45) and is symmetric.

Assume now that $\mathscr{B}_1 \cap \mathscr{B}_2$ is feedback. Clearly, the solution $\tilde{\mathscr{B}}_2 = \mathscr{B}_1 \cap \mathscr{B}_2$ will not guarantee a feedback interconnection with $\mathscr{B}_1$, so we have to construct a different one. Let $M = (\mathscr{B}_1 \cap \mathscr{B}_2)^\perp = \mathscr{B}_1^\perp \oplus \mathscr{B}_2^\perp$. Consider the canonical quasi-linear action of G on $W^*[z]$ associated with (G, T). Clearly, M and $\mathscr{B}_1^\perp$ are G-invariant R-submodules of $W^*[z]$. Consider the exact sequence of $(R - G)$-modules

$$0 \to \mathscr{B}_1^\perp \xrightarrow{i} M \xrightarrow{p} M/\mathscr{B}_1^\perp \to 0. \tag{46}$$

It follows from Proposition 2 that (46) splits as a sequence of $(R - G)$-modules. Hence, a G-invariant submodule N of M exists such that $M = N \oplus \mathscr{B}_1^\perp$. Clearly, $\tilde{\mathscr{B}}_2 = {}^\perp N$ solves the problem in the feedback case.

Assume now the interconnection between $\mathscr{B}_1$ and $\mathscr{B}_2$ is regular feedback and (G, T) is a degree-preserving symmetry. Consider again the exact sequence (46) which yields the surjection of $(R - G)$-modules

$$\mathrm{Hom}_R(M/\mathscr{B}_1^\perp, M) \xrightarrow{p\circ} \mathrm{Hom}_R(M/\mathscr{B}_1^\perp, M/\mathscr{B}_1^\perp) \to 0. \tag{47}$$

Denote

$$L = \{q \in \mathrm{Hom}_R(M/\mathscr{B}_1^\perp, M) | \deg(q \circ p)(m) \leq \deg(m), \forall m \in M\}. \tag{48}$$

It is easy to see that L is a finite-dimensional k-vector space. Moreover, it is G-invariant. Indeed, if $q \in L$, $g \in G$, and $m \in M$, we have $(g \cdot q \circ p)(m) = g \cdot ((q \circ p)(q^{-1} \cdot m))$. Since the symmetry is degree-preserving, it follows that $g \cdot q \in L$. Let

$$H = \{p \circ q | q \in L\}. \tag{49}$$

Clearly, H is a G-invariant finite-dimensional k-vector space. Notice that $Id_{M/\mathscr{B}_1^\perp} \in H$. Indeed, if we consider the splitting map q of p relative to the decomposition $M = \mathscr{B}_1^\perp \oplus \mathscr{B}_2^\perp$, we have that $q \circ p$ is the projection on the factor $\mathscr{B}_2^\perp$. Because of the regularity it easily follows that $q \in L$. On the other hand, $p \circ q = Id_{M/\mathscr{B}_1^\perp}$. Consider now the exact sequence of G-modules

$$L \xrightarrow{p\circ} H \to 0, \tag{50}$$

which yields, since G is linearly reductive,

$$L^G \xrightarrow{p\circ} H^G \to 0, \tag{51}$$

where L^G and H^G denote the subspaces consisting of G-invariant elements of, respectively, L and G. Clearly, $Id_{M/\mathscr{B}_1^\perp} \in H^G$. Let $l \in L^G$ such that $p \circ l = Id_{M/\mathscr{B}_1^\perp}$. Clearly, by the way it was constructed, l: $M/\mathscr{B}_1^\perp \to M$ is a G-equivariant splitting map. Define $N = \mathrm{Im}(l) \subseteq M$. Then $M = N \oplus \mathscr{B}_1^\perp$. Consider $\tilde{\mathscr{B}}_2 = {}^\perp N \in \mathscr{D}[W]^G$. Clearly, $\mathscr{B}_1 \cap_{\mathrm{f}} \tilde{\mathscr{B}}_2 = \mathscr{B}_1 \cap_{\mathrm{rf}} \mathscr{B}_2$. It remains to check that the interconnection is regular. We only have to check that $M_i \subseteq (\mathscr{B}_1^\perp)_i \oplus N_i$ for all i. If $m \in M_i$, we have that $m = (m - (l \circ p)(m)) + l \circ p(m)$. Since $l \in L$, the results easily follows. ■

We now have the following conclusive result.

Corollary 11. *Let G be a linearly reductive matrix group and let (G, T) be a finite rational symmetry on $\mathscr{D}[W]$. Let $\mathscr{F} \subseteq \mathscr{D}[W]$ be a class of differential behaviors which has the following properties*:

(1) *$\mathscr{F}$ is G-invariant.*
(2) *$\mathscr{F}$ is invariant by summation.*

Let $\mathscr{B}_1 \in \mathscr{D}[W]^G$ and assume that $\mathscr{B}_2 \in \mathscr{D}[W]$ exists such that

$$\mathscr{B}_1 \cap \mathscr{B}_2 \in \mathscr{F}. \tag{52}$$

Then $\tilde{\mathscr{B}}_2 \in \mathscr{D}[W]^G$ exists such that

$$\mathscr{B}_1 \cap \tilde{\mathscr{B}}_2 \in \mathscr{F}. \tag{53}$$

Moreover, if (52) *was feedback, then $\tilde{\mathscr{B}}_2$ can be chosen in such a way that* (53) *is also feedback. If* (52) *was regular feedback and* (G, T) *is a degree-preserving symmetry, then $\tilde{\mathscr{B}}_2$ can be chosen so that* (53) *is regular feedback.*

Proof. Consider

$$\mathscr{B} = (\mathscr{B}_1 \cap \mathscr{B}_2)^G. \tag{54}$$

It follows from Proposition 5 and from the remark following it, that $\mathscr{B} \in \mathscr{F}$. Since $\mathscr{B}_1 \cap \mathscr{B}_2 \subseteq \mathscr{B} \subseteq \mathscr{B}_1$, the conclusion now follows from Corollaries 7 and 9 and from Theorem 10. ∎

Remarks. (1) Corollary 11 has a simple interpretation: $\mathscr{B}_1$ can be thought of as the plant which has a certain symmetry and we want to control it by means of a feedback controller in such a way that certain goals are reached (specified by $\mathscr{F}$). The theorem then says that, under certain assumptions on these goals (specified by conditions (1) and (2) of Corollary 11), if a feedback controller exists such that by interconnecting it to the plant these goals are reached, then there is also a symmetric feedback controller which achieves this.

(2) It is important to notice that the McMillan degree of the symmetric controller $\tilde{\mathscr{B}}_2$ in general will be higher than the McMillan degree of the original controller $\mathscr{B}_2$. Unfortunately our results do not give any estimate on the growth of the McMillan degree. Nevertheless, in certain cases (e.g., for static symmetries) estimations can be obtained by using the canonical polynomial matrix representation (see [FW2]).

We now discuss some applications of Corollary 11.

Example 3 (Symmetric Stabilization and Pole Placement). Let $\mathscr{F}$ be the subclass of $\mathscr{D}[W]$ consisting of the autonomous asymptotically stable ($\lim_{t\to+\infty} w(t) = 0$ for every $w \in \mathscr{B}$) differential behaviors. Let G be any linearly reductive matrix group and let ρ be a representation of G on W. Consider the static symmetry which they induce. Clearly, the assumptions of Corollary 11 are satisfied. Therefore we have the following fact: *if a plant $\mathscr{B}$ is symmetric and asymptotically stabilizable by* (*regular*) *feedback, then it can be asymptotically stabilized in* (*regular*) *feedback by means of a controller which is also symmetric.* The same conclusion can be obtained if we replace asymptotic stability with stability. More generally, we can consider $\mathscr{F}$ consisting of the autonomous behaviors whose eigenfrequencies (those $\lambda \in \mathbf{C}$ such that the map $t \mapsto e^{\lambda t}v$ is in $\mathscr{B}$ for some $v \in W$) lay in a given subset $\Gamma \subseteq \mathbf{C}$. This evidently leads to a sort of weak (multiplicities are not taken into account) pole-placement property in a symmetric context. It is not possible to extend this result to the pole-placement property with multiplicities. This follows from analyzing a canonical polynomial matrix representation of a symmetric system obtained in [FW2]. It can be shown that certain multiplicities are intrinsically associated with the structure of the representation ρ, more precisely with the multiplicities and type of its irreducible subrepresentations. We now give a concrete example which illustrates this. Let $W = k^{Nq}$ and consider the group $G = S_q$, the permutation group on q elements, and the permutation representation ρ of degree Nq given by

$$\rho_\sigma {}^t(w_1, \ldots, w_q) := {}^t(w_{\sigma(1)}, \ldots, w_{\sigma(q)}), \qquad \sigma \in S_q, \tag{55}$$

where $w_i \in k^N$. Consider the static symmetry induced by S_q and ρ on $\mathscr{D}[W]$. We can think of this static symmetry as occurring when we model the positions ($N = 1$) or the positions and forces ($N = 2$), as in the example in Section 1, of q identical particles in k^3. In [FW2] it is proven that $\mathscr{B} \in \mathscr{D}[W]^G$ if and only if $R_{av} \in k^{h \times N}[z]$, $R_\Delta \in k^{l \times N}[z]$, both of full row rank over $k(z)$, exist such that $\mathscr{B}$ is described by the polynomial matrix representation

$$\begin{aligned} R_{av}\left(\frac{d}{dt}\right) w_{av} &= 0, \\ R_\Delta\left(\frac{d}{dt}\right) \Delta w_i &= 0, \qquad i = 1, \ldots, q-1, \end{aligned} \tag{56}$$

with $w_{av} := (1/q)(w_1 + w_2 + \cdots + w_q)$ (the center of mass of the system) and $\Delta w_i := w_i - w_{av}$ (the displacements from the center of mass). It is easy to see that the polynomial matrix representations (1) and (3) (with A and B as in (5)) of Section 1 give rise to representations such as (56) with $h = l = 3$. Notice that $\mathscr{B}$ given by (56) is autonomous if and only if $h = l = N$. In this case the eigenfrequencies of $\mathscr{B}$ are given by the union of the zeros of $\det(R_{av})$ and the zeros of $\det(R_\Delta)$. Moreover, every eigenfrequency which is a zero of $\det(R_\Delta)$ will have multiplicity at least $q - 1$. Consider now a symmetric plant $\mathscr{B}$ described by equations such as (56) with $R_{av} = 0$ and R_Δ such that $l = 1$ and the rank of $R_\Delta(\lambda)$ is 1 for every $\lambda \in \mathbf{C}$. Moreover, assume that $n(\mathscr{B}) > 0$ (equivalently that at least one component of R_Δ is not constant). $\mathscr{B}$ is then controllable, hence [W4], [W5] it has the (strong) pole-placement property. On the other hand, it is clear from previous considerations that if we interconnect it with a symmetric controller (in regular-feedback interconnection), the autonomous behavior which we obtain will have at least one eigenfrequency with multiplicity $q - 1$. For similar examples in a classical setting seee also [HLM], [HM1], [HM2], and [M]. In the case where the symmetry is not of a static type, symmetric stabilization may fail. A typical example is given by time-reversibility: indeed, no autonomous time-reversible linear differential behavior can be asymptotically stable. However, some symmetric pole-placement results can be obtained for symmetries which are not necessarily of static type. Notice, indeed, that if (G, T) is a finite rational symmetry on $\mathscr{D}[W]$ and $\Gamma \subseteq \mathbf{C}$ is such that $\tau\Gamma \subseteq \Gamma$ (see Section 3 for the definition of τ), then the class $\mathscr{F}$ of autonomous differential behaviors in $\mathscr{D}[W]$ whose eigenfrequencies lay in Γ satisfies the assumptions of Corollary 11. Of course, if we want to apply Corollary 11 in the regular case we also need to assume that the symmetry is degree-preserving.

Example 4 (Symmetric Decoupling). A general decoupling problem can be formulated in the following way. Consider a decomposition

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_l. \tag{57}$$

Given $\mathscr{B}_1 \in \mathscr{D}[W]$ we say that the *decoupling problem* (D.P.) with respect to (57) is solvable for $\mathscr{B}_1$ if $\mathscr{B}_2 \in \mathscr{D}[W]$ exists such that

$$\mathscr{B}_1 \cap \mathscr{B}_2 = \bigoplus_{i=1}^{l} P_i(\mathscr{B}_1 \cap \mathscr{B}_2), \tag{58}$$

where P_i is the projector on W_i relative to the decomposition (57). If we ask that the interconnection has to be (regular) feedback we talk about the *(regular) feedback decoupling problem* (F.D.P., R.F.D.P.).

Denote by $\mathscr{F}$ the class of behaviors $\mathscr{B}$ of $\mathscr{D}[W]$ such that $\mathscr{B} = \bigoplus_{i=1}^{l} P_{q_i}(\mathscr{B})$. Assume that G is a linearly reductive group and that (G, T) is a rational degree-preserving finite symmetry which leaves the class $\mathscr{F}$ invariant (e.g., time-reversibility or any static symmetry for which the subspaces W_i are G-invariant). The assumptions of Corollary 11 are evidently satisfied and this yields the following result: *if D.P (F.D.P., R.F.D.P.) is solvable for $\mathscr{B}_1 \in \mathscr{D}[W]^G$, then it is solvable by means of a $\tilde{\mathscr{B}}_2 \in \mathscr{D}[W]^G$.*

6. Symmetrization

Consider a rational symmetry (G, T) on $\mathscr{D}[W]$ and let $\mathscr{B} \in \mathscr{D}[W]$. We would like to study the conditions under which $\mathscr{F}(\mathscr{B})$ (or $\mathscr{R}\mathscr{F}(\mathscr{B})$) contains some symmetric differential behaviors. Since $\mathscr{B}_G$ is the largest symmetric differential behavior contained in $\mathscr{B}$ it follows from Corollaries 7 and 9 that this is equivalent to the fact $\mathscr{B}_G \in \mathscr{F}(\mathscr{B})$ (or $\mathscr{B}_G \in \mathscr{R}\mathscr{F}(\mathscr{B})$). Notice that the decoupling problems (F.D.P. and R.F.D.P.) considered in Example 4 of Section 5 are particular cases of these problems. To see this, take $G = \bigoplus_{j=1}^{k} \mathbf{Z}_2$ where $\mathbf{Z}_2 = \{1, -1\}$ and let T be given by

$$T_{(\gamma_1, \ldots, \gamma_k)}(w_1, \ldots, w_k) := (\gamma_1 w_1, \ldots, \gamma_k w_k). \tag{59}$$

Let us start considering the feedback case. Clearly, if $\mathscr{B}$ is controllable, then $\mathscr{B}_G \in \mathscr{F}(\mathscr{B})$, because of Theorem 6. In general, we have the following:

Proposition 12. *The following conditions are equivalent:*

(1) $\mathscr{B}_G \in \mathscr{F}(\mathscr{B})$.
(2) $X \in \mathscr{D}[W]^G$ *autonomous exists such that* $\mathscr{B} = \mathscr{B}_c + X$.

Proof. (2) $\Rightarrow$ (1) Obviously $\mathscr{B}_G = (\mathscr{B}_c \cap \mathscr{B}_G) + X$ which yields $\mathscr{B}_G^\perp = (\mathscr{B}_c \cap \mathscr{B}_G)^\perp \cap X^\perp$. Hence

$$\mathscr{B}_G^\perp/\mathscr{B}^\perp = (\mathscr{B}_c \cap \mathscr{B}_G)^\perp \cap X^\perp/\mathscr{B}_c^\perp \cap X^\perp \hookrightarrow (\mathscr{B}_c \cap \mathscr{B}_G)^\perp/\mathscr{B}_c^\perp. \tag{60}$$

Since the last module is free, (1) follows.

(1) $\Rightarrow$ (2) It follows from Proposition 4 that $\mathscr{B}_G = (\mathscr{B}_G)_c \oplus X$ with $X \in \mathscr{D}[W]^G$ autonomous. It then follows from Theorem 6 that $\mathscr{B} = \mathscr{B}_c + X$. ■

The regular-feedback case is far more complicated and no general result like Proposition 12 can be expected. However, our Theorem 8 which characterizes regular-feedback subbehaviors can be used as a general approach to this type of problem.

We would like to close with few remarks which illustrate this point. Assume that $\mathscr{B}$ is described by first-order differential equations; more precisely, assume that $\gamma_i(\mathscr{B}) = 0$ for all $i \neq 1$. This yields $\gamma_1(\mathscr{B}) = p(\mathscr{B}) = n(\mathscr{B})$. Notice that classical state-input linear systems are exactly of this form. It then follows from Theorem 8 and from the remark following it, that

$$\mathscr{B}_G \in \mathscr{RF}(\mathscr{B}) \quad \Leftrightarrow \quad [(\mathscr{B}_G)_0^\perp + z(\mathscr{B}_G)_0^\perp] \cap \mathscr{B}_1^\perp = \{0\}. \tag{61}$$

In particular, if $(\mathscr{B}_G)_0^\perp = \{0\}$, then $\mathscr{B}_G \in \mathscr{RF}(\mathscr{B})$. Finally, consider a more particular situation: let $\rho\colon G \to GL_k(W)$ be an irreducible representation. Consider the induced static symmetry on $\mathscr{D}[W]$. It follows that $(\mathscr{B}_G)_0^\perp$ is a G-invariant subspace of W^*, hence either $(\mathscr{B}_G)_0^\perp = \{0\}$ or $(\mathscr{B}_G)_0^\perp = W$. In the first case $\mathscr{B}_G \in \mathscr{RF}(\mathscr{B})$, in the second $\mathscr{B}_G = \{0\}$ and is not in $\mathscr{RF}(\mathscr{B})$.

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