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CONTROLLABILITY OF l^2-SYSTEMS*

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Abstract. This paper is devoted to an investigation of controllability and almost controllability of l^2-systems. These concepts are defined in terms of the possibility of steering one system trajectory to another. It is proved that a controllable l^2-system always has finite memory. The main result on almost controllability states that this is equivalent to the existence of a scattering representation. The paper ends with an investigation of the relation of almost controllability and state representations.

Key words. controllability, l^2-systems, state representation, linear systems

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1. Introduction.

Controllability has played an instrumental role in the development of control theory during the past three decades and is now a fundamental concept in mathematical systems theory. It plays a central role in control synthesis questions, related to the very possibility of exerting effective control. As such, it enters as a crucial "existence" condition in many engineering-type questions, such as stabilization and optimal control.

The notion of controllability is usually introduced for state space representations [1], [10], where it refers to the possibility of transferring the state from an initial to a terminal value. For finite-dimensional, linear, time-invariant systems, controllability then implies that any initial state can be exactly transferred to any terminal state in finite time. For nonlinear systems, we must often be satisfied with a local version of this property. For infinite-dimensional systems, on the other hand, approximate controllability and/or variations in which we allow the transfer time to go to infinity have proved to be more relevant. In fact, the question of which, and in what sense, systems described by partial differential equations are controllable is far from settled (see [9]).

Recently, a notion of controllability was introduced, where it becomes an intrinsic property of a dynamical system, and not just of a state space representation [12], [13]. The basic idea is to call a system controllable if an arbitrary past trajectory compatible with its behavior can eventually be concatenated with an arbitrary future trajectory. This notion is appealing from many points of view. It does not refer to a particular representation, and, in particular, it applies to systems that are not in state space form. In [12] and [13], mainly finite-dimensional, linear, time-invariant systems have been considered. Also, here we find that in controllable systems any past can be made exactly compatible to any future by a judicious choice of the input over a finite time interval. As may be expected, this property proves to be too demanding for infinite-dimensional systems.

The purpose of this paper is to study controllability using this vantage point for a class of infinite-dimensional systems. Specifically, we study (approximate) controllability for linear systems whose behavior is a shift-invariant, closed, linear subspace

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of $l^2(\mathbf{Z}, \mathbf{C}^q)$. We also study state representations of such systems and prove a sort of state space isomorphism theorem for almost controllable systems.

The mathematical techniques and methods of proof used here are inspired by functional analytic methods (H^∞ and the like), particularly the work of Fuhrmann [5].

To discuss systems, we follow the so-called behavioral approach, as introduced and developed in [12] and [13]. A *dynamical system* is a triple $\Sigma = (T, W, \mathcal{B})$ with $T \subset \mathbf{R}$ the *time axis*, W the *signal space*, and $\mathcal{B} \subset W^T$ the *behavior*. In this paper, we only consider *discrete-time systems* with $T = \mathbf{Z}$ or *continuous-time systems* with $T = \mathbf{R}$. Moreover, we assume that our systems are *time-invariant*; that is, that $\sigma^t \mathcal{B} = \mathcal{B}$ for every $t \in T$ (*shift-invariance*), where $\sigma^t : W^T \to W^T$ is the *t-shift* defined by $(\sigma^t f)(t') := f(t + t')$. We also only consider systems with $W = \mathbf{C}^q$ and with $\mathcal{B}$ a linear subspace of W^T (*linear systems*). For most of this paper, we focus on the following class of linear systems:

$$\mathcal{L}_q^2 := \{\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \text{ with } \mathcal{B} \text{ a closed shift-invariant linear subspace of } l_q^2\},$$

where l_q^2 indicates $l^2(\mathbf{Z}, \mathbf{C}^q)$ the Hilbert space of the $\mathbf{C}^q$-valued square-summable sequences over $\mathbf{Z}$. We often refer to a system in $\mathcal{L}_q^2$ as an *l^2-system.*

Example. (1)*l^2-systems defined by input/output maps.* Let $T : l_m^2 \to l_p^2$ be a closed linear map that commutes with the shift σ. T induces the system

$$\Sigma_T := (\mathbf{Z}, \mathbf{C}^{m+p}, G(T)) \in \mathcal{L}_{m+p}^2,$$

where $G(T)$ is the graph of the map T. These input/output systems have been widely investigated in the past (see [5] and [3]); an important case is when T is a convolution operator induced by an l^1-kernel.

(2) *l^2-systems as restrictions of other systems.* To determine how flexible it is to work with systems as a set of trajectories (the behavior), compared with simply input/output relations, suppose that we have linear input/output map $T : (\mathbf{C}^m)^{\mathbf{Z}} \to (\mathbf{C}^p)^{\mathbf{Z}}$ commuting with the shift. If $T(l_m^2) \not\subset l_p^2$, T does not induce an input/output l^2-map in the classical sense; nevertheless, we can consider the dynamical system $\Sigma = (\mathbf{Z}, \mathbf{C}^{m+p}, \mathcal{B})$, where $\mathcal{B} := G(T) \cap l_{m+p}^2$. Under certain conditions (for example, when $G(T)$ is closed in the pointwise convergence topology) we have that $\Sigma \in \mathcal{L}_{m+p}^2$ and that Σ completely determines the original behavior $G(T)$. Therefore the theory of l^2-systems can be used to analyze Σ and thus to infer properties of the map T.

For a given map $w : T \to W$, we define $w^- := w|_{T\cap(-\infty,0)}$ (the *past* of w) and $w^+ := w|_{T\cap(0,+\infty)}$ (the *future* of w). If $\mathcal{B} \subset W^T$, we indicate with $\mathcal{B}^-$ and $\mathcal{B}^+$ the sets of, respectively, the past and the future trajectories of $\mathcal{B}$.

DEFINITION 1.1. A time-invariant dynamical system $\Sigma = (T, W, \mathcal{B})$ is said to be controllable if, for every w_1 and w_2 in $\mathcal{B}$, there exist $t' \geq 0$ and $w \in \mathcal{B}$ such that

$$w^- = w_1^- \quad \text{and} \quad (\sigma^{t'} w)^+ = w_2^+.$$

This notion of controllability plays a fundamental role in the theory of linear, time-invariant, finite-dimensional, state space systems, but it proves to be very restrictive when we consider general l^2-systems, for which we propose the following.

DEFINITION 1.2. A linear time-invariant system $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$ is said to be almost controllable if there exists $K > 0$ such that, for every w_1 and w_2 in $\mathcal{B}$, there exists $v_n \in \mathcal{B}$ for $n = 1, 2, \cdots$, yielding the following:

$$(\sigma^{-n} v_n)^- \to w_1^-, \quad (\sigma^n v_n)^+ \to w_2^+, \quad \|v_n\|_2 \leq K\left(\|w_1^-\|_2 + \|w_2^+\|_2\right),$$

where $\to$ denotes limit in the l^2-topology for $n \to \infty$.

Remark. It is not obvious, from this definition, that a controllable system in $\mathcal{L}_q^2$ is almost controllable. This is indeed the case and is shown later.

Remark. The uniform boundness requirement on the v_n's in Definition 1.2 is essential. Indeed, if we drop this, then any system in $\mathcal{L}_q^2$ would satisfy the property; indeed, let $w_1, w_2 \in \mathcal{B}$ and consider $v_n := \sigma^{-n}w_2 + \sigma^n w_1$. Then $v_n \in \mathcal{B}$ for all n, and it is evident that $(\sigma^n v_n)^+ \to w_2^+$ and $(\sigma^{-n}v_n)^- \to w_1^-$ in the l^2-topology.

2. Controllable systems. The main result of this section concerning l^2-systems shows that controllable systems in the sense of Definition 1.1 have automatically finite memory.

THEOREM 2.1. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ be in $\mathcal{L}_q^2$. Then Σ is controllable if and only if $\mathcal{B}$ can be expressed as the l^2-solutions of a linear constant coefficients difference equation; that is, there exists $R(z, z^{-1}) \in \mathbf{C}^{g\times q}[z, z^{-1}]$ such that*

$$\mathcal{B} = \{w \in l_q^2 \mid R(\sigma, \sigma^{-1})w = 0\}, \tag{2.1}$$

where $R(\sigma, \sigma^{-1}) : (\mathbf{C}^q)^{\mathbf{Z}} \to (\mathbf{C}^g)^{\mathbf{Z}}$ is the operator in the shift σ induced by the polynomial matrix $R(z, z^{-1})$.

To prove Theorem 2.1, we must establish a few intermediate results, which have an interest of their own. Also, we work in a somewhat more general setting encompassing l^2-systems, since we believe that, in this way, a more complete picture of the situation can be drawn without additional effort.

For $w_1, w_2 \in W^T$ and $t \in T$, we denote by the symbol $w_1 \wedge_t w_2$ the *concatenation* of w_1 and w_2 at time t; i.e., $w_1 \wedge_t w_2(t') := w_1(t')$ for $t' < t$ and $w_1 \wedge_t w_2(t') := w_2(t')$ for $t' \geq t$. We also use the symbol $\wedge_t$ to concatenate restrictions of functions such as, for example, $w_1^- \wedge_0 w_2^+$.

DEFINITION 2.2. Let X be a linear subspace of $(\mathbf{C}^q)^{\mathbf{Z}}$ and let $\|\cdot\|_X$ be a norm on X. $(X, \|\cdot\|_X)$ is said to be a memoryless Banach space if the following hold:

(1) $(X, \|\cdot\|_X)$ is a complex Banach space,

(2) X is shift-invariant ($\sigma X = X$) and $\sigma : X \to X$ is an isometry,

(3) X is memoryless ($w_1, w_2 \in X \Rightarrow w_1 \wedge_t w_2 \in X$ for all $t \in \mathbf{Z}$).

Remark. If X is a memoryless Banach space, $w \in X$, and $I \subset \mathbf{Z}$, we often identify $w|_I$ with the trajectory in X, which is equal to w on I, and 0 outside of I. Through this identification, the spaces X^- and X^+ are seen as the subspaces of X consisting of the trajectories with support in, respectively, $(-\infty, 0)$ and $[0, +\infty)$. It follows that X^- and X^+ are closed in X and, by condition (3) of the preceding definition, $X = X^- \oplus X^+$. We indicate with P^- and P^+ the linear bounded projections from X on X^- and X^+, respectively. Once a memoryless Banach space X has been fixed, the convergence of a sequence in the norm of X is simply denoted by the symbol $\to$, with no further specification when no confusion can arise.

Example. We now present the following examples of memoryless Banach spaces, which are considered later in the paper:

(1) The space l_q^p $(1 \leq p < +\infty; q \in \mathbf{N}^+)$ of the $\mathbf{C}^q$-valued sequences over $\mathbf{Z}$ whose pth power is summable, equipped with the norm

$$\|w\|_p := \left(\sum_{-\infty}^{+\infty} |w(t)|_{\mathbf{C}^q}^p\right)^{1/p};$$

(2) The space l_q^∞ ($q \in \mathbf{N}^+$) of the bounded $\mathbf{C}^q$-valued sequences over $\mathbf{Z}$, equipped with the norm

$$\|w\|_\infty := \sup_{t\in\mathbf{Z}} |w(t)|_{\mathbf{C}^q};$$

(3) The subspace c_q^0 of l_q^∞, consisting of the sequences converging to 0 as t approaches $\pm\infty$, equipped with the norm $\|\cdot\|_\infty$.
Note the following chain of inclusions:

$$l_q^1 \subset l_q^p \subset c_q^0 \subset l_q^\infty \quad \forall p \in [1,\infty).$$

If X is a memoryless Banach space contained in $(\mathbf{C}^q)^{\mathbf{Z}}$, we consider the following class of linear systems:

$$\mathcal{L}_X := \{\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \text{ with } \mathcal{B} \text{ a closed shift-invariant linear subspace of } X\}.$$

In the case where $X = l_q^p$, we also use the notation $\mathcal{L}_q^p$ for $\mathcal{L}_X$.

Let $\Sigma = (T, W, \mathcal{B})$ be a time-invariant system and Δ a positive number. Σ is said to have *Δ-finite memory* if $w_1, w_2 \in \mathcal{B}$ and $w_1|_{[0,\Delta)} = w_2|_{[0,\Delta)}$ implies that $w_1 \wedge_0 w_2 \in \mathcal{B}$. Σ is said to have *finite memory* if it has Δ-finite memory for some Δ.

Our first goal is to study the structure of finite memory systems in $\mathcal{L}_X$. To do this, we must introduce the important system-theoretic concept of completeness. Let $\Sigma = (T, W, \mathcal{B})$ be a time-invariant system; it is said to be *complete* if, given any $w \in W^T$, we have that $w \in \mathcal{B}$ if and only if $w|_I \in \mathcal{B}|_I$ for every finite interval $I \subset T$ (with obvious meaning of $\mathcal{B}|_I$). The structure of the complete time-invariant linear systems is studied in much detail in [12] and [13]; in particular, there is the following important result.

THEOREM 2.3. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ be a linear, time-invariant system. The following conditions are then equivalent:*

(1) *Σ is complete,*

(2) *$\mathcal{B} \subset (\mathbf{C}^q)^{\mathbf{Z}}$ is closed in the pointwise convergence topology,*

(3) *There exists $R(z, z^{-1}) \in \mathbf{C}^{g\times q}[z, z^{-1}]$ such that $\mathcal{B} = \ker R(\sigma, \sigma^{-1})$.*

From (3) of Theorem 2.3, it is clear that any complete linear system over $\mathbf{Z}$ indeed has finite memory. In general, systems in $\mathcal{L}_X$ are not complete; we can actually prove that, if $\mathcal{B} \subset c_q^0$, then Σ is complete if and only if $\mathcal{B} = \{0\}$. Nevertheless, the concept of completeness proves to be useful in our investigation. In fact, we have the following result.

PROPOSITION 2.4. *Let X be a memoryless Banach space contained in c_q^0 and let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ be in $\mathcal{L}_X$. Then Σ has finite memory if and only if $\mathcal{B} = \mathcal{B}^{\text{compl}} \cap X$, where $\mathcal{B}^{\text{compl}}$ is the completion of $\mathcal{B}$ (defined as the smallest subspace of $(\mathbf{C}^q)^{\mathbf{Z}}$ that is shift-invariant, complete, and contains $\mathcal{B}$).*

Proof. Observe that $\mathcal{B} \subset \mathcal{B}^{\text{compl}} \cap X$. Assume that Σ has Δ-finite memory and let $w \in \mathcal{B}^{\text{compl}} \cap X$. Then there exists a sequence $w_n \in \mathcal{B}$ such that

$$w_n|_{[-n,n]} = w|_{[-n,n]} \quad \forall n \in \mathbf{N}. \tag{2.2}$$

Consider now the linear map $P_\Delta : \mathcal{B}^- \oplus \mathcal{B}^+ \to \mathcal{B}^-|_{[-\Delta,0)} \oplus \mathcal{B}^+|_{[0,\Delta)}$ given by

$$P_\Delta(w_1, w_2) = \left(w_1|_{[-\Delta,0)}, w_2|_{[0,\Delta)}\right).$$

Since P_Δ is surjective, there exists a linear map $Q_\Delta : \mathcal{B}^-|_{[-\Delta,0)} \oplus \mathcal{B}^+|_{[0,\Delta)} \to \mathcal{B}^- \oplus \mathcal{B}^+$ such that $P_\Delta \circ Q_\Delta = Id$. Since Σ has Δ-finite memory, we can assume that

$$\left((\sigma^{-n+\Delta}w_n)^-, (\sigma^{n-\Delta}w_n)^+\right) = Q_\Delta\left((\sigma^{-n+\Delta}w_n)^-|_{[-\Delta,0)}, (\sigma^{n-\Delta}w_n)^+|_{[0,\Delta)}\right).$$

By (2.2), for n sufficiently large, we then have that

$$\left((\sigma^{-n+\Delta}w_n)^-, (\sigma^{n-\Delta}w_n)^+\right) = Q_\Delta\left(w|_{[-n,-n+\Delta)}, w|_{[n-\Delta,n)}\right). \tag{2.3}$$

Since $w \in X \subset c_q^0$, we have that

$$\left(w|_{[-n,-n+\Delta)}, w|_{[n-\Delta,n)}\right) \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.4}$$

Q_Δ is bounded, since it acts on a finite-dimensional vector space; therefore, by (2.3) and (2.4), we have that

$$\left((\sigma^{-n+\Delta}w_n)^-, (\sigma^{n-\Delta}w_n)^+\right) \to 0 \quad \text{as} \quad n \to +\infty,$$

which implies, together with (2.2) and condition (2) of Definition 2.2, that $w_n \to w$. This yields $w \in \mathcal{B}$. The other implication follows from Theorem 2.3. □

Remark. Proposition 2.4 still holds true if $X = l_q^\infty$ and if we assume that $\mathcal{B}$ is closed in the weak*-topology of l_q^∞; the proof is identical.

Let us now state the main result of this section.

THEOREM 2.5. *Let X be a memoryless Banach space and let Σ be a controllable system in $\mathcal{L}_X$. Then Σ has finite memory.*

We first prove a proposition based on a technical lemma whose proof is omitted since it follows from a straightforward application of the Douglas factorization theorem (see [5]).

LEMMA 2.6. *Let X, Y, and Z be Banach spaces and let $A : X \to Z$ and $B : Y \to Z$ be linear bounded maps. If there exists $X_0 \subset X$ subspace of second category in X such that $\mathcal{R}(A|_{X_0}) \subset \mathcal{R}(B)$, then $\mathcal{R}(A) \subset \mathcal{R}(B)$*

We now state a result that claims that, under certain conditions, controllability may always be achieved in a uniformly bounded finite number of steps, if it can be achieved at all.

PROPOSITION 2.7. *Let X be a memoryless Banach space and let $\Sigma \in \mathcal{L}_X$ be a controllable system. Then there exists $n_0 \in \mathbf{N}$ such that, for all w_1 and w_2 in $\mathcal{B}$, there exists $w \in \mathcal{B}$ such that*

$$w^- = w_1^- \quad \text{and} \quad (\sigma^{n_0}w)^+ = w_2^+$$

Proof. Let us consider the following sequence of linear bounded maps:

$$T_n : \mathcal{B} \to X^- \oplus X^+, \tag{2.5}$$

given by $T_n(w) := (w^-, (\sigma^n w)^+)$. By controllability, we have that

$$\bigcup_{n>0} \mathcal{R}(T_n) = \mathcal{B}^- \oplus \mathcal{B}^+.$$

Proposition 2.7 will be proved if we show that there exists $n_0 \in \mathbf{N}$ such that $\mathcal{R}(T_{n_0}) = \mathcal{B}^- \oplus \mathcal{B}^+$. Let us introduce the map

$$T : \mathcal{B} \oplus \mathcal{B} \to X^- \oplus X^+$$

given by $T(w_1, w_2) := (w_1^-, w_2^+)$. Consider $M_n = T^{-1}\mathcal{R}(T_n)$. Then

$$\bigcup_{n>0} M_n = \mathcal{B} \oplus \mathcal{B},$$

and, since $\mathcal{B} \oplus \mathcal{B}$ is a Banach space, it follows, by a standard category argument (see, for example, [11]) that there exists $n_0 \in \mathbf{N}$ such that M_{n_0} is of second category in $\mathcal{B} \oplus \mathcal{B}$. Applying Lemma 2.6 to the maps T_{n_0} and T, it follows that $\mathcal{R}(T) \subset \mathcal{R}(T_{n_0})$, which implies that $\mathcal{R}(T_{n_0}) = \mathcal{B}^- \oplus \mathcal{B}^+$. □

More can be said about the range of the map T_{n_0} introduced in (2.5). In fact, consider the map

$$i : \mathcal{B} \to X^- \oplus X^+ \oplus \mathcal{B}|_{[0,n_0)}$$

with $i = T_{n_0} \oplus P_{n_0}$, where $P_{n_0} : \mathcal{B} \to \mathcal{B}|_{[0,n_0)}$ is the restriction to the interval $[0, n_0)$. It is clear that i is a linear bounded embedding (injective with closed range) and that P_{n_0} has finite-dimensional range; it is then a standard result from functional analysis (see, for example, [2]) that T_{n_0} also has closed range. This yields the following result.

PROPOSITION 2.8. *Let X be a memoryless Banach space and let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_X$ be controllable. Then $\mathcal{B}^-$ and $\mathcal{B}^+$ are closed subspaces of X.*

Proof of Theorem 2.5. Consider the following subspace of $\mathcal{B}^+$:

$$\mathcal{B}_0^+ := \{w^+ \in \mathcal{B}^+ \mid 0 \wedge_0 w^+ \in \mathcal{B}\}.$$

Define the linear map

$$R : \mathcal{B}|_{[0,n_0)} \to \mathcal{B}^+/\mathcal{B}_0^+,$$

where n_0 is the same as in Proposition 2.7, by $R(x) = v \pmod{\mathcal{B}_0^+}$, where v is any trajectory in $\mathcal{B}^+$ such that $0 \wedge_0 x \wedge_{n_0} \sigma^{-n_0} v \in \mathcal{B}$. It is easy to verify that R is a well-defined linear map, and that it is surjective. Since the domain of R is finite-dimensional, it then follows that $\mathcal{B}^+/\mathcal{B}_0^+$ is also finite-dimensional (this is actually a state space of Σ). Therefore there exists a finite-dimensional subspace N of $\mathcal{B}^+$ such that $\mathcal{B}^+ = \mathcal{B}_0^+ \oplus N$. Now consider the following decreasing sequence of subspaces of $\mathcal{B}^+$:

$$H_n := \left\{w^+ \in \mathcal{B}^+ \mid w^+|_{[0,n)} = 0\right\}.$$

Then

$$\bigcap_{n \geq 0} H_n = (0).$$

Consider $K_n := P_N H_n$, where P_N is the projection operator on the subspace N. $\{K_n\}$ is a decreasing sequence of subspaces of N with null intersection; since N has finite dimension, it then follows that there exists $\tilde{n} > 0$ such that $K_{\tilde{n}} = (0)$, which implies that $H_{\tilde{n}} \subset \mathcal{B}_0^+$. We now claim that Σ has $\tilde{n}$-finite memory; in fact, let $w_1, w_2 \in \mathcal{B}$ such that $w_1|_{[0,\tilde{n})} = w_2|_{[0,\tilde{n})}$. Then

$$(w_2 - w_1)^+ \in H_{\tilde{n}} \subset \mathcal{B}_0^+,$$

which implies that

$$w_1 \wedge_0 w_2 = w_1 + (0 \wedge_0 (w_2 - w_1)^+) \in \mathcal{B}. \qquad \square$$

We conclude this section with the following summarizing result, which encompasses Theorem 2.1, stated at the beginning of the section.

THEOREM 2.9. *Let X be a memoryless Banach space contained in c_q^0 and let $\Sigma \in \mathcal{L}_X$. Then the following conditions are equivalent:*

(1) *Σ is controllable,*
(2) *Σ has finite memory,*
(3) *there exists a polynomial matrix $R(z, z^{-1}) \in \mathbf{C}^{g\times q}[z, z^{-1}]$ such that*

$$\mathcal{B} = \{w \in X \mid R(\sigma, \sigma^{-1})w = 0\}.$$

Proof. (1)⇒(2) is Theorem 2.5. (2)⇒(3) is contained in Proposition 2.4 and Theorem 2.3. Finally, (3)⇒(1) follows from standard results of the theory of complete systems: [13] and [14] contain a proof for the case where $X = l_q^2$, which is easily generalizable to our case. □

Remark. The condition that $X \subset c_q^0$ in Theorem 2.9 is essential. In fact, it follows from the results of [13] that Theorem 2.9 is false for l_q^∞.

3. Almost controllable systems. In this section we specifically consider l^2-systems, since we believe that the Hilbert structure plays a fundamental role in this context to achieve nice representation results. We make use of frequency domain techniques including Hardy spaces theory; our main references for these matters are [4], [6], and [7].

We start with the following interesting topological characterization of almost controllability.

PROPOSITION 3.1. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ be an l^2-system. Then the following two conditions are equivalent:*

(1) *Σ is almost controllable,*
(2) *$\mathcal{B}^-$ and $\mathcal{B}^+$ are closed in l_q^2.*

Proof. (1)⇒(2). By (1), there exists $k > 0$ such that, for every $w^- \in \mathcal{B}^-$, there exists $v_n \in \mathcal{B}$ such that

$$(\sigma^n v_n)^- \to w^-, \quad (\sigma^{-n} v_n)^+ \to 0, \quad \|v_n\|_2 \le K\|w^-\|_2.$$

Consider $w_n = \sigma^n v_n$; then

$$w_n^- \to w^-, \tag{3.1}$$

$$\|w_n^+\|_2 \le \|w_n\|_2 \le K\|w^-\|_2. \tag{3.2}$$

By (3.2) we can assume, taking a subsequence if necessary, that

$$w_n^+ \to v^+ \in l_q^{2+} \quad \text{weakly}. \tag{3.3}$$

Equations (3.1) and (3.3) yield

$$w_n = w_n^- \wedge_0 w_n^+ \to w^- \wedge_0 v^+ \quad \text{weakly},$$

which implies that

$$w^- \wedge_0 v^+ \in \mathcal{B} \tag{3.4}$$

and, by (3.2),

$$\|v^+\| \leq K\|w^-\|_2. \tag{3.5}$$

From (3.4) and (3.5), it follows, by a standard argument from functional analysis, that $\mathcal{R}(P^-|_{\mathcal{B}}) = \mathcal{B}^-$ is closed. In an analogous way, we see that $\mathcal{B}^+$ is also closed.

(2)⇒(1). Since the two projections P^- and P^+ both have closed range, there exists $K > 0$ such that, for all $w^- \in \mathcal{B}^-$ and $w^+ \in \mathcal{B}^+$, there exist w_1 and w_2 in $\mathcal{B}$ such that

$$w_1^- = w^-, \qquad \|w_1\|_2 \leq K\|w^-\|_2;$$
$$w_2^+ = w^+, \qquad \|w_2\|_2 \leq K\|w^+\|_2.$$

Now consider $v_n = \sigma^n w_1 + \sigma^{-n} w_2$. Then

$$(\sigma^{-n} v_n)^- = w_1^- + \sigma^{-2n} w_2 \to w_1^-,$$
$$(\sigma^n v_n)^+ = w_2^+ + \sigma^{2n} w_1 \to w_1^+,$$
$$\|v_n\|_2 \leq K\left(\|w_1^-\| + \|w_2^+\|\right),$$

which yields (1). □

Remark. By Propositions 2.8 and 3.1, it is now evident that, for l^2-systems, controllability indeed implies almost controllability.

We now study representations of almost controllable systems, and this is the subject of the remainder of this article. We show how it is possible to represent an almost controllable system as the image of l^2-maps, while, in next section, we study state space representations. The common feature underlying these two representations is the presence of latent variables, namely, variables that are not part of the external signal, but that are introduced to express the internal structure of the system. We return to this point later.

Let $T \subset \mathbf{R}$ and W_1, W_2 be sets; consider $\mathcal{B}_1 \subset W_1^T$ and $\mathcal{B}_2 \subset W_2^T$. A map $F : \mathcal{B}_1 \to \mathcal{B}_2$ is said to be *causal* if $w_1(t) = w_2(t)$ for all $t \leq t'$ implies that $(Fw_1)(t) = (Fw_2)(t)$ for all $t \leq t'$; F is said to be *anticausal* if $w_1(t) = w_2(t)$ for all $t \geq t'$ implies that $(Fw_1)(t) = (Fw_2)(t)$ for all $t \geq t'$.

The following is the main result of this paper.

THEOREM 3.2. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ be in $\mathcal{L}_q^2$. Then the following conditions are equivalent:*

(1) *Σ is almost controllable,*
(2) *$\mathcal{B}^-$ and $\mathcal{B}^+$ are closed subspaces of l_q^2,*
(3) *There exist a number $g \in \mathbf{N}$ and two linear bounded maps*

$$F^- : l_g^2 \to l_q^2, \quad F^+ : l_g^2 \to l_q^2,$$

satisfying the following properties:

(i) *$\mathcal{R}(F^-) = \mathcal{B} = \mathcal{R}(F^+)$,*
(ii) *F^- and F^+ commute with σ,*
(iii) *F^- is anticausal and has an anticausal bounded left inverse,*
(iv) *F^+ is causal and has a causal bounded left inverse.*

Moreover, if any of the three above equivalent conditions is satisfied, then the maps F^- and F^+ in (3) *can be chosen to be isometries. If we assume that this is the case, then g is unique, and F^- and F^+ are unique up to right multiplication by unitary isomorphism on $\mathbf{C}^g$.*

The proof of Theorem 3.2 is rather involved. We first discuss some easy aspects.

Proof of Theorem 3.2 (Preamble). Note that the equivalence between (1) and (2) is proved in Proposition 3.1.

Also, it is easy to show that (3)⇒(2); in fact, consider the maps

$$A : l_g^{2-} \to l_q^{2-}, \qquad A = P^- \circ F^+ \circ P^-$$

and

$$B : l_q^{2-} \to l_g^{2-}, \qquad B = P^- \circ \tilde{F}^+ \circ P^-,$$

where $\tilde{F}^+$ is the causal left inverse of F^+, and P^- here indicates both the projection operators on l_g^{2-} and l_q^{2-}. We have that

$$B \circ A = P^- \circ \tilde{F}^+ \circ P^- \circ F^+ \circ P^- = P^- \circ \tilde{F}^+ \circ F^+ \circ P^- = Id|_{l_g^{2-}},$$

which implies that $\mathcal{R}(A) = \mathcal{B}^-$ is closed. In an analogous way, using F^-, it follows that $\mathcal{B}^+$ is closed.

It therefore remains to be proved that (2) implies (3), and the remainder of this section is devoted to this implication.

Remark. For finite memory l^2-systems, Theorem 3.2 is already obtained in [14].

We now first introduce the important frequency domain description of an l^2-system. If $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$, consider $\hat{\mathcal{B}}$ the closed subspace of $L_q^2 := L^2(\mathbf{T}, \mathbf{C}^q)$ (the Hilbert space of the $\mathbf{C}^q$- valued Lebesgue square-integrable functions on the unit circle $\mathbf{T}$) obtained as the image of $\mathcal{B}$ through the Fourier transform $\mathcal{F}_q : l_q^2 \to L_q^2$. It is well known that $\hat{\mathcal{B}}$ is a doubly invariant subspace of L_q^2 with respect to the shift $S : L_q^2 \to L_q^2$, given by $(Sw)(e^{i\theta}) := e^{i\theta} w(e^{i\theta})$. Namely, $S^n \hat{\mathcal{B}} = \hat{\mathcal{B}}$ for all $n \in \mathbf{Z}$. Doubly invariant subspaces of L_q^2 have been widely studied in the past (see [6]); we must recall only a few fundamental facts. A *range function* $J = J(e^{i\theta})$ is a function on the circle $\mathbf{T}$ taking values in $\mathcal{G}_q$ (the family of all the subspaces of $\mathbf{C}^q$); J is said to be measurable if the orthogonal projection $P(e^{i\theta})$ from $\mathbf{C}^q$ on $J(e^{i\theta})$ is measurable. If J is a measurable range function, we can consider that

$$\mathcal{M}_J = \{\hat{w} \mid \hat{w} \in L_q^2 \text{ and } \hat{w}(e^{i\theta}) \in J(e^{i\theta}) \text{ a.e. on } \mathbf{T}\},$$

and it is easy to show that $\mathcal{M}_J$ is a doubly invariant closed subspace of L_q^2. A fundamental fact is that all closed, doubly invariant subspaces of L_q^2 are of this form, and also the correspondence between J and $\mathcal{M}_J$ is one-to-one, under the convention that range functions are identified if they are equal almost everywhere. A measurable range function J is called *analytic* if there exists a finite number $\{F_1, \cdots, F_g\}$ of elements of H_q^2 (the closed subspace of L_q^2 consisting of the functions whose negative Fourier coefficients are zero) such that $J(e^{i\theta})$ is the span of $\{F_1(e^{i\theta}), \cdots, F_g(e^{i\theta})\}$ almost everywhere on $\mathbf{T}$. In a similar way, using the conjugate space $\overline{H}_q^2$, we can introduce the concept of *coanalytic* range function. If J is a range function, we can define the *orthogonal* range function $J^\perp$ by $J^\perp(e^{i\theta}) = (J(e^{i\theta}))^\perp$, where the last orthogonal must be considered in $\mathbf{C}^q$ with respect to the canonical Hermitian inner product; it can be proved that J is analytic if and only if $J^\perp$ is coanalytic.

Let us now introduce the space $L^\infty_{g\times q}$ of the $g\times q$-matrices of L^∞-functions defined on $\mathbf{T}$ and the subspace $H^\infty_{g\times q}$ consisting of those whose negative Fourier coefficients are zero. If $F\in L^\infty_{g\times q}$, we will denote by M_F the multiplicative operator induced by F, namely, $\mathrm{M}_F : L^2_q \to L^2_g$, given by $(\mathrm{M}_F w)(e^{i\theta}) := F(e^{i\theta})\, w(e^{i\theta})$. The following proposition clarifies the relation among all of these concepts. The proof is practically contained in [6]; therefore we only give a sketch of it.

PROPOSITION 3.3. *The following conditions are equivalent:*

(1) *J is an analytic range function,*

(2) *There exists $F\in H^\infty_{q\times g}$ such that $\mathcal{M}_J = \mathcal{R}(\mathrm{M}_F)$,*

(3) *There exists $L\in H^\infty_{l\times q}$ such that $\mathcal{M}_J = \ker(\mathrm{M}_L)$.*

Moreover, if any of the above equivalent conditions are satisfied, then F in (2) can be chosen to be outer ($\mathcal{M}_J\cap H^2_q = \mathcal{R}\left(\mathrm{M}_F|_{H^2_g}\right)$) and rigid ($F(e^{i\theta})$ is an isometry almost everywhere). With this choice, g is uniquely determined by the relation $g=\dim J(e^{i\theta})$ almost everywhere, and F is also uniquely determined up to right multiplication by constant unitary matrix.

Proof. (1)$\Rightarrow$(2) Consider that $\mathcal{A}_J = \mathcal{M}_J\cap H^2_q$. $\mathcal{A}_J$ is a closed S-invariant subspace of H^2_q; therefore, by the Beurling–Lax theorem (see [6] and [7]), there exist $g\in\mathbf{N}$ and $F\in H^\infty_{q\times g}$ with F rigid such that $\mathcal{A}_J = FH^2_g$. Since J is an analytic range function, it is evident that $\mathcal{M}_J=\mathcal{R}(\mathrm{M}_F)$. Moreover, F is outer by the way it has been defined.

(2)$\Rightarrow$(1) is trivial.

(3)$\Rightarrow$(1). Suppose that $\mathcal{M}_J = \ker \mathrm{M}_L$. Write L as $L=(L_1,\cdots,L_l)^t$, where $L_j\in H^\infty_{q\times 1}$. Then

$$\{w\in\mathcal{M}_J\} \Leftrightarrow \{L_j^t w = 0 \quad \forall j=1,\cdots,l\} \Leftrightarrow \{w\perp \bar{L}_j \quad \forall j=1,\cdots,l\}.$$

Let J' be the coanalytic range function spanned by the family $\{\bar{L}_1,\cdots,\bar{L}_l\}$. Since $J=(J')^\perp$, this shows that J is analytic.

Reversing this argument, we see that (1)$\Rightarrow$(3).

Uniqueness of F and the fact that $g=\dim J(e^{i\theta})$ almost everywhere simply follow from the Beurling–Lax theorem and the fact that F is outer and rigid. □

Of course, we have the following symmetric result.

PROPOSITION 3.4. *The following conditions are equivalent:*

(1) *J is a coanalytic range function,*

(2) *There exists $F\in H^\infty_{q\times g}$ such that $\mathcal{M}_J=\mathcal{R}\left(\mathrm{M}_{\overline{F}}\right)$,*

(3) *There exists $L\in H^\infty_{l\times q}$ such that $\mathcal{M}_J=\ker\left(\mathrm{M}_{\overline{L}}\right)$.*

Moreover, if any of the above equivalent conditions are satisfied, then F in (2) can be chosen to be outer and rigid. With this choice, g is uniquely determined by the relation $g=\dim J(e^{i\theta})$ almost everywhere, and F is also uniquely determined up to right multiplication by constant unitary matrix.

We are now ready to state and prove the main mathematical result.

LEMMA 3.5. *Let $\mathcal{M}$ be a closed, doubly invariant subspace of L^2_q. Then the following two conditions are equivalent:*

(1) *$\mathcal{M}^-$ (the projection of $\mathcal{M}$ on $H^{2-}_q := (H^2_q)^\perp$) is closed,*

(2) *There exist $F\in H^\infty_{q\times g}$ and $\tilde{F}\in H^\infty_{g\times q}$ such that $\mathcal{M}=\mathcal{R}(\mathrm{M}_F)$ and $\tilde{F}F = Id_g$.*

Also, if either of these two conditions is satisfied, then F in (2) can be chosen to be rigid and outer.

Proof. (1)$\Rightarrow$(2). $\mathcal{M}^-$ is closed, and it is invariant for the adjoint of the left shift acting on H^{2-}_q. Therefore, by the Beurling–Lax theorem, there exists a rigid

$\psi \in H^\infty_{q\times k}$, with $k \leq q$, such that

$$\mathcal{M}^- = \left(\overline{\psi} H_k^{2-}\right)^\perp, \tag{3.6}$$

where the orthogonal is taken with respect to the space H_q^{2-} (and not with respect to all of L_q^2!).

If $f \in L_q^2$, let us indicate by f^- and f^+ the projections of f on, respectively, H_q^{2-} and H_q^2. Now, for any $f \in \mathcal{M}$, we have that $f^- \perp \overline{\psi} H_k^{2-}$ by (3.6), and also $f^+ \perp \overline{\psi} H_k^{2-}$. Therefore $f \perp \overline{\psi} H_k^{2-}$ for all$f \in \mathcal{M}$, or, equivalently,

$$\psi^t f \perp H_k^{2-} \quad \forall f \in \mathcal{M}. \tag{3.7}$$

Since $\mathcal{M}$ is a doubly invariant subspace, (3.7) implies that $\psi^t f = 0$ for all $f \in \mathcal{M}$. We now prove that, in fact,

$$\psi^t f = 0 \Leftrightarrow f \in \mathcal{M}. \tag{3.8}$$

Let f be in L_q^2 such that $\psi^t f = 0$; it follows that $\psi^t f^+ + \psi^t f^- = 0$, and therefore $\psi^t f^- \perp H_k^{2-}$, or, equivalently, $f^- \perp \overline{\psi} H_k^{2-}$. By (3.6), it then follows that $f^- \in \mathcal{M}^-$. Since $\mathcal{M}$ is doubly invariant, we also have that

$$\psi^t S^{-n} f = 0 \quad \forall n \in \mathbf{N},$$

which, by the preceding argument, yields

$$\left(S^{-n} f\right)^- \in \mathcal{M}^- \quad \forall n \in \mathbf{N}.$$

Therefore there exists a sequence $v_n \in H_q^2$ such that

$$\left(S^{-n} f\right)^- + v_n \in \mathcal{M} \quad \forall n \in \mathbf{N}, \tag{3.9}$$

and, since $\mathcal{M}^-$ is closed, we can choose v_n such that

$$\|v_n\|_2 \leq \|f\|_2, \tag{3.10}$$

for all n. It follows immediately that $S^n \left(S^{-n} f\right)^- \to f$, and, by (3.10), we can assume—taking, if necessary, a subsequence—that $S^n v_n \to 0$ weakly. Therefore

$$S^n \left(\left(S^{-n} f\right)^- + v_n\right) \to f \quad \text{weakly},$$

which, by (3.9), implies that $f \in \mathcal{M}$. This yields (3.8), which can be equivalently expressed as $\mathcal{M} = \ker\left(\mathrm{M}_{\psi^t}\right)$. By Proposition 3.3, this implies that there exists $F \in H^\infty_{q\times g}$ such that $\mathcal{M} = \mathcal{R}\left(\mathrm{M}_F\right)$, and F can be chosen to be outer rigid.

We must still prove that F admits an H^∞ left inverse. This may be seen as follows. Consider the linear bounded map $A : H_g^{2-} \to H_q^{2-}$, given by

$$A := P^- \circ \mathrm{M}_F|_{H_q^{2-}},$$

where P^- denotes the projection onto the subspace H_q^{2-}. Consider the following adjoint of A:

$$A^* : H_q^{2-} \to H_g^{2-}, \qquad A^* = \mathrm{M}_{F^*}.$$

Since F is outer and rigid, it is easy to see that A^* is surjective. Consequently, A is injective and has closed range. We now use the fact that a function in a Hardy space (H_q^2, $H_{g\times q}^\infty$) can be holomorphically extended to the open unit disk D (see [7]); for simplicity of notation, we use the same symbol for a function on $\mathbf{T}$ and its extension to D. If $h \in H_1^\infty$ and $\alpha \in D$, we have that

$$[h(e^{i\theta}) - h(\alpha)]\left(1 - \alpha e^{-i\theta}\right)^{-1} \in H_1^2.$$

Consider that

$$f_\alpha = \left(1 - |\alpha|^2\right)^{1/2}\left(1 - \alpha e^{-i\theta}\right)^{-1}.$$

It is a matter of computation to show that $f_\alpha \in H_1^{2-}$ and $\|f_\alpha\|_2 = 1$ for all $\alpha \in D$. Let $\xi \in \mathbf{C}^g$, with $\|\xi\| = 1$. We can then show that

$$A(\xi f_\alpha) = F(\alpha)(\xi f_\alpha). \tag{3.11}$$

Assume now that there exist sequences $\{\alpha_n\} \subset D$ and $\{\xi_n\} \subset \mathbf{C}^g$, with $\|\xi_n\| = 1$, such that $F(\alpha_n)\xi_n \to 0$. By (3.11), $A(\xi_n f_{\alpha_n}) \to 0$, and $\|\xi_n f_{\alpha_n}\|_2 = 1$. This is absurd, since A is injective and has closed range. By the vectorial Corona theorem (see [5]), it then follows that there exists $\tilde{F} \in H_{g\times q}^\infty$ such that $\tilde{F}F = I_g$.

(2)⇒(1). Simply observe that A admits a left inverse given by $B := P^- \circ \mathrm{M}_{\tilde{F}}|_{H_g^{2-}}$. Therefore $\mathcal{M}^- = \mathcal{R}(A)$ is closed. □

Naturally, we also have the following symmetric result.

LEMMA 3.6. *Let $\mathcal{M}$ be a closed doubly invariant subspace of L_q^2. Then the following two conditions are equivalent:*

(1) *$\mathcal{M}^+$ (the projection of $\mathcal{M}$ on H_q^2) is closed,*

(2) *There exist $F \in H_{q\times g}^\infty$ and $\tilde{F} \in H_{g\times q}^\infty$ such that $\mathcal{M} = \mathcal{R}\left(\mathrm{M}_{\overline{F}}\right)$ and $\tilde{F}F = Id_g$.*

Also, if either of these two conditions is satisfied, then F in (2) can be chosen to be rigid and outer.

Proof of Theorem 3.2 (End). (2)⇒(3). Consider the Fourier transform $\mathcal{F}_q : l_q^2 \to L_q^2$. If $\mathcal{F}_q(\mathcal{B}) = \mathcal{M}$; then $\mathcal{F}_q(\mathcal{B}^-) = \mathcal{M}^-$, $\mathcal{F}_q(\mathcal{B}^+) = \mathcal{M}^+$, and both are closed in L_q^2. By Lemmas 3.5 and 3.6, there exist G_1 and G_2 in $H_{q\times g}^\infty$ rigid, outer, both having H^∞ left inverse, and such that

$$\mathcal{R}(\mathrm{M}_{G_1}) = \mathcal{M} = \mathcal{R}\left(\mathrm{M}_{\overline{G}_2}\right)$$

(note that g is the same for G_1 and G_2 by Propositions 3.3 and 3.4). Consider now that

$$F^+ : l_g^2 \to l_q^2, \qquad F^+ := \mathcal{F}_q^{-1} \circ \mathrm{M}_{G_1} \circ \mathcal{F}_g$$

and

$$F^- : l_g^2 \to l_q^2, \qquad F^- := \mathcal{F}_q^{-1} \circ \mathrm{M}_{\overline{G}_2} \circ \mathcal{F}_g.$$

Because of standard properties of the Fourier transform, it follows immediately that F^- and F^+ are isometries and that they satisfy properties (i)–(iv) of (3). Finally, the uniqueness of g, F^+, and F^- also follows from Propositions 3.3 and 3.4. □

Remark. The representation expressed by condition (3) of Theorem 3.2 is classically known as the *scattering representation*, and it is investigated in [8]. It is worthwhile to note that, while in [8] the scattering representation is derived from the existence of a pair of orthogonal subspaces (the *incoming* and *outgoing* subspaces) of $\mathcal{B}$

satisfying certain properties, here such a representation is independently derived from the topological assumption expressed by condition (2) of Theorem 3.2.

Remark. It is worthwhile to relate the result of Theorem 3.2 with the result of [13], which states that, if $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ is linear time-invariant and complete, then Σ is controllable if and only if $\mathcal{B} = \mathcal{R}\left(M(\sigma, \sigma^{-1})\right)$ for some polynomial matrix $M(s, s^{-1})$. Actually, if this is the case, then there exist polynomial matrices left-invertible $M_1(s)$ and $M_2(s)$ such that $\mathcal{R}\left(M_1(\sigma)\right) = \mathcal{B} = \mathcal{R}\left(M_2(\sigma^{-1})\right)$. Moreover, $M_1(\sigma)$ and $M_2(\sigma^{-1})$ can be chosen to be injective. In a sense, Theorem 3.2 generalizes this to l^2-systems.

We conclude this section with an analysis of input/output, almost controllable systems. Let $T : l^2_m \to l^2_p$ be a linear bounded map that is causal and commutes with σ; consider the induced l^2-system $\Sigma_T = (\mathbf{Z}, \mathbf{C}^{m+p}, G(T))$ (see part (1) of the example in the Introduction). We want to obtain necessary and sufficient conditions on the map T such that Σ_T is almost controllable. Consider now the *Hankel operator* $\mathcal{H}_T$ associated with the map T, namely, $\mathcal{H}_T : l^{2-}_m \to l^2_p$, given by $\mathcal{H}_T := P^+ \circ T|_{l^{2-}_m}$.

PROPOSITION 3.7. *Σ_T is almost controllable if and only if the Hankel operator $\mathcal{H}_T$ has closed range.*

Proof. It is evident that (^{t} denotes transposition)

$$\mathcal{B} = \mathcal{R}\left(\left[Id_m, T\right]^t\right). \tag{3.12}$$

Note that

$$\left[Id_m, 0\right] \circ \left[Id_m, T\right]^t = Id_m,$$

which implies, by Lemma 3.5, that $\mathcal{B}^-$ is closed. Therefore by Theorem 3.2, Σ_T is almost controllable if and only if $\mathcal{B}^+$ is closed. Therefore it suffices to show that $\mathcal{B}^+$ is closed if and only if $\mathcal{H}_T$ has closed range. Assume that $\mathcal{H}_T$ has closed range and let $f_n \in L^2_m$ be a sequence such that

$$P^+ \left[Id_m, T\right]^t f_n \to \left[\psi_1, \psi_2\right]^t \in l^2_m \oplus l^2_p.$$

Then

$$f_n^+ \to \psi_1, \qquad P^+ T f_n \to \psi_2,$$

which imply that $P^+ T f_n^- \to \psi_2 - P^+ T \psi_1$. Since $\mathcal{H}_T$ has closed range, it follows that there exists $f^- \in l^{2-}_m$ such that

$$P^+ T f^- = \psi_2 - P^+ T \psi_1,$$

which yields $\psi_2 = P^+ T\left(\psi_1 + f^-\right)$. Hence, by (3.12), $\left[\psi_1, \psi_2\right]^t \in \mathcal{B}^+$. This shows that $\mathcal{B}^+$ is closed. On the other hand, if $\mathcal{H}_T$ does not have a closed range, then there exists a sequence $f_n^- \in l^{2-}_m$ such that

$$\mathcal{H}_T f_n^- \to \phi \notin \mathcal{R}\left(\mathcal{H}_T\right). \tag{3.13}$$

There holds that $P^+ \left[Id_m, T\right]^t f_n^- \to \left[0, \phi\right]^t$. We claim that

$$[0, \phi]^t \notin \mathcal{R}\left([Id_m, T]^t\right).$$

Indeed, assume that there exists $f \in l^2_m$ such that

$$P^+ \left[Id_m, T\right]^t f = \left[0, \phi\right]^t$$

This implies that $f^+ = 0$ and $P^+Tf^- = \phi$, which by (3.13) yields a contradiction. This shows that $\mathcal{B}^+$ is not closed. □

Remark. In the scalar case (m=p=1), the condition for $\mathcal{H}_T$ to have a closed range can be expressed nicely in an equivalent way. Consider that $\hat{T} : L^2 \to L^2$, given by

$$\hat{T} := \mathcal{F} \circ T \circ \mathcal{F}^{-1}.$$

It is a standard fact that $\hat{T}$ is a multiplicative operator with symbol $H \in H^\infty$ (called the *transfer function* of Σ_T). Note that $\mathcal{H}_T$ is completely determined by H, and it can be proved that $\mathcal{H}_T$ has a closed range if and only if H admits a factorization of the kind $H = \psi \overline{K}$, where $\psi \in H^\infty$ is inner, $K \in H^\infty$, and also there exists $\delta > 0$ such that

$$|\psi(z)| + |K(z)| \geq \delta \quad \forall z \in D.$$

Consequently, a sufficient condition for the almost controllability is, in this case, that H is purely inner or, more generally, that its outer part is rational.

4. Hilbertian state models . We start this section with a few words about general latent variables models, before focusing on state models. A *dynamical system with latent variables* is defined as a quadruple

$$\Sigma_f = (T, W, L, \mathcal{B}_f),$$

with T and W as in the definition of a dynamical system given in the introduction; L is the set of *latent variables*; and $\mathcal{B}_f \subset (W \times L)^T$ the (*full*) *behavior*. As for dynamical systems, we always assume that $T = \mathbf{Z}$ (or $T = \mathbf{R}$) and that our latent variables systems are time-invariant (the definition is analogous to the one for dynamical systems); also we assume linearity, namely, that W and L are vector spaces and $\mathcal{B}_f$ is a linear subspace of $(W \times L)^T$

$$\Sigma = (T, W, P_W \mathcal{B}_f)$$

(where P_W is the projection on the first factor of $W \times L$) is said to be the *manifest* or *external* dynamical system induced by Σ_f; $P_W \mathcal{B}_f$ is called the *manifest* (or *external*) *behavior*. Σ_f is said to be a *latent variable representation* of Σ. Σ_f is said to be *externally induced* if there exists a map (called the *observability map*)

$$F : P_W \mathcal{B}_f \to P_L \mathcal{B}_f$$

such that

$$\{(w, g) \in \mathcal{B}_f\} \Leftrightarrow \{w \in P_W \mathcal{B}_f \quad \text{and} \quad g = Fw\}.$$

Σ_f is said to be *past externally induced* (*future externally induced*) if the map F is causal (anticausal).

If $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$ is an almost controllable system, the scattering representation of Σ introduced in Theorem 3.2 (3) naturally induces the following two latent variables representations of Σ:

$$\Sigma_f^{\pm} = \left(\mathbf{Z}, \mathbf{C}^q, \mathbf{C}^g, \mathcal{B}_f^{\pm}\right),$$

where $\mathcal{B}_f^{\pm} := \{(w, g) \in l_q^2 \oplus l_g^2 \ : \ F^{\pm} g = w\}$. The existence of a causal (respectively, anticausal) left inverse of F^+ (respectively, F^-) implies that Σ_f^+ (respectively, Σ_f^-) is past (respectively, future) externally induced. Note also that, in both cases, the observability map F (given by the left inverse of, respectively, F^+ and F^-, restricted to $\mathcal{B}$) is bounded. Indeed, observe that, whenever Σ and Σ_f are l^2-systems and Σ_f is an externally induced latent variable representation of Σ, the observability map F has closed graph ($G(F) = \mathcal{B}_f$) and therefore is always bounded.

If $\Sigma = (T, W, \mathcal{B})$ is a dynamical system and $\Sigma_f = (T, W, L, \mathcal{B}_f)$ is a latent variables representation of Σ, then Σ_f is said to be a *state space representation* of Σ if the following holds true:

$$[(w_1, l_1), (w_2, l_2) \in \mathcal{B}_f \text{ and } l_1(t) = l_2(t)] \Rightarrow [(w_1, l_1) \wedge_t (w_2, l_2) \in \mathcal{B}_f].$$

For state space representations, we use the notation Σ_S for Σ_f. In [12] and [13], a general theory of state space representations of a dynamical system is developed, and a notion of complexity is introduced, as well as a notion of equivalence. In particular, it is proved that, if Σ is a linear system, then the linear time-invariant state space representations of Σ, of minimal complexity, are all equivalent to each other; moreover, it is shown how to canonically construct a minimal state space representation. However, when we study dynamical systems carrying a topological structure on the behavior (as l^2-systems), then it is of interest to consider topological structures on the state space, also, and, consequently, to have notions of complexity and equivalence where these topological concepts are also considered. One of the main effects of this new setting is the loss of the equivalence of all the state space representations of minimal complexity, even for a linear system. The main result of this section is to show that, for almost controllable l^2-systems, this equivalence is actually preserved! We start with an interesting definition, which induces a topological structure on state space representations.

DEFINITION 4.1. Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$ and let $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S)$ be a time-invariant state space representation of Σ, with X a complex separable Hilbert space. Σ_S is said to be a Hilbertian state space representation of Σ if the following condition holds true: For every A open subset of l_q^2 such that $\mathcal{B} \subset A$, there exists an open neighborhood N of 0 in X such that

$$[(w_1, x_1), (w_2, x_2) \in \mathcal{B}_S \text{ and } x_1(0) - x_2(0) \in N] \Rightarrow [w_1 \wedge_0 w_2 \in A]. \tag{4.1}$$

Condition (4.1) simply says that if two trajectories in $\mathcal{B}$ have states that at $t = 0$ are "very close" to each other, then the concatenation of these two trajectories will also be "very close" to $\mathcal{B}$.

We denote by H_Σ the set of all the Hilbertian state space representations of the l^2-system Σ. If $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S) \in H_\Sigma$, define

$$X^{\text{eff}} := \{\xi \in X \mid \exists (w, x) \in \mathcal{B}_S \text{ such that } x(0) = \xi\}. \tag{4.2}$$

X^{eff} is a subspace of X (not necessarily closed), and it is called the *effective state space* of Σ_S.

DEFINITION 4.2. $\Sigma_S \in H_\Sigma$ is said to be trim if $X^{\text{eff}} = X$; it is said to be almost trim if $\overline{X^{\text{eff}}} = X$.

If $\Sigma_S \in H_\Sigma$ is externally induced, then we can define a linear map

$$g : \mathcal{B} \to X, \tag{4.3}$$

given by $g(w) := x(0)$, where $x \in X^{\mathbf{Z}}$ is such that $(w, x) \in \mathcal{B}_S$. With a slight abuse of notation, we also call g the observability map of Σ_S.

DEFINITION 4.3. $\Sigma_S \in H_\Sigma$ is said to be boundedly externally induced if g is bounded.

We now give some examples of such state space representations.

Example. (1) If $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$, consider that

$$\Sigma_S^{\text{trivial}} := \left(\mathbf{Z}, \mathbf{C}^q, \mathcal{B}, \mathcal{B}_S^{\text{trivial}}\right), \tag{4.4}$$

where

$$\mathcal{B}_S^{\text{trivial}} := \{(w,x) | w \in \mathcal{B} \text{ and } x(t) = \sigma^t w\}.$$

$\Sigma_S^{\text{trivial}}$ is usually called the *trivial state space representation* of Σ, and it is immediate to see that $\Sigma_S^{\text{trivial}} \in H_\Sigma$, is trim, and boundedly externally induced.

(2) A more important state space representation of Σ is the following:

$$\Sigma_S^{\text{c}} := \left(\mathbf{Z}, \mathbf{C}^q, \mathcal{B}/_D, \mathcal{B}_S^{\text{c}}\right), \tag{4.5}$$

where

$$D := \{w \in \mathcal{B} \mid w \wedge_0 0 \in \mathcal{B}\}$$

and

$$\mathcal{B}_S^{\text{c}} := \{(w,x) \mid w \in \mathcal{B} \text{ and } x(t) = \sigma^t w \,(\text{mod } D)\}.$$

It is called the *canonical state space representation* of Σ. It is well known [13] that Σ_S^{c} is a trim, past and future externally induced state space representation of Σ. It is easy to see that Σ_S^{c} is also boundedly externally induced. Moreover, $\Sigma_S^{\text{c}} \in H_\Sigma$ if we consider $\mathcal{B}/_D$ with the natural quotient structure, after noting that D is closed in $\mathcal{B}$. Indeed, fix (w_1, x_1) and (w_2, x_2) in $\mathcal{B}_S^{\text{c}}$, and assume that

$$\|x_1(0) - x_2(0)\| \leq \delta.$$

This means that there exists $v \in D$ such that $\|w_1 - w_2 + v\|_{\mathcal{B}} \leq \delta$, or, also, that

$$\|\,(w_1 + 0 \wedge_0 v) - (w_2 - v \wedge_0 0)\,\|_{\mathcal{B}} \leq \delta.$$

Now $w_1 \wedge_0 w_2 = (w_1 + 0 \wedge_0 v) - (w_2 - v \wedge_0 0)$, and, therefore,

$$\|w_1 \wedge_0 w_2 - w_1 + 0 \wedge_0 v\|_{\mathcal{B}} \leq \delta.$$

This shows that $\Sigma_S \in H_\Sigma$.

DEFINITION 4.4. Let $\Sigma_S^i = \left(\mathbf{Z}, \mathbf{C}^q, X_i, \mathcal{B}_S^i\right)$ be in H_Σ for $i = 1, 2$. Σ_S^1 is said to be more complex than Σ_S^2 ($\Sigma_S^1 \geq \Sigma_S^2$) if there exists a linear bounded surjective map $f : X_1 \to X_2$ such that, for every $(w, x_2) \in \mathcal{B}_S^2$, there exists $x_1 \in X_1^{\mathbf{Z}}$ such that $(w, x_1) \in \mathcal{B}_S^1$ and $f \circ x_1 = x_2$.

DEFINITION 4.5. Let Σ_S^1 and Σ_S^2 as in Definition 4.4. Σ_S^1 is said to be equivalent to Σ_S^2 ($\Sigma_S^1 \simeq \Sigma_S^2$) if there exists a linear bounded bijective map $f : X_1 \to X_2$ such that $(w, x_1) \in \mathcal{B}_S^1$ if and only if $(w, f \circ x_1) \in \mathcal{B}_S^2$.

Note that $\geq$ is a preorder on H_Σ, while $\simeq$ is an equivalence relation.

We indicate with H_Σ^* the set of all the minimal elements of H_Σ with respect to the pre-order $\geq$; namely, $\Sigma_S \in H_\Sigma^*$ if and only if $\Sigma_S \in H_\Sigma$ and $[\Sigma_S \geq \Sigma'_S] \Rightarrow [\Sigma_S \simeq \Sigma'_S]$. We later show that the canonical representation Σ_S^{c} is always minimal and that, if Σ is almost controllable, then any other minimal representation is, in fact, equivalent to Σ_S^{c}.

Let us now investigate in more detail the continuity requirement in Definition 4.1 of a Hilbertian state space representation. Assume that $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S)$ is a linear time-invariant state space representation of $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B})$, and assume that X is a complex separable Hilbert space. Define the map

$$\psi(\Sigma_S) : X^{\text{eff}} \to {}^{l_q^2}/_{\mathcal{B}} \tag{4.6}$$

by

$$\psi(\Sigma_S)(x) := w_1 \wedge_0 w_2 \pmod{\mathcal{B}},$$

where w_1, w_2 is any pair of trajectories in $\mathcal{B}$ such that there exist x_1 and x_2 in $X^{\mathbf{Z}}$ with $(w_i, x_i) \in \mathcal{B}_S$ for $i = 1, 2$ and $x_1(0) - x_2(0) = x$. To better understand how the map $\psi(\Sigma_S)$ really acts on X^{eff}, observe that the codomain of $\psi(\Sigma_S)$ can be canonically identified with ${}^{l_q^{2+}}/_{\mathcal{B}_0^+}$, where

$$\mathcal{B}_0^+ := \left\{ w^+ \in l_q^{2+} \text{ such that } 0 \wedge_0 w^+ \in \mathcal{B} \right\}.$$

Through this identification, $\psi(\Sigma_S)$ acts as follows: Given $x \in X^{\text{eff}}$, $\psi(\Sigma_S)(x)$ is the equivalence class (mod $\mathcal{B}_0^+$) of all the possible futures of the system $\mathcal{B}$ compatible with initial state at time $t = 0$ equal to x. An analogous identification can be made with respect to the past.

LEMMA 4.6. *$\psi(\Sigma_S)$ is a well-defined linear map.*

Proof. Let (w_i', x_i') and (w_i'', x_i'') be in $\mathcal{B}_S$ for $i = 1, 2$ and assume that

$$x_1'(0) - x_2'(0) = x = x_1''(0) - x_2''(0). \tag{4.7}$$

Consider $(w_i' - w_i'', x_i' - x_i'')$ for $i = 1, 2$ and observe that, by (4.7),

$$(x_1'(0) - x_1''(0)) - (x_2'(0) - x_2''(0)) = 0.$$

Therefore $(w_1' - w_1'') \wedge_0 (w_2' - w_2'') \in \mathcal{B}$, or, equivalently,

$$w_1' \wedge_0 w_2' - w_1'' \wedge_0 w_2'' \in \mathcal{B},$$

which shows that $\psi(\Sigma_S)$ is well defined. A straightforward calculation shows that $\psi(\Sigma_S)$ is linear. □

Using this lemma we can obtain the following nice characterization of Hilbertian state space models.

PROPOSITION 4.7. *Σ_S is in H_Σ if and only if $\psi(\Sigma_S)$ is bounded.*

Proof. The proof is an immediate application of the definition of H_Σ. □

If $\Sigma_S \in H_\Sigma$, then $\psi(\Sigma_S)$, being bounded on X^{eff}, can be extended in a unique way to a linear bounded map acting on $\overline{X^{\text{eff}}}$; for simplicity of notation, we denote this extension also by the symbol $\psi(\Sigma_S)$.

If $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S) \in H_\Sigma$, denote $\psi := \psi(\Sigma_S)$ and consider that

$$\Sigma_S' := \left(\mathbf{Z}, \mathbf{C}^q, {}^{X}/_{\ker \psi}, \mathcal{B}_S' \right), \tag{4.8}$$

where

$$\mathcal{B}_S' := \left\{ (w, \overline{x}) \in \left(\mathbf{C}^q \oplus {}^{X}/_{\ker \psi} \right)^{\mathbf{Z}} \mid (w, x) \in \mathcal{B}_S \right\},$$

where $\overline{x}$ denotes the equivalence class of x (mod $\ker\psi$). We have the following result.

PROPOSITION 4.8. *Σ'_S is a Hilbertian externally induced state space representation. Moreover, $\Sigma_S \geq \Sigma'_S$.*

Proof. Let us prove that Σ'_S is a state space representation of Σ. Fix (w_1, x_1) and (w_2, x_2) in $\mathcal{B}_S$, and assume that

$$x_0 := x_1(0) - x_2(0) \in \ker\psi.$$

We must only prove that

$$(w_1, \overline{x}_1) \wedge_0 (w_2, \overline{x}_2) \in \mathcal{B}'_S. \tag{4.9}$$

Since $x_0 \in \ker\psi \cap X^{\text{eff}}$, there exists $(w, x) \in \mathcal{B}_S$ such that $x(0) = x_0$ and $w \wedge_0 0 \in \mathcal{B}$. Let $x' \in X^{\mathbf{Z}}$ be such that

$$(w \wedge_0 0, x') \in \mathcal{B}_S \tag{4.10}$$

and let $x'' = x - x'$; then

$$(0 \wedge_0 w, x'') \in \mathcal{B}_S. \tag{4.11}$$

Consider now

$$(w_1 + (0 \wedge_0 w), x_1 + x'') \quad \text{and} \quad (w_2 - (w \wedge_0 0), x_2 - x').$$

These are elements of $\mathcal{B}_S$ and $(x_1 + x'')(0) - (x_2 - x')(0) = 0$. Therefore

$$\begin{aligned} &(w_1, x_1 + x'') \bigwedge_0 (w_2, x_2 - x') \\ &= (w_1 + (0 \wedge_0 w), x_1 + x'') \bigwedge_0 (w_2 - (w \wedge_0 0), x_2 - x') \in \mathcal{B}_S. \end{aligned} \tag{4.12}$$

By (4.10) and (4.11), it is evident that $\psi(x'(t)) = 0$ for all $t \geq 0$ and $\psi(x''(t)) = 0$ for all $t \leq 0$; this, together with (4.12), yields (4.9).

The fact that $\Sigma'_S \in H_\Sigma$ follows from the commutativity of the following diagram:

$$\begin{array}{ccc} X^{\text{eff}} & \xrightarrow{\psi} & l^2_q/\mathcal{B} \\ \downarrow \pi & \nearrow \psi' & \\ X^{\text{eff}}/\ker\psi \cap X^{\text{eff}} & & \end{array}$$

where $\psi' := \psi(\Sigma'_S)$.

To prove that Σ'_S is externally induced, assume that $(0, \overline{x}) \in \mathcal{B}'_S$; then $x(t) \in \ker\psi$ for all $t \in \mathbf{Z}$, which implies that $\overline{x} = 0$.

Finally, the projection

$$\pi : X \to X/\ker\psi$$

yields $\Sigma_S \geq \Sigma'_S$ in the sense of Definition 4.4. □

PROPOSITION 4.9. *Let $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S) \in H_\Sigma$ The following conditions are then equivalent:*

(1) $\Sigma_S \in H^*_\Sigma$,

(2) *Σ_S is almost trim and $\psi(\Sigma_S)$ is injective on X.*

Proof. (1)⇒(2). Consider that

$$\Sigma_S^{\text{eff}} := \left(\mathbf{Z}, \mathbf{C}^q, \overline{X^{\text{eff}}}, \mathcal{B}_S\right).$$

It is evident that $\Sigma_S \geq \Sigma_S^{\text{eff}}$, which yields $\Sigma_S \simeq \Sigma_S^{\text{eff}}$. This shows that Σ_S is almost trim, since Σ_S^{eff} is. Analogously, by Proposition 4.8, $\Sigma_S \simeq \Sigma'_S$, where Σ'_S has been defined in (4.8). Therefore there exists an isomorphism

$$f : X \to {}^{X}\!/_{\ker \psi(\Sigma_S)}$$

such that

$$\psi(\Sigma'_S) \circ f = \psi(\Sigma_S),$$

which proves that $\psi(\Sigma_S)$ is injective.

(2)⇒(1). Assume that there exists $\tilde{\Sigma}_S = \left(\mathbf{Z}, \mathbf{C}^q, \tilde{X}, \tilde{\mathcal{B}}_S\right) \in H_\Sigma$ such that $\Sigma_S \geq \tilde{\Sigma}_S$. From Proposition 4.8, it follows that Σ_S and $\tilde{\Sigma}_S$ are externally induced; moreover, we have that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{g} & X \\ \downarrow \tilde{g} & \swarrow f & \\ \tilde{X} & & \end{array} \tag{4.13}$$

where g (respectively, $\tilde{g}$) are the observability maps of Σ_S (respectively, $\tilde{\Sigma}_S$) as defined in (4.3), and f is the linear bounded surjective map yielding the preorder $\geq$ between Σ_S and $\tilde{\Sigma}_S$. Fix now $(w, x) \in \mathcal{B}_S$; by (4.13) it follows that $(w, f \circ x) \in \tilde{\mathcal{B}}_S$. It is then clear, by Definition 4.5, that to prove that $\Sigma_S \simeq \tilde{\Sigma}_S$, it suffices to prove that the map f is injective. To prove this, consider the following diagram:

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{g} & X & \xrightarrow{\psi} & {}^{l_q^2}\!/_{\mathcal{B}} \\ & \searrow \tilde{g} & \downarrow f & \nearrow \tilde{\psi} & \\ & & \tilde{X} & & \end{array}$$

where $\psi := \psi(\Sigma_S)$ and $\tilde{\psi} := \psi\left(\tilde{\Sigma}_S\right)$. It is evident that $\psi \circ g = \tilde{\psi} \circ \tilde{g}$. Using the commutativity of (4.13), we obtain that $\psi \circ g = \tilde{\psi} \circ f \circ g$, which implies that

$$\psi(x) = \left(\tilde{\psi} \circ f\right)(x) \quad \forall x \in X^{\text{eff}}.$$

Since $\overline{X^{\text{eff}}} = X$ and since all the maps involved are bounded, it follows that $\psi = \tilde{\psi} \circ f$. Since ψ is injective, this shows the injectivity of f, as desired. □

COROLLARY 4.10. *It holds that* $\Sigma_S^c \in H_\Sigma^*$.

Proof. Σ_S^c is trim; therefore, by Proposition 4.9, we must only prove that $\psi^c := \psi(\Sigma_S^c)$ is injective on X. Observe that

$$\psi^c : {}^{\mathcal{B}}\!/_{D} \to {}^{l_q^2}\!/_{\mathcal{B}}$$

is given by $\psi^c(w \pmod{D}) = (w \wedge_0 0) \pmod{\mathcal{B}}$. Therefore

$$\psi^c(w \pmod{D}) = 0 \Longleftrightarrow w \wedge_0 0 \in \mathcal{B} \Longleftrightarrow$$

$$\iff w \in D \iff w \ (\mathrm{mod}\ D) = 0. \qquad \square$$

PROPOSITION 4.11. *Let Σ_S be in H_Σ. The following conditions are then equivalent:*

(1) *Σ is minimal, trim, and boundedly externally induced,*

(2) *$\Sigma_S \simeq \Sigma_S^c$.*

Proof. (1)⇒(2). Let us denote by X the state space of Σ_S, and by A that of Σ_S^c. For $x \in X$, consider that

$$\mathcal{B}(x) := \{w \in \mathcal{B} \mid \exists z \in X^{\mathbf{Z}} : (w,z) \in \mathcal{B}_S \text{ and } z(0) = x\}.$$

Similarly, define $\mathcal{B}(a)$ for $a \in A$. It is easy to see, since $\mathcal{B}_S^c$ is past and future externally induced, that, for every $x \in X$, there exists one and only one $a \in A$ such that $\mathcal{B}(x) \subset \mathcal{B}(a)$. This yields the existence of a linear surjective map $f : X \to A$ such that $(w,z) \in \mathcal{B}_S$ if and only if $(w, f \circ z) \in \mathcal{B}_S^c$. We now prove that f is bounded. Consider the following commutative diagram:

$$(4.14) \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{g} & X \\ \downarrow g_c & \swarrow f & \\ A & & \end{array},$$

where g (respectively, g_c) are the observability maps of Σ_S (respectively, Σ_S^c). Let $C \subset A$ be an open set. We have that

$$f^{-1}(C) = g\left(g_c^{-1}(C)\right).$$

Since g_c is bounded and g is open (it is surjective and bounded by (1)), it follows that $f^{-1}(C)$ is open in X. Therefore f is bounded and $\Sigma_S \geq \Sigma_S^c$. Since Σ_S is minimal, it follows that $\Sigma_S \simeq \Sigma_S^c$.

(2)⇒(1) is contained in Corollary 4.10. $\square$

We now focus on almost controllable systems.

PROPOSITION 4.12. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$ be almost controllable. Then any minimal Hilbertian state space representation Σ_S of Σ is trim and boundedly externally induced.*

Proof. Consider that $\psi := \psi(\Sigma_S)$, as defined before. It is evident that

$$\mathcal{R}\left(\psi|_{X^{\mathrm{eff}}}\right) = \mathcal{B}^- \wedge_0 \mathcal{B}^+ / \mathcal{B},$$

which is, by the assumption of almost controllability, closed in $l_q^2/\mathcal{B}$. Since $\overline{X^{\mathrm{eff}}} = X$, it follows that

$$\mathcal{R}(\psi) = \mathcal{B}^- \wedge_0 \mathcal{B}^+ / \mathcal{B}.$$

Since ψ is injective, this implies that $X^{\mathrm{eff}} = X$.

By Proposition 4.8, Σ_S is externally induced. We then have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{g} & X \\ \downarrow \phi & \swarrow \psi & \\ \mathcal{B}^- \wedge_0 \mathcal{B}^+/\mathcal{B} & & \end{array},$$

where $\phi(w) := w \wedge_0 0 \ (\mathrm{mod}\ \mathcal{B})$. Since ϕ is bounded and ψ is an isomorphism, it follows that g is bounded. This completes the proof. $\square$

We now state the main result of this section. It consists of a state space isomorphism theorem for Hilbert space systems, with the state space isomorphism induced by bounded linear maps. Note that almost controllability plays an essential role in this result!

THEOREM 4.13. *Let $\Sigma \in \mathcal{L}_q^2$ be almost controllable and let $\Sigma_S \in H_\Sigma$. Then the following conditions are equivalent:*

(1) $\Sigma_S \in H_\Sigma^*$,

(2) $\Sigma_S \simeq \Sigma_S^c$,

(3) Σ_S *is trim and past and future externally induced.*

Proof. (1)⇒(2) follows from Propositions 4.11 and 4.12. (2)⇒(3) follows from Proposition 4.11 and the definition of Σ_S^c. Finally, (3)⇒(1) follows from Proposition 4.9 and the evident fact that, if Σ_S satisfies (3), then $\psi(\Sigma_S)$ is injective. □

As already mentioned, almost controllability is essential to have the isomorphism result expressed in Theorem 4.9. In fact, we have the following proposition.

PROPOSITION 4.14. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$ be not almost controllable. Then there exists $\Sigma_S \in H_\Sigma^*$, which is not trim.*

Proof. Consider the following state space representation of Σ:

$$\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S),$$

where

$$X := \overline{\mathcal{B}^- \wedge_0 \mathcal{B}^+}/\mathcal{B}$$

and

$$\mathcal{B}_S := \left\{(w,x) \in (\mathbf{C}^q \oplus X)^{\mathbf{Z}} \mid w \in \mathcal{B} \text{ and } x(t) = (\sigma^t w) \wedge_0 0 \ (\text{mod } \mathcal{B})\right\}.$$

It is easy to check that this is indeed a state space representation of Σ and $\psi(\Sigma_S)$ is simply the inclusion map on $l_q^2/\mathcal{B}$. It then follows that $\Sigma_S \in H_\Sigma^*$; on the other hand, Σ_S is not trim, since either $\mathcal{B}^-$ or $\mathcal{B}^+$ is not closed. □

Remark. If $\Sigma \in \mathcal{L}_q^2$ is almost controllable, then the minimal state space representation Σ_S^c can be represented in the following familiar way. There exist

$$A : X \to X, \ B : \mathbf{C}^g \to X, \qquad C : X \to \mathbf{C}^q, \ D : \mathbf{C}^g \to \mathbf{C}^q$$

linear bounded maps yielding the following representation: $(w,x) \in \mathcal{B}_S^c$ if and only if there exists $v \in l_g^2$ such that

$$\sigma x = Ax + Bv, \qquad w = Cx + Dv.$$

Such a representation is called a *driving variable representation.* The details of the construction of such a representation are not presented here, since it is completely analogous to the so-called shift realization that has been investigated for input/output systems in [5].

We close our study of state space representations by a discussion of the relation between our concepts of controllability and the classical concept of state controllability. In [12] we have defined state point controllability as the possibility of transferring the system between any two states in finite time. The appropriate version of almost state point controllability proves to be the following.

DEFINITION 4.15. Let $\Sigma \in \mathcal{L}_q^2$ and $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S) \in H_\Sigma$. Σ_S is said to be almost state point controllable if there exists $K > 0$ such that, for every pair of elements x_1 and x_2 in X, there exists a sequence $(v_n, y_n) \in \mathcal{B}_S$ yielding the following:

$$y_n(-n) \to x_1, \quad y_n(n) \to x_2, \quad \|v_n\|_2 \leq K\left(\|x_1\|^- + \|x_2\|^+\right),$$

where the convergence is in the Hilbertian topology of the space X, and where $\|\cdot\|^-$ and $\|\cdot\|^+$ are defined as follows:

$$\begin{aligned} \|x\|^- &:= \inf\{\|v^-\|_2 \mid \exists y \in X^{\mathbf{Z}} \text{ with } (v,y) \in \mathcal{B}_S \text{ and } y(0) = x\}, \\ \|x\|^+ &:= \inf\{\|v^+\|_2 \mid \exists y \in X^{\mathbf{Z}} \text{ with } (v,y) \in \mathcal{B}_S \text{ and } y(0) = x\}. \end{aligned} \tag{4.15}$$

It is possible to prove that, for minimal state space representations, almost controllability and almost state point controllability are indeed equivalent.

PROPOSITION 4.16. *Let $\Sigma = (\mathbf{Z}, \mathbf{C}^q, \mathcal{B}) \in \mathcal{L}_q^2$ and let $\Sigma_S = (\mathbf{Z}, \mathbf{C}^q, X, \mathcal{B}_S) \in H_\Sigma^*$. Then the following conditions are equivalent:*

(1) *Σ is almost controllable,*

(2) *Σ_S is almost state point controllable.*

Proof. (1)⇒(2). By Theorem 4.13 we can assume, without loss of generality, that $\Sigma_S = \Sigma_S^c$. Let x_1 and x_2 be in X. Then there exist w_1 and w_2 in $\mathcal{B}$ such that

$$x_i = w_i \ (\text{mod } D) \quad \text{for } i = 1, 2$$

and

$$\|w_1^-\| \leq 2\|x_1\|^-, \qquad \|w_2^+\| \leq 2\|x_2\|^+. \tag{4.16}$$

By (1), there exists a sequence $v_n \in \mathcal{B}$ such that

$$(\sigma^{-n} v_n)^- \to w_1^-, \qquad (\sigma^n v_n)^+ \to w_2^+ \tag{4.17}$$

and

$$\|v_n\|_2 \leq K\left(\|w_1^-\| + \|w_2^+\|\right), \tag{4.18}$$

where K is a positive constant depending only on Σ. Now, consider $y_n \in X^{\mathbf{Z}}$, given by

$$y_n(t) = \sigma^t v_n \ (\text{mod } D).$$

By (4.17) and by the fact that $\mathcal{B}^-$ is closed (see Proposition 3.1), it follows that $y_n(-n) \to x_1$. Analogously, $y_n(n) \to x_2$. By (4.16) and (4.18),

$$\|v_n\|_2 \leq 2K\left(\|x_1\|^- + \|x_2\|^+\right).$$

This yields (2).

(2)⇒(1). Let w_1 and w_2 be in $\mathcal{B}$ and let x_1 and $x_2 \in X^{\mathbf{Z}}$ be such that $(w_i, x_i) \in \mathcal{B}_S$ for $i = 1, 2$. By (2) there exists a sequence $(y_n, v_n) \in \mathcal{B}_S$ such that

$$y_n(n) \to x_2(0), \qquad y_n(-n) \to x_1(0) \tag{4.19}$$

and

$$\|v_n\|_2 \leq K\left(\|x_1(0)\|^- + \|x_2(0)\|^+\right). \tag{4.20}$$

Consider that

$$z_n = \sigma^n w_1 \wedge_{-n} v_n \wedge_n \sigma^{-n} w_2 \in l_q^2. \tag{4.21}$$

By (4.19), and by the fact that $\Sigma_S \in H_\Sigma$, there exists $\tilde{z}_n \in \mathcal{B}$ such that

$$\|\tilde{z}_n - z_n\| \to 0 \quad \text{for } n \to +\infty.$$

In particular, $(\sigma^{-n}\tilde{z}_n)^- - (\sigma^{-n}z_n)^- \to 0$, which implies, by (4.20) and (4.21), that

$$(\sigma^{-n}\tilde{z}_n)^- \to w_1^-. \tag{4.22}$$

In a similar way, we can prove that

$$(\sigma^{n}\tilde{z}_n)^+ \to w_2^+. \tag{4.23}$$

By (4.20) and (4.21), we also have that

$$\|z_n\| \le (1+K)\left(\|w_1^-\| + \|w_2^+\|\right),$$

and, on the other hand, it is not restrictive to assume that $\|\tilde{z}_n\| \le 2\|z_n\|$. This, together with (4.22) and (4.23), yields (1). □

Classically, of course, controllability is always studied for systems with inputs. We now briefly analyze the concept of almost state point controllability for state space representations of causal input/output l^2-systems, and we establish a relation with the classical notion of exact controllability as considered, for example, in [4] and [5].

Let $T : l_m^2 \to l_p^2$ be a linear bounded causal map commuting with the shift and let

$$\Sigma_T = (\mathbf{Z}, \mathbf{C}^{m+p}, G(T))$$

be the induced l^2-system as defined in part (1) of the example in the Introduction. Let Σ_S be in H_{Σ_T} and assume that it is past externally induced. It is then possible to consider the following linear map (the *reachability map* of Σ_S):

$$R : (l_m^2)^- \to X, \tag{4.24}$$

given by

$$Rv^- := g\begin{pmatrix} v^- \\ Tv^- \end{pmatrix},$$

where g is the observability map defined in (4.3). As in [5], we call a state space representation *exactly state point controllable* if R is bounded and surjective. If Σ_S is trim and almost state point controllable, then Σ_S is exactly state point controllable. In fact, in this case, Σ_T is almost controllable by Proposition 4.16, and an easy argument using the commutative diagram (4.14) shows that Σ_S is boundedly externally induced. This yields that Σ_S is exactly state point controllable. In particular, by Theorem 4.13, it follows that, if Σ_S is minimal and almost state point controllable, then Σ_S is exactly state point controllable.

On the other hand, exact state point controllability does not, in general, imply almost state point controllability; it is easy, in fact, to see that the canonical representation Σ_S^c is always exactly state point controllable, but, by Proposition 4.16, it is almost state point controllable if and only if Σ_T is almost controllable. Nevertheless, with the additional assumption that the norm $\|\cdot\|^+$ (see (4.15)) is equivalent to the original norm $\|\cdot\|_X$ of X as a Hilbert space, then exact state point controllability implies almost state point controllability. In fact, using the facts that R is an open map and that the two norms are equivalent, it is easy to prove that, for every $x \in X$, there exists $(v, y) \in \mathcal{B}_S$ such that

$$y(0) = x, \quad y(-n) \to 0, \quad \|v\|_2 \leq K\|y\|^+ \tag{4.25}$$

for a suitable constant $K > 0$. On the other hand $\mathcal{B}_-$ is closed by Proposition 3.1, and this implies that, for every $x \in X$, there exists a sequence $(v_n, y_n) \in \mathcal{B}_S$ such that

$$y_n(0) \to x, \quad y_n(n) \to 0, \quad \|v_n\|_2 \leq K'\|x\|_-, \tag{4.26}$$

where K' is a suitable positive constant. It is evident that (4.25) and (4.26) yield almost state point controllability. Let us conclude by noting that the two norms $\|\cdot\|^+$ and $\|\cdot\|_X$ are indeed equivalent for the so-called "restricted shift" state space representations, which have been investigated in [5].

5. Conclusions and extensions. In this paper we have investigated the notion of controllability as the possibility of concatenation of arbitrary trajectories. For discrete-time systems, we have seen that the possibility of concatenation of trajectories in finite time requires the system to have finite memory, which is equivalent to it having a finite-dimensional state space representation. For infinite-dimensional systems, therefore, we introduced the notion of almost controllability. Our main result is Theorem 3.2, where it is shown that almost controllability is equivalent to the existence of a scattering representation.

As a first application of almost controllability, we obtained in Theorem 4.13 a state space isomorphism result for almost controllable systems. Also, we related our notion of controllability to the classical notion of state point controllability. Under suitable conditions, these notions indeed prove to be equivalent.

Many of the results presented here for the discrete-time case are actually extendable to the continuous time case: in particular, §3 on the representation of almost controllable systems, and §4 on state models. On the other hand, the characterization of the controllable systems in the continuous-time case is more involved and still incomplete. In particular, it is reasonable to conjecture that finite memory and controllability will also be equivalent here. However, in this case, finite memory is not equivalent to the existence of a finite-dimensional state representation.

We believe there are two extensions worth investigating: (i) constructing a representation theory in the fashion of §3 and §4 for l^2-systems where autonomous phenomena are present, and investigating systems embedded in other memoryless Banach structures: of particular interest it would be to work with behaviors $\mathcal{B}$ in l_q^∞.

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