Controllability of 2-D Systems

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Abstract—**The concept of controllability of 2-D systems is introduced in a behavioral framework. This property is defined as a systems concept and is characterized in terms of system representations. Further, state-space realizations for controllable 2-D systems are presented. These realizations do not depend on any assumptions concerning causality.**

I. Introduction

In this paper, we introduce a controllability property for 2-D systems and give a characterization of this property in terms of system representations. Moreover, we consider the question of state-space realization for the class of autoregressive (AR) 2-D systems. We introduce a state-space model which can be used to describe controllable AR systems.

Our approach to 2-D systems is inspired by the behavioral approach to (1-D) dynamical systems developed in [11], [12] and it consists in viewing a 2-D system simply as a family of trajectories defined over a 2-D index set. From this point of view, it is natural to define properties of systems in a set theoretic sense, i.e., as properties of the system signals instead of as properties of its representations.

Intuitively, a system will be called controllable if it has a limited memory range, i.e., if the system is defined in $T \subseteq \mathbb{R}^2$, the values of its signals on two arbitrary subsets T_1 and T_2 of T will be independent provided that T_1 and T_2 are at sufficiently large distance.

This definition illustrates some essential points of our approach. First, no reference is made to "inputs" and "outputs" but only to "signals." Indeed, in this paper, we will not place ourselves in the classical input-output framework. Second, controllability is introduced as an adirectional property, since no restrictions are imposed on the sets T_1 and T_2. This reflects our wish to avoid the choice of a preferred direction in the plane, and to make no assumptions on the existence of causal relationships between the system variables.

Characterizing system properties in terms of the parameters of system representations is an important issue. Here we will show that, for AR 2-D systems, our concept of system controllability is equivalent to a 2-D primeness condition or, alternatively, to the existence of a moving-average (MA) representation. Moreover, it turns out that controllable AR systems correspond exactly to those systems which can be represented by transfer functions in an input-output framework. Thus, when considering controllable AR 2-D systems we will be dealing essentially with the same class of systems as considered in the classical 2-D approach [2], [3], [7], with the difference that we will not be assuming an input-output structure. This fact becomes particularly interesting when we analyze the question of state-space realizations for controllable AR systems.

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The main purpose of constructing state-space models is to provide first-order descriptions of systems, which moreover correspond to a convenient first-order updating scheme. This turns out to be a difficult problem within the input-output framework. The models introduced by Roesser [9] and Fornasini and Marchesini [3] provide a first-order recursive representation, but only for the class of 2-D quarter-plane causal transfer functions. Noncausal transfer functions are represented in [8] and [5] by first-order relations but not in recursive form. This indicates that within an input-output framework the assumption of causality is crucial for obtaining first-order recursions. As we will show, once the input-output structure is no longer imposed, this causality assumption need not be made. The state property that we introduce here for 2-D systems leads to a first-order recursive state/driving-variable model which can be used to represent any controllable AR system.

The organization of this paper is as follows. Section II deals with controllable systems, and Section III with state-space systems. In Section IV, we analyze the structure of state-space systems and introduce and characterize the notions of trimness, reachability, and observability. Section V contains concluding remarks. Proofs are collected in the Appendix.

II. Controllable 2-D Systems

Controllability is a notion of central importance in systems theory. Usually, this notion is defined as a property of the state-space realizations of a system. In this section, we give an alternative definition of controllability of a 2-D system as an intrinsic property of the system external behavior and not of its representations. Intuitively, we will say that a system defined over $T \subseteq \mathbb{R}^2$ is controllable if its memory has a limited range—thus, no matter what system signal is given on $T' \subset T$, at sufficiently large distance from T', every other system signal can occur. This generalizes the definition of controllability given in [12] for (1-D) dynamical systems. We will consider in particular the class of autoregressive 2-D systems and develop tests to check controllability for systems in this class. It turns out, that our notion of controllability is closely related to the notion of 2-D modal controllability introduced in [7] and [8].

A. The Notion of Controllability

In order to define controllable systems, we start by stating our definition of a 2-D system.

Definition 1: A *2-D system* is characterized by an index set $T \subseteq \mathbb{R}^2$, a signal space W, and a subset $\mathfrak{B}$ of W^T (the set of all functions $T \to W$), called the *behavior* of the system. The system Σ defined by T, W, and $\mathfrak{B}$ will be denoted by $\Sigma := (T, W, \mathfrak{B})$.

This definition can be interpreted as follows. A 2-D system describes a phenomenon defined on T which is characterized by attributes taking values in the set W. Thus, every manifestation of the phenomenon gives rise to a signal $w: T \to W$. The behavior $\mathfrak{B}$ species the laws which govern the phenomenon by indicating which signals are compatible with these laws.

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In this paper, we will always consider discrete 2-D systems with $T = \mathbb{Z}^2$.

Definition 2: A 2-D system $\Sigma = (\mathbb{Z}^2, W, \mathfrak{B})$ is said to be *controllable* if the following condition holds. There exists a positive real number ρ such that $\{w_1, w_2 \in \mathfrak{B};\ I_1, I_2 \subseteq \mathbb{Z}^2;\ d(I_1, I_2) \geq \rho\} \Rightarrow \{w_1|_{I_1} \wedge w_2|_{I_2} \in \mathfrak{B} \mid I_1 \cup I_2\}$. Here $d(I_1, I_2)$ stands for the Euclidean distance between I_1 and I_2; $w_1|_{I_1} \wedge w_2|_{I_2}$ denotes the signal $w \in W^{I_1 \cup I_2}$ such that $w|_{I_1} = w_1|_{I_1}$ and $w|_{I_2} = w_2|_{I_2}$, and is called the *concatenation* of w_1 with w_2 relative to (I_1, I_2).

Two aspects of this definition should be stressed. First, note that the system signals w are not split into inputs and outputs. Second, as no restrictions are imposed on the sets I_1 and I_2, controllability is not related to the choice of any preferred direction $\mathbb{Z}^2$. Roughly speaking, we may say that I_1 is surrounded by a band of width ρ beyond which all the information about the phenomenon $w_1|_{I_1}$ occurring in I_1 is lost. In this sense, ρ measures the memory range of the system.

B. Autoregressive 2-D Systems

In the sequel, we will be concerned with *autoregressive* 2-D systems. These systems have as parameter set $T = \mathbb{Z}^2$, as signal space $W = \mathbb{R}^q$, for some positive integer q, and their behavior is given as the kernel of a polynomial operator in the shifts. That is, it can be described by behavioral equations of the form $R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = o$, with $R(s_1, s_1^{-1}, s_2, s_2^{-1})$ a $g \times q$ polynomial matrix (for some positive integer g), and σ_1 and σ_2, respectively, the left- and downshift. These are defined as follows: σ_1 associates with a function $w: \mathbb{Z}^2 \rightarrow \mathbb{R}^q$ a function $\sigma_1 w: \mathbb{Z}^2 \rightarrow \mathbb{R}^q$ such that $\sigma_1 w(t_1, t_2) := w(t_1 + 1, t_2)$ for all $(t_1, t_2) \in \mathbb{Z}^2$; σ_2 acts in a similar way, with $\sigma_2 w(t_1, t_2) := w(t_1, t_2 + 1)$. Thus, an autoregressive (AR) 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is characterized by a linear operator $R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$: $(\mathbb{R}^q)^{\mathbb{Z}^2} \rightarrow (\mathbb{R}^g)^{\mathbb{Z}^2}$ such that $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$.

Clearly, the behavior $\mathfrak{B}$ of such a system is a linear *shift-invariant* subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$, i.e., $\sigma_i \mathfrak{B} = \mathfrak{B}$, $i = 1, 2$. Moreover, $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid w|_I \in \mathfrak{B}|_I$, for all finite subsets $I \subset \mathbb{Z}^2\}$. This property means that, in order to check whether or not a signal is compatible with the laws of the system, it is sufficient to check if it satisfies these laws on finite subsets of $\mathbb{Z}^2$. This is known as *completeness*. It was shown in [10] that the converse of the aforementioned also holds true, i.e., every linear, shift-invariant, and complete system is an autoregressive one.

Note that, due to shift invariance, $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ can always be represented as $\mathfrak{B} = \ker R'(\sigma_1, \sigma_2)$ for some polynomial matrix $R'(s_1, s_2)$; just take $R'(s_1, s_2) = R(s_1, s_1^{-1}, s_2, s_2^{-1}) s_1^{l_1} s_2^{l_2}$, where l_1 and l_2 are, respectively, the exponents of the highest power of s_1^{-1} and s_2^{-1} appearing in R. For simplicity, and without loss of generality, from now on we will consider only this type of AR representation.

An important distinction between 2-D and 1-D AR systems is the following. While 1-D AR systems can always be described by means of 1-D polynomial matrices of full-row rank [11], this does not hold true for the 2-D case. In particular, we may need to take $g > q$. A simple example of this situation is, for instance, a single-variable system $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$ which only allows for constant signals: the minimal number of AR equations necessary to describe $\mathfrak{B}$ is two, namely $\mathfrak{B} = \ker \begin{bmatrix} \sigma_2 - 1 \\ \sigma_1 - 1 \end{bmatrix}$, and thus $g = 2 > 1 = q$. However, we will see that controllable AR systems can always be described by a full-row rank polynomial matrix.

AR representations provide a system description in terms of relationships involving the system signals alone. There are however situations where it is convenient to introduce auxiliary variables in order to simplify the description and analysis of a system.

A *2-D system with auxiliary variables* is defined as $\Sigma^a := (T, W, A, \mathfrak{B}^a)$ with the index set $T \subseteq \mathbb{R}^2$, the external signal space W, the auxiliary variable space A, and the *internal behavior* of the system $\mathfrak{B}^a \subseteq W^T \times A^T$. The *external behavior* of Σ^a is given by $\Pi_w \mathfrak{B}^a := \{w \in W^T \mid \exists a \in A^T \text{ s.t. } (w, a) \in \mathfrak{B}^a\}$. $\Sigma^a = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^l, \mathfrak{B}^a)$ is said to be an *ARMA* (autoregressive-moving average) *system* if it can be described by behavioral equations of the form $R(\sigma_1, \sigma_2)w = M(\sigma_1, \sigma_2)a$ for some polynomial matrices $R(s_1, s_2)$ and $M(s_1, s_2)$; the left-hand side of the foregoing equations is called the autoregressive part, while the right-hand side is called the moving-average part. In particular, if $R(s_1, s_2) = I$, we will say that Σ^a is an *MA system*.

Clearly, every AR system can be viewed as an ARMA system of special type where the MA part is absent, i.e., with $M(s_1, s_2) = 0$. On the other hand, the external behavior of an ARMA system can be described in AR form. This is stated in the following proposition.

Proposition 1: Let $\Sigma^a = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^l, \mathfrak{B}^a)$ be an ARMA system, and define $\mathfrak{B} := \Pi_w \mathfrak{B}^a$. Then $\Sigma := (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is an AR system.

Proof: Appendix.

Thus AR and ARMA representations constitute alternative descriptions for the same class of systems, namely for linear, shift-invariant, and complete ones. MA systems form a strict subclass of the foregoing. In the sequel, we will show that a linear shift-invariant complete system is MA if and only if it is controllable.

C. Controllable AR 2-D Systems

In order to characterize controllable AR systems, we next introduce some preliminary definitions. Let $\mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ denote the ring of polynomials in the indeterminates $s_1, s_1^{-1}, s_2, s_2^{-1}$ with real coefficients, and $\mathbb{R}^{k_1 \times k_2}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ the set of all $k_1 \times k_2$ matrices with entries in $\mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$—note that if $k_1 = k_2$, this set is a ring. A $k \times k$ polynomial matrix $U(s_1, s_1^{-1}, s_2, s_2^{-1})$ is said to be *unimodular* if it is invertible within the ring $\mathbb{R}^{k \times k}[s_1, s_1^{-1}, s_2, s_2^{-1}]$. Given a polynomial matrix $R(s_1, s_1^{-1}, s_2, s_2^{-1})$, we will call $D(s_1, s_1^{-1}, s_2, s_2^{-1})$ a *left divisor* of R if D is square and there exists a polynomial matrix $\overline{R}(s_1, s_1^{-1}, s_2, s_2^{-1})$ such that $R = D\overline{R}$. If all the left divisors of R are also left divisors of D, we will call D a *maximal left divisor* of R. A $g \times q$ polynomial matrix $R(s_1, s_1^{-1}, s_2, s_2^{-1})$ of full-row rank is said to be *left prime* if all its maximal left divisors are unimodular. Similar definitions hold for right divisor and right primeness. In this paper, when using the aforementioned notions for polynomial matrices $R(s_1, s_2)$ in the indeterminates s_1, s_2, we will be viewing these matrices as elements of $\mathbb{R}^{k_1 \times k_2}[s_1, s_1^{-1}, s_2, s_2^{-1}]$, for some (k_1, k_2).

A subset I of $\mathbb{Z}^2$ of the form $I = \{(k, l) \in \mathbb{Z}^2 \mid k_1 < k < k_2, l_1 < l < l_2\}$ for some $-\infty \leq k_1 \leq k_2 \leq +\infty, -\infty \leq l_1 \leq l_2 \leq +\infty$, will be called an *interval* of $\mathbb{Z}^2$ and denoted by $I = (k_1, k_2) \times (l_1, l_2)$. If k_1, k_2, l_1, l_2 are finite, I will be called a *finite interval*. The subspace of compact support signals in $\mathfrak{B}$ will be denoted by $\mathfrak{B}^{compact} := \{w \in \mathfrak{B} \mid \exists I$, finite interval of $\mathbb{Z}^2$, s.t. $w|_{\mathbb{Z}^2 \setminus I} = o\}$. For $\mathfrak{L} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$, $\mathrm{cl}(\mathfrak{L})$ indicates the closure of $\mathfrak{L}$ in the topology of pointwise convergence.

We are now able to state our result on the characterization of controllable AR systems.

Theorem 1: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ be an autoregressive 2-D system. Then the following statements are equivalent.

1) Σ is controllable.

2) $\mathfrak{B} = \mathrm{cl}(\mathfrak{B}^{\text{compact}})$.

3) There exist a positive integer $g < q$ and a full-row rank $g \times q$ polynomial matrix $R(s_1, s_2)$ such that R is left prime and $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$.

4) There exist a positive integer l and a $q \times l$ polynomial matrix $M(s_1, s_2)$ such that $\mathfrak{B} = \mathrm{im}\, M(\sigma_1, \sigma_2) := \{w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid \exists a \in (\mathbb{R}^l)^{\mathbb{Z}^2} \text{ s.t. } w = M(\sigma_1, \sigma_2)a\}$, i.e., Σ corresponds to an MA system.

Proof: Appendix.

Example 1: Consider the AR 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^3, \mathfrak{B})$ with $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ and $R(s_1, s_2) := \mathrm{col}([s_2 + 1 \mid s_1 + 1 \mid 0], [s_1 \mid s_2 \mid 1])$. It is not difficult to check that $R(s_1, s_2)$ is left prime. Indeed, suppose that $D(s_1, s_2)$ is a left divisor of $R(s_1, s_2)$. Then $\det D(s_1, s_2)$ must divide all the 2×2 minors of $R(s_1, s_2)$. These minors are: $m_1(s_1, s_2) = (s_2 - s_1)(s_1 + s_2 + 1)$, $m_2(s_1, s_2) = s_1 + 1$ and $m_3(s_1, s_2) = s_2 + 1$. This implies that $\det D(s_1, s_2) = s_1^{k_1} s_2^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}$, meaning that D is unimodular, and so R is left prime.

Let now $M(s_1, s_2) := \mathrm{col}(s_1 + 1, -(s_2 + 1), (s_2 - s_1)(s_1 + s_2 + 1))$. Clearly, $RM = 0$ and any other polynomial matrix $\bar{R}(s_1, s_2)$ such that $\bar{R}M = 0$ will be of the form $\bar{R} = LR$ for some polynomial matrix $L(s_1, s_2)$, i.e., R is a *minimal left annihilator* of M. This implies that $\mathrm{im}\, M = \ker R$ and $\mathfrak{B}$ will have the MA representation $w = M(\sigma_1, \sigma_2)a$. Taking this fact into account, it is easy to see that Σ satisfies the controllability condition of Definition 2 with $\rho = 5$.

The classical approach to 2-D systems is within an input–output framework and deals with systems described by transfer functions. The connection between this class of systems and the class of autoregressive 2-D systems considered in this paper is as follows.

An autoregressive 2-D system in *full-rank input–output form* is defined as $\Sigma^{i/o} = (\mathbb{Z}^2, \mathbb{R}^m \times \mathbb{R}^p, \mathfrak{B}^{i/o})$, with $\mathbb{R}^m$ the input space, $\mathbb{R}^p$ the output space, and $\mathfrak{B}^{i/o} \subseteq (\mathbb{R}^m)^{\mathbb{Z}^2} \times (\mathbb{R}^p)^{\mathbb{Z}^2}$ the input–output behavior of the system. This behavior can be described by equations of the form $P(\sigma_1, \sigma_2)y = Q(\sigma_1, \sigma_2)u$, for some polynomial matrices $P(s_1, s_2)$, $Q(s_1, s_2)$ such that P is square and $\det P(s_1, s_2) \not\equiv 0$. Note that the matrices P and Q which describe $\mathfrak{B}^{i/o}$ are not unique, as the equations $Py = Qu$ induce the same behavior as $UPy = UQu$ if $U(s_1, s_2)$ is a unimodular polynomial matrix. It can be shown that this premultiplication by unimodular matrices is the only source of nonuniqueness in the parameters P, Q of a full-row rank input–output description.

$\Sigma^{i/o}$ is said to be an *input–output realization* of $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ if there is a permutation of the components of w, $Tw = (u, y)$ such that $w \in \mathfrak{B}$ if and only if $Tw \in \mathfrak{B}^{i/o}$. Note that not every AR 2-D system has a full-rank input–output realization.

The *transfer function* of the system $\Sigma^{i/o}$ described by $Py = Qu$ is defined as $G(s_1, s_2) := P^{-1}(s_1, s_2)Q(s_1, s_2)$. This is clearly well defined, as $(UP)^{-1}(UQ) = P^{-1}U^{-1}UQ = P^{-1}Q$ for every unimodular matrix $U(s_1, s_2)$. The converse is, however, not true: given a 2-D $p \times m$ rational matrix $G(s_1, s_2)$ there are infinitely many 2-D input–output systems $\Sigma^{i/o}$ whose transfer function is G. Indeed, if the system described by the equations $Py = Qu$ has transfer function $P^{-1}Q = G$, every system described by $LPy = LQu$, with L square and nonsingular (but not necessarily unimodular) will also have G as transfer function. We will define the *input–output system associated with G* as the smallest input–output system whose transfer function is G. This system is of the form $\Sigma^{i/o}(G) := (\mathbb{Z}^2, \mathbb{R}^m \times \mathbb{R}^p, \mathfrak{B}^{i/o}(G))$ with $\mathfrak{B}^{i/o}(G)$ described by behavioral equations $P_c(\sigma_1, \sigma_2)y = Q_c(\sigma_1, \sigma_2)u$, where $P_c^{-1}Q_c = G$ and P_c, Q_c are left coprime, i.e., $[P_c \mid -Q_c]$ is left prime. Thus $\Sigma_{i/o}(G)$ is a full-rank input–output system. A system $\Sigma^{i/o}$ is said *to have a transfer function representation* if it coincides with the input–output system associated with its transfer function.

Now, it is not difficult to prove that condition 3) of Theorem 1 implies the following.

Proposition 2: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ be an autoregressive 2-D system with nontrivial behavior, i.e., $\mathfrak{B} \neq \{o\}$. Then Σ is controllable if and only if 1) it has an input–output realization and 2) every input–output realization of Σ has a transfer function representation.

Proof: Appendix.

Thus 2-D systems described by transfer functions correspond to controllable AR systems endowed with an input–output structure.

To conclude this section, we would like to stress the difference between our definition of controllability and the classical notions of controllability of state-space realizations [7], [8], [3]. In fact, while these latter are properties of system descriptions in terms of auxiliary (state) variables, and hence internal properties, our definition is stated at the level of the external behavior of a system.

Nevertheless, for instance, the notion of 2-D modal controllability introduced in [8] can be placed in the context of our definition of controllable 2-D systems. Indeed, let $\Sigma^x = (\mathbb{Z}^2, \mathbb{R}^m \times \mathbb{R}^p, \mathbb{R}^n, \mathfrak{B}^x)$ be an auxiliary variable realization of the input–output system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^m \times \mathbb{R}^p, \mathfrak{B})$, i.e., $\mathfrak{B} = \Pi_{(u,y)}\mathfrak{B}^x$, and suppose that Σ^x can be described by the following behavioral equations:

$$\begin{cases} P(\sigma_1, \sigma_2)\; x = Q(\sigma_1, \sigma_2)u \\ \qquad\qquad\; y = S(\sigma_1, \sigma_2)x + T(\sigma_1, \sigma_2)u \end{cases}$$

with P, Q, S, T polynomial matrices, P square and nonsingular. In other words, $\mathfrak{B}^x = \ker R(\sigma_1, \sigma_2)$ with $R := \mathrm{col}([-Q \mid 0 \mid P], [-T \mid I \mid -S])$. Σ^x is said to be *modally controllable* if $[-Q \mid P]$ is left prime. It is not difficult to see that this is equivalent to the left primeness of R. Hence Σ^x is modally controllable if and only if the (u, y, x) signals (now regarded as external signals) satisfy the condition of Definition 2. This provides an interpretation of modal controllability in terms of signals instead of representation parameters.

III. State-Space Systems

We will next consider the question of state-space realizations for autoregressive 2-D systems. Our main concern is twofold: on the one hand, we aim for first-order representations which correspond to a convenient first-order updating scheme, and on the other hand, we do not want to make any assumptions on the existence of causal relationships between the system variables. At first sight, these two objectives may seem to be conflicting. A recursive updating scheme is necessarily related to a choice of direction in $\mathbb{Z}^2$, while the absence of causality assumptions translates, in a certain sense, the wish for adirectionality. Indeed, this conflict is clear within the input–output framework, where quarter-plane causal state-space models ([9], [3]) can only be used to represent quarter-plane causal transfer functions. However, this is not the case for our approach. As we will see, every controllable AR system allows for a quarter-plane causal state-space representation.

A. The State-Space Model

Let $(i, j) \in \mathbb{Z}^2$. The *past* of (i, j) will be denoted by $\mathcal{P}(i, j)$ and defined as $\mathcal{P}(i, j) := \{(k, l) \in \mathbb{Z}^2 \mid k \leq i \text{ and } l \leq j\}$. For a set $\emptyset \neq Z \subseteq \mathbb{Z}^2$, the past of Z is defined as $\mathcal{P}(Z) = \cup\{\mathcal{P}(i, j) \mid (i, j) \in Z\}$. Similarly, the *future* of (i, j) will be given by $\mathcal{F}(i, j) = \{(k, l) \in \mathbb{Z}^2 \mid k \geq i \text{ and } l \geq j\}$ and the future of Z by $\mathcal{F}(Z) = \cup\{F(i, j) \mid (i, j) \in Z\}$. The point (i', j') is said to be a *nearest neighbor* of (i, j) if $(i', j') \in \{(i+1, j), (i, j+1), (i-1, j), (i, j-1)\}$. A *path* is a sequence $(z_1, \cdots, z_r)$ in $\mathbb{Z}^2$ such that z_m is a nearest neighbor of z_{m+1} and $z_{m+1} \in \mathcal{F}(z_m)$, for all $m = 1, \cdots, r-1$. Given T_-, T_0, and T_+, subsets of $\mathbb{Z}^2$, we will say that T_0 separates T_+ and T_- if every path connecting T_- and T_+ intersects T_0. An ordered partition (T_-, T_0, T_+) of $\mathbb{Z}^2$ is said to be an *admissible partition* if T_0 separates T_- and T_+ and moreover the following condition is satisfied: $\{t_0 \in \partial T_0\} \Rightarrow \{\mathcal{P}(t_0) \setminus \{t_0\} \subseteq \mathcal{P}(T_-)$ or $\mathcal{F}(t_0) \setminus \{t_0\} \subseteq \mathcal{F}(T_+)\}$. Here ∂T_0 denotes the boundary of T_0, i.e., the set of all points in T_0 which have a nearest neighbor outside T_0.

Definition 3: The system with auxiliary variables $\Sigma^s = (\mathbb{Z}^2, W, A, \mathfrak{B}^s)$ is a *state realization* of $\Sigma = (\mathbb{Z}^2, W, \mathfrak{B})$ if $\mathfrak{B} = \Pi_w \mathfrak{B}^s := \{w \in W^{\mathbb{Z}^2} \mid \exists x \in A^{\mathbb{Z}^2} \text{ s.t. } (w, x) \in \mathfrak{B}^s\}$, and moreover if the following axiom is satisfied.

Axiom of state: $\{(T_-, T_0, T_+)$ admissible partition of $\mathbb{Z}^2$; $\emptyset \neq D_+ \subseteq T_+ \cup T_0$; $0 \neq D_- \subseteq T_- \cup T_0$; D_- and D_+ separated by $D_0 := \mathcal{P}(D_+) \cap T_0$; $D_0 \cap D_- = \emptyset$; (w_1, x_1), $(w_2, x_2) \in \mathfrak{B}^s$; $x_1|_{D_0} = x_2|_{D_0}\} \Rightarrow \{(w_1, x_1)|_{D_-} \wedge (w_2, x_2)|_{D_+} \in \mathfrak{B}^s|_{D_- \cup D_+}\}$.

The sets D_-, D_0, D_+ involved in this definition are indicated in the Fig. 1.

The concatenability condition in the axiom of state expresses the fact that the state variable x has the property of making past and future behavior conditionally independent: once the states on the separation set D_0 coincide, $\mathfrak{B}^s|_{D_-}$ and $\mathfrak{B}^s|_{D_+}$ are independent.

The following result characterizes state systems (i.e., systems which satisfy the axiom of state) in terms of their representations.

Theorem 2: Let $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ be an ARMA 2-D system. Then Σ is a state realization of $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \Pi_w \mathfrak{B}^s)$ if and only if $\mathfrak{B}^s$ can be described as $\mathfrak{B}^s = \{(w, x) \in (\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}^2} \mid \exists v \text{ s.t. (1)–(3) are satisfied}\}$.

$$\begin{cases} S(\sigma)x = o & (1) \\ \sigma_1 x = A(\sigma)x + B(\sigma)v & (2) \\ w = Cx + Dv & (3) \end{cases}$$

with $\sigma := \sigma_2^{-1}\sigma_1$, $s := s_2^{-1}s_1$, $A(s) := A_1 s + A_2$, $B(s) := B_1 s + B_2$; A_1, A_2, B_1, B_2, C, and D real matrices and $S(s)$ a polynomial matrix. The operators $A(\sigma)$, $B(\sigma)$, $S(\sigma)$ must moreover satisfy the following conditions:

1) $\ker S(\sigma)$ is $A(\sigma)$-invariant, i.e., $A(\sigma)\ker S(\sigma) \subseteq \ker S(\sigma)$,

2) $\operatorname{im} B(\sigma) \subseteq \ker S(\sigma)$.

Here $v \in (\mathbb{R}^l)^{\mathbb{Z}^2}$ (for some positive integer l) is an auxiliary variable.

Proof: Appendix.

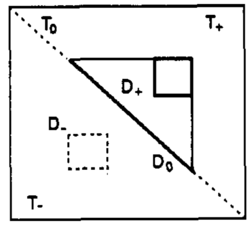

Fig. 1. D_- and D_+ are indicated, respectively, by dashed and bold lines. In this example, $D_0 \subsetneq D_+$.

The interpretation of equations (1)–(3) and conditions 1) and 2) is as follows. At every point $(i, j) \in \mathbb{Z}^2$, $v(i, j)$ can be chosen freely provided that $x|_{\mathcal{F}(i,j) \setminus \{(i,j)\}}$ and $(w, v)|_{\mathcal{F}(i,j)}$ are not given. The state components x are constrained as follows. On every diagonal line $\mathfrak{L}_k := \{(i, j) \in \mathbb{Z}^2 \mid i + j = k\}$, $k \in \mathbb{Z}$, x must satisfy (1). Given the values of x and v on $\mathfrak{L}_k$, x can be computed on $\mathfrak{L}_{k+1}$ by means of (2). Conditions 1)–2) are compatibility conditions which ensure that, on $\mathfrak{L}_{k+1}$, (1) is automatically satisfied. Thus, to compute a signal $(w^+, (x^+, v^+))$ which satisfies (1)–(3) on a half-plane $\mathcal{H}_k^+ := \{(i, j) \in \mathbb{Z}^2 \mid i + j \geq k\}$, we can use the following procedure.

1) Initialization: choose on $\mathfrak{L}_k$ a solution x_k^+ of (1).

2) For $l = k, k+1, \cdots$, choose arbitrary values v_l^+ of v on $\mathcal{L}_l$ and define

$$x_{l+1}^+ := A(\sigma)x_l^+ + B(\sigma)v_l^+$$

$$w_l^+ := Cx_l^+ + Dv_l^+.$$

3) Define $(w^+, (x^+, v^+))$ as $(w^+, (x^+, v^+))|_{\mathfrak{L}_l} := (w_l^+, (x_l^+, v_l^+))$, $l = k, k+1, \cdots$.

This provides a 2-D first-order recursive updating scheme. (See Fig. 2.)

Due to its role in the foregoing updating scheme, v will be called the *driving variable*. The variable x is the *state* variable. Equations (1)–(3) are referred to as a *state/driving variable* model, and will be denoted by $(S(s), A(s), B(s), C, D)$. The associated system $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ is denoted by $\Sigma^s(S(s), A(s), B(s), C, D)$.

Remark: In order to describe 2-D input–output systems, the following model is introduced in [3]:

$$\begin{cases} \sigma_1 x = (A_1\sigma + A_2)x + (B_1\sigma + B_2)u \\ y = Cx + Du \end{cases} \quad (4)$$

Here u is the input, y the output and x the so-called local state. This model can be considered as a special case of the more general model (1)–(3) with $S(s) = 0$ and $v = u$. This fact will be used in the next paragraph in order to obtain state representations for controllable systems.

B. Realizability

As shown in [11, Theorem 3], every autoregressive 1-D system can be represented by means of a 1-D state/driving variable model. A similar result does not hold for AR 2-D systems.

Example 2: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$ be an autoregressive 2-D system such that $\mathfrak{B} = \ker(\sigma_1 - \sigma_2)$, i.e., $\mathfrak{B}$ consists of all those signals in $\mathbb{R}^{\mathbb{Z}^2}$ which have constant value along the diagonal lines $\mathfrak{L}_k$, $k \in \mathbb{Z}$. Then Σ cannot be represented by a state/driving variable model of the form (1)–(3).

Proof: Appendix.

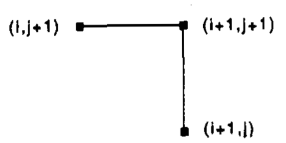

Fig. 2. The value of x in $(i+1, j+1)$ is computed from the values of (x, v) on $(i+1, j)$ and $(i, j+1)$.

However, we will show that every *controllable* AR 2-D system allows a state realization. Roughly stated, this is explained by the fact that the signals w of a controllable system can be regarded as the outputs of an "enlarged" input-output system with quarter-plane causal transfer function. The inputs of this enlarged system will constitute the driving variable in the state representation. To be precise, we introduce the following terminology.

A 2-D rational function $r(s_1, s_2) = q(s_1, s_2)/p(s_1, s_2)$—where $q(s_1, s_2) = \sum_{i=0}^{m} q_i(s_1)s_2^i$, $p(s_1, s_2) = \sum_{j=0}^{n} p_j(s_1)s_2^j$, and $p_j(s_1)$ and $q_i(s_1)$ are polynomials, for $i = 0, \cdots, m$, $j = , \cdots, n$ with $q_m(s_1) \neq 0 \neq p_n(s_1)$—is said to be *proper* if 1) $m \leq n$ and 2) the degree of $p_n(s_1)$ is not less than the degrees of $q_i(s_1)$, $i = 0, \cdots, m$, and of $p_j(s_1)$, $j = 0, \cdots, n-1$. A 2-D rational transfer matrix will be called *quarter-plane causal* if all its entries are 2-D proper rational functions.

Proposition 3 [3]: Let $\Sigma^{i/o}$ be an input-output AR 2-D system and suppose that $\Sigma^{i/o}$ has a transfer function representation $T(s_1, s_2)$. Then $\Sigma^{i/o}$ can be represented by the model (4) if and only if $T(s_1, s_2)$ is quarter-plane causal.

Now let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ be a controllable AR 2-D system. Then, by Theorem 1, Σ has an MA representation, i.e., there exists a polynomial matrix $M(s_1, s_2)$ of size $q \times l$ (for some positive integer l) such that $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid \exists a (\mathbb{R}^l)^{\mathbb{Z}^2} \text{ s.t. } w = M(\sigma_1, \sigma_2)a\}$. Note that the equation $w = Ma$ is equivalent to $\sigma_1^{d_1}\sigma_2^{d_2} w = M(\sigma_1, \sigma_2)v$, with $v := \sigma_1^{d_1}\sigma_2^{d_2} a$ and d_1 and d_2, respectively, defined as the highest degrees in s_1 and s_2 of the entries of $M(s_1, s_2)$ (regarded as polynomials in $\mathbb{R}[s_1, s_2]$). Consider now the (enlarged) input-output system $\Sigma^{i/o} = (\mathbb{Z}^2, \mathbb{R}^l \times \mathbb{R}^q, \mathfrak{B}^{i/o})$ with $\mathfrak{B}^{i/o}$ described by $\sigma_1^{d_1}\sigma_2^{d_2} w = M(\sigma_1, \sigma_2)v$. Clearly, $\Sigma^{i/o}$ has a transfer function representation $T(s_1, s_2) = s_1^{-d_1}s_2^{-d_2}M(s_1, s_2)$. Now T is quarter-plane causal and hence, by Proposition 3, $\Sigma^{i/o}$ can be represented as

$$\begin{cases} \sigma_1 x = (A_1\sigma + A_2)x + (B_1\sigma + B_2)v \\ w = Cx + Dv \end{cases}$$

for suitable real matrices A_1, A_2, B_1, B_2, C, and D. This constitutes a state representation of the form (1)–(3) with $S(s) = 0$. This proves the following result.

Theorem 3: Let Σ be a controllable autoregressive 2-D system, then it has a state representation of the form (1)–(3).

Remark: It is an immediate consequence of Proposition 3 that AR systems which have input-output realizations described by a quarter-plane causal transfer function have a state representation of the form (1)–(3). However, not every AR system allows a quarter-plane causal input-output realization (cf. Example 3). This shows the essential role of the controllability assumption in Theorem 3. All controllable systems, and not only those which can be described in input-output form by quarter-plane causal transfer functions, have a state realization.

Example 3: The controllable AR system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^2, \mathfrak{B})$ with $\mathfrak{B} = \{(w_1, w_2) \in (\mathbb{R}^2)^{\mathbb{Z}^2} \mid \sigma_1 w_1 = \sigma_2 w_2\}$ can be described by the following state/driving variable model.

$$\begin{cases} \sigma x_1 = x_2 \\ \sigma_1 x_1 = v \\ \sigma_1 x_2 = \sigma v \\ w_1 = x_1, \qquad w_2 = x_2. \end{cases}$$

Note however that none of the two input-output realizations of $\mathfrak{B}$ (which can be obtained by considering either w_1 or w_2 as input) is described by a quarter-plane causal transfer function.

Example 4 (Controllability is not a necessary condition in Theorem 3): The noncontrollable AR system $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$ with $\mathfrak{B} = \{w \in \mathbb{R}^{\mathbb{Z}^2} \mid (\sigma_1 - 1)w = 0\}$ can be trivially represented in state form as:

$$\begin{cases} \sigma_1 x = x \\ w = x. \end{cases}$$

IV. Trimness, Reachability, and Observability of State-Space Systems

In this section, we define the notions of trimness, reachability, and observability for state-space systems, and characterize these properties in terms of the system representations. Our definitions are given at a global level: they regard the restriction of the behavior to diagonal lines $\mathfrak{L}_k$, $k \in \mathbb{Z}$, instead of being concerned with the restriction to single points in $\mathbb{Z}^2$. These diagonal lines play an important role as they constitute the "propagation fronts" in our updating scheme of Section III-A.

We will present state-space algorithms for obtaining trim state-space representations and for checking observability. These algorithms closely resemble techniques developed in the geometric theory for 1-D systems.

A. Trimness

The state-space system $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ will be called *locally trim* if locally (i.e., on a point) any state value can be obtained, in other words for every $\alpha \in \mathbb{R}^n$, there exist $(t_1, t_2) \in \mathbb{R}^2$ and $x \in \Pi_x \mathfrak{B}^s$ such that $x(t_1, t_2) = \alpha$. Clearly, if Σ^s is linear and not locally trim, there exists a change of coordinates $Tx = \text{col}(x_1, x_2)$ such that $x \in \Pi_x \mathfrak{B}^s$ implies that $x_2 = 0$. In this case, x_1 is a state variable for $\Pi_w \mathfrak{B}^s$, and hence the 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \Pi_w \mathfrak{B}^s)$ has another state-space realization with lower dimension. In the sequel, we will only consider locally-trim state-space systems.

As mentioned in Section III-A, the compatibility conditions $A(\sigma)\ker S(\sigma) \subseteq \ker S(\sigma)$ and $\operatorname{im} \mathfrak{B}(\sigma) \subseteq \ker S(\sigma)$ of our state model guarantee that any initial condition $x_0 \in \ker S(\sigma)$ given on a diagonal line $\mathfrak{L}_k$ can be propagated towards the future $\mathscr{H}_k^+ := \cup\{\mathfrak{L}_j \mid j \geq k\}$. However, it may happen that x_0 cannot be extended towards the past $\mathscr{H}_k^- := \cup\{\mathfrak{L}_j \mid j < k\}$. In this case $x_0 \notin \Pi_x \mathfrak{B}^s|_{\mathfrak{L}_k} =: \mathfrak{B}_k$. Note that, due to shift invariance, $\mathfrak{B}_k = \mathfrak{B}_0$ for all $k \in \mathbb{Z}$. We define the *trim subspace* $\mathscr{T}$ as $\mathscr{T} := \mathfrak{B}_0$.

Definition 4: The representation $(S(s), A(s), B(s), C, D)$ is said to be a *trim representation* if $\mathscr{T} = \ker S(\sigma)$.

The relevance of trim representations is that they indicate explicitly which is the set of admissible initial conditions on the lines $\mathfrak{L}_k$. Given a state-space representation $(S(s), A(s), B(s), C, D)$, the trim subspace can be determined by the following algorithm.

Trimming algorithm:

Step 0: $\mathscr{T}_0 := \ker S(\sigma)$

For $k = 1, 2, \cdots$

Step k: $\mathscr{T}_k := A(\sigma)\mathscr{T}_{k-1} + \operatorname{im} B(\sigma)$.

Proposition 4: 1) The trimming algorithm is finite, i.e., there is $L \in \mathbb{N}$ such that $\mathscr{T}_k = \mathscr{T}_L$ for all integers $k \geq L$. 2) There is a polynomial matrix $T(s)$ such that $\mathscr{T}_L = \ker T(\sigma)$. 3) $\mathscr{T} = \mathscr{T}_L$. 4) $\Sigma^s(T(s), A(s), B(s), C, D) = \Sigma^s(S(s), A(s), B(s), C, D)$.

Proof: Appendix.

Note that, in order to obtain a trim representation from a nontrim one, it suffices to determine the matrix $T(s)$, as the other parameters of the representation can remain unchanged. A procedure to find $T(s)$ is given in the proof of the foregoing proposition.

B. Reachability

Let $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s) = \Sigma^s(S(s), A(s), B(s), C, D)$ be a state-space system with state variable $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ and driving-variable $v \in (\mathbb{R}^l)^{\mathbb{Z}}$. Further let $\mathfrak{B}^x := \Pi_x \mathfrak{B}^s$. The subspace $\mathscr{R}$ of $(\mathbb{R}^n)^{\mathbb{Z}}$ defined by $\mathscr{R} := \{x^* \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists x \in \mathfrak{B}^x \text{ s.t. } x|_{\mathfrak{L}_0} = 0 \text{ and } x|_{\mathfrak{L}_k} = x^* \text{ for some positive integer } k\}$ will be called the *reachable subspace of* Σ^s. Clearly, $\mathscr{R}$ is contained in the trim subspace $\mathscr{T}$ of Σ^s.

Definition 5: The state-space system $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ is said to be *reachable* if $\mathscr{R} = \mathscr{T}$.

Note that here we do not require $\mathscr{R} = (\mathbb{R}^n)^{\mathbb{Z}}$, in contrast with the definition of the related notion of global reachability given in [2]. The motivation for our definition is as follows. In considering the question of state-space realization we do not want to require that the behavior of the external variable w should be free on the diagonal lines $\mathfrak{L}_k$, $k \in \mathbb{Z}$. Note that, for instance, for the system considered in Example 3, this behavior is not free. For systems with nonfree external behavior on $\mathfrak{L}_k$, the behavior of the corresponding state variable x will in general also not be free on $\mathfrak{L}_k$ (i.e., $\mathscr{T} \neq (\mathbb{R}^n)^{\mathbb{Z}}$), and hence a natural reachability requirement is $\mathscr{R} = \mathscr{T}$. Clearly, if $\mathscr{T} = (\mathbb{R}^n)^{\mathbb{Z}}$ our reachability condition amounts to the aforementioned notion of global reachability.

Remark: We recall that while our notion of controllability is defined at the level of external behavior, reachability is defined for state-space systems, thus in terms of the internal behavior.

It is easy to show the following result.

Lemma 1: The reachable subspace of a state-space system $\Sigma^s = (S(s), A(s), B(s), C, D)$ is given by $\mathscr{R} = \operatorname{im} [B(\sigma) \mid A(\sigma)B(\sigma) \mid \cdots \mid A^{n-1}(\sigma)B(\sigma)]$, where n is the size of the state variable. In particular, if $(S(s), A(s), B(s), C, D)$ is a trim representation, then Σ^s is reachable if and only if $\ker S(\sigma) = \operatorname{im} [B(\sigma) \mid A(\sigma) \mid \cdots \mid A^{n-1}(\sigma)B(\sigma)]$.

The condition $\ker S(\sigma) = \operatorname{im} [B(\sigma) \mid A(\sigma)B(\sigma) \mid \cdots \mid A^{n-1}(\sigma)B(\sigma)]$ is equivalent to saying that $S(s)$ is a minimal left annihilator of $[B(s) \mid A(s)B(s) \mid \cdots \mid A^{n-1}(s)B(s)]$ (cf. Example 1). This property can be checked using the following lemma.

Lemma 2: Let $M(s)$ and $S(s)$ be two polynomial matrices. Let further $U(s)$ be a unimodular matrix and $F(s)$ a full-row rank matrix such that $N(s) = U(s)\operatorname{col}(F(s), 0)$. Then $S(s)$ is a minimal left annihilator of $N(s)$ if and only if there exists a unimodular matrix $V(s)$ such that $V(s)S(s)U^{-1}(s) = [0 \mid I]$, where 0 is a zero matrix with as many columns as the rows of $F(s)$.

Proof: Appendix.

Recalling the notion of controllability of Section II, it is interesting to note the following. Let $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s) = \Sigma^s(S(s), A(s), B(s), C, D)$ be a state-space system. Consider the corresponding state/driving variable and state behaviors, respectively, $\mathfrak{B}^{(x,v)} := \ker R(\sigma)$ (with $R(s) := \operatorname{col}([S(s) \mid 0], [s_1 - A(s) \mid B(s)])$) and $\mathfrak{B}^x := \Pi_x \mathfrak{B}^s$, and define the 2-D systems $\Sigma^{(x,v)} := (\mathbb{Z}^2, \mathbb{R}^{n+l}, \mathfrak{B}^{(x,v)})$, $\Sigma^x := (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B}^x)$, (here (x, v) and x are regarded as external variables). Now, it is not difficult to show that $\{\Sigma^{(x,v)} \text{ controllable}\} \Rightarrow \{\Sigma^x \text{ controllable}\} \Rightarrow \{\Sigma^s \text{ reachable}\}$. This means that Σ^s is reachable if $R(s)$ is left prime. A still weaker sufficient condition for the reachability of Σ^s is provided by the next proposition.

Proposition 5: 1) The state-space system $\Sigma^s = \Sigma^s(S(s), A(s), B(s), C, D)$ is reachable if $[s_1 - A(s) \mid B(s)]$ is left prime. 2) Every controllable AR 2-D system has a reachable state-space realization.

Proof: Appendix.

C. Observability

As before for $k \in \mathbb{Z}$ let $\mathfrak{L}_k := \{(i, j) \in \mathbb{Z}^2 \mid i + j = k\}$ and define the half planes $\mathscr{H}_k^+ := \cup\{\mathfrak{L}_l \mid l \geq k\}$, $\mathscr{H}_k^- := \cup\{\mathfrak{L}_l \mid l < k\}$. A state-space system $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ with state variable x will be called *future observable* if $\{w|_{\mathscr{H}_k^+} = o\} \Rightarrow \{x|_{\mathfrak{L}_k} = o\}$, and *past observable* if $\{w|_{\mathscr{H}_k^-} = o\} \Rightarrow \{x|_{\mathfrak{L}_k} = o\}$.

Definition 6: Σ^s is said to be *observable* if $\{w = o\} \Rightarrow \{x = o\}$.

Note that both past and future observability imply observability, but that the converse does not hold true.

To investigate observability, we introduce the following subspaces. Suppose that $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s)$ has state variable $x \in (\mathbb{R}^n)^{\mathbb{Z}^2}$ and driving variable $v \in (\mathbb{R}^l)^{\mathbb{Z}^2}$. Define subspaces $\mathscr{V}^+$ and $\mathscr{V}^-$ of $(\mathbb{R}^n)^{\mathbb{Z}}$ as follows:

$$\mathscr{V}^+ := \left\{x_0 \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists (w, x) \in \mathfrak{B}^s \text{ s.t. } x|_{\mathfrak{L}_0} = x_0 \text{ and } w|_{\mathscr{H}_0^+} = o\right\}$$

$$\mathscr{V}^+ := \left\{x_0 \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists (w, x) \in \mathfrak{B}^s \text{ s.t. } x|_{\mathfrak{L}_0} = x_0 \text{ and } w|_{\mathscr{H}_0^-} = o\right\}.$$

Then, clearly, Σ^s is future observable if and only if $\mathscr{V}^- = \{0\}$, past observable iff $\mathscr{V}^+ = \{0\}$ and observable iff $\mathscr{V}^- \cap \mathscr{V}^+ = \{0\}$.

Let $(S(s), A(s), B(s), C, D)$ be a trim representation of Σ^s, and define $\bar{C}(s) := \operatorname{col}(C, S(s))$ and $\bar{D} := \operatorname{col}(D, 0)$. Then Σ^s is described by the following behavioral equations

$$\begin{cases} \sigma_1 x = A(\sigma)x + B(\sigma)v \\ \begin{bmatrix} w \\ o \end{bmatrix} = \bar{C}(\sigma)x + \bar{D}v \end{cases} \tag{5}$$

The subspaces $\mathscr{V}^+$ and $\mathscr{V}^-$ can be calculated in terms of $A(\sigma)$, $B(\sigma)$, $\bar{C}(\sigma)$, and $\bar{D}$ by means of the following algorithms.

$\mathscr{V}^+$ Algorithm:

Step 0: $\mathscr{V}_0^+ := (\mathbb{R}^n)^{\mathbb{Z}}$

For $k = 1, 2, \cdots$

Step k: $\mathscr{V}_k^+ := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v \in (\mathbb{R}^l)^{\mathbb{Z}} \text{ s.t. } A(\sigma)x + B(\sigma)v \in \mathscr{V}_{k-1}^+ \text{ and } \bar{C}(\sigma)x + \bar{D}v = o\}$.

Remark: Note that, for the system described by (5) the subspace $\mathscr{V}^+$ is the 2-D version of what is called the *largest output nulling subspace* in geometric theory of 1-D systems.

$\mathscr{V}^-$ Algorithm:
Step 0: $\mathscr{V}_0^- := (\mathbb{R}^n)^{\mathbb{Z}}$
For $k = 1, 2, \cdots$
Step k: $\mathscr{V}_k^- := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v \in (\mathbb{R}^l)^{\mathbb{Z}} \exists \bar{x} \in \mathscr{V}_{k-1}^-$ s.t. $A(\sigma)\bar{x} + B(\sigma)v = x$ and $\bar{C}(\sigma)\bar{x} + \bar{D}v = o\}$.

Explicit procedures to compute $\mathscr{V}_k^+$ and $\mathscr{V}_k^-$ are given in the proof of the following proposition.

Proposition 6: 1) Both the $\mathscr{V}^+$ and the $\mathscr{V}^-$ algorithms are finite, i.e., there exist L^+ and $L^- \in \mathbb{N}$ such that $\mathscr{V}_k^+ = \mathscr{V}_{L^+}^+$ for all $k \geq L^+$ and $\mathscr{V}_k^- = \mathscr{V}_{L^-}^-$ for all $k \geq L^-$. 2) $\mathscr{V}_{L^+}^+ = \ker V^+(\sigma)$ and $\mathscr{V}_{L^-}^- = \ker V^-(\sigma)$ for some polynomial matrices $V^+(s)$ and $V^-(s)$. 3) $\mathscr{V}^+ = \mathscr{V}_{L^+}^+$ and $\mathscr{V}^- = \mathscr{V}_{L^-}^-$.

Proof: Appendix.

This proposition provides a method of checking observability in terms of polynomial matrices instead of in terms of infinite-dimensional linear spaces.

We next show that observability can also be checked without invoking the subspaces $\mathscr{V}^+$ and $\mathscr{V}^-$. This method is based on the elimination of the driving variable v on the state-space representation, in order to obtain a description only in terms of the variables x and w, which are the relevant variables for the notion of observability. The procedure of elimination of v is given as follows.

Driving-Variable Elimination: Let $\Sigma^s = \Sigma^s(S(s), A(s), B(s), C, D)$, and define $E(s, s_1) := \text{col}\,(S(s), s_1 - A(s), C)$, $F := \text{col}\,(0, 0, -I)$ and $G(s) := \text{col}\,(0, B(s), -D)$. Then the behavioral equations for Σ^s become the following:

$$\left[E(\sigma, \sigma_1) \mid F\right]\begin{bmatrix} x \\ w \end{bmatrix} = G(\sigma)v. \tag{6}$$

Further, let $\bar{U}(s)$ be a unimodular matrix such that $\bar{U}(s)G(s) = \text{col}\,(G_F(s), 0)$, with $G_F(s)$ full-row rank, and partition $\bar{U}(s) = \text{col}\,(U_1(s), U(s))$, with $U_1(s)G(s) = G_F(s)$ and $U(s)G(s) = 0$. Now, premultiplying (6) by $\bar{U}(\sigma)$ yields the following equivalent equations:

$$\begin{cases} U_1(\sigma)\left[E(\sigma, \sigma_1) \mid F\right]\begin{bmatrix} x \\ w \end{bmatrix} = G_F(\sigma)v & (7) \\ U(\sigma)\left[E(\sigma, \sigma_1) \mid F\right]\begin{bmatrix} x \\ w \end{bmatrix} = o. & (8) \end{cases}$$

Finally, $G_F(\sigma)$ is surjective, as $G_F(s)$ is full-row rank, which implies that (8) describes the behavior only in terms of x and w.

Using the description of given by (8), observability can be characterized as follows.

Proposition 7: Let $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B}^s) = \Sigma^s(S(s), A(s), B(s), C, D)$ be a state-space system. Define the matrices $E(s, s_1)$, F, $G(s)$, $\bar{U}(s)$, and $U(s)$ as previously stated. Then Σ^s is observable if and only if $U(s)E(s, s_1)$ is a zero-right-prime matrix, i.e., for all $(\lambda, \lambda_1) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ $U(\lambda)E(\lambda, \lambda_1)$ has full-column rank.

Proof: Appendix.

V. Conclusion

In this paper, we have considered autoregressive 2-D systems and, in particular, the class of controllable AR systems. We defined controllability as an external property of systems and characterized it in terms of system representations. Moreover, we showed that controllable systems constitute exactly the class of AR systems which can be described in input–output form by means of a 2-D transfer function.

Further, we defined a concept of state for 2-D systems and derived the corresponding state-space model. This model is a first-order state/driving variable representation which leads to a first-order recursive updating scheme. Although not every AR system has such a representation, we proved that every controllable AR system can be represented in state-space form. This shows that, in contrast with the case of input–output systems, the realizability of controllable AR systems does not depend on the existence of 2-D causal relationships between the system variables. Finally, we introduced the notions of trimness, reachability, and observability for state-space systems and gave conditions for these properties in terms of the system describing parameters.

Appendix

Proofs

Proof of Proposition 1: Clearly, if $\Sigma^a = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^l, \mathfrak{B}^a)$ is an ARMA system, $\Pi_w \mathfrak{B}^a =: \mathfrak{B}$ is a linear and shift-invariant subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$. Moreover $\Pi_w: (\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^l)^{\mathbb{Z}^2} \to (\mathbb{R}^q)^{\mathbb{Z}^2}$ is a linear and continuous map with respect to the topology of pointwise convergence. Hence, since with this topology $(\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^l)^{\mathbb{Z}^2}$ and $(\mathbb{R}^q)^{\mathbb{Z}^2}$ are both linearly compact spaces (cf. [6]), it follows that Π_w maps closed linear subspaces into closed linear subspaces. Thus, as $\mathfrak{B}^a$ is a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^l)^{\mathbb{Z}^2}$ with the pointwise convergence topology, we conclude that $\mathfrak{B}$ is a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$. By [10, Theorem 2.1], together with linearity and shift invariant this means that $\mathfrak{B}$ can be described as $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ for some 2-D polynomial matrix $R(s_1, s_2)$. Hence $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is an AR system. ■

Proof of Theorem 1: We will show that the following implications hold: 3) ⇒ 4) ⇒ 1) ⇒ 2) ⇒ 3).

3) ⇒ 4): Suppose that $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ with $R(s_1, s_2)$ a full-row rank left-prime polynomial matrix. Without loss of generality we can assume that $R = [P \mid Q]$ with $P(s_1, s_2)$ a $g \times g$ polynomial matrix with nonzero determinant. The left primeness of R implies that $P(s_1, s_2)$ and $Q(s_1, s_2)$ have only unimodular common left divisors. Hence (P, Q) is a left-coprime factorization of the rational matrix $G(s_1, s_2) = P^{-1}(s_1, s_2)Q(s_1, s_2)$. Let $(\bar{Q}, \bar{P})$ be a right-coprime factorization of G, i.e., $\bar{P}$ is square and nonsingular, $\bar{Q}$ and $\bar{P}$ have only unimodular common right divisors and $G = \bar{Q}\bar{P}^{-1}$. Then, $P\bar{Q} - Q\bar{P} = 0$. Moreover, if (P^*, Q^*) also satisfy $P^*\bar{Q} - Q^*\bar{P} = 0$, there exists a polynomial matrix $L(s_1, s_2)$ such that $P^* = LP$ and $Q^* = LQ$. Define now $M(s_1, s_2) := \text{col}\,(\bar{Q}(s_1, s_2), -\bar{P}(s_1, s_2))$ and let $\mathscr{M} := \text{im}\, M(\sigma_1, \sigma_2)$. It follows from Proposition 1 that there exists a polynomial matrix $R^*(s_1, s_2)$ such that $\mathscr{M} = \ker R^*(\sigma_1, \sigma_2)$. Thus $R^*(s_1, s_2)\, M(s_1, s_2) = 0$. Partitioning $R^* = [P^* \mid Q^*]$ such that $R^*M = P^*\bar{Q} - Q^*\bar{P}$, the foregoing is equivalent to $P^*\bar{Q} - Q^*\bar{P} = 0$, and hence there exists L such that $R^* = [P^* \mid Q^*] = L[P \mid Q] = LR$. Consequently $\ker R(\sigma_1, \sigma_2) \subset \ker R^*(\sigma_1, \sigma_2)$, i.e., $\mathfrak{B} \subseteq \mathscr{M}$. On the other hand, it is clear that $\mathscr{M} \subseteq \mathfrak{B}$. So $\mathfrak{B} = \mathscr{M}$ showing that 4) holds true.

4) ⇒ 1): Let $\mathfrak{B} = \text{im}\, M(\sigma_1, \sigma_2)$ for some $q \times l$ polynomial matrix $M(s_1, s_2)$. Define the *radius* $r(M)$ of $M(s_1, s_2)$ as the maximum of the degrees in s_1 and s_2 of the entries of M. We will see that the condition of Definition 2 holds with $\rho > 2r(M)$. Indeed, let $w_1, w_2 \in \mathfrak{B}$ and $I_1, I_2 \subseteq \mathbb{Z}^2$ be such that $d(I_1, I_2) \geq 2r(M) + 1$. Let further $a_1, a_2 \in (\mathbb{R}^l)^{\mathbb{Z}^2}$ be such that $w_1 = Ma_1$ and $w_2 = Ma_2$, and construct $a^* \in (\mathbb{R}^l)^{\mathbb{Z}^2}$ as follows: $a^*(t_1, t_2) = a_1(t_1, t_2)$ if $d((t_1, t_2), I_1) \leq r(M)$ and $a^*(t_1, t_2) = a_2(t_1, t_2)$ if $d((t_1, t_2), I_2) \leq r(M)$. It is not difficult to check that the element $w^* \in \mathfrak{B}$ defined by $w^* := Ma^*$ satisfies $w^*|_{I_1} = w_1$ and $w^*|_{I_2} = w_2$. This shows that Σ is controllable.

1) ⇒ 2): Suppose that Σ is controllable and let $w \in \mathfrak{B}$. Consider a sequence $(I_k)_{k\in\mathbb{N}}$ of finite intervals of $\mathbb{Z}^2$ satisfying $I_k \subseteq \mathbb{Z}^2$ and such that for every finite subset $C \subseteq \mathbb{Z}^2$ there is $N \in \mathbb{N}$ s.t. $C \subset I_N$. By the controllability assumption, for each $k \in \mathbb{N}$, there exist $w_k \in \mathfrak{B}$ and a finite interval $\bar{I}_k$ of $\mathbb{Z}^2$, with $I_k \subset \bar{I}_k$, such that $w_k|_{I_k} = w|_{I_k}$ and $w_k|_{\mathbb{Z}^2 \setminus \bar{I}_k} = 0$. Clearly, the sequence $(w_k)_{k\in\mathbb{N}}$ converges to w in the topology of pointwise convergence. So $\mathfrak{B} \subseteq \mathrm{cl}(\mathfrak{B}^{\text{compact}})$. To see that $\mathrm{cl}(\mathfrak{B}^{\text{compact}}) \subseteq \mathfrak{B}$ we invoke the fact that autoregressive behaviors are closed subspaces in the topology of pointwise convergence [10]. Thus, as $\mathfrak{B}^{\text{compact}} \subseteq \mathfrak{B}$, $\mathrm{cl}(\mathfrak{B}^{\text{compact}}) \subseteq \mathrm{cl}\,\mathfrak{B} = \mathfrak{B}$.

2) ⇒ 3): As $\mathfrak{B}$ is an autoregressive behavior, there exists a 2-D polynomial matrix $\bar{R}(s_1, s_2)$ such that $\mathfrak{B} = \ker \bar{R}(\sigma_1, \sigma_2)$. If $\bar{R}$ is not full-row rank, there is a square full-rank polynomial $U(s_1, s_2)$ such that $U\bar{R} = \mathrm{col}(R^*, 0)$ with R^* full-row rank. Let $D(s_1, s_2)$ be a maximal left divisor of $R^*(s_1, s_2)$ and $R(s_1, s_2)$ be such that $R^* = DR$. *Thus,* $U\bar{R} = \mathrm{col}(DR, 0)$. Note that R is left prime and hence, as 3) implies 2), $\mathfrak{B}' := \ker R(\sigma_1, \sigma_2)$ satisfies the condition $\mathfrak{B}' = \mathrm{cl}((\mathfrak{B}')^{\text{compact}})$. Moreover, since the matrices U and D are square and full rank, we conclude that $\mathfrak{B}^{\text{compact}} = (\mathfrak{B}')^{\text{compact}}$. This implies that $\mathfrak{B} = \mathfrak{B}'$, showing that $\mathfrak{B}$ allows the full-row rank left-prime description $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$. ■

Proof of Proposition 2:

The "*if*" part is obvious.

"only if": Suppose that $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is a nontrivial controllable AR 2-D system. Then, by condition 3) in Theorem 1, there is a full-row rank left-prime $g \times q$ polynomial matrix $R(s_1, s_2)$ (with $g < q$) such that $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$. Let T be a permutation matrix such that $RT^{-1} = [P \mid -Q]$ with $P(s_1, s_2)$ square ($g \times g$) and nonsingular, and define $Tw =:$ col (y, u), where y has size g and u size $q-g$. Then, the input–output system $\Sigma^{i/o} = (\mathbb{Z}^2, \mathbb{R}^g \times \mathbb{R}^{q-g}, \mathfrak{B}^{i/o})$, with $\mathfrak{B}^{i/o}$ described by $Py = Qu$, is an input–output realization of Σ. Moreover, as R is left prime, so will be RT^{-1}. This means that P and Q are left-coprime polynomial matrices, and so $\Sigma^{i/o}$ can be represented by the transfer function $G := P^{-1}Q$. Thus, Σ has an input–output realization which is representable by means of a transfer function. Let now $\Sigma_*^{i/o} = (\mathbb{Z}^2, \mathbb{R}^g \times \mathbb{R}^{q-g}, \mathfrak{B}_*^{i/o})$ be an arbitrary input–output realization of Σ. Then it is not difficult to see that $\mathfrak{B}_*^{i/o}$ can be described by equations of the form $P^*y^* = Q^*u^*$, with P^* and Q^* such that there exists a permutation matrix T^* satisfying $R = [P^* \mid -Q^*]T^*$. This once more implies that P^* and Q^* are left coprime and hence $\Sigma_*^{i/o}$ can be represented by the transfer function $G^* = P^{*-1}Q^*$. So every input–output realization of Σ is representable by a transfer function. ■

Proof of Theorem 2: The "if" part of the statement is easily verified. The reciprocal implication follows from the lemma below.

Lemma: Let $\mathfrak{L}_k$ denote the diagonal line $\mathfrak{L}_k := \{(i, j) \in \mathbb{Z}^2 \mid i + j = k\}$, $k \in \mathbb{Z}$. Define $\mathfrak{B}_k^s := \mathfrak{B}^s|_{\mathfrak{L}_k} := \mathfrak{B}^s(\mathfrak{L}_k)$ and $\mathfrak{B}_k^x := \mathfrak{B}^x(\mathfrak{L}_k)$ with $\mathfrak{B}^x := \Pi_x \mathfrak{B}^s$. Further, define the operator π: $(\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^n)^{\mathbb{Z}^2} \to (\mathbb{R}^n)^{\mathbb{Z}^2} \times (\mathbb{R}^n)^{\mathbb{Z}^2} \times (\mathbb{R}^n)^{\mathbb{Z}^2} \times (\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^q)^{\mathbb{Z}^2}$ by $\pi(w, x) := (\sigma_1 x, x, \sigma x, w, \sigma w)$ for all $(w, x) \in (\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^n)^{\mathbb{Z}^2}$. Then

1) $\mathfrak{B}_k^s = \mathfrak{B}_0^* := \{(w, x) \in (\mathbb{R}^q)^{\mathfrak{L}_k} \times (\mathbb{R}^n)^{\mathfrak{L}_k} \mid x \in \mathfrak{B}_k^x$ and $\forall \tau \in \mathfrak{L}_k (w(\tau), x(\tau)) \in \mathfrak{B}_k^s(\tau)\}$

2) $\mathfrak{B}^s = \mathfrak{B}^* := \{(w, x) \in (\mathbb{R}^q)^{\mathbb{Z}^2} \times (\mathbb{R}^q)^{\mathbb{Z}^2} \mid (w, x)|_{\mathfrak{L}_k} \in \mathfrak{B}_k^s$ $\forall k \in \mathbb{Z}$ and $\pi(w, x)(\tau) =: (\sigma_1 x(\tau), x(\tau), \sigma x(\tau), w(\tau), \sigma w(\tau)) \in \pi\mathfrak{B}^s(\tau)$ $\forall \tau \in \mathbb{Z}^2\}$.

The proof of this result will be given at the end. Its interpretation is as follows. The behavior $\mathfrak{B}^s$ is characterized by its restriction $\mathfrak{B}_k^s$ to diagonal lines $\mathfrak{L}_k$ as well as by "three-point laws" corresponding to the relationship satisfied by $\sigma_1 x(\tau)$, $x(\tau)$, $\sigma x(\tau)$, $w(\tau)$, and $\sigma w(\tau)$ for every $\tau \in \mathbb{Z}^2$. Moreover, $\mathfrak{B}_k^s$ is characterized by the restriction $\mathfrak{B}_k^x$ of the x-behavior to the line $\mathfrak{L}_k$ together with static laws corresponding to the relationships satisfied by $x(\tau)$ and $w(\tau)$ for every $\tau \in \mathbb{Z}^2$.

Note that for all $\tau \in \mathbb{Z}^2$ $\mathfrak{B}_k^s(\tau) = \mathfrak{B}_k^s(0, 0)$ and $\pi\mathfrak{B}^s(\tau) = \pi\mathfrak{B}^s(0, 0)$. Clearly, $\mathfrak{B}_k^s(0, 0)$ and $\pi\mathfrak{B}^s(0, 0)$ are linear subspaces of, respectively, $\mathbb{R}^q \times \mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q$. Moreover, for $k \in \mathbb{Z}$, $\mathfrak{B}_k^x = \mathfrak{B}_0^x$ is a 1-D AR behavior and hence can be represented as $\mathfrak{B}_0^x = \ker S(\sigma)$ for a suitable polynomial matrix $S(s)$. Thus it is easily seen that there exist linear subspaces $\mathscr{V}_0$ of $\mathbb{R}^q$ and $\mathscr{V}_1$ of $\mathbb{R}^n$, and real matrices $\bar{A}_0$, $\bar{A}_1$, $\bar{B}_0$, $\bar{B}_1$, and C such that $(w, x) \in \mathfrak{B}^s$ if and only if the following is satisfied:

$$\begin{cases} S(\sigma)x = o \\ \sigma_1 x = \bar{A}_0 x + \bar{A}_1 \sigma x + \bar{B}_0 w + \bar{B}_1 \sigma w + \mathscr{V}_1 \\ w = Cx + \mathscr{V}_0. \end{cases}$$

Defining $A_0 := \bar{A}_0 + \bar{B}_0 C$, $A_1 := \bar{A}_1 + \bar{B}_1 C$, $B_0 := [\bar{B}_0 | I]V$, $B_1 := [\bar{B}_1 | 0]V$, and $D := [I | 0]V$, with V such that $\mathrm{im}\, V = \begin{bmatrix} \mathscr{V}_0 \\ \mathscr{V}_1 \end{bmatrix}$, the previous equations become.

$$\begin{cases} S(\sigma)x = o \\ \sigma_1 x = (A_0 + A_1\sigma)x + (B_0 + B_1\sigma)v \\ w = Cx + Dv \end{cases}$$

where v is an auxiliary free variable.

In order to check that condition 1) of the theorem is verified define $A(s) := A_0 + A_1 s$ and suppose that $v|_{\mathfrak{L}_k} = 0$ for some $k \in \mathbb{Z}$. Then $x|_{\mathfrak{L}_{k+1}} = A(\sigma)x\ |_{\mathfrak{L}_k}$. As $x|_{\mathfrak{L}_{k+1}} \in \mathfrak{B}^x\ |_{\mathfrak{L}_{k+1}} = \mathfrak{B}_0^x = \ker S(\sigma)$, this implies that $S(\sigma)A(\sigma)x|_{\mathfrak{L}_k} = 0$. Thus, since also $\mathfrak{B}_k^x = \mathfrak{B}_0^x = \ker\ S(\sigma)$, there must hold $\ker S(\sigma)A(\sigma) \subseteq \ker S(\sigma)$, or equivalently, $A(\sigma) \ker S(\sigma) \subseteq \ker S(\sigma)$. To prove 2) assume that $x|_{\mathfrak{L}_k} = 0$ for some $k \in \mathbb{Z}$. Then $x|_{\mathfrak{L}_{k+1}} = B(\sigma)v$ (with $B(s) := B_1 s + B_0$). Consequently, $S(\sigma)B(\sigma)v = o$. As v is a free variable, this means that $S(s)B(s) = 0$, i.e., $\mathrm{im}\, B(\sigma) \subseteq \ker S(\sigma)$.

$x|_{\mathfrak{L}_{k+1}} = B(\sigma)v$ (with $B(s) := B_1 s + B_0$). Consequently, $S(\sigma)B(\sigma)v = o$. As v is a free variable, this means that $S(s)B(s) = 0$, i.e., $\mathrm{im}\, B(\sigma) \subseteq \ker S(\sigma)$.

Proof of the Lemma: To prove (1), we consider $k = 0$ (the arguments for other $k \in \mathbb{Z}$ are similar). Clearly, $\mathfrak{B}_0^s \subseteq \mathfrak{B}_0^*$. Consider now an element $(w, x) \in \mathfrak{B}_0^*$. Then $x \in \mathfrak{B}_0^x$ and therefore there exists an element $\bar{w} \in (\mathbb{R}^q)^{\mathfrak{L}_0}$ such that $(\bar{w}, x) \in \mathfrak{B}_0^s$. Moreover, for every $j \in \mathbb{Z}$, $(w(j, -j), x(j, -j)) \in \mathfrak{B}^s(j, -j)$ and hence there exists $(w^j, x^j) \in \mathfrak{B}_0^s$ such that $(w^j(j, -j), x^j(j, -j)) = (w(j, -j), x(j, -j))$. We next use the axiom of state in order to prove that in every point $(j, -j) \in \mathfrak{L}_0$ the value of $(\bar{w}, x)$ can be replaced by $(w^j(j, -j), x^j(j, -j))$ yielding still an element in $\mathfrak{B}_0^s$. This implies that $(w, x) \in \mathfrak{B}_0^s$ as desired. Starting with $j = 0$, let (w', x') and (w'', x'') be elements of $\mathfrak{B}^s$ such that $(w', x')|_{\mathfrak{L}_0} = (\bar{w}, x)$ and $(w'', x'')|_{\mathfrak{L}_0} = (w^0, x^0)$. Define the partition (T_-, T_0, T_+) of $\mathbb{Z}^2$ by $T_- := \cup\{\mathfrak{L}_k \mid k < 0\}$, $T_0 := \mathfrak{L}_0$ and $T_+ = \cup\{\mathfrak{L}_k \mid k > 0\}$. Let $D_+ = \{(0, 0)\}$ and $D_- := T_- \cup T_0 \setminus D_+$. Then $D_0 := P(D_+) \cap T_0 = D_+$ obviously separates D_- and D_+. Thus, since $x'|_{D_0} = x(0, 0) = x^0(0, 0) = x''|_{D_0}$, it follows from the axiom of state that $(w', x')|_{D_-} \wedge (w'', x'')|_{D_+} \in \mathfrak{B}^s|_{D_- \cup D_+}$. Consequently, $(\bar{w}, x)|_{\mathcal{Q}_0 \setminus \{(0,0)\}} \wedge (w^0, x^0)|_{\{(0,0)\}} = (\bar{w}, x)|_{\mathcal{Q}_0 \setminus \{(0,0)\}} \wedge (w, x)|_{\{(0,0)\}} \in \mathfrak{B}_0^s$.

Repeating the aforementioned reasoning with $(\bar{w}, x)$ replaced by $(\bar{w}, x)|_{\mathcal{Q}_0 \setminus \{(0,0)\}} \wedge (w, x)|_{\{(0,0)\}}$, $j = 1$ and $D_+ = \{(1, -1)\}$ yields that $(\bar{w}, x)|_{\mathcal{Q}_0 \setminus \{(0,0), (1,-1)\}} \wedge (w, x)|_{\{(0,0),(1,-1)\}} \in \mathfrak{B}_0^s$. So, $(w, x)|_{\{(0,0),(1,-1)\}} \in \mathfrak{B}_0^s|_{\{(0,0),(1,-1)\}}$. In this way it is not difficult to show that for every $i \in \mathbb{Z}_+ (w, x)|_{I_i} \in \mathfrak{B}_0^s|_{I_i}$ with $I_i := \{(-i, i), \cdots, (0, 0), \cdots (i, -i)\}$. Taking into account that $\mathfrak{B}_0^s$ is a complete 1-D behavior (cf. [11, Definition 3]), this implies that $(w, x) \in \mathfrak{B}_0^s$ proving claim (1) of the Lemma.

In order to show that (2) is satisfied, note that $\mathfrak{B}^s \subseteq \mathfrak{B}^*$. We next prove that the reciprocal inclusion also holds true. Let $(w, x) \in \mathfrak{B}^*$. Then there exist elements $(w', x'), (w'', x'') \in \mathfrak{B}^s$ such that $(w', x')|_{\mathcal{Q}_0} = (w, x)|_{\mathcal{Q}_0}$ and $\pi(w'', x'')(0, 0) = \pi(w, x)(0, 0)$. Clearly, $(x'(0,0), x'(1,-1)) = (x(0,0), x(1,-1)) = (x'(0,0), x'(1,-1))$. Therefore, the axiom of state implies that $(\tilde{w}, \tilde{x})|_{\mathcal{Q}_0 \cup \{(1,0)\}} := (w', x')|_{\mathcal{Q}_0} \wedge (w'', x'')|_{\{(1,0)\}} \in \mathfrak{B}^s|_{\mathcal{Q}_0 \cup \{(1,0)\}}$. Now, since $(\tilde{w}, \tilde{x})(1, 0)$ and $(w, x)(1, 0)$ are both elements of $\mathfrak{B}^s(1, 0)$, and $\tilde{x}(1, 0) = x''(1, 0) = x(1, 0)$, it follows from the state axiom that the value of $\tilde{w}$ at $(1, 0)$ can be replaced by $w(1, 0)$ yielding still a trajectory in $\mathfrak{B}^s|\mathcal{Q}_0 \cup \{(1, 0)\}$ (cf. proof of (1)). Thus, $(w', x')|_{\mathcal{Q}_0} \wedge (w, x'')|_{\{(1,0)\}} = (w, x)|_{\mathcal{Q}_0} \wedge (w, x)|_{\{(1,0)\}} = (w, x)|_{\mathcal{Q}_0 \cup \{(1,0)\}} \in \mathfrak{B}^s|_{\mathcal{Q}_0 \cup \{(1,0)\}}$. By successively considering trajectories $(w^j, x^j) \in \mathfrak{B}^s$ such that $\pi(w^j, x^j)(j, -j) = \pi(w, x)(j, -j)$, $j = \mp 1, \mp 2, \cdots$, it can be shown (using the foregoing argument) that for every $i \in \mathbb{Z}_+$ $(w, x)|_{\mathcal{Q}_0 \cup J_i} \in \mathfrak{B}^s|_{\mathcal{Q}_0 \cup J_i}$, where $J_i := \{(1, 0) + (-i, i), \cdots, (1, 0), \cdots, (1, 0) + (i, -i)\}$. Consequently, $(w, x)|_{\mathcal{Q}_0 \cup \mathcal{Q}_1} \in \mathfrak{B}^s|_{\mathcal{Q}_0 \cup \mathcal{Q}_1}$. Using the same kind of arguments as previously mentioned, it is not difficult to see that $(w, x)|_{\mathcal{D}_k} \in \mathfrak{B}^s|_{\mathcal{D}_k}$, with $\mathcal{D}_k := \bigcup_{i=-k}^{k} \mathcal{Q}_i$. Hence, as $\mathfrak{B}^s$ is complete, we conclude that $(w, x) \in \mathfrak{B}^s$ proving claim (2). ∎

Proof of Example 2: Let $\Sigma := (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$, with $\mathfrak{B} := \ker(\sigma_2 - \sigma_1) = \ker(\sigma - 1)$. Suppose that Σ has a state realization $\Sigma^s = (\mathbb{Z}^2, \mathbb{R}, \mathbb{R}^n, \mathfrak{B}^s)$ described by the following behavioral equations:

$$\begin{cases} S(\sigma)x = o & \text{(A1)} \\ \sigma_1 x = A(\sigma)x + B(\sigma)v & \text{(A2)} \\ w = Cx + Dv. & \text{(A3)} \end{cases}$$

Let further $\bar{D}(s)$ be a maximal left divisor of $S(s)$, and $R(s)$ a polynomial matrix such that $S(s) = \bar{D}(s)R(s)$. Define the following 1-D behaviors. $\tilde{\mathfrak{B}} := \{\tilde{x} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid S(\sigma)\tilde{x} = o\} = \ker S(\sigma)$ and $\bar{\mathfrak{B}} := \{\bar{x} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists \tilde{x} \in \tilde{\mathfrak{B}} \text{ s.t. } \bar{x} = R(\sigma)\tilde{x}\}$. It is easily checked that $\bar{\mathfrak{B}} = \ker \bar{D}(\sigma)$ and as $\bar{D}(s)$ is square and full rank, $\bar{\mathfrak{B}}$ is a finite-dimensional subspace of $(\mathbb{R}^n)^{\mathbb{Z}}$. We will call a finite-dimensional behavior an *autonomous behavior*, and refer to variables whose behavior is autonomous as *autonomous variables*.

To prove the desired result, we will use the following fact.

Fact: Let $\tilde{x} \in \tilde{\mathfrak{B}}$ and suppose that, for some polynomial matrix $N(s)$, the variable $\xi := N(\sigma)\tilde{x}$ is an autonomous variable. Then, there exists a polynomial matrix $K(s)$ such that $\xi = K(\sigma)\bar{x}$, with $\bar{x} := R(\sigma)\tilde{x}$ (as above).

Proof of the Fact: Recall that $S(s) = \bar{D}(s)R(s)$ with $\bar{D}(s)$ square, full rank, and maximal left divisor of $S(s)$, and $R(s)$ full-row rank. Note that $R(s)$ is left prime, and so there is a unimodular polynomial matrix $U(s)$ such that $R(s)U^{-1}(s) = [I | 0]$. For $\tilde{x} \in \tilde{\mathfrak{B}}$, let $\hat{x} := U(\sigma)\tilde{x}$ and partition $\hat{x} := \text{col}(\hat{x}_1, \hat{x}_2)$ such that $R(\sigma)U^{-1}(\sigma)\hat{x} = \hat{x}_1$. Define $\hat{\mathfrak{B}} := \{\hat{x} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \bar{D}(\sigma)\hat{x}_1 = o\}$; then $\tilde{\mathfrak{B}} = \{\tilde{x} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists \hat{x} \in \hat{\mathfrak{B}} \text{ s.t. } \tilde{x} = U^{-1}(\sigma)\hat{x}\}$. Consider now $\xi := N(\sigma)\tilde{x} = N(\sigma)U^{-1}(\sigma)\hat{x}$. If ξ is autonomous, none of its components can depend on $\hat{x}_2$. Thus $N(s)U^{-1}(s) = [K(s) | 0]$, for some polynomial matrix $K(s)$, with $N(\sigma)U^{-1}(\sigma)\hat{x} = K(\sigma)\hat{x}_1$. So $\xi = K(\sigma)\hat{x}_1 = K(\sigma)R(\sigma)U^{-1}(\sigma)\hat{x} = K(\sigma)R(\sigma)\tilde{x}$, i.e., $\xi = K(\sigma)\bar{x}$ with $\bar{x} := R(\sigma)\tilde{x}$.

Now let $\bar{x} \in (\mathbb{R}^n)^{\mathbb{Z}^2}$ be such that $\bar{x} = R(\sigma)x$ and $x \in \Pi_x \mathfrak{B}^s$, and define $w_k := w|_{\mathcal{Q}_k}$, $\bar{x}_k := \bar{x}|_{\mathcal{Q}_k}$ $(K \in \mathbb{Z})$. Clearly, for all $k \in \mathbb{Z}$, $\bar{x}_k$, and w_k are autonomous variables. This implies that, in (A3), $D = 0$. Moreover, it follows from (A2) that $R(\sigma)\sigma_1 x = R(\sigma)A(\sigma)x + R(\sigma)B(\sigma)v$, and thus also $R(s)B(s) = 0$. This yields: $\sigma_1\bar{x} = R(\sigma)A(\sigma)x$, $w = Cx$. Now, invoking the previous fact it is not difficult to see that there exists polynomial matrices $E(s)$ and $F(s)$ such that: $\sigma_1\bar{x} = E(\sigma)\bar{x}$ and $w = F(\sigma)\bar{x}$. Hence, the variables w and $\bar{x}$ satisfy the following equations:

$$\begin{cases} D(\sigma)\bar{x} = o \\ \sigma_1\bar{x} = E(\sigma)\bar{x} \\ w = F(\sigma)\bar{x}. \end{cases} \tag{A4}$$

Let $\mathfrak{B}^*$ be the w-behavior induced by (A4). Then $\mathfrak{B} \subseteq \mathfrak{B}^*$, and as $\mathfrak{B}^*$ is (obviously) finite-dimensional, so will be $\mathfrak{B}$. This contradicts the fact that $\mathfrak{B} = \ker(\sigma - 1)$. We conclude in this way that Σ cannot have a state realization. ∎

Proof of Proposition 4: It is easily seen that, for $k = 0, 1, 2, \cdots$: 1) $\mathcal{T}_k$ is $A(\sigma)$-invariant and im $B(\sigma)\mathcal{T}_k$, and 2) $\mathcal{T}_{k+1} \subseteq \mathcal{T}_k$. We will show that: 3) there exists a polynomial matrix $T_k(s)$ such that $\mathcal{T}_k = \ker T_k(\sigma)$ and 4) there is $L \in \mathbb{N}$ such that $\mathcal{T}_k = \mathcal{T}_L$ for all $k \geq L$.

It follows from here that $\mathcal{T}_L = A(\sigma)\mathcal{T}_L + \text{im}\, B(\sigma) = \ker T_L(\sigma)$, and consequently $\mathcal{T} = \mathcal{T}_L$, $T(s) = T_L(s)$ and $\Sigma^s(T(s), A(s), B(s), C, D) = \Sigma^s(S(s), A(s), B(s), C, D)$. This yields the desired result.

To prove 3), note that this holds for $k = 0$, with $T_0(s) = S(s)$. Suppose that for $k \in \mathbb{N}$ there exists $T_k(s)$ such that $\mathcal{T}_k = \ker T_k(\sigma)$. Then $\mathcal{T}_{k+1}$ can be described as $\mathcal{T}_{k+1} = \{x_1 \mid \exists x_0, v \text{ s.t. (I) is satisfied}\}$, with (I) given by

$$\begin{bmatrix} T_k(\sigma) & 0 \\ A(\sigma) & B(\sigma) \end{bmatrix} \begin{pmatrix} x_0 \\ v \end{pmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} x_1. \tag{I}$$

Let $U(s) := \text{col}(U_1(s), U_2(s))$, be a unimodular matrix such that $U_1(s)R(s) = F(s)$, $U_2(s)R(s) = 0$, with $R(s) := \text{col}([T_k(s)\ 0], [A(s)\ B(s)])$ and $F(s)$ a full-row rank matrix. Denoting $T_{k+1}(s) := U_2(s)\begin{bmatrix} 0 \\ I \end{bmatrix}$, (I) is equivalent to

$$\begin{cases} F(\sigma)\begin{pmatrix} x_0 \\ v \end{pmatrix} = U_1(\sigma)\begin{bmatrix} 0 \\ I \end{bmatrix} x_1 \\ T_{k+1}(\sigma)x_1 = o \end{cases}$$

and as $F(\sigma)$ is surjective it is clear that $\mathcal{T}_{k+1} = \ker T_{k+1}(\sigma)$.

To see that (4) holds true, note that if $\mathcal{T}^*$ and $\mathcal{T}^{**}$ are two autoregressive behaviors in $(\mathbb{R}^p)^{\mathbb{Z}}$, for some $p \in \mathbb{N}$, such that $\mathcal{T}^{**} \subseteq \mathcal{T}^*$ then the number m^{**} of free variables in $\mathcal{T}^{**}$ cannot exceed the number m^* of free variables in $\mathcal{T}^*$. Thus, for the sequence $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \cdots \supseteq \mathcal{T}_k \supseteq \mathcal{T}_{k+1} \supseteq \cdots$ there exists $M \in \mathbb{N}$ such that for $k \geq M$ the number of free variables m_k of $\mathcal{T}_k$ is $m_k = m_M$. Moreover, for every autoregressive behavior $\mathcal{T}$ there holds that for $\tau \in \mathbb{N}$ sufficiently large dim $\mathcal{T}|_{[0,\tau)} = \bar{n} + \bar{m}\tau$, where $\bar{m}$ is the number of free variables in

$\mathcal{T}$ and $\bar{n}$ is the state-space dimension in a minimal state-space realization of $\mathcal{T}$ (cf. [11]). So, if $\mathcal{T}^{**} \subseteq \mathcal{T}^{*}$ and $m^{**} = m^{*}$, there must be $\dim \mathcal{T}^{**}|_{(0,\tau)} = n^{**} + m^{**}\tau \le n^{*} + m^{**}\tau = \dim \mathcal{T}^{*}|_{(0,\tau)}$, for $\tau \in \mathbb{N}$ sufficiently large, implying that $n^{**} \le n^{*}$ (here n^{*} and n^{**} have the obvious meaning). Thus, for the sequence $\mathcal{T}_M \supseteq \mathcal{T}_{M+1} \supseteq \cdots \mathcal{T}_k \supseteq \mathcal{T}_{k+1} \supseteq \cdots$, we will have $n_M \ge n_{M+1} \ge \cdots \ge n_k \ge n_{k+1} \ge \cdots$ implying that there exists $L \in \mathbb{N}$ such that $n_k = n_L$ for all $k \ge L$. This means that $\mathcal{T}_L$ and $\mathcal{T}_k$ $(k \ge L)$ have the same number of free variables and the same minimal state-space dimension, and as $\mathcal{T}_k \subseteq \mathcal{T}_L$ it follows that $\mathcal{T}_k = \mathcal{T}_L$ for all $k \ge L$ (cf. [11]). This concludes the proof of Proposition 4. ■

Proof of Lemma 2: Assume that $M(s)$ and $F(s)$ are, respectively, $k \times l$ and $j \times l$ matrices $(j \le k)$. Let I denote the $(k-j) \times (k-j)$ identity matrix and denote by 0 the $(k-j) \times j$ zero matrix. Clearly, $[0\; I]U^{-1}$ is a minimal left annihilator of M. Thus S is also a minimal left annihilator of M iff it is unimodularly equivalent to $[0\; I]U^{-1}$, i.e., if and only if there exists $V(s)$ unimodular such that $S(s) = V(s)[0\; I]U^{-1}(s)$. ■

Proof of Proposition 5:

1) Suppose that for the state-space system $\Sigma^s = \Sigma^s(S(s), A(s), B(s), C, D)$, the polynomial matrix $[s_1 - A(s) \mid B(s)]$ is left prime. It is not difficult to see that this implies that $\bar{\Sigma}^s = \Sigma^s(0, A(s), B(s), C, D)$ is reachable. This means that starting with $x_0 = o$ on $\mathfrak{L}_0$ it is possible to reach any x^* in the trim subspace $\bar{\mathcal{T}}$ of $\bar{\Sigma}^s$ on $\mathfrak{L}_k$, for k sufficiently large. Clearly, the trim subspace $\mathcal{T}$ of Σ^s is contained on $\bar{\mathcal{T}}$, and thus also Σ^s will be reachable.

2) As mentioned in Section III, if $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is a controllable AR 2-D system, $\mathfrak{B}$ is the output behavior of an input–output system described by a quarter-plane causal rational transfer function. Moreover, according to [1] every such transfer function can be realized by means of a state-space model $(0, A(s), B(s), C, D)$, with $[s_1 - A(s) \mid B(s)]$ left prime (over $\mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$). By 1) this means that $\Sigma^s = \Sigma^s(0, A(s), B(s), C, D)$ is a reachable state-space realization of Σ. ■

Proof of Proposition 6: We will first prove the statements about $\mathcal{V}^+$. In order to do so, we will show that for all $k = 0, 1, 2, \cdots$ there is a polynomial matrix $V_k^+(s)$ such that $\mathcal{V}_k^+ = \ker V_k^+(\sigma)$, and moreover that $\mathcal{V}_{k+1}^+ \subseteq \mathcal{V}_k^+$. By similar arguments as the ones used in the proof of Proposition 4, this implies that 1) and 2) hold true for $\mathcal{V}^+$. Clearly $\mathcal{V}_1^+ \subseteq \mathcal{V}_0^+$. Suppose now that $\mathcal{V}_k^+ \subseteq \mathcal{V}_{k-1}^+$. Then $\mathcal{V}_{k+1}^+ := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v$ s.t. $A(\sigma)x + B(\sigma)v \in \mathcal{V}_k^+$ and $\bar{C}(\sigma)x + \bar{D}v = o\} \subseteq \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v$ s.t. $A(\sigma)x + B(\sigma)v \in \mathcal{V}_{k-1}^+$ and $\bar{C}(\sigma)x + \bar{D}v = o\} =: \mathcal{V}_k^+$. So $\mathcal{V}_{k+1}^+ \subseteq \mathcal{V}_k^+$, for $k = 0, 1, 2, \cdots$. Now, $\mathcal{V}_0^+ = (\mathbb{R}^n)^{\mathbb{Z}}$ and so there exists a polynomial matrix $V_0^+(s)$ such that $\mathcal{V}_0^+ = \ker V_0^+(s)$, namely, $V_0^+(s) = 0$. Suppose that for $k \in \mathbb{N}$ there is a polynomial matrix $V_k^+(s)$ such that $\mathcal{V}_k^+ = \ker V_k^+(\sigma)$. Then $\mathcal{V}_{k+1}^+ = \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v$ s.t. $A(\sigma)x + B(\sigma)v \in \ker V_k^+(\sigma)$ and $\bar{C}(\sigma)x + \bar{D}v = o\}$, i.e., $x \in \mathcal{V}_{k+1}^+$ iff

$$\begin{bmatrix} V_k^+(\sigma)A(\sigma) \\ \bar{C}(\sigma) \end{bmatrix} x = \begin{bmatrix} -V_k^+(\sigma)B(\sigma) \\ -\bar{D} \end{bmatrix} v.$$

Eliminating the variable v from this description (cf. Proof of Proposition 4) shows that $\mathcal{V}_{k+1}^+ = \ker V_{k+1}^+(\sigma)$, with $V_{k+1}^+(s)$ a suitable polynomial matrix. So, for $k = 0, 1, 2, \cdots$, there exists a polynomial matrix $V_k^+(s)$ such that $\mathcal{V}_k^+ = \ker V_k^+(\sigma)$. As mentioned previously, this implies that 1) and 2) hold true. Consider now $\mathcal{V}_{L^+}^+$ with L^+ as in 1). Then $\mathcal{V}_{L^+}^+ = \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v$ s.t. $A(\sigma)x + B(\sigma)v \in \mathcal{V}_{L^+}^+$ and $\bar{C}(\sigma)x + \bar{D}v = o\}$. Let $x_0 \in \mathcal{V}_{L^+}^+$. Then $x_0 \in \Pi_x \mathfrak{B}^s|_{\mathfrak{L}_0}$ and moreover there exists $x_k \in \Pi_x \mathfrak{B}^s|_{\mathfrak{L}_k}$ and v_k, $k = 0, 1, 2, \cdots$, such that $x_{k+1} = A(\sigma)x_k + B(\sigma)v_k$, $Cx_k + Dv_k = o$. It can be shown that this implies that there is $(w, x, v) \in \mathfrak{B}^s$ such that $(w, x, v)|_{\mathfrak{L}_k} = (o, x_k, v_k)$, for $k = 0, 1, 2 \cdots$. So clearly $\mathcal{V}_{L^+}^+ \subseteq \mathcal{V}^+$. Finally, it is not difficult to see that $\mathcal{V}^+ \subseteq \mathcal{V}_k^+$ for all $k = 0, 1, 2, \cdots$ and so $\mathcal{V}_{L^+}^+ = \mathcal{V}^+$.

To prove the statements about $\mathcal{V}^-$, we will first see that for all $k = 0, 1, 2, \cdots$, $\mathcal{V}_{k+1}^- \subseteq \mathcal{V}_k^-$. Clearly, $\mathcal{V}_1^- \subseteq \mathcal{V}_0^-$. Suppose that for $k \in \mathbb{N}$ $\mathcal{V}_k^- \subseteq \mathcal{V}_{k-1}^-$. Then $\mathcal{V}_{k+1}^- := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v\; \exists \bar{x} \in \mathcal{V}_k^-$ s.t. $A(\sigma)\bar{x} + B(\sigma)v = x$ and $\bar{C}(\sigma)\bar{x} + \bar{D}v = o\} \subseteq \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v\; \exists \bar{x} \in \mathcal{V}_{k-1}^-$ s.t. $A(\sigma)\bar{x} + B(\sigma)v = x$ and $\bar{C}(\sigma)\bar{x} + \bar{D}v = o\} =: \mathcal{V}_k^-$. So $\mathcal{V}_{k+1}^- \subseteq \mathcal{V}_k^-$ for all $k = 0, 1, 2, \cdots$. Next, we will show that for all $k = 0, 1, 2, \cdots$ there exists a polynomial matrix $V_k^-(s)$ such that $\mathcal{V}_k^- = \ker V_k^-(\sigma)$. This holds for $k = 0$ with $V_0^-(s) = 0$. Suppose that it also holds for some $k \in \mathbb{N}$. Then $\mathcal{V}_{k+1}^- = \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \exists v\; \exists \bar{x} \in \ker V_k^-(\sigma)$ s.t. $A(\sigma)\bar{x} + B(\sigma)v = x$ and $\bar{C}(\sigma)\bar{x} + \bar{D}v = o\}$, and so $x \in \mathcal{V}_{k+1}^-$ iff there exists $\bar{x}$ and v s.t.

$$\begin{bmatrix} V_k^-(\sigma) & 0 \\ A(\sigma) & B(\sigma) \\ \bar{C}(\sigma) & \bar{D} \end{bmatrix} \begin{pmatrix} \bar{x} \\ v \end{pmatrix} = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} x.$$

By similar arguments as previously mentioned, we conclude that there is a polynomial matrix $V_{k+1}^-(s)$ such that $\mathcal{V}_{k+1}^- = \ker V_{k+1}^-(\sigma)$. This shows the desired result. Consequently 1) and 2) hold true for $\mathcal{V}^-$.

Note, finally, that $\mathcal{V}^- \subseteq \mathcal{V}_k^-$ for all $k = 0, 1, 2, \cdots$. Moreover, if $x_0 \in \mathcal{V}_{L^-}^-$, $x_0 \in \Pi_x \mathfrak{B}^s|_{\mathfrak{L}_0}$ and there exist v_{-k} and x_{-k}, $k = 1, 2, \cdots$ such that $x_{-k} \in \Pi_x \mathfrak{B}^s|_{\mathfrak{L}_{-k}}$, $A(\sigma)x_{-k} + B(\sigma)v_{-k} = x_{-k+1}$, and $Cx_{-k} + Dv_{-k} = o$. This implies that there is $(w, x) \in \mathfrak{B}^s$ such that $(w, x)|_{\mathfrak{L}_{-k}} = (o, x_{-k})|_{\mathfrak{L}_{-k}}$, $k = 1, 2, \cdots$ and $x|_{\mathfrak{L}_0} = x_0$. So $x_0 \in \mathcal{V}^-$ and we conclude that $\mathcal{V}^- = \mathcal{V}_{L^-}^-$. ■

Proof of Proposition 7: For the given state-space system Σ^s, the (w, x)-behavior $\mathfrak{B}^s$ is described by $-U(\sigma)Fw = U(\sigma)E(\sigma, \sigma_1)x$. Thus, $\{w = o\} \Rightarrow \{x = o\}$ iff $\ker U(\sigma)E(\sigma, \sigma_1) = \{0\}$. Now, it is a well-known result that given a 2-D polynomial matrix $R(s, s_1)$, $\ker R(\sigma, \sigma_1) = \{0\}$ iff $R(\lambda, \lambda_1)$ has full-column rank for all $(\lambda, \lambda_1) \in (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. This yields the desired result. ■

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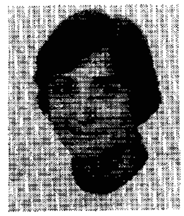

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