State for 2-D Systems

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ABSTRACT

A new definition of state for N-D systems is given in a noncausal context. This definition is based on a deterministic Markovian-like property. It is shown that, for the particular case of (AR) 2-D systems, it yields systems that can be described by a special kind of first-order equations. The solutions of these equations can be simulated by means of a local line-by-line computational scheme.

1. INTRODUCTION

The main motivation of this paper is to examine the concept of state for N-D systems. However, for simplicity of exposition, we will concentrate mainly on discrete 2-D systems. The theory of dynamical systems has been mainly concerned with 1-D systems, with phenomena evolving in time. 2-D systems have been introduced to describe phenomena depending on two independent variables, often regarded as spatial variables, as in image analysis.

Our approach to 2-D systems is inspired by some of the recent work [5] in the area of 1-D dynamical systems. However, an important difference which we will emphasize is the following. Whereas the 1-D systems considered in [5] are defined over time, and have therefore a natural preferred direction (namely forward time, past and future), we will not view 2-D systems as having a preferred direction. In fact, when considering 2-D systems there are

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problems in which it is indeed natural to consider both, one, or none of the independent variables as having a preferred forward direction. Just as in 1-D systems, it may or may not be natural to introduce a preferred direction. All these situations are valid. In this paper we will consider systems without preferred directions, since we believe that most situations in image analysis or with spatial coordinates are connected to this case. Note that this point of view is rather different from many of the papers on 2-D systems, where a sector of the plane is chosen as representing the "past". Another difference of our approach is that, as in [5], we will not take the input-output structure as our initial vantage point. We will view a discrete 2-D system simply as a family of functions defined on $\mathbb{Z}^2$ and view the state as a convenient set of latent variables.

Most of the contributions in the area of 2-D systems deal with 2-D systems in input-output form and with state-space representations of these systems. The main ideas in this field were introduced by Attasi [1], Roesser [4], and Fornasini and Marchesini [2]. The theory of discrete 2-D systems developed by these authors is based on the notion of 2-D causality with respect to a prespecified (partial) ordering of $\mathbb{Z}^2$. It is in this context that, in [2], a definition of a 2-D system in input-output form is given and state concepts are introduced.

In this paper we will propose a new definition of discrete 2-D system as well as a new concept of state. In this framework we will discuss the question of state-space representations.

The paper is organized as follows.

In Section 2 we give a definition of 2-D system and specify the class of systems to be considered in the sequel.

In Section 3 we introduce the notion of Markovian system.

In Section 4 we consider Markovian (AR) systems and present a theorem on the representation of such systems. This constitutes the basic result of the paper.

State-space systems are introduced in Section 5.

Finally, in Section 6 we discuss some aspects of the problem of simulating 2-D systems in state-space form. We will pay special attention to the recursive computation of solutions.

Concluding remarks are presented in Section 7.

All proofs are given in the appendix.

2. 2-D SYSTEMS

In this section we will define the class of linear, shift-invariant, and complete 2-D systems and relate this class to the family of what we will call autoregressive 2-D systems. The notions we introduce here are the 2-D analogues of the notions studied in [5] for 1-D systems. For generality, we will start with a basic definition for an N-D system.

DEFINITION 2.1. An *N-D system* is characterized by a *parameter set* $T \subset \mathbb{R}^N$, a *signal space* W, and a subset $\mathfrak{B}$ of W^T which will be called the *behavior* of the system. The system Σ defined by T, W, and $\mathfrak{B}$ is denoted by $\Sigma := (T, W, \mathfrak{B})$.

The intuitive content of this definition is as follows. An N-D system is defined by N independent variables taking their values in $T \subset \mathbb{R}^N$. For 1-D dynamical systems T is the time set. In image analysis T consists of the spatial variables, usually $\mathbb{R}^2$, $\mathbb{Z}^2$, or $\{1,2,\dots,L\}^2$ (if we assume that the images appear as $L \times L$ pixels), etc. The phenomenon which we are describing is specified by attributes which take their values in the space W. Often W is the space $\mathbb{R}^q$: the phenomenon is described by q real-valued attributes. In image analysis W may be the discrete set $\{0,1,\dots,k\}$ if we assume k grey levels, etc. Now, each realization of the phenomenon yields a trajectory $w: T \to W$. We assume that the phenomenon is governed by certain laws. These laws let us conclude that a certain trajectory can and others cannot occur. This yields the behavior $\mathfrak{B} \subset W^T$; $\mathfrak{B}$ consists of those trajectories compatible with the laws governing the phenomenon.

EXAMPLES. In [5, 6] many examples of 1-D systems are given. We will now present two examples of 2- and 3-D systems.

1. *Image processing*. Let $w_1: \mathbb{Z}^2 \to \mathbb{R}$ denote the unprocessed picture and $w_2: \mathbb{Z}^2 \to \mathbb{R}$ the processed picture. Assume that w_2 is a convolution $w_2 = G * w_1$ of w_1 with $G: \mathbb{Z}^2 \to \mathbb{R}$ a specified kernel (think for simplicity of G as having compact support) [that is, $w_2(k,l) = \sum_{l'=-\infty}^{+\infty}\sum_{k'=-\infty}^{+\infty} G(k-k', l-l')w_1(k',l')$]. This defines a 2-D system $(\mathbb{Z}^2, \mathbb{R}^2, \mathfrak{B})$ with $\mathfrak{B} := \{(w_1, w_2): \mathbb{Z}^2 \to \mathbb{R}^2 \mid w_2 = G * w_1\}$.

2. *Maxwell's equations*. Consider the static version of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

$$\nabla \times \mathbf{E} = \vec{0},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0},$$

$$\nabla \cdot \mathbf{j} = 0, \tag{M}$$

where $\mathbf{E}:\mathbb{R}^3 \to \mathbb{R}^3$ is the electric field, $\mathbf{B}:\mathbb{R}^3 \to \mathbb{R}^3$ the magnetic field, $\mathbf{j}:\mathbb{R}^3 \to \mathbb{R}^3$ the electric current density, and $\rho:\mathbb{R}^3 \to \mathbb{R}$ the electric charge density. $\nabla\cdot$ denotes the divergence, $\nabla\times$ the curl, ϵ_0 the dielectric constant of free space, and c the speed of light. This defines the 3-D system $(\mathbb{R}^3, \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}, \mathfrak{B})$ with $\mathfrak{B} := \{(\mathbf{E},\mathbf{B},\mathbf{j},\rho):\mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times \mathbb{R} \,|(\mathbf{E},\mathbf{B},\mathbf{j},\rho)$ satisfies (M)$\}$.

In the sequel we consider only the special class of 2-D systems with $T = \mathbb{Z}^2$ and $W = \mathbb{R}^q$, for some integer q.

Definition 2.2. The 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is *linear* if $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$.

Of the many symmetries which one can study for dynamical systems, time invariance is one of the most important and most elementary ones. For 2-D systems the analogous property is shift invariance. We will consider two operators σ_1 and σ_2, on $(\mathbb{R}^q)^{\mathbb{Z}^2}$, defined as follows. $\sigma_i:(\mathbb{R}^q)^{\mathbb{Z}^2} \to (\mathbb{R}^q)^{\mathbb{Z}^2}$ $(i = 1,2)$; σ_1 associates with $w:\mathbb{Z}^2 \to \mathbb{R}^q$ a function $\sigma_1 w:\mathbb{Z}^2 \to \mathbb{R}^q$ such that $\sigma_1 w(t_1,t_2) := w(t_1+1,t_2)$, and will be called the *left shift*; the action of the *down shift* σ_2 is given by $\sigma_2 w(t_1,t_2) := w(t_1,t_2+1)$ $\forall(t_1,t_2)\in\mathbb{Z}^2$. The operators σ_1^{-1} and σ_2^{-1} are respectively the *right shift* and *up shift*, and σ_1^k, σ_2^k for $k\in\mathbb{Z}$ are defined in the obvious way. Observe that the operators σ_1 and σ_2 commute.

Definition 2.3. $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is a *shift-invariant* 2-D system if $\sigma_i\mathfrak{B} = \mathfrak{B}$, $i = 1,2$.

Our examples 1 and 2 are both linear and shift-invariant.

In this paper we will consider the following class of 2-D systems described by the behavioral equations involving the signal w and its shifts:

$$\sum_{k'=-k_1}^{k_2} \sum_{l'=-l_1}^{l_2} R_{k'l'} w(k+k', l+l') = 0 \qquad \forall k,l\in\mathbb{Z},$$

where $R_{k'l'}$ is a real $g\times q$ matrix for $-k_1 \leqslant k' \leqslant k_2$ and $-l_1 \leqslant l' \leqslant l_2$. Introducing the polynomial matrix in two variables $R\in\mathbb{R}^{g\times q}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ defined by

$$R(s_1, s_1^{-1}, s_2, s_2^{-1}) := \sum_{l'}\sum_{k'} R_{k'l'} s_1^{k'} s_2^{l'}$$

allows us to write this behavioral equation as

$$R\left(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}\right) w = 0 \tag{AR}$$

This defines the 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ with $\mathfrak{B} = \mathfrak{B}(R) := \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ and $R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ viewed as a map from $(\mathbb{R}^q)^{\mathbb{Z}^2}$ to $(\mathbb{R}^g)^{\mathbb{Z}^2}$.

We will call the systems described by such behavioral equations 2-D *AR systems*. (AR stands for auto regressive.) Thus each AR system $(\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is described by a $g \in \mathbb{N}$ and a polynomial operator in two variables, $R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) : (\mathbb{R}^q)^{\mathbb{Z}^2} \to (\mathbb{R}^g)^{\mathbb{Z}^2}$.

The 1-D version of AR systems has been studied in detail in [5]. It was shown in particular that AR systems are characterized by linearity, time invariance, and completeness (or equivalently, *closure* in the topology of pointwise convergence).

A similar result holds, in fact, for 2-D systems. A subset of $\mathbb{Z}^2$ of the form $I = \{(k, l) \in \mathbb{Z}^2 | k_1 \leqslant k \leqslant k_2,\ l_1 \leqslant l \leqslant l_2\}$ for some $-\infty \leqslant k_1 \leqslant k_2 \leqslant +\infty$, $-\infty \leqslant l_1 \leqslant l_2 \leqslant +\infty$, is called an *interval* in $\mathbb{Z}^2$. If k_1, k_2, l_1, and l_2 are finite, then we will call I a *finite interval.*

DEFINITION 2.4. A shift-invariant 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is *complete* if $\{w \in \mathfrak{B}\} \Leftrightarrow \{w|_I \in \mathfrak{B}|_I$ for all finite intervals $I \subset \mathbb{Z}^2\}$.

Our first result establishes the connection between the class of linear, shift-invariant, and complete 2-D systems and the class of 2-D AR systems.

THEOREM 2.1. *Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ be a 2-D system. Then the following conditions are equivalent*:

(1) Σ is an AR system.

(2) Σ is a linear, shift-invariant, and complete system.

(3) $\mathfrak{B}$ is a linear, shift-invariant, closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$ equipped with the topology of pointwise convergence.

Proof. See Appendix. ∎

REMARK 1. It is important to remark that, whereas $R(s_1, s_1^{-1}, s_2, s_2^{-1})$ uniquely determines $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$, there are for a given $\mathfrak{B}$ many polynomial matrices describing this behavior. Simply take $R' = UR$ with U any unimodular polynomial matrix [that is, $U(s_1, s_1^{-1}, s_2, s_2^{-1})$ is square and $\det U(s_1, s_1^{-1}, s_2, s_2^{-1}) = s_1^{d_1} s_2^{d_2}$ for some $d_1, d_2 \in \mathbb{Z}$]. Then clearly $\ker R'(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$.

REMARK 2. Contrary to the 1-D case (where we can always assure that the number g of scalar autoregressive equations necessary to describe a 1-D AR system is not greater than q), nothing can be said about the number of scalar 2-D AR equations which are necessary to represent a linear, shift-invariant, and complete 2-D system. This is illustrated in the example below.

EXAMPLE.

Let

$$R(s_1, s_1^{-1}, s_2, s_2^{-1}) = \mathrm{col}\left[(s_1+1)^g(s_2+1), (s_1+1)^{g-1}(s_2+1)^2, \dots, (s_1+1)(s_2+1)^g\right]$$

for some $g \in \mathbb{N}$, and consider the 2-D system Σ defined by $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$ with $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$. It is not difficult to see that Σ cannot be represented by less that g scalar 2-D AR equations.

3. MARKOVIAN SYSTEMS

Let $\Sigma = (T, W, \mathfrak{B})$ be an N-D system. When would we want to call $\mathfrak{B}$ a "state" behavior, or, to borrow a term from the theory of stochastic processes, when would we want to call Σ "Markovian"? Is there a relation between $\mathfrak{B}$ being a state behavior and the fact that $\mathfrak{B}$ can be described by behavioral equations which are first-order in σ_1 and σ_2? Those are the questions which we will examine in the sequel of this paper. Because it is customary to do so in mathematical system theory, we will use the notation X for the signal space of a Markovian system. Also, we will concentrate on N-D systems where the parameter set T is an interval of $\mathbb{Z}^N$, that is, $T = \{(k_1, k_2, \dots, k_N) \in \mathbb{Z}^N | l_1^i \leqslant k_i \leqslant l_2^i,\ i = 1, \dots, N\}$ for some $-\infty \leqslant l_1^i \leqslant l_2^i \leqslant +\infty$. However, all definitions immediately generalize to intervals in $\mathbb{R}^N$.

Let $\Sigma = (T, W, \mathfrak{B})$ be an N-D system, and let $T' \subset T$. Then $\Sigma|_{T'}$, the restriction of Σ to T', is defined by $\Sigma|_{T'} := (T', W, \mathfrak{B}|_{T'})$ with, as usual, $\mathfrak{B}|_{T'} := \{w' \in W^{T'} | \exists w \in \mathfrak{B}$ such that $w|_{T'} = w'\}$. Let $T_1, T_2 \subset T$. Then we will say that $\mathfrak{B}$ is (T_1, T_2)-*concatenable* if

$$\left\{ w_1 \in \mathfrak{B}|_{T_1},\ w_2 \in \mathfrak{B}|_{T_2}, w_1|_{T_1 \cap T_2} = w_2|_{T_1 \cap T_2} \right\} \Rightarrow \left\{ w_1 \underset{(T_1, T_2)}{\wedge} w_2 \in \mathfrak{B}|_{T_1 \cup T_2} \right\}.$$

Here $w_1 \underset{(T_1,T_2)}{\wedge} w_2$ denotes the element $w \in W^{T_1 \cup T_2}$ such that $w|_{T_1} = w_1$ and $w|_{T_2} = w_2$.

For $t' = (t'_1, t'_2, \ldots, t'_N) \in \mathbb{Z}^N$, $t'' = (t''_1, t''_2, \ldots, t''_N) \in \mathbb{Z}^N$, define the distance $d(t', t'') = \sum_{i=1}^N |t'_i - t''_i|$. We will call $(t_1, t_2, \ldots, t_k)$, $t_i \in \mathbb{Z}^N$, $i \in \{1, \ldots, k\}$, a *path* if $d(t_i, t_{i+1}) = 1$ for $i \in \{1, \ldots, k-1\}$. We will call it a *straight path* if $d(t_1, t_k) = k - 1$.

As an aside, observe the following relation between convex sets and intervals. We will call $T \subset \mathbb{Z}^N$ *convex* if every straight path $(t_1, \ldots, t_k)$ with $t_1, t_k \in T$ is completely contained in T, that, is $t_i \in T$ $\forall i \in \{1, \ldots, k\}$. It is easy to see that T is convex iff it is an interval.

Let $T_+, T_0, T_- \subset \mathbb{Z}^N$. Then T_+ and T_- are said to be *separated* by T_0 if each straight path $(t_1, t_2, \ldots, t_k)$ with $t_1 \in T_+$ and $t_k \in T_-$ must contain a point $t_i \in T_0$ for some $i \in \{1, \ldots, k\}$. This is illustrated in Figure 1.

We now come to our crucial definition of the Markov property.

DEFINITION 3.1. Let T be an interval in $\mathbb{Z}^N$. The N-D system $\Sigma = (T, X, \mathfrak{B})$ is said to be *Markovian* if for all triples of subsets (T_+, T_0, T_-), $T_+, T_0, T_- \subset T$, such that T_+ and T_- are separated by T_0, there holds that $\mathfrak{B}$ is $(T_+ \cup T_0, T_- \cup T_0)$-concatenable.

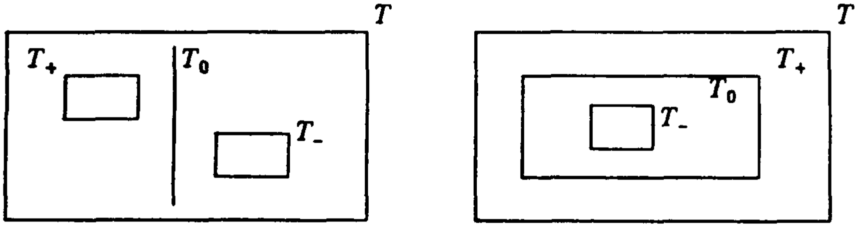


FIG. 1.

Observe that

PROPOSITION 3.1. *Let T be an interval of $\mathbb{Z}^N$, $T' \subset T$ be a subinterval of T, and $\Sigma = (T, X, \mathfrak{B})$ be an N-D system. Then, if Σ is Markovian, so is $\Sigma|_{T'}$.*

Let us now explain the intuitive content of the above definition. First note that the crucial idea of state—given what is happening on the boundary, then what can happen on one side and what can happen on the other side are independent—is expressed by the concatenability condition of our definition. Note however that (contrary to what is often done in the theory of Markov 2-D stochastic processes [7]) we do not require one of the sets T_- or T_+ to be bounded. Such an assumption is in fact difficult to accept, since it is not even required in 1-D systems. Second, note that the property expressed in Proposition 3.1 is very much a part of our definition: we want the concatenability property not only for partitions T_+, T_0, T_- of the whole parameter set $T = T_+ \cup T_0 \cup T_-$, but also for all triples T_+, T_0, T_- with $T_+ \cup T_0 \cup T_-$ possibly a strict subset of T. In fact, we shall see later on that one can restrict the sets $T_+ \cup T_0 \cup T_-$ for which we want the concatenability property of Definition 3.1 to hold a great deal further, in particular to intervals, but in any case it does not suffice to consider only partitions of T.

For 1-D systems it is easy to prove the following

PROPOSITION 3.2. *Let $\Sigma = (T, W, \mathfrak{B})$ be a 1-D dynamical system, with T an interval in $\mathbb{Z}$. Then Σ has the Markov property iff*

$$\{w_1, w_2 \in \mathfrak{B},\ t_0 \in T,\ w_1(t_0) = w_2(t_0)\} \Rightarrow \left\{ w_1 \underset{t_0}{\wedge} w_2 \in \mathfrak{B} \right\}.$$

Here $w_1 \underset{t_0}{\wedge} w_2$ denotes the element w of W^T such that $w|_{(-\infty, t_0] \cap T} = w_1|_{(-\infty, t_0] \cap T}$ and $w|_{[t_0, \infty) \cap T} = w_2|_{[t_0, \infty) \cap T}$.

Proof. See Appendix. ∎

The upshot of this proposition is that for 1-D systems it actually suffices in Definition 3.1 to look at sets $T_- = T \cap (-\infty, t_0)$, $T_0 = \{t_0\}$, $T_+ = T \cap (t_0, \infty)$. In particular $T_- \cup T_0 \cup T_+ = T$. We would like to emphasize that such simple partitions are not sufficient, for N-D systems with $N > 1$, to guarantee that a Markovian system is describable by first-order behavioral equations. This is shown in the following example.

EXAMPLE. Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B})$ be the 2-D system described by the equations

$$\left(\sigma_2^3 - \sigma_2^2 - \sigma_2 - 1\right) w = 0,$$

$$\left(\sigma_1 - \sigma_2^3\right) w = 0.$$

It can be shown that, for every partition (T_-, T_0, T_+) of $\mathbb{Z}^2$ such that T_0 separates T_- and T_+, $\mathfrak{B}$ is $(T_- \cup T_0, T_0 \cup T_+)$-concatenable. However, the above equations cannot be reduced to a first order description (i.e. to equations involving only $w, \sigma_1 w, \sigma_2^{\pm 1} w$ or $w, \sigma_1^{-1} w, \sigma_2^{\pm 1} w$).

For the sake of simplicity, from now on we will concentrate our attention on discrete 2-D systems. We will first show that, for this particular case, it is enough to restrict $T_+ \cup T_0 \cup T_-$ in Definition 3.1 to a special kind of subsets of T, namely elementary squares and lines.

A subset $\mathscr{S}$ of $\mathbb{Z}^2$ will be called an *elementary square* if $\mathscr{S} = \{(t_1, t_2), (t_1+1, t_2), (t_1+1, t_2+1), (t_1, t_2+1)\}$ for some $(t_1, t_2) \in \mathbb{Z}^2$. A *line* in T is a subset $\mathscr{L} \subset T$ such that $\mathscr{L} = \{(t', t'') | (t', t'') = (t_1, t_2) + z(v_1, v_2),\ z \in \mathbb{Z}\} \cap T$ for some $(t_1, t_2) \in \mathbb{Z}^2$ and $(v_1, v_2) \in \{(0,1), (1,0)\}$.

We will say that $\Sigma = (T, X, \mathfrak{B})$ (and $\mathfrak{B}$) are *square-concatenable* if $\Sigma|_{\mathscr{S}}$ is Markovian for all elementary squares $\mathscr{S}$. Analogously Σ will be called *line-concatenable* if $\Sigma|_{\mathscr{L}}$ is Markovian for all lines $\mathscr{L}$. Further Σ (and $\mathfrak{B}$) are said to be *square-complete* if $\{w \in \mathfrak{B}\} \Leftrightarrow \{w|_{\mathscr{S}} \in \mathfrak{B}|_{\mathscr{S}}$ for all elementary squares $\mathscr{S} \subset T\}$.

PROPOSITION 3.3. *Let $\Sigma = (\mathbb{Z}^2, X, \mathfrak{B})$ be a shift-invariant and complete 2-D system. Then Σ is Markovian iff $\mathfrak{B}$ is*

(1) *square-complete,*
(2) *square-concatenable, and*
(3) *line-concatenable.*

Proof. See Appendix. ■

In [6] it is proven that a complete, time-invariant 1-D system $\Sigma = (\mathbb{Z}, X, \mathfrak{B})$ is Markovian iff it can be described by a behavioral equation which is first-order, that is, iff there exists $f: X \times X \to \mathbb{R}$ such that $\{x \in \mathfrak{B}\} \Leftrightarrow \{f(x(t+1), x(t)) = 0$ for all $t \in \mathbb{Z}\}$. Proposition 3.3 allows to conclude that a similar result holds for 2-D systems. In fact, square-completeness for $\Sigma = (\mathbb{Z}^2, X, \mathfrak{B})$ means that there is $f: X \times X \times X \times X \to \mathbb{R}^g$ such that $\{x \in \mathfrak{B}\} \Leftrightarrow \{f(x(t_1, t_2), x(t_1+1, t_2), x(t_1+1, t_2+1), x(t_1, t_2+1)) = 0$ for all $(t_1, t_2) \in$

$\mathbb{Z}^2\}$. Moreover, square-concatenability means that the four-point law expressed by f can be decomposed into two three-point laws in the following way. There are $f_i: X \times X \times X \to \mathbb{R}^{g_i}$, $i = 1,2,3,4$, such that

$$\{f(x(t_1,t_2),x(t_1+1,t_2),x(t_1+1,t_2+1),x(t_1,t_2+1))=0 \text{ for all } (t_1,t_2)\in\mathbb{Z}^2\}$$

$$\Leftrightarrow \left\{\begin{array}{l} f_1(x(t_1,t_2),x(t_1+1,t_2),x(t_1,t_2+1)) = 0 \text{ and} \\ f_2(x(t_1+1,t_2+1),x(t_1+1,t_2),x(t_1,t_2+1)) = 0 \text{ for all } (t_1,t_2)\in\mathbb{Z}^2 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} f_3(x(t_1+1,t_2),x(t_1,t_2),x(t_1+1,t_2+1)) = 0 \text{ and} \\ f_4(x(t_1,t_2+1),x(t_1,t_2),x(t_1+1,t_2+1)) = 0 \text{ for all } (t_1,t_2)\in\mathbb{Z}^2. \end{array}\right\}$$

Thus, every complete, shift-invariant, Markovian 2-D system can be described by pairs of 2-D first-order behavioral equations. Note however that, unlike for 1-D systems, the converse is not necessarily true, as the condition of line concatenability must also be satisfied.

4. MARKOVIAN AR SYSTEMS

The fact that a complete, time-invariant 1-D system is Makovian iff it can be described by first-order behavioral equations implies, in particular, that an AR system $\Sigma = (\mathbb{Z}, \mathbb{R}^n, \mathfrak{B})$ is Markovian iff there exists a first-order matrix $R(s) = R_0 + R_1 s$ such that $\mathfrak{B} = \mathfrak{B}(R)$. Thus, linear, time-invariant, and complete Markovian systems are exactly those which can be described by first-order equations

$$R_1 x(t+1) + R_0 x(t) = 0 \qquad \forall t \in \mathbb{Z},$$

with $R_0, R_1 \in \mathbb{R}^{g\times n}$ (it is always possible to choose $g \leqslant n$).

A similar result holds, it turns out, for 2-D systems. First, however, we will prove some auxiliary results concerning square and line concatenability.

LEMMA 4.1. *Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B})$ be a square-complete AR system. Then, if Σ is square-concatenable, it can be described by a behavioral AR equation $Ex + F\sigma_1 x + G\sigma_2 x + H\sigma_1\sigma_2 x = 0$, where $E, F, G, H \in \mathbb{R}^{g\times n}$ are* *such that*:

(1) $[E\ \ F\ \ G\ \ H]$ *has full row rank*,
(2) $\mathrm{im}(E)\cap\mathrm{im}(H)=\{0\}$,
(3) $\mathrm{im}(F)\cap\mathrm{im}(G)=\{0\}$.

Proof. See Appendix. ∎

REMARK. Conditions 2 and 3 of the lemma express the fact that Σ can be described by two pairs of first-order behavioral equations [i.e. equations of the form $(R_0+R_1\sigma_1^i+R_2\sigma_2^j)x=0$, with $j,i=1,-1$ and $R_0,R_1,R_2\in\mathbb{R}^{l\times n}$ for some positive integer l]. More concretely, condition (2) means that Σ can be described by behavioral equations

$$(A_1+B_1\sigma_1+C_1\sigma_2)x=0,$$

$$\left(A_2+B_2\sigma_1^{-1}+C_2\sigma_2^{-1}\right)x=0 \tag{BE}$$

for suitable real matrices A_i, B_i, C_i ($i=1,2$), while condition (3) means that Σ can also be described by

$$\left(\bar{A}_1+\bar{B}_1\sigma_1^{-1}+\bar{C}_1\sigma_2\right)x=0,$$

$$\left(\bar{A}_2+\bar{B}_2\sigma_1+\bar{C}_2\sigma_2^{-1}\right)x=0 \tag{$\overline{\mathrm{BE}}$}$$

for some real matrices $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ ($i=1,2$).

In order to characterize the line-concatenability of an AR system $\Sigma=(\mathbb{Z}^2,\mathbb{R}^n,\mathfrak{B})$, we will consider the horizontal and the vertical behavior of Σ. These can be defined as follows. For $t\in\mathbb{Z}$, denote by $\mathscr{L}_1(t)$ the horizontal line $\mathscr{L}_1(t):=\{(t_1,t)|t_1\in\mathbb{Z}\}$ and by $\mathscr{L}_2(t)$ the vertical line $\mathscr{L}_2(t):=\{(t,t_2)|t_2\in\mathbb{Z}\}$. Clearly, as Σ is shift-invariant, $\mathfrak{B}|_{\mathscr{L}_1(t)}\cong\mathfrak{B}|_{\mathscr{L}_1(0)}$ and $\mathfrak{B}|_{\mathscr{L}_2(t)}\cong\mathfrak{B}|_{\mathscr{L}_2(0)}$ for all $t\in\mathbb{Z}$. Let $\mathfrak{B}_1:=\mathfrak{B}|_{\mathscr{L}_1(0)}$ and $\mathfrak{B}_2:=\mathfrak{B}|_{\mathscr{L}_2(0)}$. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are both subspaces of $(\mathbb{R}^n)^{\mathbb{Z}}$ and can be viewed respectively as the behaviors of the 1-D "line" systems $\Sigma_1:=(\mathbb{Z},\mathbb{R}^n,\mathfrak{B}_1)$ and $\Sigma_2:=(\mathbb{Z},\mathbb{R}^n,\mathfrak{B}_2)$. We will say that $\mathfrak{B}_1$ is the *horizontal behavior* of Σ, while $\mathfrak{B}_2$ is its *vertical behavior*.

Obviously, Σ is line-concatenable iff Σ_1 and Σ_2 are Markovian. Thus

LEMMA 4.2. *Let* $\Sigma=(\mathbb{Z}^2,\mathbb{R}^n,\mathfrak{B})$ *be a square-complete AR system*, $\mathfrak{B}_1:=\mathfrak{B}|_{\mathscr{L}_1(0)}$, *and* $\mathfrak{B}_2:=\mathfrak{B}|_{\mathscr{L}_2(0)}$. *Then* Σ *is line-concatenable iff it can be* *described by behavioral AR equations of the form*:

$$E_1 x + F_1 \sigma_1 x = 0, \tag{1.1}$$

$$E_2 x + G_2 \sigma_2 x = 0, \tag{1.2}$$

$$E_3 x + F_3 \sigma_1 x + G_3 \sigma_2 x + H_3 \sigma_1 \sigma_2 x = 0 \tag{1.3}$$

such that (1.1) *describes* $\mathfrak{B}_1$ *and* (1.2) *describes* $\mathfrak{B}_2$.

EXAMPLE. Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^2, \mathfrak{B})$ be the system described by the following behavioral equations:

$$x_1 - \sigma_2 x_1 = 0 \tag{2.1}$$

$$x_2 - \sigma_1 x_1 + \sigma_2 x_2 = 0 \tag{2.2}$$

with $x = \mathrm{col}(x_1, x_2)$. (2.1) and (2.2) clearly imply that $\sigma_2^2 x_2 = x_2$. Therefore (2.1) does not describe $\mathfrak{B}_2$. Moreover it is not difficult to check that $\mathfrak{B}_2 = \{x : \mathbb{Z} \to \mathbb{R}^2 | \sigma_2 x_1 = x_1 \text{ and } \sigma_2^2 x_2 = x_2\}$, implying that there is no first-order description for this behavior.

This example shows that the fact that (1.1) must represent (i.e. describe) $\mathfrak{B}_1$ precisely and (1.2) must represent $\mathfrak{B}_2$ precisely imposes some conditions on the matrices $E_1, F_1, E_2, G_2, E_3, F_3, G_3, H_3$. Unfortunately these conditions are somewhat involved.

Given $E_1, F_1 \in \mathbb{R}^{g_1 \times n}$, $E_2, G_2 \in \mathbb{R}^{g_2 \times n}$, $E_3, F_3, G_3, H_3 \in \mathbb{R}^{g_3 \times n}$, define the matrix $[E\ F\ G\ H]$ as follows:

$$[E\ F\ G\ H] := \begin{bmatrix} E_1 & F_1 & 0 & 0 \\ 0 & 0 & E_1 & F_1 \\ E_2 & 0 & G_2 & 0 \\ 0 & E_2 & 0 & G_2 \\ E_3 & F_3 & G_3 & H_3 \end{bmatrix}.$$

$[E\ F\ G\ H]$ is said to satisfy *condition* $C1$ if, whenever there exists a unimodular matrix $U \in \mathbb{R}^{g \times g}[s, s^{-1}]$ such that

$$U(\sigma_1, \sigma_1^{-1})[E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2] = \begin{bmatrix} D(\sigma_1, \sigma_1^{-1}) \\ 0 \end{bmatrix} + \sigma_2 \begin{bmatrix} M(\sigma_1, \sigma_1^{-1}) \\ \Pi(\sigma_1, \sigma_1^{-1}) \end{bmatrix}$$

or

$$U\left(\sigma_1, \sigma_1^{-1}\right)\left[E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2\right] = \begin{bmatrix} M\left(\sigma_1, \sigma_1^{-1}\right) \\ \Pi\left(\sigma_1, \sigma_1^{-1}\right) \end{bmatrix} + \sigma_2 \begin{bmatrix} D\left(\sigma_1, \sigma_1^{-1}\right) \\ 0 \end{bmatrix}$$

and $D(s, s^{-1})$ has full row rank, then the polynomial matrix $\Pi(s, s^{-1})$ is a left multiple of $E_1 + F_1 s$. Analogously, $[E\ \ F\ \ G\ \ H]$ is said to satisfy *condition* $C2$ if, whenever there is a unimodular matrix $U \in \mathbb{R}^{g \times g}[s, s^{-1}]$ such that

$$U\left(\sigma_2, \sigma_2^{-1}\right)\left[E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2\right] = \begin{bmatrix} D\left(\sigma_2, \sigma_2^{-1}\right) \\ 0 \end{bmatrix} + \sigma_1 \begin{bmatrix} M\left(\sigma_2, \sigma_2^{-1}\right) \\ \Pi\left(\sigma_2, \sigma_2^{-1}\right) \end{bmatrix}$$

or

$$U\left(\sigma_2, \sigma_2^{-1}\right)\left[E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2\right] = \begin{bmatrix} M\left(\sigma_2, \sigma_2^{-1}\right) \\ \Pi\left(\sigma_2, \sigma_2^{-1}\right) \end{bmatrix} + \sigma_1 \begin{bmatrix} D\left(\sigma_2, \sigma_2^{-1}\right) \\ 0 \end{bmatrix}$$

and $D(s, s^{-1})$ has full row rank, then the polynomial matrix $\Pi(s, s^{-1})$ is a left multiple of $E_2 + G_2 s$.

It is possible to prove that, in Lemma 4.2, (1) is equivalent to saying that $[E\ \ F\ \ G\ \ H]$ satisfies condition C1, while (2) is equivalent to say that $[E\ \ F\ \ G\ \ H]$ satisfies C2 (see Appendix).

Intuitively, condition C1 expresses the fact that all restrictions imposed on $\mathfrak{B}_1$ by (1.2) and (1.3) are implied by (1.1). Similarly, condition C2 means that all the restrictions imposed on $\mathfrak{B}_2$ by (1.1) and (1.3) are already implied by (1.2).

The next result characterizes a Markovian AR 2-D system in terms of the corresponding behavioral equations, and is a corollary of Proposition 3.3 and Lemmas 4.1 and 4.2.

THEOREM 4.1. *Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B})$ be an AR 2-D system. Then Σ is Markovian iff it can be described by behavioral equations of the following type*:

$$E_1 x + F_1 \sigma_1 x = 0, \tag{3.1}$$

$$E_2 x + G_2 \sigma_2 x = 0, \tag{3.2}$$

$$E_3 x + F_3 \sigma_1 x + G_3 \sigma_2 x + H_3 \sigma_1 \sigma_2 x = 0, \tag{3.3}$$

where $E_1, F_1 \in \mathbb{R}^{g_1 \times n}$, $E_2, G_2 \in \mathbb{R}^{g_2 \times n}$, $E_3, F_3, G_3, H_3 \in \mathbb{R}^{g_3 \times n}$ *(for some positive integers* g_1, g_2, *and* g_3*), such that the following conditions hold*:

(1) (i) $[E\ F\ G\ H]$ *has full row rank*,
(ii) $\mathrm{im}(E) \cap \mathrm{im}(H) = \{0\} = \mathrm{im}(G) \cap \mathrm{im}(F)$,
(2) (i) (3.1) *represents* $\mathfrak{B}_1$ *precisely*,
(ii) (3.2) *represents* $\mathfrak{B}_2$ *precisely*.

Proof. See Appendix. ∎

When $E_3 = F_3 = G_3 = H_3 = 0$, simpler sufficient conditions can be derived. In fact, it is not difficult to prove that:

COROLLARY 4.1. *Suppose that the 2-D system* $\Sigma = (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B})$ *is described by the following behavioral equations*:

$$(E_1 + F_1\sigma_1)x = 0, \tag{4.1}$$

$$(E_2 + F_2\sigma_2)x = 0. \tag{4.2}$$

Then, if the polynomial matrices $\mathrm{col}[E_i + F_i s, E_j], \mathrm{col}[E_i + F_i s, F_j]$ $(i, j = 1,2;\ i \neq j)$ *have full row rank*, Σ *is Markovian. Moreover* (4.1) *describes* $\mathfrak{B}_1$ *and* (4.2) *describes* $\mathfrak{B}_2$.

5. STATE SPACE SYSTEMS

In this section state-space systems will be defined as a special type of Markovian systems. As a motivation we will start with the definition of 1-D state-space systems.

The case of 1-D systems where a preferred time direction (i.e. past and future) is recognized has been widely studied in [5, 6]. The following definition was introduced.

DEFINITION 5.1. A *discrete 1-D system in state-space form* is defined as $\Sigma := (T, W, X, \mathfrak{B})$, where $T = \mathbb{Z}$ is the time set, W the signal space, X the state space, and $\mathfrak{B} \subset (W \times X)^T$ the behavior of the system. $\mathfrak{B}$ is required to satisfy the *axiom of state*:

$$\{(w_1, x_1), (w_2, x_2) \in \mathfrak{B}, t_0 \in \mathbb{Z}, x_1(t_0) = x_2(t_0)\}$$
$$\Rightarrow \left\{(w_1, x_1) \underset{t_0}{\wedge} (w_2, x_2) \in \mathfrak{B}\right\}.$$

Here $(w_1,x_1)\underset{t_0}{\wedge}(w_2,x_2)$ denotes the element $(w,x)\in(W\times X)^{\mathbb{Z}}$ such that $(w,x)|_{(-\infty,t_0)}=(w_1,x_1)|_{(-\infty,t_0)}$ and $(w,x)|_{[t_0,\infty)}=(w_2,x_2)|_{[t_0,\infty)}$. The interval notation is used to denote intervals in $\mathbb{Z}$.

The preference for a certain time direction is clearly reflected by the asymmetry in the axiom of state: if the states x_1 and x_2 of w_1 and w_2 coincide at t_0, then w_1 and w_2 are concatenable with respect to the partition $\{(-\infty,t_0),[t_0,\infty)\}$, but the same does not necessarily hold for the partition $\{(-\infty,t_0],(t_0,\infty)\}$.

To deal with systems with no privileged time direction, a "symmetrized" version of the above axiom of state should be considered.

Definition 5.2. A *discrete* 1-*D bilateral state-space system* is defined as $\Sigma:=(T,W,X,\mathfrak{B})$, where $T=\mathbb{Z}$ is the parameter set, W the signal space, X the state space, and $\mathfrak{B}\subset(W\times X)^T$ the behavior of the system. $\mathfrak{B}$ is required to satisfy the *bilateral axiom of state*:

$$\{(w_1,x_1),(w_2,x_2)\in\mathfrak{B},\ t_0\in\mathbb{Z},\ x_1(t_0)=x_2(t_0),\ T_1=(-\infty,t_0),\ T_2=[t_0,\infty)\}$$

$$\Rightarrow\left\{(w_1,x_1)\underset{(T_1,T_2)}{\wedge}(w_2,x_2)\in\mathfrak{B}\right\},$$

$$\{(w_1,x_1),(w_2,x_2)\in\mathfrak{B},\ t_0\in\mathbb{Z},\ x_1(t_0)=x_2(t_0),\ T_1=(-\infty,t_0],\ T_2=(t_0,\infty)\}$$

$$\Rightarrow\left\{(w_1,x_1)\underset{(T_1,T_2)}{\wedge}(w_2,x_2)\in\mathfrak{B}\right\}.$$

The first condition of the bilateral axiom of state is nothing more than the axiom of state in Definition 5.1; the second is the axiom of state for the "time-reversed" system corresponding to Σ. This is, the variable x is a state variable both forwards and backwards in time.

It is not difficult to prove that

Proposition 5.1. *Let* $\Sigma=(\mathbb{Z},W,X,\mathfrak{B})$, *with* $\mathfrak{B}\subset(W\times X)^{\mathbb{Z}}$. *Then* $\mathfrak{B}$ *satisfies the bilateral axiom of state iff for every interval* $I\subset\mathbb{Z}$ *and for all triples of subsets* (T_+,T_0,T_-) *of* I *such that* T_+ *and* T_- *are separated by* T_0, *the following holds*:

$$\{T_1=T_+\cup T_0,T_2=T_-,(w_1,x_1)\in\mathfrak{B}|_I,(w_2,x_2)\in\mathfrak{B}|_I,x_1|_{T_0}=x_2|_{T_0}\}$$

$$\Rightarrow\left\{(w_1,x_1)\underset{(T_1,T_2)}{\wedge}(w_2,x_2)\in\mathfrak{B}|_{T_1\cup T_2}\right\}.$$

Here, $(w_1, x_1) \underset{(T_1,T_2)}{\wedge} (w_2, x_2)$ *denotes the element* $(w, x) \in (W \times X)^{\mathbb{Z}}|_{T_1 \cup T_2}$ *such that* $(w, x)|_{T_1} = (w_1, x_1)|_{T_1}$ *and* $(w, x)|_{T_2} = (w_2, x_2)|_{T_2}$.

Note that the concatenability property expressed in this proposition does not privilege any special choice of (T_1, T_2), as the roles of T_+ and T_- can be interchanged. The concept of state that we next introduce for N-D systems $(N > 1)$ is a generalization of this property.

DEFINITION 5.3. A *discrete N-D state-space system* is defined as $\Sigma := (\mathbb{Z}^N, W, X, \mathfrak{B})$, where W is the signal space, X the state space, and $\mathfrak{B} \subset (W \times X)^{\mathbb{Z}^N}$ the behavior of the system. $\mathfrak{B}$ is required to satisfy the *Axiom of N-D state*: For every interval $I \subset \mathbb{Z}^N$ and for all triples (T_+, T_0, T_-) of subsets of I such that T_+ and T_- are separated by T_0, the following holds:

$$\left\{T_1 = T_+ \cup T_0,\ T_2 = T_-,\ (w_1, x_1) \in \mathfrak{B}|_I,\ (w_2, x_2) \in \mathfrak{B}|_I,\ x_1|_{T_0} = x_2|_{T_0}\right\}$$

$$\Rightarrow \left\{(w_1, x_1) \underset{(T_1,T_2)}{\wedge} (w_2, x_2) \in \mathfrak{B}|_{T_1 \cup T_2}\right\}.$$

The behaviors $\mathfrak{B}^x := \Pi_x \mathfrak{B} := \{x \in X^{\mathbb{Z}^N} | \exists w \in W^{\mathbb{Z}^N} \text{ s.t. } (w, x) \in \mathfrak{B}\}$ and $\mathfrak{B}^w := \Pi_w \mathfrak{B}$ are respectively called the *state behavior* and the *external behavior* of Σ. It is not difficult to see that both Σ and $\Sigma^x := (\mathbb{Z}^N, X, \mathfrak{B}^x)$ are Markovian systems.

The system Σ is said to be a state-space realization of $\Sigma^w := (\mathbb{Z}^N, W, \mathfrak{B}^w)$. Clearly, every N-D system $\Sigma = (\mathbb{Z}^N, W, \mathfrak{B}^w)$ admits a trivial state-space realization $\Sigma = (\mathbb{Z}^N, W, \mathfrak{B}^w, \mathfrak{B})$, where $\mathfrak{B} = \{(w, x) \in (W \times \mathfrak{B}^w)^{\mathbb{Z}^N} | w \in \mathfrak{B}^w$ and $x(t) = w\ \forall t \in \mathbb{Z}^N\}$, i.e., the state x associated with w consists, at every point, in the whole function w. As a consequence, the state space will often be infinite-dimensional. However, if Σ^w is an AR system, there will also exist a state-space realization with finite-dimensional state space. This is stated in our next result. For simplicity, only the case of 2-D systems has been considered.

PROPOSITION 5.2. *Let* $\Sigma^w = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B}^w)$ *be an autoregressive 2-D system. Then there is a positive integer n and a subspace* $\mathfrak{B}$ *of* $(\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}^2}$ *such that* $\Sigma := (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ *is a state-space realization of* Σ^w *and, moreover, the behavior* $\mathfrak{B}$ *of* Σ *is described by AR behavioral equations.*

Proof. See Appendix. ∎

It was proven in [6] that an AR 1-D system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ is a state-space system iff there are equations of the form

$$\sigma x = Ax + Bu,$$
$$w = Cx + Du \tag{S}$$

[with $u \in (\mathbb{R}^l)^{\mathbb{Z}}$ for some l, and A, B, C, D real matrices of suitable dimensions], such that $\mathfrak{B} = \{(w, x) \in (\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}} | \exists u \in (\mathbb{R}^l)^{\mathbb{Z}}$ s.t. (w, x, u) satisfies $(S)\}$.

As for the bilateral state property, there holds:

PROPOSITION 5.3. *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ be an AR 1-D system. Then Σ is a bilateral state-space system iff $\mathfrak{B}$ can be described by the behavioral equations*

$$Ex + F\sigma x = 0,$$
$$Nx + Mw = 0$$

for some real matrices E, F, M, and N.

Proof. See Appendix. ■

In other words, Σ is a bilateral state-space system iff its state behavior is Markovian and the external behavior is related to the state behavior by means of a static relation. This also holds for N-D systems ($N > 1$). In particular, for the case of AR 2-D systems:

THEOREM 5.1. *Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ be an AR 2-D system. Then Σ is a state-space system iff $\mathfrak{B}$ can be described by the behavioral equations*

$$E_1 x + F_1 \sigma_1 x = 0, \tag{5.1}$$
$$E_2 x + G_2 \sigma_2 x = 0, \tag{5.2}$$
$$E_3 x + F_3 \sigma_1 x + G_3 \sigma_2 x + H_3 \sigma_1 \sigma_2 x = 0, \tag{5.3}$$
$$Nx + Mw = 0 \tag{5.4}$$

for some real matrices $E_1, F_1, E_2, G_2, E_3, F_3, G_3, H_3, M, N$ such that Equations (5.1)–(5.3) *satisfy the conditions of Theorem* 4.1 *for the system* $\Sigma^x = (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B}^x)$.

Proof. See Appendix. ■

REMARK. Equation (5.4) can also be written as

$$w_2 = N'x + Lw_1,$$

$$w_1 \text{ free},$$

$$w = T\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where T is a permutation matrix, or equivalently as

$$w_2 = N'x,$$

$$w_1 \text{ free},$$

$$w = T\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for some invertible matrix T.

To illustrate our concept of state, we next give some examples of state-space realizations.

EXAMPLE 1. Let $\Sigma^w = (\mathbb{Z}^2, \mathbb{R}, \mathfrak{B}^w)$ be described by the following behavioral equation:

$$\left[\sigma_2 + \left(\sigma_2^3 + 1\right)\sigma_1 + \sigma_2\sigma_1^2\right] w = 0. \tag{6}$$

Let $x = \text{col}[x_1, x_2, x_3, x_4, x_5]$ be defined by $x_1 := \sigma_2^2\sigma_1 w_1$, $x_2 := \sigma_2\sigma_1 w$, $x_3 := \sigma_2^{-1}\sigma_1 w$, $x_4 := \sigma_1 w$, $x_5 := w$. Then, if w is a solution of (6), (w, x) satisfies the following equations:

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\sigma_2 + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}\right)x = 0, \tag{7.1}$$

$$\left(\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\sigma_1 + \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right)x = 0, \tag{7.2}$$

$$w = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} x \tag{7.3}$$

Conversely, given any situation (w, x) of (7), w will satisfy Equation (6). Moreover, it is not difficult to check that (7.1) and (7.2) fulfill the conditions of Corollary 4.1. Therefore they describe a Markovian system. This shows that $\Sigma = (\mathbb{Z}^2, \mathbb{R}, \mathbb{R}^5, \mathfrak{B})$, with $\mathfrak{B}$ described by Equations (7), is a state-space realization of Σ^w.

EXAMPLE 2. Let $\mathfrak{B}^w \subset (\mathbb{R}^2)^{\mathbb{Z}^2}$ be the behavior described by the following equation:

$$\left[(1+\sigma_2^2)+\sigma_2\sigma_1\right] w_1 + \left[\sigma_2 + (1+\sigma_2^2)\sigma_1\right] w_2 = 0. \tag{8}$$

It is not difficult to verify that the system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^2, \mathbb{R}^7, \mathfrak{B})$ described by the behavioral equations

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sigma_2 + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}\right) x = 0,$$

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sigma_1 + \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right) x = 0,$$

$$w = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x$$

is a state-space realization of $\Sigma^w = (\mathbb{Z}^2, \mathbb{R}^2, \mathfrak{B}^w)$.

6. RECURSIVE COMPUTATIONS

Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ be a 2-D state-space system, and consider a representation of $\mathfrak{B}$:

$$E_1 x + F_1 \sigma_1 x = 0$$

$$E_2 x + G_2 \sigma_2 x = 0$$

$$E_3 x + G_3 \sigma_2 x + F_3 \sigma_1 x + H_3 \sigma_1 \sigma_2 x = 0 \tag{9}$$

and

$$Mw + Nx = 0,$$

as derived in Theorem 5.1. As we shall see, this representation is very useful in order to specify what initial conditions are required to compute a solution, and provides a recursive method for computing solutions. This method consists basically in what we will call a line-by-line computational scheme. In order to compute the values of a solution (w, x) we will proceed as follows. Let $\mathscr{L}_2(i)$ be a vertical line in $\mathbb{Z}^2$ as defined in Section 4. As an initialization step, the values on $\mathscr{L}_2(0)$ are computed from initial conditions given on this line. For $k = 1, 2, \ldots,$ the values on $\mathscr{L}_2(k)$ will be computed from the values on $\mathscr{L}_2(k-1)$ and from initial conditions given on $\mathscr{L}_2(k)$ itself. For $k = -1, -2, \ldots,$ the values on $\mathscr{L}_2(k)$ will be computed from the values on $\mathscr{L}_2(k+1)$ and initial conditions on $\mathscr{L}_2(k)$. Of course, an analogous scheme can be implemented considering horizontal lines instead of vertical ones. We will only be concerned with equations (9) which describe the state behavior $\mathfrak{B}^x$. Once the values of x are computed, the values of w can easily be obtained through the relation $Mw + Nx = 0$.

Define the following matrices, with entries in $\mathbb{R}[s]$:

$$P(s) := E_2 + G_2 s,$$

$$Q(s) := Q_0 + Q_1 s,$$

where $Q_0 := \mathrm{col}[F_1, F_3]$ and $Q_1 := \mathrm{col}[0, H_3]$, and

$$R(s) := R_0 + R_1 s$$

with $R_0 := \mathrm{col}[-E_1, -E_3]$ and $R_1 := \mathrm{col}[0, -G_3]$. It is easy to check that (9) is equivalent to

$$P(\sigma_2)x = 0, \tag{10.1}$$

$$Q(\sigma_2)\sigma_1 x = R(\sigma_2)x. \tag{10.2}$$

As the conditions of Theorem 5.1. are supposed to hold, (10.1) describes the vertical behavior $\mathfrak{B}_2^x$. Moreover, it will follow that:

Lemma 6.1. *Consider the equations*

$$P(\sigma)v = 0, \tag{11.1}$$

$$Q(\sigma)z = R(\sigma)v, \tag{11.2}$$

$$P(\sigma)z = 0 \tag{11.3}$$

in $(\mathbb{R}^n)^{\mathbb{Z}}$, *with* P, Q, *and* R *as in* (10). *Then, for every solution* v *of* (11.1), *there exists a* z *such that* (11.2) *is satisfied. Further, any such* z *will also satisfy* (11.3). *Analogously, for every solution* z *of* (11.3), *there exists a* v *such that* (11.2) *is satisfied; moreover,* v *will also satisfy* (11.1).

This result implies that the values of any solution of (10) can be computed by the following *line-by-line computational scheme*:

0. Compute a solution x_0 of $P(\sigma)x_0 = 0$.
   Define $x(0, \cdot) = x_0$.

For $k = 1, 2, \ldots$

k. Compute a solution x_k of

$$Q(\sigma)x_k = R(\sigma)x_{k-1}. \tag{12$_k$}$$

Compute a solution x_{-k} of

$$R(\sigma)x_{-k} = Q(\sigma)x_{-(k-1)}. \tag{12$_{-k}$}$$

Define $x(k, \cdot) = x_k$ and $x(-k, \cdot) = x_{-k}$.

Lemma 6.1 guarantees that Equations $(12)_k$ and $(12)_{-k}$ are solvable $(k = 1, 2, \ldots)$ and that their solutions satisfy $P(\sigma)x_k = 0 = P(\sigma)x_{-k}$.

In view of the above, it is now clear that in order to compute solutions of (10) it is enough to obtain a method for computing solutions of (11.1)–(11.2) and of (11.2)–(11.3). We will only analyze Equations (11.1)–(11.2). The case of Equations (11.2)–(11.3) can be dealt with in a similar way. Note, moreover, that as Equation (11.2) does not restrict the solutions of (11.1), these equations can be solved separately.

The structure of the solution set of (11.1) has been studied in detail in [6, Chapter 4]. We will describe the results here only to the extent which they are relevant for our purposes.

Recalling that $P(\sigma) = P_1\sigma + P_0$, consider first Equation (11.1)

$$P_1\sigma v + P_0 v = 0$$

with $v: \mathbb{Z} \to \mathbb{R}^n$ and $P_1, P_0 \in \mathbb{R}^{g \times n}$. This equation defines a Markovian 1-D system $(\mathbb{Z}, \mathbb{R}^n, \ker(P_1\sigma + P_0))$, with $P_1\sigma + P_0$ viewed as an operator from $(\mathbb{R}^n)^{\mathbb{Z}}$ to $(\mathbb{R}^g)^{\mathbb{Z}}$. The structure of the solutions of (11.1) is as follows. Some components are free; some components are determined by these free variables and initial conditions. Also, all solutions will take values in a subspace of

$\mathbb{R}^n$. In general, this subspace is a strict subspace, resulting in components which are identically zero. More precisely, there will exist integers n_1, n_2, n_3, $n_1 + n_2 + n_3 \leqslant n$, an injective matrix $C \in \mathbb{R}^{n\times(n_1+n_2+n_3)}$, and matrices $A^+ \in \mathbb{R}^{(n_1+n_3)\times(n_1+n_3)}$, $A^- \in \mathbb{R}^{(n_2+n_3)\times(n_2+n_3)}$, $B^+ \in \mathbb{R}^{(n_1+n_2)\times n_3}$, $B^- \in \mathbb{R}^{(n_2+n_3)\times n_1}$ such that (11.1) is equivalent to both

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(t+1) = A^+\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(t) + B^+ v_3(t),$$

$$v(t) = C\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}(t)$$

and

$$\begin{pmatrix} v_3 \\ v_2 \end{pmatrix}(t-1) = A^-\begin{pmatrix} v_3 \\ v_2 \end{pmatrix}(t) + B^- v_1(t),$$

$$v(t) = C\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}(t).$$

This shows that in order to compute a solution, one can choose freely $v_3(t)$ for t nonnegative and $v_1(t)$ for t nonpositive, in addition to $v_2(0)$.

Next consider Equation (11.2)

$$Q(\sigma)z = R(\sigma)v, \tag{11.2}$$

where we will consider $v: \mathbb{Z} \to \mathbb{R}^n$ as given, and wish to compute $z: \mathbb{Z} \to \mathbb{R}^n$.

Regarding $Q(\sigma) = Q_1\sigma + Q_0$ as a matrix pencil, it follows from [3] that there are real invertible matrices U and V such that:

$$UQ(\sigma)V = \begin{bmatrix} 0 & & & & \\ & D_1(\sigma) & & & \\ & & D_2(\sigma) & & \\ & & & \ddots & \\ & & & & D_l(\sigma) \end{bmatrix}$$

with

$$D_i(\sigma) = D_i^0 + D_i^1\sigma, \qquad i = 1,\dots, l.$$

Further, the blocks $D_i(\sigma)$ can essentially be of five types:

1. $(-J + K\sigma)$ with J and K $k_1 \times k_1$ invertible matrices,

2. $\begin{bmatrix} \sigma & & \\ 1 & \ddots & \\ & \ddots & \sigma \\ & & 1 \end{bmatrix}$ of size $(k_2+1)\times k_2$,

3. $\begin{bmatrix} 1 & \sigma & & \\ & \ddots & \ddots & \\ & & \ddots & \sigma \\ & & & 1 \end{bmatrix}$ of size $k_3 \times k_3$,

4. $\begin{bmatrix} \sigma & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \sigma \end{bmatrix}$ of size $k_4 \times k_4$,

5. $\begin{bmatrix} \sigma & 1 & & \\ & \ddots & \ddots & \\ & & \sigma & 1 \end{bmatrix}$ of size $k_5 \times (k_5+1)$.

Thus, to derive the structure of the solutions of (11.2) it is enough to study the structure of the solutions of $D(\sigma)z^* = S(\sigma)v$, where v is given, $D(\sigma)$ is of one of the types 1–5, and $S(\sigma) = S_0 + S_1\sigma$.

Suppose first that $D(\sigma) = -J + K\sigma$ as in type 1. Without loss of generality K can be taken to be the identity matrix, yielding

$$z^*(t+1) = Jz^*(t) + S_0v(t) + S_1v(t+1). \tag{13}$$

To compute a solution of this equation, $z^*(0)$ can be chosen freely and the other values will be completely determined by this initial condition and by v.

If $D(\sigma)$ is of type 2, it can be reduced to the form

$$\begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad I \in \mathbb{R}^{k_2 \times k_2},$$

by premultiplication by the unimodular matrix

$$U(\sigma)=\begin{bmatrix} 1 & -\sigma & \sigma^2 & \cdot & \cdot & \cdot & (-1)^{k_2}\sigma^{k_2} \\ & \cdot & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \sigma^2 \\ & & & & & \cdot & -\sigma \\ & & & & & & 1 \end{bmatrix}.$$

In this case, recalling that by assumption the restrictions on v are redundant, the equation $D(\sigma)z^* = S(\sigma)v$ will be equivalent to

$$z^*(t) = S_0 v(t) + S_1 v(t+1) \cdots + S_{k_2} v(t+k_2) \tag{14}$$

for some real matrices $S_0, \dots, S_{k_2}$.

If $D(\sigma)$ is of type 3, then

$$[D(\sigma)]^{-1}=\begin{bmatrix} 1 & -\sigma & \sigma^2 & \cdot & \cdot & \cdot & (-1)^{k_3-1}\sigma^{k_3-1} \\ & \cdot & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \sigma^2 \\ & & & & & & -\sigma \\ & & & & & & 1 \end{bmatrix},$$

implying that the equation $D(\sigma)z^* = S(\sigma)v$ is equivalent to

$$z^*(t) = S_0' v(t) + \cdots + S_{k_3}' v(t+k_3) \tag{15}$$

for suitable real matrices $S_0', \dots, S_{k_3}'$. The structure of the solution of (15) and (14) is similar.

The case when $D(\sigma)$ is of type 4 is analogous to the previous case, now with σ replaced by σ^{-1}, yielding

$$z^*(t) = S_0'' v(t) + S_1'' v(t-1) + \cdots + S_{k_4}'' v(t-k_4). \tag{16}$$

Finally, suppose that $D(\sigma)$ is of type 5, and that $z^* = \mathrm{col}[z_1^*, \dots, z_l^*]$. It is easy to see that now the equation $D(\sigma)z^* = S(\sigma)v$ will be equivalent to both

$$\begin{bmatrix} z_1^* \\ \vdots \\ z_{l-1}^* \end{bmatrix}(t+1) = -\begin{bmatrix} z_2^* \\ \vdots \\ z_l^* \end{bmatrix}(t) + S_0 v(t) + S_1 v(t+1) \tag{17}$$

and

$$\begin{bmatrix} z_2^* \\ \vdots \\ z_l^* \end{bmatrix}(t-1) = -\begin{bmatrix} z_1^* \\ \vdots \\ z_{l-1}^* \end{bmatrix}(t) + S_0 v(t-1) + S_1 v(t). \tag{18}$$

This shows that in order to compute a solution z^*, given v, one can choose freely $z_l^*(t)$ for t nonnegative and $z_1^*(t)$ for t nonpositive in addition to $(z_2^*, \dots, z_{l-1}^*)(0)$.

Coming back to the structure of the solutions of (11.2), with v given, we can conclude the following. Some components are free, some components are determined by the free variables and initial conditions together with the values of v, and some other components are uniquely determined by v.

REMARK. The updating scheme for z is not necessarily first-order [cf. Equations (14)–(16)]. However, it is a local updating scheme in the sense

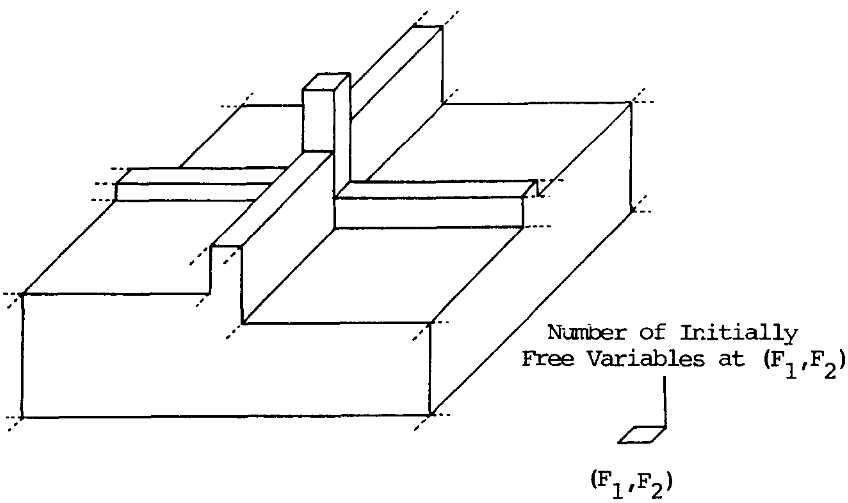


FIG. 2. Number of free variables and initial conditions for unique solution.

that, if $v, z \in (R^n)^{\mathbf{Z}}$, the only values of v necessary to compute $z(t)$ are $v(t-n), \ldots, v(t), \ldots v(t+n)$ (note that $k_2, k_3, k_4 \leqslant n$).

An analogous situation occurs with the solutions v of (11.2) if z is considered to be given.

This gives the updating structure for the proposed line-by-line computational scheme, and hence shows how the solutions of (9) can be computed. The required initial conditions are illustrated in Figure 2.

7. CONCLUSION

In this paper N-D systems have been considered in a noncausal framework. Following the behavioral approach, a definition of N-D system was given which does not involve an *a priori* distinction between input and output. A new concept of state was introduced with basis on the Markov property for deterministic N-D systems—this property can be seen as the generalization to the N-D case of the Markovian property introduced in [6] for deterministic 1-D systems.

It was shown that for the particular case of AR 2-D systems, the state property is equivalent to the existence of a representation of the system in which the state behavior is described by means of a special kind of first-order behavioral equations, and the external behavior is connected with the state behavior by means of a static (zero-order) relation. Moreover, it turns out that this special representation can be used to simulate the solutions of state-space systems by means of a local line-by-line computational scheme.

APPENDIX

Proof of Theorem 2.1. The equivalence of (2) and (3) is an immediate generalization of Proposition 4 in [5] to the 2-D case. To prove that any linear, shift-invariant, and complete 2-D system $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ is an AR system, we introduce the following notation and definitions:

$L := (\mathbb{R}^q)^{\mathbf{Z}^2}$.

$L^* := \{ v \in (\mathbb{R}^{1 \times q})^{\mathbf{Z}^2} | v \text{ has compact support}\}$.

$\tilde{s}_1$ and $\tilde{s}_2$ are operators in L^* defined by $\tilde{s}_1 v(t_1, t_2) = v(t_1 - 1, t_2)$, $\tilde{s}_2 v(t_1, t_2) = v(t_1, t_2 - 1)$ $\forall (t_1, t_2) \in \mathbb{Z}^2$ $\forall v \in L^*$.

$\langle \cdot, \cdot \rangle : L^* \times L \to \mathbb{R}$ is defined by $\langle a, b \rangle = \Sigma_{(t_1, t_2) \in \mathbf{Z}} a(t_1, t_2) b(t_1, t_2)$ $\forall (a, b) \in L^* \times L$.

Clearly $\langle a, \sigma_i b\rangle = \langle \tilde{s}_i a, b\rangle$, $i = 1,2$, $\forall (a,b) \in L^* \times L$, i.e., $\tilde{s}_i$ is the dual of σ_i.

L^* can be identified with $\mathbb{R}^{1\times q}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ by means of the isomorphism $\Phi(v) = \sum_{(t_1,t_2)\in \mathbb{Z}^2} v(t_1,t_2) s_1^{t_1} s_2^{t_2}$ $\forall v \in L^*$. It is easy to see that $\Phi(\tilde{s}_i v) = s_i \Phi(v)$ $(i = 1,2)$, and therefore the operator $\tilde{s}_i$ in L^* is identified with multiplication by s_i in $\mathbb{R}^{1\times q}[s_1, s_1^{-1}, s_2, s_2^{-1}]$. From now on we will identify L^* with $\Phi(L^*)$ and $\tilde{s}_i$ with s_i.

Our proof will be based on the following elementary facts.

Fact 1. *L^* is a finitely generated free module over the Noetherian ring $\mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$. More concretely, $L^* = \sum_{i=1}^{q} \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}] e_i$, where $(e_1, \ldots, e_q)$ is the standard basis of $\mathbb{R}^{1\times q}$.*

Fact 2. *$\mathfrak{B}^{\perp} := \{ v \in L^* | \langle v, w\rangle = 0 \ \forall w \in \mathfrak{B}\}$ is an $\mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$-submodule of L^*.*

Fact 3. *Let N be a Noetherian ring, $M = Nm_1 + \cdots + Nm_l$ $(m_i \in M$, $i = 1, \ldots, l)$ a finitely generated free module over N, and S and N-submodule of M. Then there are ideals $J_1, \ldots, J_l$ in N such that $S = J_1 m_1 + \cdots + J_l m_l$.*

Fact 4 [8]. *Every ideal J in a Noetherian ring N is finitely generated, i.e., there are $j_1, \ldots, j_k \in J$ such that $J = Nj_1 + \cdots + Nj_k$.*

It follows from this that there exist $r_1, \ldots, r_g \in L^*$ such that $\mathfrak{B}^{\perp} = \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}] r_1 + \cdots + \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}] r_g$.

Now, let $\mathfrak{X} := \{ w \in (\mathbb{R}^q)^{\mathbb{Z}^2} | \langle s_1^{t_1} s_2^{t_2} r_i, w\rangle = \langle r_i, \sigma_1^{t_1}\sigma_2^{t_2} w\rangle = 0 \ \ \forall (t_1, t_2) \in \mathbb{Z}^2$, $i = 1, \ldots, g\}$. We will see that $\mathfrak{X} = (\mathfrak{B}^{\perp})^{\perp}$. Since $s_1^{t_1} s_2^{t_2} r_i \in \mathfrak{B}^{\perp}$ for all $(t_1, t_2) \in \mathbb{Z}^2$ and $i = 1, \ldots, g$, it is obvious that $(\mathfrak{B}^{\perp})^{\perp} \subset \mathfrak{X}$. To verify that $\mathfrak{X} \subset (\mathfrak{B}^{\perp})^{\perp}$, let $w \in \mathfrak{X}$, and b be an arbitrary element of $\mathfrak{B}^{\perp}$. Then b can be written as $b = b^1 r_1 + \cdots + b^g r_g$, where for $i = 1, \ldots, g$, $b^i = \sum_{(t_1,t_2)\in\mathbb{Z}^2} b^i_{t_1,t_2} s_1^{t_1} s_2^{t_2}$ $(b^i_{t_1,t_2} \in \mathbb{R})$. Thus,

$$\langle b, w\rangle = \sum_{i=1}^{g} \langle b^i r_i, w\rangle = \sum_{i=1}^{g} \left\langle \sum_{(t_1,t_2)\in\mathbb{Z}^2} b^i_{t_1,t_2} s_1^{t_1} s_2^{t_2} r_i, w \right\rangle$$

$$= \sum_{i=1}^{g} \sum_{(t_1,t_2)\in\mathbb{Z}^2} b^i_{t_1,t_2} \langle s_1^{t_1} s_2^{t_2} r_i, w\rangle = 0, \qquad \text{as} \quad w \in \mathfrak{X}.$$

Therefore $w \in (\mathfrak{B}^{\perp})^{\perp}$, yielding $\mathfrak{X} = (\mathfrak{B}^{\perp})^{\perp}$.

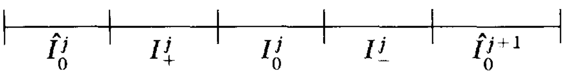
Finally, as $\mathfrak{B}$ is a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$, we have $(\mathfrak{B}^{\perp})^{\perp} = \mathfrak{B}$, and we conclude in this way that $\mathfrak{B} = \{w \in (\mathbb{R}^q)^{\mathbb{Z}^2} | R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w = 0\}$, where R is the $g \times q$ polynomial matrix defined by $R := \mathrm{col}[r_1, \dots, r_g]$. This shows that condition (2) of the theorem implies (1). The reciprocal implication is obvious. ■

Proof of Proposition 3.2. "Only if": Let $t_0 \in T$, and consider the partition $\{T_+, T_0, T_-\}$ of T defined by $T_- = (-\infty, t_0) \cap T$, $T_0 = \{t_0\}$, $T_+ = (t_0, +\infty) \cap T$. Clearly T_- and T_+ are separated by T_0. Therefore, as Σ is supposed to be Markovian, $\mathfrak{B}$ is $(T_- \cup T_0, T_+ \cup T_0)$-concatenable. This clearly implies that $\{w_1, w_2 \in \mathfrak{B},\ w_1(t_0) = w_2(t_0)\} \Rightarrow \{w_1 \underset{t_0}{\wedge} w_2 \in \mathfrak{B}\}$.

"If": Suppose that $\{w_1, w_2 \in \mathfrak{B},\ t_0 \in T,\ w_1(t_0) = w_2(t_0)\} \Rightarrow \{w_1 \underset{t_0}{\wedge} w_2 \in \mathfrak{B}\}$, and let (T_-, T_0, T_+) be a triple of subsets of T such that $T_- \cup T_0 \cup T_+ = T$ and such that T_+ and T_- are separated by T_0. Assume further, without loss of generality, that $T = \hat{I}_0^1 \cup I_+^1 \cup I_0^1 \cup I_-^1 \cup \hat{I}_0^2 \cup \cdots \cup \hat{I}_0^k \cup I_+^k \cup I_0^k \cup I_-^k \cup \hat{I}_0^{k+1}$, with

$$\bigcup_{j=1}^{k} I_0^j \cup \bigcup_{j=1}^{k+1} \hat{I}_0^j = T_0, \qquad \bigcup_{j=1}^{k} I_-^k = T_-, \qquad \bigcup_{j=1}^{k} I_+^k = T_+,$$

and where the subintervals $\hat{I}_0^j, I_+^j, I_0^j, I_-^j, \hat{I}_0^{j+1}$ $(j = 1, \dots, k)$ are as indicated below:



Let $I_j = I_+^j \cup I_0^j \cup I_-^j$ $(j = 1, \dots, k)$. It is not difficult to see that $\mathfrak{B}|_{I_j}$ is $(I_+^j \cup I_0^j, I_-^j \cup I_0^j)$-concatenable, and $\mathfrak{B}|_{I_l \cup \hat{I}_0^l \cup I_{l+1}}$ is $(I_l \cup \hat{I}_0^l, \hat{I}_0^l \cup I_{l+1})$-concatenable $(l = 1, \dots, k-1)$. This implies that $\mathfrak{B}$ is $(T_- \cup T_0, T_+ \cup T_0)$-concatenable. The fact that Σ is Markovian follows now easily by noticing that the condition $T_+ \cup T_0 \cup T_- = T$ can be dropped. ■

Proof of Proposition 3.3. Suppose that Σ is Markovian. Since elementary squares and lines are subintervals of $\mathbb{Z}^2$, it is clear (by Proposition 3.1) that Σ is square and line-concatenable. To see that Σ is square-complete, define ∂ and $\mathfrak{B}(\partial)$ as follows: $\partial := \{(u, v) : \mathbb{Z} \to X \times X | \forall T \in \mathbb{Z}\ \ \exists x \in \mathfrak{B}$ s.t. $(x(t, \cdot), x(t+1, \cdot)) = (u, v)\}$, $\mathfrak{B}(\partial) := \{x : \mathbb{Z}^2 \to X | \forall t \in \mathbb{Z}\ (x(t, \cdot), x(t+1, \cdot)) \in \partial\}$. Note that $\Sigma(\partial) := (\mathbb{Z}, X \times X, \partial)$ is a shift-invariant and complete 1-D system. Moreover, defining $\mathscr{L}(t) := \{(t, t_2) | t_2 \in \mathbb{Z}\}$, we will clearly have that $\Sigma|_{\mathscr{L}(t) \cup \mathscr{L}(t+1)}$ is Markovian, implying that $\Sigma(\partial)$ is $(1-\mathrm{D})$ Markovian.

We will first see that $\mathfrak{B} = \mathfrak{B}(\partial)$. The inclusion $\mathfrak{B} \subset \mathfrak{B}(\partial)$ is obvious. To prove the reciprocal one, let $x \in \mathfrak{B}(\partial)$. Then, there are elements x_1 and x_2 in $\mathfrak{B}$ such that $(x(-1,\cdot), x(0,\cdot)) = (x_1(-1,\cdot), x_1(0,\cdot))$ and $(x(0,\cdot), x(1,\cdot)) = (x_2(0,\cdot), x_2(1,\cdot))$. Let $I(t) := \bigcup_{k=-t}^{t} \mathscr{L}(t)$ $(t \in \mathbb{N})$. As $I(1)$ is an interval of $\mathbb{Z}^2$, $\Sigma|_{I(1)}$ is Markovian. This implies that

$$x_1 \underset{(\mathscr{L}(-1) \cup \mathscr{L}(0),\, \mathscr{L}(0) \cup \mathscr{L}(1))}{\wedge} x_2 \in \mathfrak{B}|_{I(1)}$$

[because $\mathscr{L}(0)$ separates $\mathscr{L}(-1)$ and $\mathscr{L}(1)$, and $x_1|_{\mathscr{L}(0)} = x_2|_{\mathscr{L}(0)}$]. Therefore, also $x|_{I(1)} \in \mathfrak{B}|_{I(1)}$. A simple inductive reasoning shows that, for all $t \in \mathbb{N}$, $x|_{I(t)} \in \mathfrak{B}|_{I(t)}$, and, as Σ is complete, this means that $x \in \mathfrak{B}$.

Now, as $\Sigma(\partial)$ is Markovian and complete, it is possible to prove (by similar arguments to the above) that

$$\{(u, v) \in \partial\} \Leftrightarrow \{(u, v)|_{\{t, t+1\}} \in \partial|_{\{t, t+1\}}\ \forall t \in \mathbb{Z}\}.$$

Therefore, it follows from the definition of $\mathfrak{B}(\partial)$ that $\{x \in \mathfrak{B}(\partial)\} \Leftrightarrow \{x|_{\mathscr{S}} \in \mathfrak{B}(\partial)|_{\mathscr{S}}$ for all elementary squares $\mathscr{S} \subset \mathbb{Z}^2\}$, yielding that $\mathfrak{B} = \mathfrak{B}(\partial)$ is square-complete.

Note that in order to prove that conditions (1), (2), and (3) imply that Σ is Markovian, it is enough to show that the concatenability property holds for partitions (T_+, T_0, T_-) of intervals $T \subset \mathbb{Z}^2$. We will only treat in detail the case where $T = [a, b] \times [c, d]$ is a finite interval of $\mathbb{Z}^2$ and the separation set T_0 is as indicated in Figure 3. (The interval notation will be used to denote intervals in $\mathbb{Z}$.)

All the other cases can be analyzed using the same kind of arguments.

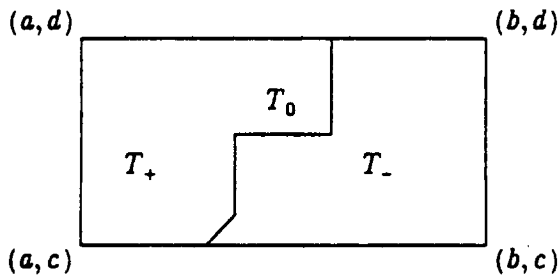


FIG. 3. $\{(b, t) | c \leqslant t \leqslant d\} \subset T_-$; $\{(a, t) | c \leqslant t \leqslant d\} \subset T_+$.

Let $T=[a,b]\times[c,d]$. Suppose that T_+, T_0, and T_- are as indicated in the figure and that $x_1, x_2 \in \mathfrak{B}$ are such that $x_2|_{T_0}=x_1|_{T_0}$. As Σ is square-concatenable [by assumption (1)],

$$w := x_1 \underset{(T_+\cup T_0,\, T_0\cup T_-)}{\wedge} x_2$$

satisfies $w|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ for all elementary squares $\mathcal{S}\subset T$.

Denote $I:=(-\infty,a-1]\times[c,d]$ and $J:=[b+1,\infty)\times[c,d]$, and consider $\tilde{x}\in W^{I\cup T\cup J}$ defined by

$$\tilde{x} := x_1 \underset{(I,T)}{\wedge} w \underset{(T,J)}{\wedge} x_2.$$

Clearly, also $\tilde{x}$ satisfies $\tilde{x}|_{\mathcal{S}}\in\mathfrak{B}|_{\mathcal{S}}$ for all elementary squares $\mathcal{S}\subset I\cup T\cup J$.

Let $\mathcal{H}(t):=\{(t_1,t)|t_1\in\mathbb{Z}\}$ $(t\in\mathbb{Z})$. As Σ is also assumed to be line-concatenable, $\tilde{x}|_{\mathcal{H}(c)}\in\mathfrak{B}|_{\mathcal{H}(c)}$ and $\tilde{x}|_{\mathcal{H}(d)}\in\mathfrak{B}|_{\mathcal{H}(d)}$. This means that there are elements $x^-, x^+\in\mathfrak{B}$ such that $x^-|_{\mathcal{H}(c)}=\tilde{x}|_{\mathcal{H}(c)}$ and $x^+|_{\mathcal{H}(d)}=\tilde{x}|_{\mathcal{H}(d)}$. Denote $\mathcal{H}^- := \bigcup_{t\leqslant c}\mathcal{H}(t)$, $\mathcal{H}^+ := \bigcup_{t\geqslant d}\mathcal{H}(t)$ (where all t's are integers), and define

$$x^* := x^- \underset{(\mathcal{H}^-,\, I\cup T\cup J)}{\wedge} \tilde{x} \underset{(I\cup T\cup J,\, \mathcal{H}^+)}{\wedge} x^+.$$

Clearly $x^*|_{\mathcal{S}}\in\mathfrak{B}|_{\mathcal{S}}$ for all elementary squares $\mathcal{S}$. Thus, as $\mathfrak{B}$ is square-complete [assumption (3)], we conclude that $x^*\in\mathfrak{B}$.

Finally, as by construction

$$x^*|_T = x_1 \underset{(T_1\cup T_0,\, T_0\cup T_-)}{\wedge} x_2,$$

this yields that $\mathfrak{B}$ is $(T_+\cup T_0, T_-\cup T_0)$-concatenable. ∎

Proof of Lemma 4.1. First note that, as $\Sigma=(\mathbb{Z}^2,\mathbb{R}^n,\mathfrak{B})$ is supposed to be linear and shift-invariant, there is a linear subspace L of $\mathbb{R}^{4n}$ such that $\mathfrak{B}|_{\mathcal{S}}=L$ for all elementary squares $\mathcal{S}\subset\mathbb{Z}^2$. Moreover, there exist matrices $E,F,G,H\in\mathbb{R}^{g\times n}$ such that $[E\ F\ G\ H]$ has full row rank and $L=\ker[E\ F\ G\ H]$. Thus, by the square completeness of Σ, $x\in\mathfrak{B}$ iff

$$Ex(t_1,t_2)+Fx(t_1+1,t_2)+Gx(t_1,t_2+1)+Hx(t_1+1,t_2+1)=0 \qquad \forall(t_1,t_2)\in\mathbb{Z}^2.$$

Let now $\mathrm{col}[a,b,c,d], \mathrm{col}[a',b,c,d'] \in L$. As Σ is square-concatenable, also $\mathrm{col}[a',b,c,d]$ and $\mathrm{col}[a,b,c,d']$ will be elements of L. This means that $L = L_1 \cap L_2$, where $L_1 := \{\mathrm{col}[a,b,c,d^*] \in \mathbb{R}^{4n} | d^* \in \mathbb{R}^n$ and $\exists d \in \mathbb{R}^n$ s.t. $\mathrm{col}[a,b,c,d] \in L\}$ and $L_2 := \{\mathrm{col}[a^*,b,c,d] \in \mathbb{R}^{4n} | a^* \in \mathbb{R}^n$ and $\exists a \in \mathbb{R}^n$ s.t. $\mathrm{col}[a,b,c,d] \in L\}$. The subspaces L_1 and L_2 can be written as $L_1 = \ker[A_1\, B_1\, C_1\, 0]$ and $L_2 = \ker[0\, B_2\, C_2\, D_2]$, for suitable matrices $A_1, B_1, C_1 \in \mathbb{R}^{l_1 \times n}$ and $B_2, C_2, D_2 \in \mathbb{R}^{l_2 \times n}$, yielding

$$\ker[E \quad F \quad G \quad H] = \ker\begin{bmatrix} A_1 & B_1 & C_1 & 0 \\ 0 & B_2 & C_2 & D_2 \end{bmatrix}.$$

This implies that there is a matrix $T \in \mathbb{R}^{(l_1+l_2)\times g}$ such that

$$T[E \quad F \quad G \quad H] = \begin{bmatrix} A_1 & B_1 & C_1 & 0 \\ 0 & B_2 & C_2 & D_2 \end{bmatrix}.$$

Moreover, $\mathrm{rank}\, T[E\ F\ G\ H] = \mathrm{rank}[E\ F\ G\ H] = g$, implying that $\mathrm{rank}\, T = g$. Further, as $TE = \mathrm{col}[A_1, 0]$ and $TH = \mathrm{col}[0, D_2]$, $\mathrm{im}(TE) \cap \mathrm{im}(TH) = \{0\}$ Finally, let $z \in \mathrm{im}(E) \cap \mathrm{im}(H)$. Then there exist $z_1, z_2 \in \mathbb{R}^n$ such that $Ez_1 = z = Hz_2$. Thus $TEz_1 = Tz = THz_2$ and $Tz \in \mathrm{im}(TE) \cap \mathrm{im}(TH)$. This implies that $Tz = 0$, and as T is injective, $z = 0$. We conclude in this way that $\mathrm{im}(E) \cap \mathrm{im}(H) = \{0\}$. The proof that $\mathrm{im}(F) \cap \mathrm{im}(G) = \{0\}$ is analogous. ∎

FACT. *Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^n, \mathfrak{B})$ be described by the following behavioral equations*:

$$E_1 x + F_1 \sigma_1 x = 0, \tag{1}$$

$$E_2 x + G_2 \sigma_2 x = 0, \tag{2}$$

$$E_3 x + F_3 \sigma_1 x + G_3 \sigma_2 x + H_3 \sigma_1 \sigma_2 x = 0. \tag{3}$$

Consider the conditions C1 and C2 defined in Section 4. Then (1) *describes $\mathfrak{B}_1$ iff condition C1 holds. Analogously* (2) *describes $\mathfrak{B}_2$ iff condition C2 holds.*

Proof. We will only show the first equivalence. The proof of the second one is similar.

"Only if": Let $U(s, s^{-1})$ be a unimodular matrix such that $U(\sigma_1, \sigma_1^{-1})$ $[E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2] = \mathrm{col}[D(\sigma_1, \sigma_1^{-1})\sigma_2 + M(\sigma_1, \sigma_1^{-1}),\ \Pi(\sigma_1, \sigma_1^{-1})]$, with $D(s, s^{-1})$ full row rank. Then (1), (2), (3) are equivalent to

$$(E_1 + F_1\sigma_1)x = 0, \tag{1}$$

$$\Pi(\sigma_1, \sigma_1^{-1})x = 0, \tag{4}$$

$$D(\sigma_1, \sigma_1^{-1})\sigma_2 x + M(\sigma_1, \sigma_1^{-1})x = 0. \tag{5}$$

Consequently, the elements of $\mathfrak{B}_1$ also satisfy Equation (4). This means that the solutions of (1) are contained in the solutions of (4), and therefore there will be a polynomial matrix $L(s, s^{-1})$ such that $\Pi(s, s^{-1})$ is a left multiple of $E_1 + F_1 s$.

"If": Note that Equations (1), (2), (3) are equivalent to

$$(E_1 + F_1\sigma_1)x = 0, \tag{1}$$

$$(E_1 + F_1\sigma_1)\sigma_2 x = 0, \tag{6}$$

$$Q(\sigma_1)\sigma_2 x = R(\sigma_1)x, \tag{7}$$

where $Q(s) = \text{col}[G_2, G_2 s, G_3 + H_3 s]$ and $R(s) = \text{col}[-E_2, -E_2 s, -E_3 - F_3 s]$. Moreover, (6) and (7) can be written as

$$\begin{bmatrix} 0 & E_1 + F_1\sigma_1 \\ -R(\sigma_1) & Q(\sigma_1) \end{bmatrix} \begin{bmatrix} x \\ \sigma_2 x \end{bmatrix} = 0, \tag{8}$$

and there is a unimodular matrix $U(s) = \text{col}[(U_1(s)\ U_2(s)), (U_3(s)\ U_4(s))]$ such that

$$U(\sigma_1)\begin{bmatrix} E_1 + F_1\sigma_1 \\ Q(\sigma_1) \end{bmatrix} = \begin{bmatrix} U_1(\sigma_1)(E_1 + F_1\sigma_1) + U_2(\sigma_1)Q(\sigma_1) \\ U_3(\sigma_1)(E_1 + F_1\sigma_1) + U_4(\sigma_1)Q(\sigma_1) \end{bmatrix} = \begin{bmatrix} D(\sigma_1) \\ 0 \end{bmatrix},$$

with $D(s)$ full row rank. [Note that $D(s)$ is a g.c.d. of $E_1 + F_1 s$ and $Q(s)$.] Thus, $U^*(s) = \text{col}[(0\ U(s)), (I\ 0)]$ is a unimodular matrix such that $U^*(\sigma_1)(E + F\sigma_1 + G\sigma_2 + H\sigma_1\sigma_2) = \text{col}[D(\sigma_1)\sigma_2 - U_2(\sigma_1)R(\sigma_1),\ -U_4(\sigma_1)R(\sigma_1),\ E_1 + F_1\sigma_1]$, and condition C1 will imply that $\text{col}[-U_4(s)R(s), E_1 + Fs]$ is a left multiple of $E_1 + F_1 s$. Consequently there exists a polynomial matrix $L(s)$ such that $L(s)(E_1 + F_1 s) = -U_4(s)R(s)$. This implies that Equations (1), (6),

(7) are equivalent to

$$(E_1 + F_1\sigma_1)x = 0, \tag{1}$$

$$D(\sigma_1)\sigma_2 x = U_2(\sigma_1)R(\sigma_1)x. \tag{9}$$

Consider now the following 1-D equations:

$$(E_1 + F_1\sigma)v_1 = 0, \tag{10}$$

$$D(\sigma)v_2 = U_2(\sigma)R(\sigma)v_1. \tag{11}$$

As $D(s)$ has full row rank, $D(\sigma)$ is surjective. Thus, for every solution v_1 of (10) there is v_2 such that (11) is satisfied. Further, v_2 is a solution of (10). To prove this, let $U^{-1}(s) = \text{col}[(\bar{U}_1(s)\;\; \bar{U}_2(s)), (\bar{U}_3(s)\;\; \bar{U}_4(s))]$. Then $\bar{U}_1(s)U_2(s) + \bar{U}_2(s)U_4(s) = 0$ and $\bar{U}_1(s)D(s) = E_1 + F_1 s$. This implies that $-\bar{U}_1(s)U_2(s)R(s) = \bar{U}_2(s)U_4(s)R(s) = -\bar{U}_2(s)L(s)(E_1 + F_1 s)$. Thus, premultiplying both sides of (11) by $\bar{U}_1(\sigma)$ yields $\bar{U}_1(\sigma)D(\sigma)v_2 = \bar{U}_2(\sigma)L(\sigma)(E_1 + F_1\sigma)v_1$. As v_1 is a solution of (10), this becomes $(E_1 + F_1\sigma)v_2 = 0$, i.e., v_2 also satisfies (10). In terms of Equations (1) and (9) this has the following interpretation. Let $x \in (\mathbb{R}^n)^{\mathbb{Z}^2}$, $\mathscr{L} = \{(t,0) \in \mathbb{Z}^2 | t \in \mathbb{Z}\}$, and suppose that $x|_{\mathscr{L}}$ satisfies (1). Then $x|_{\mathscr{L}}$ admits an extension $x^+ \in (\mathbb{R}^n)^{\mathbb{Z}^2}$ such that $x^+|_{\mathscr{H}^+}$ satisfies (1) and (9) [here $\mathscr{H}^+ := \{(t_1, t_2) \in \mathbb{Z}^2 | t_2 \geq 0\}$]. A similar reasoning shows that $x|_{\mathscr{L}}$ also admits an extension x^- such that $x^-|_{\mathscr{H}^-}$ satisfies the behavioral equations (1), (9) (here $\mathscr{H}^-$ is defined in the obvious way). This implies that $x|_{\mathscr{L}}$ can be extended to $\mathbb{Z}^2$ as an element of $\mathfrak{B}$, yielding the fact that (1) induces $\mathfrak{B}_1$. ■

Proof of Theorem 4.1. To prove this result it is enough to note that, when Σ is line-concatenable [i.e., (2) is satisfied], condition (1) (ii) implies that Σ is square-concatenable. Let $\mathscr{S} = \{(t_1, t_2), (t_1 + 1, t_2), (t_1, t_2 + 1), (t_1 + 1, t_2 + 1)\}$ and $x_1, x_2 \in \mathfrak{B}|_{\mathscr{S}}$ such that $x_1(t_1 + 1, t_2) = x_2(t_1 + 1, t_2)$ and $x_1(t_1, t_2 + 1) = x_2(t_1, t_2 + 1)$. As $\text{im}(E) \cap \text{im}(H) = \{0\}$,

$$Ex_1(t_1, t_2) + Fx_1(t_1 + 1, t_2) + Gx_1(t_1 + 1, t_2) + Hx_2(t_1 + 1, t_2 + 1) = 0,$$

i.e.,

$$x := x_1 \underset{(T_+ \cup T_0, T_0 \cup T_-)}{\wedge} x_2$$

(with $T=\{(t_1,t_2)\}$ $T_0=\{(t_1+1,t_2),(t_1,t_2+1)\}$, $T_-=\{(t_1+1,t_2+1)\}$) satisfies the behavioral equations in $\mathcal{S}$. Now, if Σ is line-concatenable, the same kind of argument as in the proof of Proposition 3.3 ("if" part) shows that x can be extended to $\mathbb{Z}^2$ as an element of $\mathfrak{B}$. Therefore $x\in\mathfrak{B}|_{\mathcal{S}}$, and $\mathfrak{B}|_{\mathcal{S}}$ is $(T_+\cup T_0, T_-\cup T_0)$-concatenable. Analogously, we can prove that $\mathfrak{B}|_{\mathcal{S}}$ is $(T_+'\cup T_0', T_-'\cup T_0')$-concatenable, with $T_+'=\{(t_1,t_2+1)\}$, $T_-'=\{(t_1+1,t_2)\}$, and $T_0'=\{(t_1,t_2),(t_1+1,t_2+1)\}$. This suffices to show that Σ is square-concatenable. ■

Proof of Proposition 5.2. Let $\Sigma^w=(\mathbb{Z}^2,\mathbb{R}^q,\mathfrak{B}^w)$ be an AR system, and suppose that $\mathfrak{B}^w$ is described by the behavioral equation $R(\sigma_1,\sigma_2)w=0$, where $R(s_1,s_2):=\sum_{i=0}^{a}\sum_{j=0}^{b}R_{ij}s_1^is_2^j$ and $R_{ij}\in\mathbb{R}^{g\times q}$, $i=0,\dots,a$, $j=0,\dots,b$. Define the variable $z:=S(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w$, with $S(s_1,s_1^{-1},s_2,s_2^{-1})=\operatorname{col}[s_1^{-a}s_2^{-b},\dots,s_1^{-a}s_2^{b},\dots,s_1^as_2^{-b},\dots,s_1^as_2^b]$. The behavior $\mathfrak{B}^z:=S\mathfrak{B}$ can be described by autoregressive equations. Moreover it is easy to verify that $\mathfrak{B}^z$ is $(T_+\cup T_0,T_-\cup T_0)$-concatenable for all triples of subsets T_+,T_-,T_0 of $\mathbb{Z}^2$ such that $\{T_+,T_0,T_-\}$ is a partition of $\mathbb{Z}^2$, T_0 is a vertical or horizontal line, and T_+ and T_- are separated by T_0. This will imply that $\mathfrak{B}^z$ can be described by behavioral equations of a special form. In fact, taking T_0 to be a vertical line in $\mathbb{Z}^2$, it is possible to see that $\mathfrak{B}^z$ can be described by a behavioral equation

$$[M_0(\sigma_2)+M_1(\sigma_2)\sigma_1]z=0, \tag{1}$$

for some polynomial matrices $M_0(s)$ and $M_1(s)$. Analogously, taking T_0 to be a horizontal line in $\mathbb{Z}^2$ shows that $\mathfrak{B}^z$ can also be described by

$$(N_0(\sigma_1)+N_1(\sigma_1)\sigma_2)z=0, \tag{2}$$

for some polynomial matrices $N_0(s)$ and $N_1(s)$. Suppose that the vertical and horizontal behavior associated with $\mathfrak{B}^z$ are given, respectively, by the following equations:

$$P_2(\sigma_2)z=0, \tag{3}$$

$$P_1(\sigma_1)z=0. \tag{4}$$

Assume further that $M_i(s)=\sum_{j=0}^{m_i}M_i^js^j$, $N_i(s)=\sum_{j=0}^{n_i}N_i^js^j$, $P_k(s)=\sum_{j=0}^{p_k}p_k^js^j$ $(i=0,1;\ k=1,2)$, and let $l=\max\{m_0,m_1,n_0,n_1,p_1,p_2\}$. Define a new variable $x:=\bar{S}(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})z$, where $\bar{S}=\operatorname{col}[s_1^{-l}s_2^l,\dots,s_1^ls_2^{-l},\dots,s_1^ls_2^l]$, and consider $\mathfrak{B}^{z,x}:=\{(z,x)|z\in\mathfrak{B}^z \text{ and } x=\bar{S}(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})z\}$. It can be verified that $\mathfrak{B}^{z,x}$ satisfies the axiom of 2-D state. Finally, as z incorporates w—i.e., there is a projection operator Π such that $w = \Pi z$—also $\mathfrak{B} := \{(w,x) | w \in \mathfrak{B}^w \text{ and } x = \bar{S}w\}$ satisfies the axiom of state. Thus $\Sigma := (\mathbb{Z}^2, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ (where n is the size of x) is a state-space realization of Σ^w. Moreover, by construction, Σ is an AR system. ■

Proof of Proposition 5.3. "Only if": Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathbb{R}^n, \mathfrak{B})$ be an AR state-space system. It follows from the axiom of state that if $(w,x) \in \mathfrak{B}$, then the value of w at any point $t \in \mathbb{Z}$ does not depend on $(w,x)|_{\mathbb{Z}\setminus\{t\}}$ once $x(t)$ is given. Thus $\mathfrak{B} = \{(w,x) \in (\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}} | x \in \mathfrak{B}^x \text{ and } \forall t \in \mathbb{Z}\ (w,x)(t) \in S(t)\}$, with $S(t) \subset \mathbb{R}^q \times \mathbb{R}^n\ \forall t \in \mathbb{Z}$. As $\mathfrak{B}$ is a shift-invariant and linear subspace of $(\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}}$, there is a linear subspace L of $\mathbb{R}^q \times \mathbb{R}^n$ such that $S(t) = L\ \forall t \in \mathbb{Z}$. Moreover, L can be represented as the kernel of a linear operator defined on $\mathbb{R}^q \times \mathbb{R}^n$. Therefore there will exist matrices M and N, of suitable sizes, such that $\mathfrak{B} = \{(w,x) \in (\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}} | x \in \mathfrak{B}^x \text{ and } Mw + Nx = 0\}$. Now to complete our proof it is enough to show that $\mathfrak{B}^x$ can be described by a first-order behavioral equation $Ex + Fx = 0$. Let $\partial := \{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n | \forall t \in \mathbb{Z}\ \exists x \in \mathfrak{B} \text{ s.t. } (x(t), x(t+1)) = (a,b)\}$. As ∂ is a linear subspace of $\mathbb{R}^n \times \mathbb{R}^n$, there are matrices E and F such that $(a,b) \in \partial$ iff $Ea + Fb = 0$. Define $\mathfrak{B}(\partial) := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} | \forall t \in \mathbb{Z}\ (x(t), x(t+1)) \in \partial\}$. We will see that $\mathfrak{B}^x = \mathfrak{B}(\partial)$. The inclusion $\mathfrak{B}^x \subset \mathfrak{B}(\partial)$ is obvious. To prove that $\mathfrak{B}(\partial) \subset \mathfrak{B}^x$, let $x \in \mathfrak{B}(\partial)$. Then, there exist elements x_1 and $x_2 \in \mathfrak{B}$ such that $(x(0), x(1)) = (x_1(0), x_1(1))$ and $(x(-1), x(0)) = (x_2(-1), x_2(0))$. Now, as $x_1(0) = x(0) = x_2(0)$ and $\mathfrak{B}^x$ is Markovian, $\hat{x} := x_2 \underset{0}{\wedge} x_1 \in \mathfrak{B}$. As a consequence, $x|_{[-1,1]} = \hat{x}|_{[-1,1]} \in \mathfrak{B}^x|_{[-1,1]}$. Analogously, we can prove that $x|_{[-k,k]} \in \mathfrak{B}^x|_{[-k,k]}$ for all $k \in \mathbb{N}$, and, because $\mathfrak{B}^x$ is complete, we conclude that $x \in \mathfrak{B}^x$. Therefore $x \in \mathfrak{B}^x$ iff $Ex(t) + Fx(t+1) = 0\ \forall t \in \mathbb{Z}$, yielding the desired result.

The reciprocal implication is obvious. ■

Proof of Theorem 5.1. To prove the "only if" part it is enough to show that $\mathfrak{B} = \{(w,x) \in (\mathbb{R}^q \times \mathbb{R}^n)^{\mathbb{Z}^2} | x \in \mathfrak{B}^x \text{ and } Mw + Nx = 0\}$ for some suitable matrices M and N. Let $(t_1^\circ, t_2^\circ) \in \mathbb{Z}^2$, and consider the partition $\{T_+, T_0, T_-\}$ of $\mathbb{Z}^2$ such that $T_+ = \varnothing$, $T_0 = \{(t_1^\circ, t_2^\circ)\}$, and $T_- = \mathbb{Z}^2 \setminus T_0$. Given $(w_1, x_1), (w_2, x_2) \in \mathfrak{B}$, define w' and w'' in $(\mathbb{R}^q)^{\mathbb{Z}^2}$ as follows: $w'(t_1, t_2) = w_2(t_1, t_2)$ if $(t_1, t_2) \neq (t_1^\circ, t_2^\circ)$, $w'(t_1^\circ, t_2^\circ) = w_1(t_1^\circ, t_2^\circ)$, $w''(t_1, t_2) = w_1(t_1, t_2)$ if $(t_1, t_2) \neq (t_1^\circ, t_2^\circ)$, and $w''(t_1^\circ, t_2^\circ) = w_2(t_1^\circ, t_1^\circ)$. The axiom of state implies that if $x_1(t_1, t_2) = x_2(t_1, t_2)$, then $(w', x_2), (w'', x_1) \in \mathfrak{B}$. This means that if $(w,x) \in \mathfrak{B}$, given $x(t_1, t_2)$, the value $w(t_1, t_2)$ is independent of $(w,x)|_{\mathbb{Z}^2 \setminus \{(t_1, t_2)\}}$. Now the fact that the structure of $\mathfrak{B}$ is as indicated above easily follows from the same kind of arguments as in the previous proof.

The "if" statement is easy to verify. ■

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