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SINGULAR OPTIMAL CONTROL: A GEOMETRIC APPROACH*

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Abstract. Linear quadratic singular optimal control problem is solved for nonminimum phase and noninvertible systems. A state space decomposition is obtained and a reduced order nonsingular subproblem is solved. The optimal stabilizing input of the singular problem has been found when there are no transmission zeros on the imaginary axis.

Key words. optimal control, singular optimal control, geometric control, distributions

1. Introduction. This paper is concerned with linear quadratic problems in which the cost functional is not positive definite in the control. These are called singular problems. In [1], the finite horizon problem and the infinite horizon problem have been solved when the system is minimum phase. It was also shown that the regular part of the optimal input is feedback implementable.

The geometric theory of linear systems added a great deal of insight into the structure of the solution of such singular problems. In fact, it could be claimed that the theory of (almost) controlled invariant and controllable subspaces are the generic tools for studying this class of problems as demonstrated in [1].

In the present paper, we will investigate the problem further and obtain algorithms for actually computing the optimal control. The nonminimum phase case is also considered and results are found by solving reduced order algebraic Riccati equations. As is well known, the optimal control may not exist in the class of regular control functions and indeed, our optimal trajectory and the ensuing state trajectory lies in the class of distributions. In addition, for positive times, the optimal trajectory is smooth and lies on a predetermined linear subspace of the state space.

We will be using standard notation: $\mathbb{R}^m$ for m-dimensional Euclidean space, $\mathbb{R}^+ := [0, \infty)$, $\mathscr{D}'$ for the distributions with support on $\mathbb{R}^+$, $A\backslash B$ for $A \cap B^{\text{complement}}$, $\langle A|\mathscr{L}\rangle$ for the largest A-invariant subspace containing the subspace $\mathscr{L}$, and $\langle \mathscr{L}|A\rangle$ for the smallest A-invariant subspace contained in $\mathscr{L}$. Of course, for the familiar $\dot{x} = Ax + Bu$, $y = Cx$, $\langle A|\text{im } B\rangle$ is the reachable subspace, while $\langle \ker C|A\rangle$ is the nonobservable subspace.

2. Problem statement. In this paper we will study the full linear quadratic problem with nonnegative cost functional. The usual formulation is to consider, for the system $\dot{x} = Ax + Bu$, the cost functional $\int q(\underset{\sim}{x}, \underset{\sim}{u})\, dt$ with q a quadratic form (in x and u jointly). However, since we will only be concerned with the situation in which $q \geqq 0$, we can always introduce the output $y = Cx + Du$ such that $\|y\|^2 = q(x, u)$. As in [1] we are thus led to consider the linear system

$$(1) \qquad \Sigma \;\; \dot{x} = Ax + Bu, \qquad y = Cx + Du$$

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with state space $\mathscr{X} = \mathbb{R}^n$, input space $\mathscr{U} = \mathbb{R}^m$, and output space $\mathscr{Y} = \mathbb{R}^p$, and with cost $\int \|\underset{\sim}{y}\|^2\, dt$.

Consider the following spaces of inputs:

(i) *Regular inputs*:

$$\mathscr{U}^{\text{reg}} = \mathscr{L}_2^{\text{loc}}(\mathbb{R}^+; \mathbb{R}^m)$$
$$= \left\{ \underset{\sim}{u}: \mathbb{R}^+ \to \mathbb{R}^m \;\middle|\; \underset{\sim}{u} \text{ is measurable and } \int_0^T \|\underset{\sim}{u}\|^2\, dt < \infty \text{ for all } T \in \mathbb{R}^+ \right\}.$$

(ii) *Distributional inputs*: Even though we could consider general distributions on $\mathbb{R}^+$, we will, as in [1], restrict our attention to Bohl type distributions (those whose Laplace transform is rational). Thus

$$\mathscr{U}^{\text{dist}} := \{\underset{\sim}{u} \in \mathscr{D}'_+ \mid \underset{\sim}{u} = \underset{\sim}{u}^{\text{imp}} + \underset{\sim}{u}^{\text{reg}} \text{ with } \underset{\sim}{u}^{\text{imp}} \text{ an impulsive distribution, and } \underset{\sim}{u}^{\text{reg}} \in \mathscr{U}^{\text{reg}}\}.$$

An impulsive distribution is one with support in 0, i.e., a distribution of the form $\sum_{i=0}^N a_i \underset{\sim}{\delta}^{(i)}$ with $a_i \in \mathbb{R}^m$, $\underset{\sim}{\delta}$ the Dirac delta, and (i) the i-th derivative.

Let $\mathscr{X}^{\text{reg}}$, $\mathscr{X}^{\text{dist}}$, $\mathscr{Y}^{\text{reg}}$ and $\mathscr{Y}^{\text{dist}}$ be similarly defined. Obviously $\mathscr{U}^{\text{dist}} \supset \mathscr{U}^{\text{reg}}$. Now for any given initial condition $\underset{\sim}{x}(0) = x_0$ and any $\underset{\sim}{u} \in \mathscr{U}^{\text{dist}}$, Σ generated in the standard way unique solutions $\underset{\sim}{x} \in \mathscr{X}^{\text{dist}}$ and $\underset{\sim}{y} \in \mathscr{Y}^{\text{dist}}$ (for details, see [1, § 3]). In order to display the dependence on x_0 and $\underset{\sim}{u}$ we will denote these unique solutions by $\underset{\sim}{x}(x_0, \underset{\sim}{u})$ and $\underset{\sim}{y}(x_0, \underset{\sim}{u})$. Of course, if $\underset{\sim}{u} \in \mathscr{U}^{\text{reg}}$ then also $\underset{\sim}{x}(x_0, \underset{\sim}{u}) \in \mathscr{X}^{\text{reg}}$ and $\underset{\sim}{y}(x_0, \underset{\sim}{u}) \in \mathscr{Y}^{\text{reg}}$. However, it is important to observe that some $\underset{\sim}{u} \in \mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{reg}}$ may lead to solutions $\underset{\sim}{y}(x_0, \underset{\sim}{u}) \in \mathscr{Y}^{\text{reg}}$.

Now consider the cost function $\int_0^\infty \|\underset{\sim}{y}\|^2\, dt$. Formally, define

$$\mathscr{J}: \mathscr{X} \times \mathscr{U}^{\text{dist}} \to \mathbb{R}^e$$

by

$$\mathscr{J}(x_0, \underset{\sim}{u}) := \int_0^\infty \|\underset{\sim}{y}(x_0, \underset{\sim}{u})\|^2\, dt \tag{2}$$

where we will agree to set $\mathscr{J}(x_0, \underset{\sim}{u}) = \infty$ when

$$\underset{\sim}{y}(x_0, \underset{\sim}{u}) \in \mathscr{Y}^{\text{dist}} \backslash \mathscr{Y}^{\text{reg}} \quad \text{or when} \quad \underset{\sim}{y}(x_0, \underset{\sim}{u}) \in \mathscr{L}_2^{\text{loc}} \backslash \mathscr{L}_2.$$

We will be interested in minimizing $\mathscr{J}$ with or without stability conditions on the state. Let

$$\mathscr{U}^{\text{dist}}_{\text{stab}}(x_0) := \{\underset{\sim}{u} \in \mathscr{U}^{\text{dist}} \mid \lim_{t \to \infty} \underset{\sim}{x}(x_0, \underset{\sim}{u})(t) = 0\}$$

and let $\mathscr{U}^{\text{reg}}_{\text{stab}}(x_0)$ be similarly defined. Now define

$$\mathscr{J}^*(x_0) := \underset{u \in \mathscr{U}^{\text{dist}}}{\text{infimum}}\, \mathscr{J}(x_0, \underset{\sim}{u})$$

and

$$\mathscr{J}^*_{\text{stab}}(x_0) := \underset{\underset{\sim}{u} \in \mathscr{U}^{\text{dist}}_{\text{stab}}(x_0)}{\text{infimum}}\, \mathscr{J}(x_0, \underset{\sim}{u}).$$

We will study a number of aspects of the cost minimization problem introduced above. In particular we shall answer the following questions:

(i) How can $\mathscr{J}^*$ and $\mathscr{J}^*_{\text{stab}}$ be evaluated? When are $\mathscr{J}^*(x_0)$ and $\mathscr{J}^*_{\text{stab}}(x_0)$ finite? When are they zero?

(ii) Find, if it exists, $\underset{\sim}{u}^* \in \mathscr{U}^{\text{dist}}$ such that $\mathscr{J}(x_0, \underset{\sim}{u}^*) = \mathscr{J}^*(x_0)$. Is $\underset{\sim}{u}^*$ unique? When is $\underset{\sim}{u}^* \in \mathscr{U}^{\text{reg}}$?

(iii) Same questions for $\underset{\sim}{u}^* \in \mathscr{U}^{\text{dist}}_{\text{stab}}(x_0)$ and $\mathscr{J}^*_{\text{stab}}(x_0)$.

Example. Before jumping into the details of the analysis, let us consider the special case in which we consider the controllable system Σ: $\dot{x}=Ax+Bu$ and are asked to minimize $\int_0^\infty \|\underline{x}\|^2\,dt$, i.e., $y=x$. Now set $\mathscr{X}=\mathscr{X}_1\oplus\mathscr{X}_2$ with $\mathscr{X}_1:=\operatorname{im} B$ and $\mathscr{X}_2:=(\operatorname{im} B)^\perp$ In this basis, Σ becomes:

$$\dot{x}_1=A_{11}x_1+A_{12}x_2+u,$$

$$\dot{x}_2=A_{21}x_1+A_{22}x_2,$$

and

$$\mathscr{J}(x_0)=\int_0^\infty (\|\underline{x}_1\|^2+\|\underline{x}_2\|^2)\,dt,$$

where we have chosen also the basis in $\mathscr{U}$ suitably and we have assumed that B is injective.

Note that (A, B) controllable implies (A_{22}, A_{21}) controllable. Let $x_0=(x_{1,0}, x_{2,0})$ be given. Now solve the classical linear quadratic problem which asks to minimize

$$\int_0^\infty (\|\underline{v}\|^2+\|\underline{x}_2\|^2)\,dt$$

with $\underline{v}$ as control, for $\dot{x}_2=A_{22}x_2+A_{21}v$, $\underline{x}_2(0)=x_{2,0}$. This yields $v^*=Fx_2$ as the optimal control law and $x_{2,0}^TKx_{2,0}$ as the minimal cost. There K is the unique positive definite symmetric solution of the appropriate algebraic Riccati equation and $F=-A_{21}^TK$. Now it is easy to see that $\mathscr{J}(x_{1,0}, x_{2,0})\geqq x_{2,0}^TKx_{2,0}$, and that $\mathscr{J}^*(x_{1,0}, x_{2,0})=\mathscr{J}^*_{\text{stab}}(x_{1,0}, x_{2,0})=x_{2,0}^TKx_{2,0}$ provided $x_{1,0}=Fx_{2,0}$. If, however, $x_{1,0}\neq Fx_{2,0}$ then we can use the impulsive control $\underline{u}=(Fx_{2,0}-x_{1,0})\underline{\delta}$ in order to obtain $\underline{x}_1(0^+)=F\underline{x}_2(0^+)=Fx_{2,0}$. This impulse derives the state to the desired subspace.

The optimal control law then looks like

$$\underline{u}=(Fx_{2,0}-x_{1,0})\underline{\delta}\quad \text{for } t=0,$$

$$\underline{u}=F(A_{21}x_1+A_{22}x_2)-A_{11}x_1-A_{12}x_2\quad \text{for } t>0.$$

Our purpose is to generalize this picture: the optimal control consists of an impulse part at $t=0$. This brings us to a subspace where the rest of the motion takes place and where a classical LQ problem needs to be solved. This surface (a linear subspace) in $\mathscr{X}$ is the *regular subspace.* The computation of this subspace and the control law to be used on it can be carried out by solving a classical algebraic Riccati equation. The computation of the impulses which bring us on this surface involves linear equations only.

3. Some notions from geometric control. The analysis of the singular LQ-problem defined by Σ via (1) and (2) needs the full power of the geometric theory of linear systems as exposed in [2], generalized to "almost" versions and distributional inputs in [3], and further generalized and made relevant to linear quadratic problems in [1]. In this section we will introduce these notions in a self-contained way and recall some relevant facts regarding them.

Consider for the system

$$\Sigma: \dot{x}=Ax+Bu,\qquad y=Cx+Du.$$

The following line-up of subspaces:

(i) $\mathscr{V}^*$, the output nulling subspace, defined as

$$\mathscr{V}^*:=\{x_0\in\mathscr{X}\,|\,\exists \underline{u}\in\mathscr{U}^{\text{reg}} \text{ such that } \underline{y}(x_0, \underline{u})=\underline{0}\};$$

(ii) $\mathcal{R}^*$, the controllable output nulling subspace, defined as

$$\mathcal{R}^* := \{x_0 \in \mathcal{X} \mid \exists \underline{u} \mathcal{U}^{\text{reg}} \text{ such that } \underline{y}(x_0, \underline{u}) = \underline{0} \text{ and such that } \underline{u}(x_0, \underline{u}) \text{ has compact support}\};$$

(iii) $\mathcal{V}^*_{\mathcal{D}}$, the distributional output nulling subspace, defined as

$$\mathcal{V}^*_{\mathcal{D}} := \{x_0 \in \mathcal{X} \mid \exists \underline{u} \in \mathcal{U}^{\text{dist}} \text{ such that } \underline{y}(x_0, \bar{u}) = \underline{0} \text{ distribution}\};$$

(iv) $\mathcal{R}^*_{\mathcal{D}}$, the controllable distributional output nulling subspace, defined as

$$\mathcal{R}^*_{\mathcal{D}} := \{x_0 \in \mathcal{X} \mid \exists \underline{u} \in \mathcal{U}^{\text{dist}} \text{ such that } \underline{y}(x_0, \underline{u}) = \underline{0} \text{ and } \underline{x}(x_0, \underline{u}) \text{ has compact support}\};$$

(v) $\mathcal{V}^*_a$, the L_∞-almost output nulling subspace, defined as

$$\mathcal{V}^*_a := \{x_0 \in \mathcal{X} \mid \forall \varepsilon > 0, \exists \underline{u} \in \mathcal{U}^{\text{reg}} \text{ such that } \|\underline{y}(x_0, \underline{u})\|_{L_\infty} \leqq \varepsilon\};$$

(vi) $\mathcal{R}^*_a$, the controllable L_∞-almost output nulling subspace, defined as

$$\mathcal{R}^*_a := \{x_0 \in \mathcal{X} \mid \exists T > 0 \text{ such that } \forall \varepsilon > 0, \exists \underline{u} \in \mathcal{U}^{\text{reg}}, \text{ such that } \|\underline{y}(x_0, u)\|_{L_\infty} \leqq \varepsilon \text{ and support } \underline{x}(x_0, \underline{u}) \subset [0, T]\};$$

(vii) $\mathcal{V}^*_b$, the L_2-almost output nulling subspace, defined as

$$\mathcal{V}^*_b := \{x_0 \in \mathcal{X} \mid \forall \varepsilon > 0, \exists \underline{u} \in \mathcal{U}^{\text{reg}}, \text{ such that } \|\underline{y}(x_0, \underline{u})\|_{L_2} \leqq \varepsilon\};$$

(viii) $\mathcal{R}^*_b$, the controllable L_2-almost output nulling subspace, defined as

$$\mathcal{R}^*_b := \{x_0 \in \mathcal{X} \mid \exists T > 0 \text{ such that } \forall \varepsilon > 0, \exists \underline{u} \in \mathcal{U}^{\text{reg}}, \text{ such that } \|\underline{y}(x_0, \underline{u})\|_{L_2} \leqq \varepsilon \text{ and support } \underline{x}(x_0, \underline{u}) \subset [0, T]\}.$$

These subspaces have been studied in [3] for the case $D = 0$, and much of it has been generalized to the case $D \neq 0$ in [1]. Actually the case $D \neq 0$ is easily reduced to the case $D = 0$. Indeed, by choosing the bases in $\mathcal{U}$ and $\mathcal{Y}$ properly, we may always write Σ as

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \qquad y_1 = C_1 x + u_1, \qquad y_2 = C_2 x$$

with $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$, $\mathcal{U}_2 = \ker D$, $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, $\mathcal{Y}_1 = \operatorname{im} D$. Now define

$$\Sigma' \colon \dot{x} = A'x + B_2 u_2, \qquad y_2 = C_2 x$$

with $A' := A - B_1 C_1$. It is easy to see that the subspaces (i)–(viii) are identical for Σ (with input u and output y) and for Σ' (with input u_2 and output y_2). The properties desired below are easily obtained from this observation and the results of [3]. However, it is convenient to express the relations in terms of Σ directly.

PROPOSITION 1. *There holds*
1. $\mathcal{V}^*_b = \mathcal{V}^*$, $\mathcal{R}^*_b = \mathcal{R}^*_{\mathcal{D}}$;
2. $\mathcal{V}^*_a = \mathcal{V}^* + \mathcal{R}^*_a$, $\mathcal{V}^*_b = \mathcal{V}^* + \mathcal{R}^*_b$;
3. $\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{R}^*_a = \mathcal{V}^* \cap \mathcal{R}^*_b$;
4. $\mathcal{R}^*_a = \mathcal{R}^*_b \cap C^{-1} \operatorname{im} D$, $\mathcal{R}^*_b = [A \mid B]((\mathcal{R}^*_a \oplus \mathcal{U}) \cap \ker [C \mid D])$.

We particularly draw attention to property 4 which yields a simple way of deriving $\mathcal{R}^*_a$ from $\mathcal{R}^*_b$ and vice versa.

In [1], [3] simple algorithms have been derived for the computation of the subspaces (i)-(viii). For the situation at hand, these are

$$\mathcal{V}_0 := \mathcal{X}, \qquad \mathcal{V}_{i+1} = \begin{bmatrix} A \\ \hline C \end{bmatrix}^{-1} \left((\mathcal{V}_i \oplus \{0\}) + \text{im} \begin{bmatrix} B \\ \hline D \end{bmatrix} \right);$$

$$\mathcal{R}_0 := \{0\}, \mathcal{R}_{i+1} = (C^{-1} \text{ im } D) \cap [A \mid B]((\mathcal{R}_i \oplus \mathcal{U}) \cap \ker [C \mid D]),$$

$$\mathcal{S}_0 := \{0\}, \mathcal{S}_{i+1} = [A \mid B]((\mathcal{S}_i \oplus \mathcal{U}) \cap \ker [C \mid D]).$$

These recursive algorithms compute the desired subspaces. In fact,

$$\mathcal{V}_i \downarrow \mathcal{V}_n = \mathcal{V}^*,$$

$$\mathcal{R}_i \uparrow \mathcal{R}_n = \mathcal{R}_a^*,$$

$$\mathcal{S}_i \uparrow \mathcal{S}_n = \mathcal{R}_b^*,$$

$$(\mathcal{V}_i \cap \mathcal{R}_i) \uparrow (\mathcal{V}_n \cap \mathcal{R}_n) = \mathcal{R}^*,$$

$$(\mathcal{V}_i \cap \mathcal{J}_i) \uparrow (\mathcal{V}_n \cap \mathcal{J}_n) = \mathcal{R}^*.$$

These algorithms immediately yield the following.

PROPOSITION 2. *There hold*

1. $\mathcal{R}_a^* = (C^{-1} \text{ im } D) \cap [A \mid B]((\mathcal{R}_a^* \oplus \mathcal{U}) \cap \ker [C \mid D]);$
2. $\mathcal{R}_b^* = [A \mid B]((\mathcal{R}_b^* \oplus \mathcal{U}) \cap \ker [C \mid D]).$

The subspaces introduced allow to decide invertibility of Σ. We quote some results to this effect from [1]. We will say that Σ is *right invertible* if for every $\underline{y} \in \mathcal{Y}^{\text{dist}}$ there exists an $\underline{u} \in \mathcal{U}^{\text{dist}}$ such that $\underline{y}(0, \underline{u}) = \underline{y}$. (In [1] it is actually assumed in the definition that $\underline{y} \in \mathcal{Y}^{\text{reg}}$, but the above definition defines an equivalent and perhaps a more natural property.)

PROPOSITION 3. *The following statements are equivalent*:

(i) Σ *is right invertible.*

(ii) $\mathcal{V}_b^* = \mathcal{X}$ *and* $\text{im} [C \mid D] = \mathcal{Y}$.

(iii) *The transfer function* $T(s) = D + C(Is - A)^{-1}B$ *is right invertible over the field of rational functions.*

Also, left invertibility is readily desired from the notions introduced above. The system Σ is called *left invertible* if $\{0 \neq u \in \mathcal{U}^{\text{dist}}\} \Rightarrow \{\underline{y}(0, \underline{u}) \neq \underline{0}\}$. (In [1] it is actually assumed in the definition that $\underline{y}(0, \underline{u}) \in \mathcal{Y}^{\text{reg}}$ but the above definition defines an equivalent and perhaps a more natural property.)

PROPOSITION 4. *The following statements are equivalent*:

(i) Σ *is left invertible.*

(ii) $\mathcal{R}^* = \{0\}$ *and* $\ker [{B \atop D}] = \{0\}$.

(iii) *The transfer function* $T(s) = D + C(Is - A)^{-1}B$ *is left invertible over the field of rational functions.*

Note that left invertibility immediately implies that $\{\underline{y}(0, \underline{u}_1) = \underline{y}(0, \underline{u}_2)\} \Rightarrow \{\underline{u}_1 = \underline{u}_2\}$.

Actually, using the results of [4] we can also classify the transfer functions with a polynomial inverse.

PROPOSITION 5. *The following statements are equivalent*:

(i) $\mathcal{R}_b^* = \mathcal{X}$.

(ii) $T(s) = D + C(Is - A)^{-1}B$ *has a right inverse which is a polynomial matrix.*

Finally, the equivalence of the open loop definitions of the spaces (i)-(viii) and their feedback counterparts lies at the basis of many control theoretic applications of these notions. We will only need the following here.

PROPOSITION 6. *There exist a feedback matrix* $F: \mathscr{X} \to \mathscr{U}$ *and a chain* $B_i \subset B$ *such that*
(i) $(A+BF)\mathscr{V}^* \subset \mathscr{V}^*$ *and* $(C+DF)\mathscr{V}^* = \{0\}$;
(ii) $(A+BF)\mathscr{R}^* \subset \mathscr{R}^*$;
(iii) $\mathscr{R}_b^* = B_1 \oplus A_F B_2 \oplus \cdots \oplus A_F^{n-1} B_n$.

Proof. The proof follows from [6, Thm. 1] where the subspaces $\mathscr{R}_b^*$ and $\mathscr{V}^*$ are called strongly reachable and weakly unobservable subspaces and denoted as $\mathscr{W}$ and $\mathscr{V}$.

4. A suitable basis choice and preliminary feedback. By means of an appropriate choice of the basis in the input, state, and output space, and by applying a preliminary feedback, it is possible to simplify the analysis considerably.

Decompose $\mathscr{Y} = \mathscr{Y}_1 \oplus \mathscr{Y}_2$ with $\mathscr{Y}_1 = \text{im } D$ and $\mathscr{Y}_2 = \mathscr{Y}_1^\perp$. Now choose $\mathscr{U}_2 = \ker D$ and $\mathscr{U}_1$ such that $\mathscr{U}_1 = \mathscr{U}_1 \oplus \mathscr{U}_2$. By suitably choosing the basis we obtain $D = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. This yields for Σ:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad y_1 = C_1 x + u_1, \quad y_2 = C_2 x,$$

with $\|y\|^2 = \|y_1\|^2 + \|y_2\|^2$. It is easy to see that the spaces (i)–(viii) introduced in § 3 are identical for the system Σ as for

$$\dot{x} = A'x + B_2 u_2, \qquad y_2 = C_2 x \text{ with } A' := A - B_1 C_1$$

where we consider u_2 as input and y_2 as output.

Now decompose the state space as follows:

$$\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \mathscr{X}_3 \oplus \mathscr{X}_4 \oplus \mathscr{X}_5$$

with

$$\mathscr{X}_3 = \mathscr{R}^*, \quad \mathscr{X}_2 \oplus \mathscr{X}_3 = \mathscr{V}^*, \quad \mathscr{X}_3 \oplus \mathscr{X}_4 = \mathscr{R}_a^*,$$

and

$$\mathscr{X}_3 \oplus \mathscr{X}_4 \oplus \mathscr{X}_5 = \mathscr{R}_b^*.$$

Now choose feedback F such that $(A' + B_2 F)\mathscr{V}^* \subset \mathscr{V}^*$, $C_2 \mathscr{V}^* = \{0\}$, $(A' + B_2 F)\mathscr{R}^* \subset \mathscr{R}^*$ and $\mathscr{R}_b^* = B_{20} \oplus (A' + B_2 F) B_{21} \oplus (A' + B_2 F)^2 B_{22} \oplus \cdots \oplus (A' + B_2 F)^{n-1} B_{2n}$ where $B_{20} = B_2$ and B_{2i} is a chain in B_2 (see Proposition 5). This yields $\mathscr{V}^* \cap \text{im } B_2 \subset \mathscr{R}^*$. Also from Proposition 1.4 we know that $\mathscr{R}_a^* = \mathscr{R}_b^* \cap \ker C_2$ and $\mathscr{R}_b^* = (A' + B_2 F)\mathscr{R}_1^* + \text{im } B_2$.

The choice of basis indicated and the feedback

$$u_1 = u_1' - C_1 x, \tag{3a}$$

$$u_2 = u_2' + Fx, \tag{3b}$$

reduces our system to $\bar{\Sigma}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{15} \\ A_{21} & A_{22} & 0 & 0 & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & 0 & 0 & A_{44} & A_{45} \\ A_{51} & 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \\ B_{15} \end{bmatrix} u_1' + \begin{bmatrix} 0 \\ 0 \\ B_{23} \\ B_{24} \\ B_{25} \end{bmatrix} u_2',$$

(4) $$y_1 = u_1', \qquad y_2 = [C_{21} \;\; 0 \;\; 0 \;\; 0 \;\; C_{25}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

The problem is to

(5) $$\text{minimize} \int_0^\infty (\|u_1'\|^2 + \|y_2\|^2)\, dt.$$

We have the following.

PROPOSITION 7.

1. $\ker C_{25} = \{0\}$.
2. *The transfer function associated with*

$$\left\{ \begin{bmatrix} A_{33} & A_{34} & A_{35} \\ 0 & A_{44} & A_{45} \\ 0 & A_{54} & A_{55} \end{bmatrix}, \begin{bmatrix} B_{23} \\ B_{24} \\ B_{25} \end{bmatrix}, [0 \;\; 0 \;\; I] \right\}$$

has a right inverse which is a polynomial matrix.

3. *The system*

$$\left\{ \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix}, \begin{bmatrix} B_{24} \\ B_{25} \end{bmatrix}, [0 \;\; I] \right\}$$

is left invertible.

4. (A_{33}, B_{23}) *is a controllable pair.*

Proof.

1. It follows from $\mathscr{R}_b^* \cap \ker C_2 = \mathscr{R}_a^*$.

2. It follows from Proposition 4 that this transfer function with the output matrix replaced by $[0 \;\; 0 \;\; C_{25}]$ has a polynomial right inverse. The result then follows using 1.

3. By Proposition 4 we need to show that the controllability output nulling subspace associated with this system is zero. Assume that this is not the case and add this subspace to $\mathscr{X}_3$. Clearly the subspace obtained in this way will be in the controllability output nulling subspace for the original system Σ, proving the claim.

4. Follows from $\langle A' + B_2F | \text{im } B_2 \cap \mathscr{R}^* \rangle = \mathscr{R}^*$. □

We will use the feedback F and the chain B_{2i} to simplify the system representation given in (4). We obtain

PROPOSITION 8. *There exists a coordinate transformation such that* $\bar{\bar{\Sigma}}_T = \Sigma^*$ *where*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} A_{11}^* & 0 & 0 & 0 & A_{15} \\ A_{21}^* & A_{22} & 0 & 0 & A_{25} \\ 0 & A_{32} & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \\ B_{15} \end{bmatrix} u_1' + \begin{bmatrix} 0 \\ 0 \\ B_{23} \\ B_{24} \\ B_{25} \end{bmatrix} u_2',$$

$$y_2 = [C_{21}^* \;\; 0 \;\; 0 \;\; 0 \;\; C_{25}]\underline{x}, \qquad y_1 = u_1'$$

where (C_{21}^*, A_{11}^*) *is an observable pair.*

Proof. Follows from Lemma 1 and the Column Elimination Algorithm given in [6]. □

5. Regular *LQ* problems. Our approach in studying singular *LQ*-problems will be to reduce them to regular *LQ*-problems. Regular *LQ*-problems are those for which distributional inputs are not candidates for optimal controls since they always lead to infinite cost:

DEFINITION. The *LQ* problem Σ (as defined by (1), (2)) will be called *regular* if $\{x_0 \in \mathscr{X},\ \underline{u} \in \mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{reg}}\} \Rightarrow \{\mathscr{J}(x_0, \underline{u}) = \infty\}$.

It is easy to decide regularity as follows.

THEOREM 1. *The following conditions are equivalent*:

(i) Σ *defines a regular LQ-problem.*

(ii) $\{x_0 \in \mathscr{X},\ \underline{u} \in \mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{reg}}\} \Rightarrow \{\underline{y}(x_0, \underline{u}) \in \mathscr{Y}^{\text{dist}} \backslash \mathscr{Y}^{\text{reg}}\}$.

(iii) $\ker D = \{0\}$.

Proof. (i)$\Rightarrow$(ii): Since $\underline{y}(x_0, \underline{u}) = \underline{y}(x_0, 0) + \underline{y}(0, \underline{u})$, (ii) is equivalent to

$$\{\underline{u} \in \mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{reg}}\} \Rightarrow \{\underline{y}(0, \underline{u}) \in \mathscr{Y}^{\text{dist}} \backslash \mathscr{Y}^{\text{reg}}\}.$$

Now, if this were not the case, $\exists \underline{u} \in \mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{reg}}$ such that $\underline{y}(0, \underline{u}) \in \mathscr{Y}^{\text{reg}}$. The corresponding $\underline{x}(0, \underline{u})$ will satisfy $\underline{x}(0, \underline{u})(t) \in \langle A|\text{im } B\rangle$ for all $t > 0$. In particular, $\underline{x}(0, \underline{u})(1) \in \langle A|\text{im } B\rangle$. Hence since $\underline{x}(0, u)(1)$ belongs to the controllable subspace of Σ, we can modify, if needed, $\underline{u}(t)$ to $\underline{u}^{\text{new}}(t)$ for $t \geqq 1$ such that $y(0, \underline{u}^{\text{new}}) \in \mathscr{L}(0, \infty)$. This $\underline{u}^{\text{new}}$ is still in $\mathscr{U}^{\text{dist}} \backslash \mathscr{U}^{\text{new}}$, but $\mathscr{J}(x_0, \underline{u}^{\text{new}}) < \infty$. Hence {not (ii)}$\Rightarrow${not (i)}. The implication (ii)$\Rightarrow$(i) is obvious.

To show the equivalence of (ii) and (iii), observe that by a suitable choice of the basis in $\mathscr{U}$ and $\mathscr{Y}$, Σ may be represented as

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad y_1 = C_1 x + u_1, \quad y_2 = C_2 x,$$
$$u = (x_1, u_2), \qquad y = (y_1, y_2)$$

with $u_1 \in \mathbb{R}^{m_1}$, $m_1 = \text{codim} \ker D = \dim \text{im } D$ and $u_2 \in \mathbb{R}^{m_2}$, $m_2 = m - m_1$. Now (ii)$\Leftrightarrow$ $\{m_2 = 0\}\Leftrightarrow$(iii) is obvious. $\square$

Regular *LQ* problems may thus be reduced by a simple basis change and a feedback transformation to the standard *LQ* problems.

Recall the *LQ* problem standard if it is regular and if the associated $\mathscr{V}^* = \{0\}$, i.e., if $\ker D = \{0\}$ and if $\langle C^{-1} \text{im } D|A - (D^T D)^{-1} D^T C\rangle = \{0\}$.

Let Σ define a regular *LQ*-problem. By Theorem 1 this is equivalent to $\ker D = \{0\}$. By choosing the basis in $\mathscr{U}$ appropriately and making an orthogonal basis change in $\mathscr{Y}$ we can then bring D into the form $[{}^I_0]$. Σ becomes

$$\dot{x} = Ax + Bu, \quad y_1 = C_1 x_2 + u, \quad y_2 = C_2 x, \quad y = (y_1, y_2).$$

Now use the preliminary feedback $u = v - C_1 x_2$. This yields the system

$$\dot{x} = A'x + Bv, \qquad y_2 = C_2 x$$

with $A' := A - B_1 C_1$ and $\mathscr{J} = \int_0^\infty (\|\underline{v}\|^2 + \|\underline{y}_2\|^2)\, dt$.

We obtain the familiar standard $L\tilde{Q}$ problem

$$\text{minimize} \int_0^\infty (\|\tilde{u}\|^2 + \|\tilde{y}\|^2)\, dt$$

for $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$; $\tilde{y} = \tilde{C}\tilde{x}$ with the simple basis change such that $\tilde{C} = C/L$ where $L = (\mathscr{V}^*)^\perp$.

6. The singular *LQ*-problem without stability constraints. At this point it is convenient to study the *LQ*-problem introduced in § 2 with and without stability separately.

We will reduce the general singular problem to regular ones, and regular problems to standard ones. In addition, we will assume throughout that (A, B) is asymptotically stabilizable. We have the well-known proposition as follows.

PROPOSITION 9. *Let (A, B) be asymptotically stabilizable and assume that Σ defines a standard LQ-problem. Then the control law $u^* = F_0 x$ generates $\mathcal{J}(x_0, \underline{u}^*) = \mathcal{J}^*(x_0)$. Here*

$$F_0 = -(D^T D)^{-1}(B^T P_0 + D^T C) \tag{6}$$

and P_0 is the unique positive semi-definite solution of the algebraic Riccati equation

$$A^T P + PA - (PB + C^T D)^T (D^T D)^{-1}(PB + C^T D) + C^T C = 0. \tag{7}$$

In fact, $P_0 > 0$ and $\mathcal{J}^*(x_0) = x_0^T P_0 x_0$. Moreover, the closed loop system $\dot{x} = (A + BF_0)x$ is asymptotically stable.

It is easy to extend Proposition 9 to the regular case.

PROPOSITION 10. *Let (A, B) be asymptotically stabilizable and assume that Σ defines a regular LQ-problem. Then the control law (6) with P_0 the infimal positive semidefinite solution of the algebraic Riccati equation (7) generates $\mathcal{J}(x_0, u^*) = \mathcal{J}^*(x_0)$,* and *$\mathcal{J}^*(x_0) = x_0^T P_0 x_0$. Further $\{\mathcal{J}^*(x_0) = 0\} \Leftrightarrow \{x_0 \in \ker P_0\} \Leftrightarrow \{x_0 \in \mathcal{V}^* = \langle C^{-1}$ im $D | (D^T D)^{-1} D^T C \rangle\}$. Finally, the closed loop system $\dot{x} = (A + BF_0)x$ will be asymptotically stable if and only if $\mathcal{V}^* \subset \mathcal{L}^-(A - B(D^T D)^{-1} D^T C)$ (i.e., detectability).*

Proof. See [1] with the sign change on (6.2) at page 23, [7].

Note that in order to solve for P_0 and F_0 in Proposition 10 it suffices to solve a standard algebraic Riccati equation of dimension = the codimension of $\mathcal{V}^*$, since $P_0 = P_0^T \geqq 0$ and $\ker P_0 = \mathcal{V}^*$.

We now have all the preliminary results which go into the solution of the general singular LQ problem *without* stability. We will assume that the problem is already in the form (4)-(5).

THEOREM 2. *Assume that (A, B) is asymptotically stabilizable and consider the singular LQ-problem* (4)-(6). *Then*

(i) *$\mathcal{J}^*(x_0) < \infty$. In fact,*

$$\mathcal{J}^*((x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}, x_{5,0})) = x_{1,0}^T P_0 x_{1,0}$$

with P_0 the unique positive semidefinite solution of the algebraic Riccati equation

$$\begin{aligned} A_{11}^T P + PA_{11} - (PA_{15} + C_{21}^T C_{25})^T (C_{25}^T C_{25})^{-1}(PA_{15} + C_{21}^T C_{25}) \\ - PB_{11} B_{11}^T P + C_{21}^T C_{21} = 0. \end{aligned} \tag{8}$$

Moreover, $P_0 > 0$.

(ii) *$\forall x_0 \in \mathcal{X}$, there exists an $u^* \in \mathcal{U}^{\text{dist}}$ such that $\mathcal{J}(x_0, u^*) = \mathcal{J}^*(x_0)$. This optimal control is generated as follows*

$$u_1'^* = -B_{11}^T P_0 x_1 \tag{9}$$

and $\underline{u}_2'^*$ such that $\underline{x}_5^*$ is regular and satisfies

$$x_5^* = -(C_{25}^T C_{25})^{-1}(A_{15}^T P_0 + C_{25}^T C_{21}) x_1. \tag{10}$$

There always exists a distribution $\underline{u}_2'^*$ such that (10) *will be satisfied as a distribution.*

(iii) *The optimal trajectory lies on the linear subspace*

$$x_5 = -(C_{25}^T C_{25})^{-1}(A_{15}^T P_0 + C_{25}^T C_{21}) x_1$$

for $t > 0$.

(iv) *The optimal trajectory*

$$\underline{x}_1^*, \underline{x}_2^*, \underline{x}_3^*, \underline{x}_4^*, \underline{x}_5^*$$

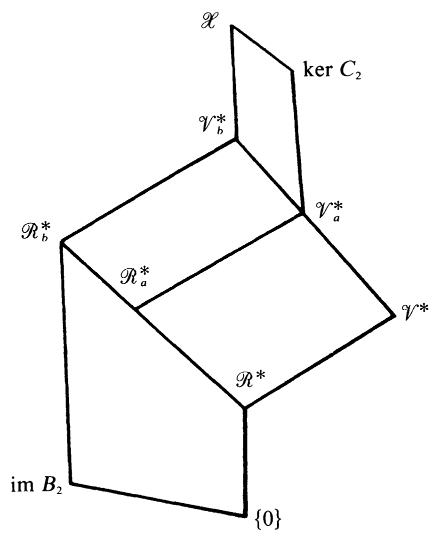
is such that

$$\underline{x}_1^* \text{ and } \underline{x}_2^* \text{ are regular}$$

and $\underline{x}_3^*$, $\underline{x}_4^*$, *and* $\underline{x}_5^*$ *may be distributions. Moreover,* $\underline{x}_1^*$, $\underline{x}_2^*$, $\underline{x}_4^*$ *and* $\underline{x}_5^*$ *are uniquely defined, while* $\underline{x}_3^*$ *is not.*

The proof of this theorem is given in Appendix A.

Theorem 1 allows us to recognize several interesting special cases of the singular LQ-problem. Recall the following lattice diagram (B_2 and C_2 are as defined in § 4):



The problem is *standard*:
$\Leftrightarrow$ {the optimal control is a regular function and $\mathscr{J}^*(x_0) > 0$ for $x_0 \neq 0$}
$\Leftrightarrow \{\mathscr{V}_b^* = \{0\}\}$.

The problem is *regular*:$\Leftrightarrow$ {the optimal control is a regular function}
$\Leftrightarrow \{\mathscr{R}_b^* = \{0\}\}$
$\Leftrightarrow \{\text{Ker } D = \{0\}\}$.

The problem is *cheap*:$\Leftrightarrow \{\mathscr{J}^*(x_0) = 0$ for all $x_0\}$
$\Leftrightarrow \{\mathscr{V}_b^* = \mathscr{X}\}$.

The problem is *totally singular*:$\Leftrightarrow$ {the optimal control has zero regular part}
$\Leftrightarrow \{\mathscr{R}_b^* = \mathscr{X}$ and $\mathscr{R}^* = \{0\}\}$.

The problem is *potentially singular*:
$\Leftrightarrow$ {there always is an optimal control with regular part zero}
$\Leftrightarrow \{\mathscr{R}_b^* = \mathscr{X}\}$.

7. The singular LQ-problem with stability. In this section we will generalize the ideas of § 6 in order to study the singular LQ-problem with the stability constraint $\lim_{t\to\infty} \underline{x}(t) = 0$ as a side condition. We start by analyzing the geometric structure of Σ as given in § 2 in still a bit more detail and derive a refinement of the decomposition (4).

7.1. A further decomposition of $\mathscr{V}^*$. Our approach to solve the singular LQ-problem with stability needs a further decomposition of the output nulling subspace

$\mathcal{V}^*$. Consider $\mathcal{V}^-$, $\mathcal{V}^0$, and $\mathcal{V}^+$ the output nulling subspaces with respectively asymptotic stability, neutral stability, and exponential instability. These are defined and computed as follows.

Take any F such that $(A+BF)\mathcal{V}^* \subset \mathcal{V}^*$ (solutions converge neither for $t \to \pm\infty$) and $(C+DF)\mathcal{V}^* = \{0\}$. Then $(A+BF)\mathcal{R}^* \subset \mathcal{R}^*$. Now there exists such an F with the property that the characteristic polynomial of $(A+BF)|\mathcal{R}^*$ is equal to any given monic polynomial of suitable degree. However, the eigenvalues of $(A+BF)$ (mod $\mathcal{R}^*$) are independent of the F which we choose. Now choose an F such that the spectra of $(A+BF)|\mathcal{R}^*$ and $(A+BF)$ (mod $\mathcal{R}^*$) are disjoint. This yields a decomposition of $\mathcal{V}^*$ into $\mathcal{V}^* = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{R}^*$ with $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ $(A+BF)$-invariant and such that $(A+BF)|\mathcal{V}_1$, $(A+BF)|\mathcal{V}_2$ and $(A+BF)|\mathcal{V}_3$ have their spectra in the open right half plane, on the imaginary axis, and in the open left half plane, respectively. In terms of these, set $\mathcal{V}^+ = \mathcal{R}^* \oplus \mathcal{V}_1$, $\mathcal{V}^0 = \mathcal{R}^* \oplus \mathcal{V}_2$, and $\mathcal{V}^- = \mathcal{R}^* \oplus \mathcal{V}_3$.

Using such an F and the above decomposition of $\mathcal{X}_2 \oplus \mathcal{X}_3$ in (4) yields a decomposition of $\mathcal{X}_2$ into $\mathcal{X}_2 = \mathcal{X}_{21} \oplus \mathcal{X}_{22} \oplus \mathcal{X}_{23}$ with an associated partitioning of A_{22}, A_{25}, and B_{12} into

$$(11) \qquad A_{22} = \begin{bmatrix} A_{22,1} & 0 & 0 \\ 0 & A_{22,2} & 0 \\ 0 & 0 & A_{22,3} \end{bmatrix} \quad A_{25} = \begin{bmatrix} A_{25,1} \\ A_{25,2} \\ A_{25,3} \end{bmatrix} \quad B_{12} = \begin{bmatrix} B_{12,1} \\ B_{12,2} \\ B_{12,3} \end{bmatrix}$$

with $\sigma(A_{22,1})$, $\sigma(A_{22,2})$, and $\sigma(A_{22,3})$ in the open right half plane, on the imaginary axis, and in the open left half plane, respectively. Furthermore, $A_{32} = 0$.

7.2. The regular LQ-problem with stability. In the *standard* problem we obtain asymptotic stability of the closed loop system as a consequence of the minimization of $\mathcal{J}$. This shows that the standard problem has the same solution with or without the side constraint $\lim_{t\to\infty} \underline{x}(t) = 0$. The difference starts when we consider the regular problem.

Consider now the regular LQ problem: Σ with $\ker D = \{0\}$. By Theorem 1 we may restrict attention to $\mathcal{U}^{\text{reg}}$. Now consider the subspaces $\mathcal{V}^+$, $\mathcal{V}^0$, and $\mathcal{V}^-$ introduced in § 7.1. Because of regularity, these may be computed in more detail:

$$\mathcal{V}^* = \langle C^{-1} \operatorname{im} D | A' \rangle \quad \text{with } A' := A - (D^TD)^{-1}D^TC$$

and, also because of regularity, $\mathcal{R}^* = \{0\}$. Now make a spectral decomposition of $\mathcal{V}^*$ corresponding to the decomposition of the spectrum of $A'|\mathcal{V}^*$ into its open right half plane, its imaginary axis, and its open left half plane parts. This yields $\mathcal{V}^+$, $\mathcal{V}^0$, and $\mathcal{V}^-$ respectively.

We obtain the following proposition.

PROPOSITION 11. *Consider the regular LQ-problems*: Σ *with* $\ker D = \{0\}$, *with the stability condition* $\lim_{t\to\infty} \underline{x}(t) = 0$. *Then*

(i) *For all* $x_0 \in \mathcal{X}$, *there exists a control* $\underline{u} \in \mathcal{U}^{\text{reg}}_{\text{stab}}(x_0)$ *such that* $\mathcal{J}(x_0, \underline{u}) < \infty$ *if and only if* (A, B) *is asymptotically stabilizable. Assume this to be the case.*

(ii) *There exists a supremal nonnegative definite symmetric solution,* P_+, *to the algebraic Riccati equation* (7). *We have* $\inf_{u \in \mathcal{U}^{\text{reg}}_{\text{stab}}} \mathcal{J}(x_0, \underline{u}) = \inf_{u \in \mathcal{U}^{\text{dist}}_{\text{stab}}} \mathcal{J}(x_0, \underline{u}) = x_0^T P_+ x_0$.

(iii) *For all* $x_0 \in \mathcal{X}$ *there exists an optimal control* $u^* \in \mathcal{U}^{\text{reg}}_{\text{stab}}(x_0)$ (*hence* $\mathcal{J}(\underline{u}^*, x_0) = \underline{x}_0^T P_+ x_0$) *if and only if* $\mathcal{V}^0 = \{0\}$.

(iv) *Assuming this to be the case, then* $u^* = F_+ x$ *with* $F_+ := -(D^TD)^{-1}(B^TP_+ + D^TC)$ *generates the optimal control.*

(v) $\{\inf_{u \in \mathcal{U}^{\text{reg}}_{\text{stab}}} \mathcal{J}(x_0, \underline{u}) = 0\} \Leftrightarrow \{x_0 \in \ker P_+\} \Leftrightarrow \{x_0 \in \mathcal{V}^0 + \mathcal{V}^-\}$.

(vi) $P_+ = P_0$ (*and consequently the nonnegative definite solution of* (7) *is unique*) *if and only if* $\mathcal{V}^+ = \{0\}$ (*i.e., exponential detectability*).

(vii) $u^* = F_0 x$ *will yield also asymptotic stability if and only if* $\mathcal{V}^+ + \mathcal{V}^0 = \{0\}$. *In this case, the LQ problem* Σ *with and without stability give identical answers.*

Proof. Of course, part (i) is obvious. Assume thus that (A, B) is asymptotically stabilizable using the representation derived at the end of § 5. It follows that we should prove this proposition for the LQ-problem in which we are asked to minimize $\int_0^\infty (\|u\|^2 + \|y\|^2)\,dt$ for the system $\dot{x} = Ax + Bu$; $y = Cx$. Then $\mathcal{V}^* = \langle \ker C | A \rangle$, while $\mathcal{V}^+$, $\mathcal{V}^0$, $\mathcal{V}^-$ correspond to the decomposition of $\mathcal{V}^*$ associated with the partition of the spectrum of $A|\mathcal{V}^*$ into its components in the open right half plane, on the imaginary axis, and in the open left half plane, respectively. The associated algebraic Riccati equation is

$$A^T P + PA - PBB^T P + C^T C = 0.$$

Let P_0 be the infimal nonnegative definite symmetric solution of this algebraic Riccati equation. Since (A, B) is asymptotically stabilizable, such a solution exists. Using standard calculations, it is easy to see that, whenever $\lim_{t\to\infty} \underline{x}(t) = 0$, then $\mathcal{J}(x_0, \underline{u}) = x_0^T P_0 x_0 + \int_0^\infty \|\underline{u} + B^T P_0 \underline{x}(x_0, \underline{u})\|^2\,dt$. Now use the preliminary feedback $u = v - B^T P_0 x$. The problem then requires the minimization of $\int_0^\infty \|\underline{v}\|^2\,dt$ for $\dot{x} = A_0 x + Bv$ with $A_0 := A - BB^T P_0$, under the stability constraint $\lim_{t\to\infty} \underline{x}(x_0, \underline{v})(t) = 0$. Let $\mathcal{L}^+$, $\mathcal{L}^0$, $\mathcal{L}^-$ be the decomposition of $\mathcal{X}$ corresponding to the partition of the spectrum of A_0 into its components in the open right half plane, on the imaginary axis, and in the open left half plane. By Proposition 9, we know that $\ker P_0 = \mathcal{V}^* = \langle \ker C | A \rangle$. Further $\mathcal{V}^*$ is A_0-invariant and $A_0 \pmod{\mathcal{V}^*}$ has its eigenvalues in the open left half plane.

Now minimize $\mathcal{J}(\underline{x}_0, \underline{u}) = \int_0^\infty \|\underline{v}\|^2\,dt$ subject to $\dot{x} = A_0 x + Bv$, $\underline{x}(0) = x_0$, and $\lim_{t\to\infty} \underline{x}(x_0, \underline{v})(t) = 0$. Clearly if $x_0 \in \mathcal{L}^-$, the optimal control $\underline{v}^* = 0$, and $\min \mathcal{J}'(x_0, \underline{v}^*) = 0$. Next, if $0 \neq x_0 \in \mathcal{L}^0$, $\inf \mathcal{J}'(x_0, \underline{v}^*) = 0$ (see [5, Lemma 3.2]) but no optimal control exists since $\underline{v}^* = 0$ does not meet the condition $\lim_{t\to\infty} \underline{x}(x_0, \underline{v}^*) = 0$. Consider now the situation $x_0 \in \mathcal{L}^+$.

Define $A_+ := A_0 | \mathcal{L}^+$ and $B_+ := QB$ with Q the projection of π onto $\mathcal{L}^+$ along $\mathcal{L}^0 \oplus \mathcal{L}^-$. Note that the stabilizability of (A, B) implies that (A_+, B_+) is controllable. Further the eigenvalues of A_+ are in the open right half plane. Now solve the minimization of $\int_0^\infty \|\underline{v}\|^2\,dt$ for $\dot{x}_+ = A_+ x_+ + B_+ v$ with $\underline{x}_+(0) = x_{+,0}$ and $\lim_{t\to\infty} \underline{x}_+(t, \underline{v}) = 0$. The optimal control for this problem is $v^* = -B^T W_+^{-1} x_+$ with W_+, related to the controllability Grammian, defined as the unique solution of $A_+ W_+ + W_+ A_+^T = B_+ B_+^T$. Clearly $W_+ = W_+^T > 0$, and hence $\pi_+ = W_+^{-1}$ is the supremal (alternatively, the unique positive definite) symmetric solution of $\pi_+ A_+ + A_+^T \pi_+ - \pi_+ B_+ B_+^T \pi_+ = 0$. Now combine the solution which we obtained for $\mathcal{L}^-$, $\mathcal{L}^0$, and $\mathcal{L}^+$. Define

$$\pi = \begin{bmatrix} \pi_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

to conform with the partition of $\mathcal{X}$ into $\mathcal{X} = \mathcal{L}^+ \oplus \mathcal{L}^0 \oplus \mathcal{L}^-$. This yields $x_0^T \pi x_0$ as $\inf \mathcal{J}'(x_0, \underline{v})$. If $\mathcal{X} = \mathcal{L}^+ \oplus \mathcal{L}^-$ then $v^* = -B^T \pi x$ is the optimal control law.

Combining this solution with the preliminary feedback yields $x_0^T (P_0 + \pi) x_0$ for $\inf \mathcal{J}(x_0, \underline{u})$ and $u^* = -B^T (P_0 + \pi) x$. Define now $P_+ = P_0 + \pi$ and unify all the statements of Proposition 11.

7.3. The singular case with stability. We are now in a position to state the solution of the general singular LQ problems with stability. We will assume that the problem is already in the form (4)-(5) with the refinement of $\mathscr{X}_1$ leading to the partition of A_{22} as given in (11).

THEOREM 3. *Consider the singular LQ-problem* (2) *with the stability constraint* $\lim_{t\to\infty} \underline{x}(t)=0$. *Assume that by a preliminary feedback and a proper choice of the bases,* Σ *is already in the form* (4)-(5), *with* $\mathscr{X}_2$ *further decomposed so as to induce the form* (11). *Then*

(i) *For all* $x_0 \in \mathscr{X}$, *there exists a control* $\underline{u} \in \mathscr{U}^{\text{dist}}_{\text{stab}}(x_0)$ *with* $\mathscr{J}(x_0, \underline{u}) < \infty$ *if and only if* (A, B) *is asymptotically stabilizable. Assume this to be the case.*

(ii) *Let* P_0 *be as defined in Theorem* 2 (i). *Now let* W_+ *be the solution of*

$$A_{22,1}W_+ + W_+A_{22,1}^T = B_{12,1}B_{12,1}^T + A_{25,1}A_{25,1}^T. \tag{12}$$

Then $W_+ = W_+^T > 0$, *and* $\mathscr{J}^*_{\text{stab}}(x_0) = x_{1,0}^T P_0 x_{1,0} + x_{21,0}^T W_+^{-1} x_{21,0}$.

(iii) *For all* $x_0 \in \mathscr{X}$, *there exists an optimal control* $\underline{u}^* \in \mathscr{U}^{\text{dist}}_{\text{stab}}(x_0)$ *if and only if* $\mathscr{X}_{22}=0$ (*in the notation of* § 7.1 *this means* $\mathscr{V}^* = \mathscr{V}^+ + \mathscr{V}^-$). *This optimal control is generated as follows*

$$u_1^* = -B_{11}^T P_0 x_1 - B_{21,1}^T W_+^{-1} x_{21} \tag{13}$$

and $u_2'^*$ *such that* x_5^* *is regular and satisfies*

$$x_5^* = -(C_{25}^T C_{25})^{-1}(A_{15}^T P_0 + C_{25}^T C_{21})x_1 - A_{21,1}^T W_+^{-1} x_{21}. \tag{14}$$

7.4. Computation of optimal input. In this part we will discuss the computation of the optimal input. Using Theorem 3 (iii), we will obtain $u_1'^*$, x_5^*, x_1^*, x_{21}^* for initial conditions x_{10} and $x_{21,0}$. To compute $u_2'^*$ we consider the differential equation

$$\begin{bmatrix} \dot{x}_{2,3} \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} A_{22,3} & 0 & 0 & A_{25} \\ 0 & A_{33} & A_{34} & A_{35} \\ 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_{2,3} \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} B_{12,3} \\ B_{13} \\ B_{14} \\ B_{15} \end{bmatrix} u_1' + \begin{bmatrix} 0 \\ B_{23} \\ B_{24} \\ B_{25} \end{bmatrix} u_2'.$$

We will first compute $\bar{x}_{2,3}$, $\bar{x}_3$, $\bar{x}_4$, $\bar{x}_5$ by taking $u_1' = u_1^*$ and $u_2' = 0$. Since from Theorem 3 part (iii) $\mathscr{X}_{22}=0$ we can conclude

$$\begin{bmatrix} A_{22,3} & 0 & 0 & A_{25} \\ 0 & A_{33} & A_{34} & A_{35} \\ 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & A_{54} & A_{55} \end{bmatrix}, \begin{bmatrix} 0 \\ B_{23} \\ B_{24} \\ B_{25} \end{bmatrix}$$
$$[\, 0 \quad 0 \quad 0 \quad I \,]$$

has $\mathscr{V}^* + \mathscr{R}_b^* = \mathscr{X}$ (see Proposition 6.2). Therefore, we will compute $u_2'^*$ such that $\Delta x_5(t) = \bar{x}_5(t) - x_5^*(t)$ will be zero. If $\Delta x_5(0) \neq 0$, $u_2'^*$ contains impulses. Using the results of [6] one can directly compute the impulsive and regular parts of $u_2'^*$. We will first find the feedback F and chain B_i which are defined by Proposition 5 by using [6, Theorem 1] for $A' = A - B_1C_1$, B_2 and C_2. By applying the nested version of the left structure algorithm one can find output transformation Q, input transformation G and feedback F such that $A_F = A' + B_2F$, $C_2^T = [(C_1^*)^T \cdots (C_\alpha^*)^T]$ and $B_2 = [B_{21} \cdots B_{2\alpha+1}]$ where $(C_i^*)^T$, $B_{2i} \in \mathscr{R}^{nx(q_i - q_{i-1})}$. If $Q \neq I$ one has to introduce Q into ARE in Theorem 3. We will choose $[T_3 \; T_4] = \operatorname{Im}[B_{22} \; B_{23} \; A_FB_{23} \cdots B_{\alpha+1} \cdots A_F^{\alpha-2}B_{\alpha+1}]$ and $T_5 = [B_{21} \; A_FB_{22} \cdots A_F^{\alpha-1}B_{2\alpha}]$ where columns of $[T_3 \; T_4]$ and T_5 are basis vectors for x_3, x_4 and x_5 respectively. With this special basis selection we will obtain the matrix

$$N=\begin{bmatrix} A_{33} & A_{34} & A_{35} & B_{23} \\ 0 & A_{44} & A_{45} & B_{24} \\ 0 & A_{54} & A_{55} & B_{25} \end{bmatrix}$$

where

$$N=\begin{bmatrix} N_{11} & \cdots & N_{1\theta_1} \\ \vdots & & \vdots \\ N_{\theta_2 1} & \cdots & N_{\theta_2\theta_1} \end{bmatrix}$$

and for each $i\ \exists a j^*(i)$ such that $N_{i,j^*(i)}=I$ where $j^*(i)<i$. Then we will apply the column elimination algorithm [6] to eliminate nonzero elements of each row. Let $J=\{j \mid \exists i \ni j^*(i)=j\}$. For each $j\in J$, $\dot{x}_{j^*(i)}+A_{55}\Delta x_5(t)$. We start the procedure with $i=\theta_2$, at each step we know x_i and compute $x_{j^*(i)}$ since $j^*(i)<i$. Recursively one can find $x_i(t)\ \forall i$ and $u_2'(t)$. From the special selection of the basis vectors in $\mathscr{X}_5$ it is not hard to prove that the impulsive part of u_2' has the following property: $u_{2i}'=[\zeta_1^T\delta(t)\ \zeta_2^T\dot{\delta}(t)\cdots\zeta_{\alpha-1}^T\delta^{\alpha-1}(t)]$. The numerical aspects of the computations are investigated in [6].

Appendix A. Proof of Theorem 2. We start with the system in the form (4)–(5). The idea is now to consider the subsystem

$$\mathscr{J}'=\int_0^\infty (\|\underline{u}_1'\|^2+\|C_{21}\underline{x}_1+C_{25}\underline{x}_5\|^2)\,dt$$

with x_5 considered as a control (i.e., as being unconstrained). Obviously

$$\inf \mathscr{J}'(x_{1,0})\leqq \inf \mathscr{J}(x_{1,0},x_{2,0},x_{3,0},x_{4,0},x_{5,0}).$$

Since the LQ-problem thus obtained is regular, we can apply Proposition 10. Observe that asymptotic stabilizability of (A,B) implies asymptotic stabilizability of $(A_{11},(A_{15}\,|\,B_{11}))$. The resulting optimal control $(u_1'^*, x_5^*)$ is then given by (9)–(10) and consequently in order to prove statements (i), (ii), and (iii) of Theorem 2 it suffices to show that there always exists a distribution $\underline{u}_2^*$ such that (10) will be satisfied. More explicitly, define $L:=-(C_{25}^TC_{25})^{-1}(A_{15}^TP_0+C_{25}^TC_{21})$. Then we should generate $\underline{x}_5^*=L\underline{x}_1^*$ with $\underline{x}_1^*$ defined by

$$\dot{x}_1^*=(A_{11}+A_{15}L-B_{11}B_{11}^TP_0)\underline{x}_1^*,\qquad \underline{x}_1^*(0)=x_{1,0}.$$

The fact that the desired $\underline{u}_2'^*$ exists is an immediate consequence of Proposition 7.2. Note that since in particular $x_{5,0}$ may be unequal to $-Lx_{1,0}$, we obtain in general distributions for $u_2'^*$ and hence for $\underline{x}_3^*$, $\underline{x}_4^*$, and $\underline{x}_5^*$. The uniqueness claim (iv) of Theorem 2 may be shown as follows. From the original construction of $\underline{x}_5^*$ and $\underline{u}_1'^*$ it follows that they are unique. From Proposition 4 it follows that $\underline{x}_3^*$ is not unique, while from Proposition 7.5 it follows that $\underline{x}_4^*$ and $\underline{x}_5^*$ are unique.

Appendix B. Proof of Theorem 3. We start with the problem in the form (4)–(5)–(11) and consider first the subsystem

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_{11} & 0\\ A_{12} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}A_{15}\\ A_{25}\end{bmatrix}x_5+\begin{bmatrix}B_{11}\\ B_{12}\end{bmatrix}x_1'$$

for which we will minimize

$$\mathcal{J}'_{\text{stab}} = \int_0^\infty (\|\underline{u}'_1\|^2 + \|C_{21}\underline{x}_1 + C_{25}\underline{x}_5\|^2)\,dt$$

with x_5 considered as a control and with the constraint $\lim_{t\to\infty}(\underline{x}_1(t), \underline{x}_2(t)) = a$. Since this is a regular problem, we can apply Proposition 9. This yields the optimal trajectory $\underline{x}_1^*, \underline{x}_2^*, \underline{x}_5^*$ which converges to zero at $t \to \infty$. Using the ideas of Appendix A, this will yield Theorem 3 provided we can show that there exists a $\underline{u}_2'^*$ which generates $\underline{x}_3^*$, $\underline{x}_4^*$ which also converge to zero. This, however, immediately follows from the fact that in Proposition 7.2 a right inverse with a polynomial transfer function is obtained.

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