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From Time Series to Linear System—Part III. Approximate Modelling*

JAN C. WILLEMS†

An optimal approximate model, defined in terms of the complexity of a model and the misfit between a model and the observed data, yields algorithms for computing an optimally fitting model with a maximal admissible complexity or, alternatively, a minimally complex model which explains an observed time series up to a maximally tolerated misfit.

Key Words—Linear systems; time series analysis; modelling; identification; least squares approximation; correlation methods; data reduction and analysis; (complexity); (misfit).

Abstract—In the third part of this paper the problem of finding a linear time invariant complete system which models an observed time series will be continued. However, in this part it will be assumed that the model is required to explain the observations only approximately. This raises the question of what is meant by an approximate model and how one should judge a model against data which, in a strict sense, falsify the model. The initial sections of this paper introduce the methodology: either the maximal admissible complexity is fixed and the best fitting model in the model class is sought, or the maximally tolerated misfit is fixed and the least complex model in the model class is sought. This is illustrated by means of finite dimensional models, where natural complexity and misfit functions suggest themselves. Subsequently what is meant by the complexity of a (linear time invariant complete) system is discussed. A concrete complexity function is proposed which is defined in terms of the richness of the behaviour on finite time intervals and is in one-to-one correspondence with the number of inputs, the number of states, and the observability indices of the underlying minimal i/s/o system. The next definition is that of the misfit between an observed time series and a system. A concrete misfit function is proposed which takes into consideration the lag structure of the system and in how far the observations fail to corroborate the (AR) equations which define the system's behaviour. How the misfit may be computed by means of the correlation function of the data is then shown. These concepts are subsequently used in order to set up two algorithms for approximate modelling of an observed q-dimensional time series. The first algorithm fixes the maximal complexity—in a sense this comes down to fixing the number and the lags of the required AR equations. The second algorithm fixes the tolerated misfit. It is adaptive in nature in the sense that the number and the structure of the resulting (AR) equations will depend on the observations. These ideas and algorithms are illustrated by means of two simulated examples.

19. INTRODUCTION

THE CLASSICAL view of modelling is the descriptive one of the physicist: nature functions consistently according to some universal laws and the task is to discover them.

In some cases, it may, in principle, be possible to obtain such laws by deduction or extrapolation from observed data. In Willems (1979) it has been shown how one can for example view Newton's laws as a logical extrapolation of Kepler's laws. However, the practice in the descriptive sciences is really not this; it is much more the concept of *falsification* than that of *deduction* which is the central idea. This observation, in fact, has formed the cornerstone of the philosophy of science since Popper. Models and laws are postulated, often on the basis of an Aristotelian philosophical view and aesthetic appeal, and it is only later that one discovers that, to some extent, they could also have been deduced from already existing knowledge and observed facts. In this sense, models are obtained neither by deduction, nor by induction, but by inspiration.

Against this background one should, as far as the descriptive sciences are concerned, not expect too much from the methodology put forward in Part II, where a theory of modelling has been proposed on the basis of observations. However, in the prescriptive sciences, as, for example, control engineering, signal processing, etc., particularly in the *adaptive* aspects of these fields, the view of the modelling process is much closer in spirit to the framework which was outlined in Part II of this paper: one has a family of measurements, and one postulates a model class; the modelling process then consists of selecting, on the basis of the measurements, an appropriate element from the model class. In Part II a methodology was set up, by means of which this process can be approached, under the premise that one wants models which explain the data exactly, and which looks for the most powerful model which is unfalsified.

As is apparent from the derived algorithms, the

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† Mathematics Institute, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands.

results of Part II may not be particularly useful in the case of "real" data corrupted by noise, round-off errors etc. As an example, consider Algorithm 2 with $q = 1$ and $T = \mathbb{Z}_+$, i.e. assume that a scalar time series $\tilde{\mathbf{w}} = (\tilde{w}(0), \tilde{w}(1), \ldots, \tilde{w}(t), \ldots)$ is observed which is to be modelled by means of a linear time invariant complete system. General considerations, or Algorithm 2 for that matter, show that two possible conclusions emerge: either $\tilde{\mathbf{w}}$ was an input, which means that it is not explained, or there exists $n \in \mathbb{Z}_+$ and $p_{n-1}, p_{n-2}, \ldots, p_0 \in \mathbb{R}$ such that $\tilde{\mathbf{w}}$ satisfies the recursion $\tilde{w}(t+n) + p_{n-1}w(t+n+1) + \cdots + p_0\tilde{w}(t) = 0$ (assume that n is as small as possible), in which case $p(\sigma)\mathbf{w} = \mathbf{0}$ with $p(s) := s^n + p_{n+1}s^{n-1} + \cdots + p_0$ is chosen as the model which explains the measurements. In that case the conclusion is that $\tilde{\mathbf{w}}$ was the output generated by an autonomous system. With raw data and infinite precision arithmetic, the first situation always results, and hence, in trying to obtain a model which explains the measurements exactly, a model results which explains everything and hence teaches nothing about the underlying phenomenon: an unfalsifiable and, therefore, a worthless model.

On the positive side, things are not as bad as they may seem from all this, since carrying out rank determinations and solving linear equations *approximately* is well within the expertise of numerical linear algebra techniques. As such, it can be claimed that the algorithms in Part II may already provide useful methods for realistic applications.

Nevertheless, it is much more logical to let the approximation question enter not on the level of the numerical implementation of an algorithm, but, if possible, to incorporate this aspect in the very problem definition. This is the subject of the present Part III of this paper.

As already mentioned in the introduction of Part II, the main motivation for treating the exact modelling issue first is that in order to make progress on the approximate modelling question, it is absolutely necessary to have a good understanding of the exact modelling question and to have a suitable mathematical vocabulary for it available. As such, the first task will be to refine the language of Part II so as to incorporate the approximation issue. As motivated in Part II, a stochastic approach is not used. As such, the paradigm on which the model selection is based will be *low complexity* and *high accuracy* (not unbiasedness, consistency and efficiency as would be the case in the familiar statistical framework).

The key concepts in this approach will be the *complexity* of a model and the *misfit* between a model and the measurements. These concepts will be treated in the next sections. Approximate modelling will consist of an implementation of the principle that the desired optimal model is simply the most accurate model within a preassigned tolerated complexity level or the least complex model in a given model class which approximates the observed data up to a preassigned tolerated misfit. The idea of formalizing complexity considerations in system identification has also been pursued in Rissanen (1985) and Caines (1986). Also, the idea that identification algorithms (should) have an interpretation as optimal approximation is basic in the point of view put forward by Ljung (Ljung and Söderström, 1983; Ljung, 1985).

20. METHODOLOGY

In this section the methodology which underlies the approximate modelling procedures will be outlined. Actually, many classical identification algorithms can be interpreted nicely in this context.

In the language of Section 13, let S be a phenomenon, $\mathscr{M} \subset 2^S$ be a model set, and $\mathscr{Z} \subset 2^S$ be a set of measurements. The *complexity* c is a mapping $c: \mathscr{M} \to \mathscr{C}$ with $\mathscr{C}$, the *complexity level space*, a partially ordered space. The *misfit* ε is a mapping $\varepsilon: \mathscr{Z} \times \mathscr{M} \to \mathscr{E}$ with $\mathscr{E}$ the *misfit level space*, also a partially ordered space. It is logical to demand that $\{M_1, M_2 \in M, M_1 \subset M_2\} \Rightarrow \{c(M_1) \leqslant c(M_2)$ and $\varepsilon(Z, M_1) \geqslant \varepsilon(Z, M_2)\ \forall Z \in \mathscr{Z}\}$. In other words, the partial ordering is isotone with respect to the usual ordering induced by inclusion. However, in some situations one may want to use complexity or misfit functions which do not satisfy these conditions.

The complexity can be viewed as the inverse of the power of a model and is hence a quantitative measure for expressing how powerful a model actually is. The misfit $\varepsilon(Z, M)$ indicates how far the model M fails to explain the measurements Z. Large complexity and large misfit are both undesirable properties of a model. Models with large complexity explain too much, while models for which the misfit is large explain the observations poorly and therefore do not inspire much confidence.

Now assume that S, $\mathscr{M}$, $\mathscr{Z}$, $c: \mathscr{M} \to \mathscr{C}$, and $\varepsilon: \mathscr{Z} \times \mathscr{M} \to \mathscr{E}$ is such a modelling set-up. The approximate modelling methods used here proceed along one of the following lines.

First methodology

Fix the *maximal admissible complexity*, c^{adm}. Then call $M^* \in \mathscr{M}$ the *optimal approximate model* in the model class $\mathscr{M}$ for $Z \in \mathscr{Z}$ if it satisfies the following conditions:

(i) $c(M^*) \leqq c^{\text{adm}}$;

(ii) $\{M \in \mathscr{M}, c(M) \leqq c^{\text{adm}}\} \Rightarrow \{\varepsilon(Z, M^*) \leqq \varepsilon(Z, M)\}$;

(iii) $\{M \in \mathscr{M}, c(M) \leqq c^{\text{adm}}, \varepsilon(Z, M) = \varepsilon(Z, M^*)\} \Rightarrow \{c(M^*) \leqq c(M)\}$.

In this methodology the optimal approximate model has an allowed complexity level (i) and, within this class, a minimal misfit (ii). However, if there are many models achieving this minimum, then it is logical to choose the one which has smallest complexity (iii). This often induces uniqueness, while (i) and (ii) alone may not.

Second methodology

Fix the *maximal tolerated misfit*, ε^{tol}. Then $M^* \in \mathcal{M}$ is called the *optimal approximate model* in the model class $\mathcal{M}$ for $Z \in \mathcal{Z}$ if it satisfies the following conditions:

(i) $\varepsilon(Z, M^*) \leqq \varepsilon^{\text{tol}}$;

(ii) $\{M \in \mathcal{M}, \varepsilon(Z, M) \leqq \varepsilon^{\text{tol}}\} \Rightarrow \{c(M^*) \leqq c(M)\}$;

(iii) $\{M \in \mathcal{M}, \varepsilon(Z, M) \leqq \varepsilon^{\text{tol}}, c(M) = c(M^*)\} \Rightarrow \{\varepsilon(Z, M^*) \leqq \varepsilon(Z, M)\}$.

In this methodology the optimal approximate model has a tolerated error level (i) and, within this class, a minimal complexity (ii). However, if there are many models achieving this minimum, then it is logical to choose the one which has smallest misfit (iii). This often induces uniqueness, while (i) and (ii) alone may not.

There are a number of minor variations on the above procedures which can have consequences on these algorithms. In particular, the results may differ slightly if c^{adm} and/or ε^{tol} are interpreted in the sense that they require $c(M) < c^{\text{adm}}$ or $\varepsilon(Z, M) < \varepsilon^{\text{tol}}$ in (i) (with obvious modifications carried through in (ii) and (iii)).

Note that when $\mathcal{C}$ and $\mathcal{E}$ are partially, and not totally, ordered, then it will be very unlikely that (unique) optimal approximate models exist. So, it is principally the totally ordered spaces $\mathcal{C}$ and $\mathcal{E}$ which are of interest. However, for dynamical systems it is most natural to view both $\mathcal{C}$ and $\mathcal{E}$ initially as partially ordered. In order to make them totally ordered, isotone maps $C: \mathcal{C} \to \mathcal{C}'$ and $E: \mathcal{E} \to \mathcal{E}'$ will then be applied with $\mathcal{C}'$ and $\mathcal{E}'$ totally ordered, and $c' := Cc$ and $\varepsilon' := E\varepsilon$ viewed as the new complexity and misfit functions.

The usual approach in system identification is to fix a parametrized class of models and then choose the parameters such that some criterion is minimized. As such these methods may be viewed as an application of the first methodology which is regarded as a *parameter fitting* method with the number of free parameters the maximal admissible complexity. The second methodology, which is more appealing, is *adaptive* in nature since basically the structure (i.e. the complexity) of the chosen model is adjusted so that the required fit is achieved.

Of course, most appealing of all is to have a methodology in which a combination of the complexity and the misfit is used in a *utility function* u: $\mathcal{C} \times \mathcal{E} \to \mathcal{U}$ yielding $u(M, Z)$ to be maximized. However, this will not be pursued here since it seems difficult to come up with an intuitively justifiable utility function.

As already emphasized in Part II, only situations in which it is assumed that realizations of the phenomenon are being directly observed will be considered. More specifically, linear time invariant means that it will be assumed that observations consist of a vector time series which is interpreted by means of a linear time-invariant system, incorporating as its signal variables the observed time series only. Of course the observations could also be interpreted by means of auxiliary variables, if it is logical in that case to keep (the norm of) these auxiliary variables small. This leads to the notion of the credibility of the model: it makes sense to prefer (accept as more credible) an interpretation with small auxiliary variables above one in which the auxiliary variables are large in norm. These interesting ramifications of the theory will be pursued at a later stage.

En passant, the classical system identification methods which are based on the premises of statistics, such as AR, ARMA, or ARMAX identification, prediction error methods, minimum description length methods, maximum likelihood identification etc., are basically methods which are based on the minimization of the complexity, and perhaps, the credibility (likelihood), but do not explicitly allow for a misfit. Thus Akaike's criterion (Akaike, 1977) should be viewed as a proposal of a utility which is a combination of the credibility and the complexity of a model. Ljung's work mentioned earlier introduces approximation in a more explicit way. The question remains in many of these stochastic approaches, of course, of how reasonable it is to use stochastic models, where the need comes from to insist on a stochastic interpretation of the lack of fit between data and model, and why—if one accepts stochastic models as an interpretation of observed phenomena—it is desirable to have a small noise term.

In the remainder of this paper, these ideas will be developed in detail for finite dimensional spaces and linear time invariant complete systems. For illustrative purposes, however, the case of finite sets will be treated first.

Example 8. Let S be a finite set and let $|\cdot|$ denote the cardinality. Take $\mathcal{M} = \mathcal{Z} = \{\text{all non-empty subsets of } S\}$, $\mathcal{C} = \mathbb{R}_+$, $\mathcal{E} = [0, 1]$. Define $c(M) = {}^2\log M$ (this complexity measure is discussed for example in Rissanen (1985) and Kolmogorov (1965)) and $\varepsilon(Z, M) = \dfrac{|Z \cap M^{\text{complement}}|}{|Z|}$ (clearly, this situation in which ε is introduced without at least having a distance measure, is not very subtle).

If the first approximate modelling methodology is used on this example, then to begin with one should decide on the maximal number of elements which the chosen model is allowed to contain. This yields for the optimal approximate model

$$M^* = \begin{cases} Z' & \text{if } {}^2\log|Z| \leqslant c^{\text{adm}} \\ Z' & \text{if } {}^2\log|Z| > c^{\text{adm}}. \end{cases}$$

Here Z' is any subset of Z such that $|Z'| = \text{ent}(2^{c^{\text{adm}}})$.

If the second approximate modelling methodology is used on this example, then one should decide on the maximal percentage of falsified elements which one is willing to tolerate. This yields for the optimal approximate model $M^* = Z'$ with Z' any non-empty subset of Z such that $|Z - Z'| = \text{ent}(\varepsilon^{\text{tol}}|Z|)$.

In the case at hand a reasonable utility function would be $-u(Z, M) = {}^2\log|M| + {}^2\log|Z \cap M^{\text{complement}}|$. Suitably interpreted, this yields $M^* = Z$ as the unique optimal model.

Example 9 (*Factor analysis*). Many data analysis problems consist of fitting a linear subspace to a set of observed vectors. In this terminology, $S = \mathbb{R}^n$, $\mathcal{M} = \{M \subset \mathbb{R}^n | M \text{ linear}\}$ and $\mathcal{Z} = \{\text{all finite subsets of } \mathbb{R}^n\}$. The observation set will be denoted $Z = \{z_1, z_2, \ldots, z_N\}$, $z_i \in \mathbb{R}^n$, $i \in \mathbb{N}$.

The most common approach to this subspace fitting problem is *regression*. This assumes that the observation consists of two components, $z_i \approx \text{col}(u_i, y_i)$, with $u_i \in \mathbb{R}^{n_1}$ and $y_i \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$. We call u the *regressor* and y the *regressand*. Now look for a matrix $L \in \mathbb{R}^{n_2 \times n_1}$ such that the graph of $y = Lu$ fits the data in a suitable sense. In regression, L is chosen such that $\dfrac{1}{N}\sum_{i=1}^{N} \|y_i - Lu_i\|^2$ is minimized. Using Euclidean norms and assuming that $\dfrac{1}{N}\sum_{i=1}^{N} u_i u_i^T$ is invertible, yields

$L^* = \left(\frac{1}{N}\sum_{i=1}^{N} y_i u_i^T\right)\left(\frac{1}{N}\sum_{i=1}^{N} u_i u_i^T\right)^{-1}$ as the optimal L. In our modelling language, the structure—and hence the complexity—has been completely fixed by the regression set-up. The graph of $y = Lu$ is thought of as the model M—which is hence an n_1-dimensional subspace of $\mathbb{R}^n$—and one has the misfit $\varepsilon(M,Z) := \sqrt{\frac{1}{N}\sum_{i=1}^{N} \|e_i\|^2}$ with $e_i := y_i - Lu_i$. The misfit is to be minimized. Note that the optimal L^* is completely specified by the fact that the error e_i is uncorrelated by the data: $\frac{1}{N}\sum_{i=1}^{N} e_i u_i^T = 0$. Basically regression assumes that the u_is have been observed without error, but that the measurements have reproduced the model outcomes Lu_i only up to the error $y_i - Lu_i$. The model is then guessed by minimizing the average squared error.

Another approach to this problem is *factor analysis*. In there, it is again assumed that in the observation there is a component of the model and an error. The model produces n-vectors of the form Lu with $u \in \mathbb{R}^m$, the *factors*, and $L \in \mathbb{R}^{n\times m}$, the *loading matrix*. The observations are obtained by adding an error term $e = \text{col}(e_1, e_2, \ldots, e_N)$, yielding $z = Lu + e$. The problem now is to deduce from the data z_i, $i \in \mathbf{N}$, the model parameters (m, L) and, if possible, also the factors u_i, $i \in \mathbf{N}$, or at least the explained part of the model $m_i = Lu_i$, $i \in \mathbf{N}$. The assumption which makes this problem a well-defined one is that the factors u and the components of the error $e^1, e^2, \ldots, e^n$, are independent. This independence is taken to mean $\frac{1}{N}\sum_{i=1}^{N} u_i e_i^k = 0$ and $\frac{1}{N}\sum_{i=1}^{N} e_i^k e_i^{k'} = 0$ for all $k, k' \in \mathbf{n}, k \neq k'$.

It is possible to phrase the problem of finding (m, L) as a matrix analysis problem. Introduce the data covariance matrix $\Sigma_z := \frac{1}{N}\sum_{i=1}^{N} z_i z_i^T$. Obviously $\Sigma_z = \Sigma_z^T \geqslant 0$. Now assume that Σ_z can be written as $\Sigma_z = L\Sigma_u L^T + \Sigma_e$ with $0 \leqslant \Sigma_u = \Sigma_u^T \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{n\times m}$, and $\Sigma_e = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$. Then, assuming that $N \geqslant 2n$, it may be shown that there exist u_i and e_i, $i \in \mathbf{N}$, such that $z_i = Lu_i + e_i$, $i \in \mathbf{N}$, $\frac{1}{N}\sum_{i=1}^{N} u_i u_i^T = \Sigma_u$, $\frac{1}{N}\sum_{i=1}^{N} e_i e_i^T = \Sigma_e$, and $\frac{1}{N}\sum_{i=1}^{N} u_i e_i^T = 0$, iff $\Sigma_z = \frac{1}{N}\sum_{i=1}^{N} z_i z_i^T$ equals $L\Sigma_u L^T + \Sigma_e$. Consequently, the factor analysis problem reduces to writing $\Sigma_z := \frac{1}{N}\sum_{i=1}^{N} z_i z_i^T$ in the form $L\Sigma_u L^T + \Sigma_e$ with $\Sigma_u = \Sigma_u^T \geqslant 0$ and Σ_e diagonal and non-negative definite.

Define the complexity of the factor analysis model

$$z_i = Lu_i + e_i \qquad i \in \mathbf{N} \tag{M}$$

as $c(M) := m$, the dimension of the factor space. Denote $\Sigma_z = \frac{1}{N}\sum_{i=1}^{N} z_i z_i^T$, $\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} Lu_i (Lu_i)^T$, and $\Sigma_e = \frac{1}{N}\sum_{i=1}^{N} e_i e_i^T$. With this choice of the complexity, the factor analysis modelling question with minimal complexity leads to the problem of determining, for a given $\Sigma_z = \Sigma_z^T \geqslant 0$, a minimal rank matrix $\hat{\Sigma} = \hat{\Sigma}^T \geqslant 0$ such that $\Sigma_e = \Sigma - \hat{\Sigma}$ is diagonal with non-negative entries, i.e. of trying to model the observations Σ_z with a minimally complex model.

There always exist many admissible models with complexity $(n-1)$. Simply take any non-zero non-negative definite diagonal matrix Λ, determine the smallest non-negative root $\tilde{\mu}$ of $\det(\Sigma_z - \mu\Lambda) = 0$, and choose $\hat{\Sigma} = \Sigma_z - \tilde{\mu}\Lambda$. Taking for $\Lambda = \text{diag}\underset{\substack{\uparrow \\ i\text{th element}}}{(0, \ldots, 0, 1, 0, \ldots, 0)}$ corresponds to least squares regression with the ith observed variable as regressand. Taking for $\Lambda = \text{diag}(1, 1, \ldots, 1)$ corresponds to what is called *orthogonal regression*. This may be interpreted as assuming that the errors on all variables are equal. The cases for which 1 or $(n-1)$ are the minimal achievable complexity can be identified. Indeed, it can be shown that $\{c^*(\Sigma_z) = 1\} \Leftrightarrow \{\Sigma_z$ is a *Spearman matrix*, i.e. up to change of sign of the variables z_i, all elements of Σ_z are strictly positive and $\sigma_{ik}\sigma_{jl} - \sigma_{ij}\sigma_{jk} = 0$; $\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk} \leqslant 0$ for all $i \neq j \neq k \neq l\}$ and $\{c^*(\Sigma_z) = n-1\} \Leftrightarrow \{\Sigma_z^{-1}$ is a *Frobenius-like matrix*, i.e. up to a change of sign of the variables z_i, Σ_z^{-1} has strictly positive elements$\}$. For a discussion of this, see Kalman (1984) and Bekker and De Leeuw (1985). Determining explicit conditions for c^* to be $2, \ldots, n-2$, appears very difficult. The case $c^*(\Sigma_z) = n-1$ is particularly interesting. The ith column of Σ_z^{-1} determines the regression coefficients if the ith variable is taken to be the regressand. Hence, if Σ_z^{-1} is Frobenius-like, then regression against any variable as regressand will yield the same sign pattern for the weights in the regression equation obtained by taking any of the variables as regressand.

It is logical to define the misfit of the factor analysis model m as $\varepsilon(M,Z) = \Sigma_e = \frac{1}{N}\sum_{i=1}^{N} e_i e_i^T$ or as $\varepsilon(M,Z) = \text{Trace}\ \Sigma_e = \frac{1}{N}\sum_{i=1}^{N} \|e_i\|^2$. It is worthwhile to pursue the resulting approximate modelling problem with m as the complexity and $\frac{1}{N}\sum_{i=1}^{N} \|e_i\|^2$ as the misfit.

Both regression and factor analysis can also be viewed in a stochastic context as an attempt to approximate an n-dimensional (normally distributed) random vector by means of one which has its support on a subspace. Actually regression may then be explained either as writing $y = Lu + e$ with u and e independent or such that $\mathcal{E}\{\|e\|^2\}$ is minimized. The details will not be pursued here.

Regression has the advantage over factor analysis that it is explicitly solvable (without even requiring that algebraic equations be solved), whereas factor analysis as a general problem appears untractable. However, factor analysis has important invariance properties: indeed, it is invariant under a reordering of the components of z. Explicitly, if Π is a permutation matrix and if (m, L) is a minimally complex factor model for the data z_i, $i \in \mathbf{N}$, then $(m, \Pi L)$ will be a minimally complex factor model for the data Πz_i, $i \in \mathbf{N}$. Regression on the other hand is severely prejudiced by the separation of z into col(u, y), a separation which is very artificial in most applications. Also, if Σ_z^{-1} is not Frobenius-like, then regression may not be very sensible to start with since already the sign of the coefficients of the resulting linear relation will depend on the choice of the regressand.

Another approach to the subspace fitting problem which is invariant under permutation of the components of the observation vector is the total least squares approach. It is explicitly solvable using a singular value decomposition (SVD) and hence it requires solving algebraic equations. This approach will be pursued in the next section.

21. TOTAL LEAST SQUARES

In this section the methodology explained in the previous section will be illustrated in order to fit a linear subspace to a finite number of points in $\mathbb{R}^n$ by means of a total least squares criterion. As will be shown, this problem can be solved by means of SVD. The results of this section will serve as an important tool when the dynamic case is discussed.

Let $S = \mathbb{R}^n$, equipped with the Euclidean inner product, $\mathcal{M} = \{$all linear subspaces of $\mathbb{R}^n\}$, and $\mathcal{Z} = \{$all finite subsets of $\mathbb{R}^n\}$. Take $\mathcal{C} = [0, 1]$, equipped with the natural ordering and define the complexity c: $\mathcal{M} \to \mathcal{C}$ as $c(M) = \frac{\dim M}{n}$. Let $\mathcal{E} = \mathbb{R}_+$, endowed with the lexicographic ordering. Now define the misfit $\varepsilon(Z, M)$. Considering the measurement matrix $Z = [z_1, z_2, \ldots, z_N]$ and, for $a \in \mathbb{R}^n$,

define $d(Z, a)$—the misfit between the data and the law $\langle a, z\rangle = 0$—as

$$\mathrm{d}(Z,a) := \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N} |\langle a, z_i\rangle|^2}}{\|a\|} = \underset{a_i \perp \operatorname{span} a}{\text{minimum}} \sqrt{\frac{1}{N}\sum_{i=1}^{N} \|z_i - a_i\|^2}.$$

Now define the (total) misfit $\varepsilon(Z, M)$ as

$$\varepsilon(Z,M) := \max_{0 \neq a \perp M} \mathrm{d}(Z,a).$$

This definition of complexity chooses as the measure of complexity the number of linearly independent constraints—in other words, the number of laws—by which a model restricts the behaviour. The choice of the misfit function can be explained as follows. If $a \perp M$ then the basic claim is that $\langle z, a\rangle = 0$ is a law which holds for phenomena consistent with the model M. Consequently $d(Z, a)$ measures how far the measurements violate this law. Hence the misfit $\varepsilon(Z, M)$ gives the degree in which the worst law which is implied by accepting the model M fails to be corroborated by the measurements Z. This expresses the fact that a model obtained by a chain of laws is judged by its weakest link: the combination of a truth and a half truth remains a half truth.

The following proposition shows how to compute the misfit.

Proposition 23. Let $Z = \{z_1, z_2, \ldots, z_N\} \in \mathscr{Z}$ and denote by Z also the measurement matrix $Z = [z_1, z_2, \ldots, z_N] \in \mathbb{R}^{n \times N}$. Define the covariance matrix of the measurements by $\Pi_Z := \frac{1}{N} ZZ^T$. Let $M \in \mathscr{M}$ and let $Q_M = I - P_M$, with P_M orthogonal projection onto M. Then $\varepsilon(Z, M) = \sigma_{\max}\left(Q_M \frac{Z}{\sqrt{N}}\right) = \sqrt{\sigma_{\max}(Q_M \Pi_Z Q_M)}$.

Proof. See Appendix P.

The solution of the approximate modelling question for the case at hand is discussed next.

Theorem 24. Let $U\Sigma V^T$ be a SVD of $\frac{1}{\sqrt{N}} Z$ with SVs $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ and left singular vectors u_i, $i \in \mathbf{n}$. Let c^{adm}, the maximal admissible complexity, or alternatively, $\varepsilon^{\mathrm{tol}}$, the maximal tolerated misfit, be given. Then an optimal approximate model M^* is obtained as follows. Assume c^{adm} given. Then,

if $\mathrm{ent}(n \cdot c^{\mathrm{adm}}) = 0$, $M^* = 0$;

if $\mathrm{ent}(n \cdot c^{\mathrm{adm}}) \geqslant r$, $M^* = \operatorname{span}(z_1, z_2, \ldots, z_N)$;

otherwise, $M^* = \operatorname{span}(u_1, u_2, \ldots, u_k)$, with the integer k defined such that

$$\sigma_k > \sigma_{k+1} = \sigma_{\mathrm{ent}(n \cdot c^{\mathrm{adm}})+1}.$$

Assume $\varepsilon^{\mathrm{tol}}$ given. Then,

if $\varepsilon^{\mathrm{tol}} \geqslant \sigma_1$, $M^* = 0$;

if $\varepsilon^{\mathrm{tol}} < \sigma_r$, $M^* = \operatorname{span}(z_1, z_2, \ldots, z_N)$;

otherwise, $M^* = \operatorname{span}(u_1, u_2, \ldots, u_k)$, with the integer k defined such that

$$\sigma_k > \varepsilon^{\mathrm{tol}} \geqslant \sigma_{k+1}.$$

The corresponding complexity and misfit are $c(M^*) = k$ and $\varepsilon(Z, M^*) = \sigma_{k+1}$.

Proof. See Appendix P.

It is actually possible, by modifying the definition of the error level in the spirit of Proposition SV(V), to obtain M^* as the unique optimal approximate model.

A few words regarding the sensitivity of the model M^* with respect to the data follow. This sensitivity is clearly related to the sensitivity of the span of the left singular vectors of the data matrix Z. Now if $Z \in \mathbb{R}^{n \times N}$ is the data matrix and $u_1, u_2, \ldots, u_k$ are its left singular vectors corresponding to its first k singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k \geqslant \sigma_{k+1} \geqslant \cdots$ then it is known that (see, e.g. Golub and Van Loan, 1984, p. 287) the sensitivity of span $(u_1, u_2, \ldots, u_k)$ is proportional to $\frac{1}{\sigma_k - \sigma_{k+1}}$. As expected, this sensitivity becomes infinite when $\sigma_k \cong \sigma_{k+1}$. Even when c^{adm} is imposed, however, this difficulty will not be encountered, due to the choice of ε and the secondary minimization (iii) in the methodology for choosing M^*. All this implies is that it is important, when applying Theorem 24, to consider the σ_k and σ_{k+1} corresponding to the optimal approximate model and, if σ_k is nearly equal to σ_{k+1}, to decrease c^{adm} or increase $\varepsilon^{\mathrm{tol}}$ somewhat. This will yield a model which is much more robust, which will be less complex, but which fits the data with almost the same accuracy as the original model.

The results of Theorem 24 depend of course on the chosen expression for ε. However, similar results can be obtained for other least squares type misfit functions.

For example, if $\varepsilon(Z, M)$ is defined as

$$\min_{\hat{z}_i \in M} \sqrt{\frac{1}{N}\sum_{i=1}^{N} \|z_i - \hat{z}_i\|^2}$$

then a similar result to Theorem 24 is obtained with k such that

$$\sqrt{\sum_{i=k}^{r} \sigma_i^2} \leqslant \varepsilon^{\mathrm{tol}} < \sqrt{\sum_{i=k+1}^{r} \sigma_i^2}.$$

Note, however, that this error measure will not give uniqueness of M^* in the case of repeated singular values $\sigma_k = \sigma_{k+1}$. Also, for the case at hand nothing changes, clearly, if c^{adm} is interpreted as requiring $c(M) < c^{\mathrm{adm}}$: simply replace $\mathrm{ent}(n \cdot c^{\mathrm{adm}}) + 1$ by $\mathrm{ENT}(n \cdot c^{\mathrm{adm}})$ in the rule for the choice of k. However, if $\varepsilon^{\mathrm{tol}}$ is interpreted as requiring $\varepsilon(Z, M) < \varepsilon^{\mathrm{tol}}$, then k should be determined by $\sigma_k \geqslant \varepsilon^{\mathrm{tol}} > \sigma_{k+1}$.

In the example under consideration of least squares fitting of a linear subspace to a cloud of points in $\mathbb{R}^n$, the choice of the complexity of the model is rather compelling and equals the dimension of the subspace which defines the model. The choice of the misfit, however, is not as compelling. The misfit

$$\varepsilon(Z,M) = \max_{a \perp M} \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N} |\langle a, z_i\rangle|^2}}{\|a\|}$$

used in Theorem 24 is basically an *equation error* oriented misfit

measure since it expresses the fit by checking how far the equations defining the model fail to be satisfied. The misfit

$$\varepsilon(Z, M) = \min_{\hat{z}_i \in M} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \|z_i - \hat{z}_i\|^2}$$

on the other hand is a *behaviour error* oriented misfit measure since it expresses how far the data can be reproduced by the model. However, for the case at hand these two measures lead to solutions which are essentially equivalent. Later in the paper it will be shown that the distinction between both situations is of much more consequence in the dynamic case.

A recurrent question concerning the approach presented here is how fixing the maximally admissible complexity compares with methods which fix a maximal number of free parameters and subsequently fit these parameters to the data. This number of free parameters could then be interpreted as being related to the complexity. The two ideas are not unrelated, but there are important differences. For one thing one insists on talking about the complexity of a model class and it is awkward to judge *a posteriori* a chosen model by the class out of which it was selected. Secondly, the number of free parameters in a model class is not always mathematically a well-defined concept. The relation between the notion of complexity and the number of free parameters becomes visible in determination of the dimension of the manifold of models having a maximally admissible complexity. For the modelling set-up discussed in the present section the set $\left\{M \in \mathscr{M} \mid c(M) = \frac{n'}{n}\right\}$ is a manifold of dimension $n(n-n')$. It is reasonable to identify this number with the number of free parameters. This expression shows however that there is not even a monotonic relationship between the number of free parameters and the definition of complexity. As such it is a difficult matter to speak about the number of free parameters in the set $\{M \in \mathscr{M} \mid c(M) \leqslant c^{\text{adm}}\}$ since this set will in general not even be a differentiable manifold. In conclusion, there is only a vague relation between complexity and *parsimony* (in the number of free parameters).

22. THE COMPLEXITY OF DYNAMICAL SYSTEMS

Up to now, complexity spaces which are totally ordered have been considered. However for dynamical systems a partially ordered complexity measure will be introduced.

Recall that $\mathbf{L} := \{$all linear shift invariant closed subspaces of $(\mathbb{R}^q)^T$, $T = \mathbb{Z}$ or $\mathbb{Z}_+$, equipped with the topology of pointwise convergence$\}$. Take $\mathscr{C} := [0,1]^{\mathbb{Z}_+}$, i.e. each element $\mathbf{c} \in \mathscr{C}$ is a map $\mathbf{c}$: $\mathbb{Z}_+ \to [0,1]$. Endow $\mathscr{C}$ with the partial order of pointwise domination: $\{\mathbf{c}' \geqslant \mathbf{c}''\} :\Leftrightarrow \{\mathbf{c}'_t \geqslant \mathbf{c}''_t, \forall t \in \mathbb{Z}_+\}$.

The notion of complexity will be defined as follows.

Definition 5. The *complexity* of a linear time invariant complete system $\{T, \mathbb{R}^q, \mathscr{B}\}$ with $\mathscr{B} \in \mathbf{L}$ is defined as $\mathbf{c}$: $\mathbf{L} \to \mathscr{C}$ with

$$\mathbf{c}_t(\mathscr{B}) := \frac{\dim \mathscr{B}_t}{q(t+1)}$$

where $\mathscr{B}_t := \mathscr{B}\big|_{T \cap [0,t]}$.

This definition of complexity simply measures the relative number of degrees of freedom which are visible in the system's behaviour in each time window, i.e. the relative number of independent sequences which the system can conceivably generate in a time interval of a given length. The more sequences, the fewer constraints they are required to satisfy, the higher the complexity.

This definition is very similar to what one can take as the complexity of a real number $a \in [0,1]$. Assume that a is written in binary expansion as $a = a_1 a_2 \ldots a_t \ldots$. Define N_t as the number of distinct subsequences of length t appearing somewhere in this binary expansion, and $\mathbf{c}_t := \frac{N_t}{2^t}$ as the relative number of such distinct sequences. It is reasonable to define the sequence $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_t, \ldots)$ as the *complexity* of the number a. *It is easy to see that* $\mathbf{c}_t$ is non-increasing. Hence $c_\infty := \lim_{t \to \infty} \mathbf{c}_t$ exists. If $c_\infty > 0$ then a can be thought of as being 'random'. Random numbers are certainly irrational in this sense. If the a_is are stochastically generated and independent, with $P(a_i = 0) = p$, $p \neq 0, 1$, then $c_\infty = 1$ with probability 1, as such numbers with $0 < c_\infty < 1$ are random in a non-probabilistic sense.

Recall that associated with an element $\mathscr{B} \in \mathbf{L}$ there are a number of important related integers. First consider $\mathbf{d}(\mathscr{B})$: $\mathbb{Z}_+ \to \mathbb{Z}_+$ defined by $\mathbf{d}_t(\mathscr{B}) := \dim \mathscr{B}_t$. Its derivative $\boldsymbol{\rho}(\mathscr{B})$: $\mathbb{Z}_+ \to \mathbb{Z}_+$ is defined by $\boldsymbol{\rho}_t(\mathscr{B}) := \mathbf{d}_t(\mathscr{B}) - \mathbf{d}_{t-1}(\mathscr{B})$ (define $\mathbf{d}_{-1}(\mathscr{B}) := 0$), and the negative of its second derivative, the *structure indices*, $\boldsymbol{\gamma}(\mathscr{B})$: $\mathbb{Z}_+ \to \mathbb{Z}_+$ are defined by $\boldsymbol{\gamma}_t(\mathscr{B}) := \boldsymbol{\rho}_{t-1}(\mathscr{B}) - \boldsymbol{\rho}_t(\mathscr{B})$ (define $\boldsymbol{\rho}_{-1}(\mathscr{B}) := 0$). From the results of Section 7 it follows that $\mathbf{d}(\mathscr{B})$ is non-negative ($\mathbf{d} \geqslant \mathbf{0}$), monotone non-decreasing ($\mathbf{d}' = \boldsymbol{\rho} \geqslant \mathbf{0}$), and concave ($\mathbf{d}'' = -\boldsymbol{\gamma} \leqslant \mathbf{0}$). These indices can also be approached from the AR descriptions. Let $\mathscr{B} \in \mathbf{L}$ be described by $R(\sigma)\mathbf{w} = \mathbf{0}$ with $R = \text{col}(r_1, r_2, \ldots, r_g)$. Then $\partial(R) = (\partial_1, \partial_2, \ldots, \partial_g)$ with $\partial_i := \partial(r_i)$ (assume without loss of generality $0 \leqslant \partial_1 \leqslant \partial_2 \leqslant \cdots \leqslant \partial_g$) is called the *lag structure* of R, while $\zeta_1, \zeta_2, \ldots, \zeta_t, \ldots$, with $\zeta_t :=$ the number of ∂_is equal to t, is called the *equation structure* of R. Since many polynomial matrices R define the same behaviour $\mathscr{B} \in \mathbf{L}$ (for Rs with full row rank this family is generated in the case $T = \mathbb{Z}_+$ by the transformation group $R \to UR$ where U ranges over the unimodular polynomial matrices—this statement should be suitably modified for the case $T = \mathbb{Z}$), this leads to the definition of the *shortest lag AR description* of $\mathscr{B}$ defined with $g = g^*(\mathscr{B})$ as small as possible ($g^* =$ normal rank $R(s)$) and, within this class, $\partial_1^*(\mathscr{B}), \partial_2^*(\mathscr{B}), \ldots, \partial_g^*(\mathscr{B})$ lexicographically minimal (see Section 7 for a more precise formulation). The associated equation structure is called the *tightest equation structure*. This sequence $\zeta_0^*(\mathscr{B}), \zeta_1^*(\mathscr{B}), \ldots, \zeta_t^*(\mathscr{B}), \ldots$ has the significance that $\zeta_t^*(\mathscr{B})$ represents the number of equations of lag t when the equations are expressed with minimal possible lags. The specification of all these indices

can also be approached from the state space point of view. Indeed in a state space model for $\mathscr{B}$ with a minimal number of states and driving inputs let $m^*(\mathscr{B})$ be the minimal number of driving inputs and $v_1^*(\mathscr{B}), v_2^*(\mathscr{B}), \ldots, v_p^*(\mathscr{B})$ the observability indices of the associated pair (A, C).

In Section 7 it was shown that there is a one-to-one relation between

$$(\mathbf{d}_0(\mathscr{B}), \mathbf{d}_1(\mathscr{B}), \ldots, \mathbf{d}_t(\mathscr{B}), \ldots)$$

$$(\rho_0(\mathscr{B}), \rho_1(\mathscr{B}), \ldots, \rho_t(\mathscr{B}), \ldots)$$

$$(\gamma^0(\mathscr{B}), \gamma_1(\mathscr{B}), \ldots, \gamma_t(\mathscr{B}), \ldots)$$

$$(g^*(\mathscr{B}); \partial_1^*(\mathscr{B}), \ldots, \partial_{g^*}^*(\mathscr{B}))$$

$$(\zeta_0^*(\mathscr{B}), \zeta_1^*(\mathscr{B}), \ldots, \zeta_t^*(\mathscr{B}), \ldots)$$

and $(m^*(\mathscr{B}); v_1^*(\mathscr{B}), \ldots, v_p^*(\mathscr{B}))$.

Now, since $\mathbf{c}_t(\mathscr{B}) = \dfrac{\mathbf{d}_t(\mathscr{B})}{q(t+1)}$, the complexity

$$\mathbf{c}(\mathscr{B}) = (\mathbf{c}_0(\mathscr{B}), \mathbf{c}_1(\mathscr{B}), \ldots, \mathbf{c}_t(\mathscr{B}), \ldots)$$

can be added to this list.

Theorem 25. The complexity function introduced in Definition 5 has the following properties:

(i) $\{\mathscr{B}', \mathscr{B}'' \in \mathbf{L}, \quad \mathscr{B}' \subset \mathscr{B}''\} \Rightarrow \{\mathbf{c}(\mathscr{B}') \leqslant \mathbf{c}(\mathscr{B}'')\}$; $0 \leqslant \mathbf{c}_t(\mathscr{B}) \leqslant 1$; and $\mathbf{c}_t'(\mathscr{B}) \leqslant 0$: $\mathbf{c}(\mathscr{B})$ is monotone non-increasing;

(ii) For t sufficiently large $\mathbf{c}_t(\mathscr{B}) = \dfrac{m}{q} + \dfrac{1}{t+1} \cdot \dfrac{n}{q}$, with m the dimension of the input space in any i/o or i/s/o representation of $\mathscr{B}$, and n the dimension of the state space in any minimal state space representation of $\mathscr{B}$. Hence $c_\infty(\mathscr{B}) := \lim_{t \to \infty} \mathbf{c}_t(\mathscr{B})$ equals $\dfrac{m}{q}$, and the rate at which this limit is approached, $v_\infty(\mathscr{B}) := \lim_{t \to \infty} t(\mathbf{c}_t(\mathscr{B}) - c_\infty(\mathscr{B}))$, equals n/q.

(iii) There is a one-to one relation between $\mathbf{c}(\mathscr{B})$ and the structure indices, or, equivalently, with the shortest lag indices, with the tightest equation structure, or the number of inputs and the observability indices of a minimal input/minimal state i/s/o representation. Let $\mathscr{B}', \mathscr{B}'' \in \mathbf{L}$, m', m'' be the corresponding number of inputs, n', n'' be the corresponding number of states in any minimal i/s/o representation of $\mathscr{B}$, and $\zeta' = (\zeta_0', \zeta_1', \ldots)$, $\zeta'' = (\zeta_0'', \zeta_1'', \ldots)$ be the corresponding tightest equation structure. Then $\{\mathbf{c}(\mathscr{B}') \geqslant \mathbf{c}(\mathscr{B}'')\} \Rightarrow \{m' \geqslant m''\}$, and

$$\{\mathbf{c}(\mathscr{B}') \geqslant \mathbf{c}(\mathscr{B}'')\} \Leftrightarrow \left\{ \sum_{i=0}^{k} (i+1)\zeta_i' \leqslant \sum_{i=0}^{k} (i+1)\zeta_i'' \text{ for all } k \in \mathbb{Z}_+ \right\}.$$

In particular, $\{\mathbf{c}(\mathscr{B}') \geqslant \mathbf{c}(\mathscr{B}'')$ and $m' = m''\} \Rightarrow \{n' \geqslant n''\}$.

Proof. See Appendix P.

Statement (iii) illustrates in perhaps the most direct way how to interpret the notion of complexity. Complex systems are described by few and high order equations: they obey a small number of laws and those laws which they obey are rather "loose" in the sense that they involve large lags and hence lead to hard constraints only if large segments of the time series involved are considered.

The above theorem shows the relation between the notion of complexity and the number of parameters in the class of linear time invariant finite dimensional models with a given complexity. There the complexity fixes the number of equations describing $\mathscr{B}$—this corresponds to fixing the number of outputs (and thus the number of inputs)—and the degrees of the equations describing $\mathscr{B}$. By looking at shortest lag descriptions much of the freedom in this representation is exhausted but some parameters can still be eliminated while respecting the degrees of the polynomials of the (AR) equations describing the behaviour. In any case it shows that high complexity means *few* and *high order* (AR) relations. This implies that there will be no simple monotone relation between the complexity and the number of parameters characterizing the class of systems with a given complexity. This shows again that there is only a vague relation between complexity and parsimony.

As already mentioned, there also exists a one-to-one relation between the complexity with its partial ordering and a combination of the number of inputs and the observability indices of any minimal i/s/o representation. However, the exact expression is not particularly illuminating. If, on the other hand, one concentrates on systems with a fixed number of inputs and states, then the connection becomes more direct. The next theorem explores this relationship. In particular it will be shown that the generic elements are the most complex ones. This formalizes in yet another way the intuitive statement: non-generic $\Leftrightarrow$ structured $\Leftrightarrow$ not complex.

Recall that if the i/s/o system $\sigma\mathbf{x} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, $\mathbf{w} = \text{col}(\mathbf{u}, \mathbf{y})$ is considered as being parametrized by the matrices $(A, B, C, D) \in \mathbb{R}^{n^2+nm+pn+pm}$, then the elements which yield an observable pair (A, C), with $(n)\text{MOD}(p)$ observability indices equal to $\text{ENT}\left(\dfrac{n}{p}\right)$, and p-$(n)\text{MOD}(p)$ observability indices equal to $\text{ent}\left(\dfrac{n}{p}\right)$, form an open, dense, and measure exhausting (the Lebesgue measure of its complement being zero) subset of

$\mathbb{R}^{n^2+nm+pn+pm}$. These elements have *generic* observability indices.

Theorem 26. Let $\mathscr{B}', \mathscr{B}'' \in \mathbf{L}$, m', m'' be the corresponding number of inputs in any i/o representation, n', n'' be the corresponding minimal number of states, and $\nu' = (\nu_1', \nu_2', \ldots)$, $\nu'' = (\nu_1'', \nu_2'', \ldots)$ be the corresponding observability indices in any minimal i/s/o representation. Then

(i) $\{\mathbf{c}(\mathscr{B}') \geq \mathbf{c}(\mathscr{B}'')\} \Rightarrow \{m' \geq m''\}$

(ii) $\{\mathbf{c}(\mathscr{B}') \geq \mathbf{c}(\mathscr{B}'')$ and $c_\infty(B') = c_\infty(\mathscr{B}'')$ (equivalently, $m' = m''$)$\} \Rightarrow \{n' \geq n''\}$

(iii) $\{\mathbf{c}(\mathscr{B}') \geq \mathbf{c}(\mathscr{B}''), \ c_\infty(\mathscr{B}') = c_\infty(\mathscr{B}'')$ and $v_\infty(\mathscr{B}') = v_\infty(\mathscr{B}'')$ (equivalently, $m' = m''$ and $n' = n''$)$\}$

$$\Rightarrow \left\{ \sum_{i \mid \nu_i' \leq k} \nu_i' \geq \sum_{i \mid \nu_i'' \leq k} \nu_i'' \text{ for all } k \in \mathbb{Z}_+ \text{ (equivalently, } \sum_{i \mid \nu_i' \geq k} \nu_i' \leq \sum_{i \mid \nu_i'' \geq k} \nu_i'' \text{ for all } k \in \mathbb{Z}_+) \right\}.$$

Proof. See Appendix P.

The following corollary is particularly interesting.

Corollary 27. Consider the class of finite dimensional linear time invariant i/s/o systems with a fixed number of inputs, states, and outputs. Then the elements with generic observability indices are precisely those with maximal complexity.

The following diagrams show typical graphs for a complexity function $\mathbf{c}(\mathscr{B})$ and $\mathbf{d}(\mathscr{B})$ with $\mathbf{d}_t(\mathscr{B}) = \dim \mathscr{B}_t = q(t+1)\mathbf{c}_t(\mathscr{B})$ (see Diagrams 1 and 2).

The partial ordering which has been taken on the complexity space implies that $\mathscr{B}'$ will be more complex than $\mathscr{B}''$ iff, on each time interval, $\mathscr{B}'$ allows more sequences than $\mathscr{B}''$. This is a very strong ordering and in approximate modelling total orderings will be considered which are isotonic with but weaker than the above ones. In the algorithms in Section 25, the lexicographic ordering: $\{\mathbf{c}' \geq \mathbf{c}''\} :\Leftrightarrow \{$either $\mathbf{c}' = \mathbf{c}''$ or there exists $t \in \mathbb{Z}$ such that $\mathbf{c}_t' > \mathbf{c}_t''$ and $\mathbf{c}_{t'}' = \mathbf{c}_{t'}''$ for $0 \leq t' < t\}$ will be used. This total ordering expresses strong preference for short order lag relations (further comments in Section 25). At this point, however, the reverse lexicographic ordering is, in view of Theorem 26, perhaps more appealing. The reverse lexicographic ordering is defined by $\{\mathbf{c}' \geq \mathbf{c}''\} :\Leftrightarrow \{$either $\mathbf{c}' = \mathbf{c}''$ or there exists $t \in \mathbb{Z}$ such that $\mathbf{c}_t' > \mathbf{c}_t''$ and $\mathbf{c}_{t'}' = \mathbf{c}_{t'}''$ for $t' > t\}$. This ordering tries to keep the number of inputs (unexplained variables) and the number of states small and is a total ordering on $\mathbf{c}(L)$. However the development of approximate modelling algorithms in this case is much more complicated and is a topic of future research. In any case, $\mathbf{c}(\mathscr{B})$ itself appears as quite a compelling measure of complexity of a linear time invariant finite dimensional system.

23. THE MISFIT BETWEEN AN OBSERVED TIME SERIES AND A LINEAR TIME INVARIANT SYSTEM

In order to measure the misfit between an observed time series and a model, norms will have to be introduced. This will be done by assuming that the components of the observed time series belong to a linear subspace $\mathscr{K} \subset \mathbb{R}^T$ which is assumed to be a shift invariant ($\sigma\mathscr{K} \subset \mathscr{K}$ if $T = \mathbb{Z}_+$ and $\sigma\mathscr{K} = \mathscr{K}$ if $T = \mathbb{Z}$) inner product space with inner product and norm denoted by $\langle .,. \rangle_{\mathscr{K}}$ and $\|\cdot\|_{\mathscr{K}}$, respectively. The subscript $\mathscr{K}$ will be dropped whenever there is no danger of confusion. Also assume that σ in non-expansive on $\mathscr{K}$, i.e. that

$$\|\sigma \mathbf{k}\|_{\mathscr{K}} \leq \|\mathbf{k}\|_{\mathscr{K}} \text{ for all } \mathbf{k} \in \mathscr{K}.$$

Examples of spaces $\mathscr{K}$ are:

(1) (l_2-spaces): $\mathscr{K} = l_2(T; \mathbb{R}^q)$.

(2) (Weighted l_2-spaces): Let ρ: $T \to \mathbb{R}_+$ be given with ρ positive and non-increasing and define $\mathscr{K}$ by $\mathscr{K} := \{\mathbf{k} \in \mathbb{R}^T \mid \|\mathbf{k}\|_{\mathscr{K}}^2 := \sum_{t \in T} \mathbf{k}^2(t)\rho(t) < \infty\}$. Particularly important is the case $\rho(t) = \alpha^t$ for some $0 < \alpha \leq 1$ (exponential weighting).

(3) (Filtered l_2-spaces): Let $0 \neq R \in \mathbb{R}[\sigma]$, $\mathscr{K} = l_2(T; \mathbb{R}^q)$ with norm $\|\mathbf{k}\|_{\mathscr{K}}^2 = \|R(\sigma)\mathbf{k}\|_{l_2}$.

(4) (Almost periodic sequences). Let $T = \mathbb{Z}_+$ or $\mathbb{Z}$ and $\mathscr{K}$ consist of the almost periodic sequences (i.e. all $\mathbf{k}$: $T \to \mathbb{R}$ such that $\forall \varepsilon > 0$ there exists $T(\varepsilon) \in \mathbb{Z}_+$ such that $|\mathbf{k}(t + T(\varepsilon)) - \mathbf{k}(t)| \leq \varepsilon$ for all $t \in T$). The almost periodic sequences are countable sums of the type $\sum_{i \in \mathbb{Z}_+} a_i \cos(\omega_i t + \varphi_i)$ with a_i, $\omega_i \in \mathbb{R}_+$, $\varphi_i \in [0, 2\pi)$ (and the infinite sum suitably defined). Take $\langle \mathbf{k}_1, \mathbf{k}_2 \rangle_{\mathscr{K}} = \lim_{t \to \infty} \frac{1}{t+1} \sum_{\tau=0}^{t} \mathbf{k}_1(\tau)\ \mathbf{k}_2(\tau)$ for $T = \mathbb{Z}_+$ and $\langle \mathbf{k}_1, \mathbf{k}_2 \rangle_{\mathscr{K}} = \lim_{t \to \infty} \frac{1}{2t+1} \sum_{\tau=-t}^{t} \mathbf{k}_1(\tau)\mathbf{k}_2(\tau)$ for $T = \mathbb{Z}$.

The misfit measure which will be considered is an *equation error* oriented misfit function. It is inspired by the misfit for linear relations used in Section 21. However, for the case at hand the model $\mathscr{B} \in \mathbf{L}$ will induce, through $\mathscr{B}^\perp$, an *infinite* number of linearly independent laws. Nevertheless, this infinite family is in effect generated by a *finite* number of laws: let $T = \mathbb{Z}_+$ and $R = \text{col}[r_1, r_2, \ldots, r_g]$ with $r_i(s) \in \mathbb{R}^{1 \times q}[s]$ be such that $\mathscr{B} = \mathscr{B}(R)$. Then all elements of $\mathscr{B}^\perp$ are of the form $\sum_{i=1}^{g} p_i(s) r_i(s)$ with $p_i(s) \in \mathbb{R}[s]$. The same holds, with $p_i(s) \in \mathbb{R}[s, s^{-1}]$, for $T \in \mathbb{Z}$. It is this finiteness which will be exploited in order to come up with

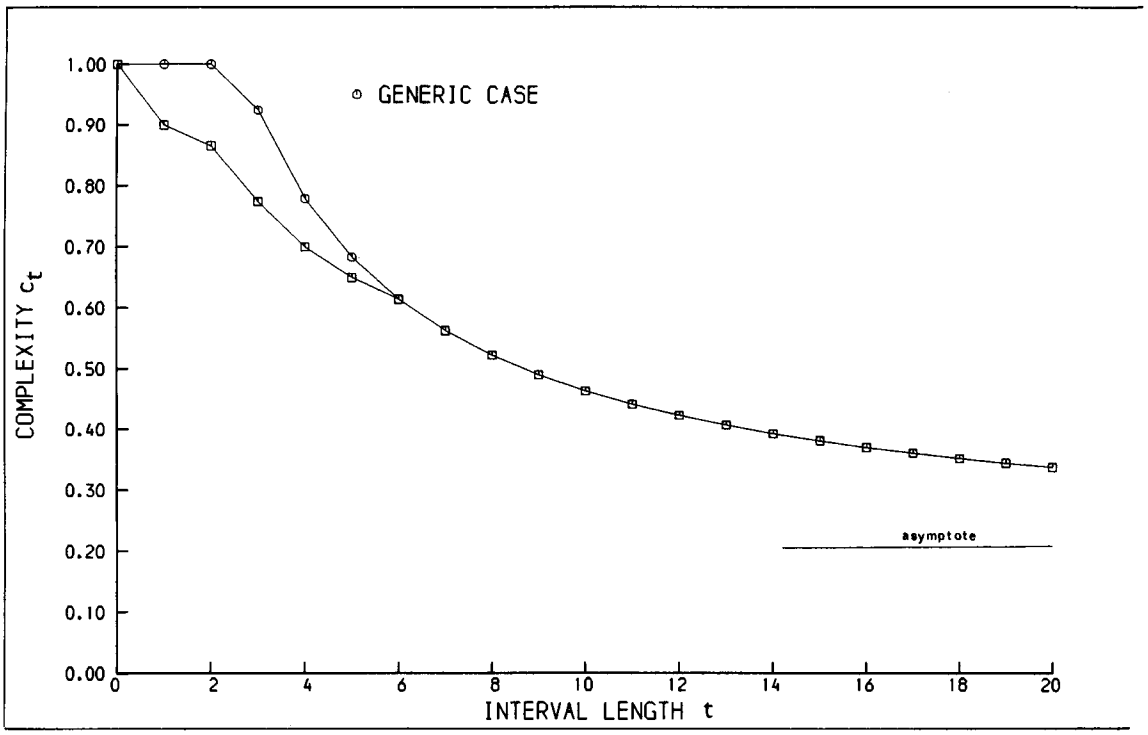


DIAGRAM 1.

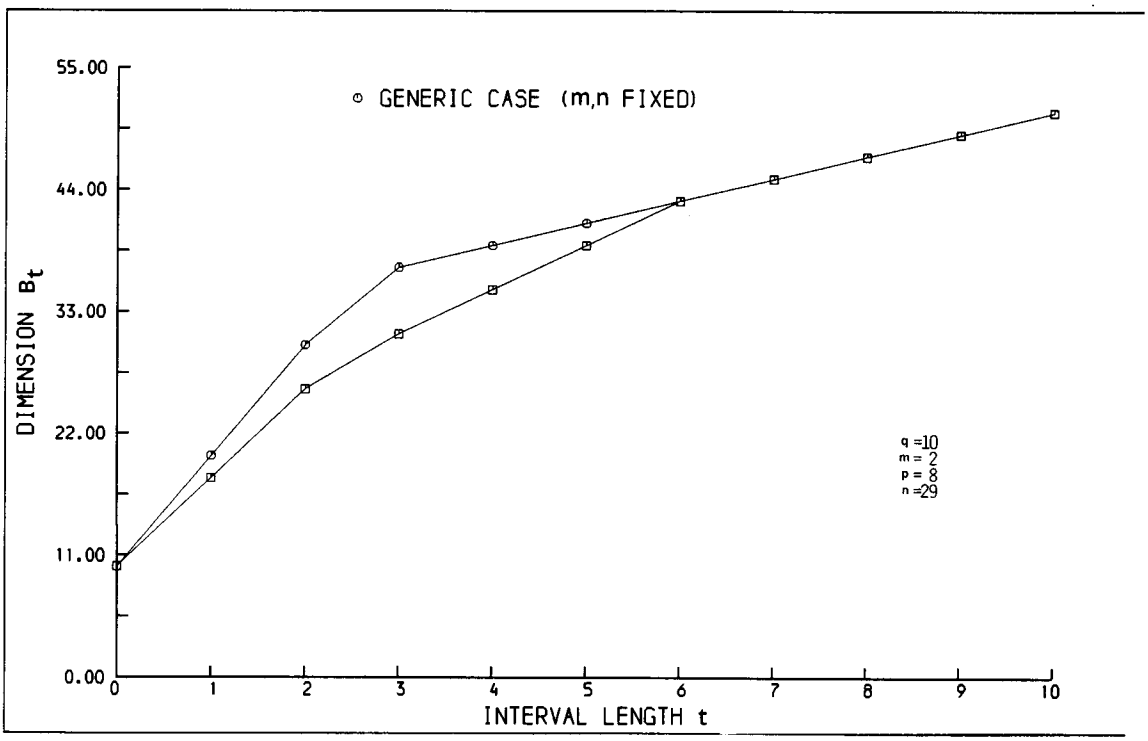


DIAGRAM 2.

a misfit measure which will consist of a *finite* set of non-zero real numbers. In analogy with Section 22 where a linear subspace M imposing $\dim M$ independent linear laws gives rise to $\dim M$ misfit figures (the square root of the largest eigenvalues of $Q_M \Pi_Z Q_M$), in first instance g misfit numbers will be associated with an (AR) system of equations specified, in the above sense, by g generating independent (AR) laws. Recall that this $g = p = q - m$ with p the number of outputs and m the number of inputs in any i/o representation of $\mathscr{B}$.

Let $\mathscr{B} \in \mathbf{L}$ and define

$$\mathscr{N}'_0 := \mathscr{B}_0^\perp$$
$$\mathscr{N}'_1 := \mathscr{B}_1^\perp \cap (\mathscr{B}_0^\perp + \sigma^* \mathscr{B}_0^\perp)^\perp$$
$$\vdots$$
$$\mathscr{N}'_t := \mathscr{B}_t^\perp \cap (\mathscr{B}_{t-1}^\perp + \sigma^* \mathscr{B}_{t-1}^\perp)^\perp$$
$$\vdots$$

Note the following significance of $\mathscr{N}'_t$. Since $\mathscr{B}_t^\perp$ represents the tth order (AR) laws satisfied by $\mathscr{B}$, $\mathscr{B}_{t-1}^\perp + \sigma^* \mathscr{B}_{t-1}^\perp$ represents the tth order (AR) laws which can be obtained by simply shifting the $(t-1)$th order laws, since if $a(s) \in \mathscr{B}_{t-1}^\perp$, then $(\alpha_0 + \alpha_1 s) a(s) \in \mathscr{B}_t^\perp$ for all $\alpha_0, \alpha_1 \in \mathbb{R}$. Hence $\mathscr{N}'_t$ consists of all (AR) laws satisfied on $\mathscr{B}$ and which are "truly" of tth order.

Let $r_t := \dim \mathscr{N}'_t$. Assume that $\mathscr{L}$ consists of all singletons $\{\tilde{\mathbf{w}}\}$ with $\tilde{\mathbf{w}} = \mathrm{col}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \ldots, \tilde{\mathbf{w}}_q)$ and $\tilde{\mathbf{w}}_i \in \mathscr{K}$ for $i \in \mathbf{q}$. Define $\sum_{t \in \mathbb{Z}_+} r_t$ non-negative real numbers as follows:

$$\eta_k^t(\tilde{\mathbf{w}}, \mathscr{B}) := \underset{\substack{\mathscr{N}' \supset \mathscr{N}'_t \\ \dim \mathscr{N}' \geqslant r_t - k + 1}}{\text{minimum}} \ \underset{a \in \mathscr{N}'}{\text{maximum}} \frac{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\|a\|_{\mathscr{L}_t^*}}$$

for $k = 1, 2, \ldots, r_t$ and $t = 0, 1, 2, \ldots$. If $\mathscr{N}' = \{0\}$ then define $\dfrac{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\|a\|_{\mathscr{L}_t^*}} = 0$.

These numbers can be interpreted as follows. They correspond to the lack of fit with which $\tilde{\mathbf{w}}$ fails to corroborate $p = \sum_{t \in \mathbb{Z}_+} r_t$ well chosen representative orthogonal AR relations specifying $\mathscr{B}$. These relations are chosen in such a way that the first r_0 of them for an orthogonal basis for $\mathscr{B}_0^\perp$, the next r_1 form an orthogonal basis for an orthogonal complement of $\mathscr{B}_0^\perp + \sigma^* \mathscr{B}_0^\perp$ in $\mathscr{B}_1^\perp$, etc. As misfit the sequence $\eta^0, \eta^1, \ldots, \eta^t, \ldots$ will be chosen.

Definition 6. The misfit between $\{\tilde{\mathbf{w}}\} \in \mathscr{L}$ and the dynamical model $\{T, \mathbb{R}^q, \mathscr{B}\}$ with $\mathscr{B} \in \mathbf{L}$ is defined as $\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) \in E$ with $E := (\mathbb{R}_+^g)^{\mathbb{Z}_+}$ and

$$\varepsilon_t^k(\tilde{\mathbf{w}}, \mathscr{B}) := \eta_k^t(\tilde{\mathbf{w}}, \mathscr{B})$$

for $k \leqslant r_t$ and $\varepsilon_t^k(\tilde{\mathbf{w}}, \mathscr{B}) = 0$ otherwise.

This misfit measure may be explained as follows: $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B})$ is a measure for the lack of fit of the tth order lag (AR) relations specifying $\mathscr{B}$, obtained by considering only 'truly' tth order lag relations and disregarding the $(t-1)$th order relations and their shifts.

As a first illustrative example, assume $T = \mathbb{Z}$, $q = 1$, and $\mathscr{K} = l_2(\mathbb{Z}; \mathbb{R})$. Let $\mathscr{B} = \mathscr{B}(p)$ with $p(s) = p_l s^l + p_{l-1} s^{l-1} + \cdots + p_0$, $p_0 p_l \neq 0$, and $\sum_{i=0}^{l} p_i^2 = 1$. Then $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) = 0$ for $t \neq l$, while $\varepsilon_l(\tilde{\mathbf{w}}, \mathscr{B}) = \|p_l \sigma^l \tilde{\mathbf{w}} + p_{l-1} \sigma^{l-1} \tilde{\mathbf{w}} + \cdots + p_0 \tilde{\mathbf{w}}\|_{l_2}$ is the equation error for the measurement $\tilde{\mathbf{w}}$ when the model $\mathscr{B}$ is postulated.

As a second example, let $\mathscr{B}$ be defined by the (AR) relations $(\sigma + 1)\mathbf{y}_1 = \mathbf{u}$; $(\sigma^2 + \sigma + 1)\mathbf{y}_2 = \mathbf{u}$ on the variables $w = \mathrm{col}(u, y_1, y_2)$. Then $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) = 0$ for $t \neq 1, 2$, $\varepsilon_1^1(\tilde{\mathbf{w}}, \mathscr{B}) = \dfrac{1}{\sqrt{3}} \|(\sigma + 1)\tilde{\mathbf{y}}_1 - \tilde{\mathbf{u}}\|_{l_2}$, and

$$\varepsilon_2^1(\tilde{\mathbf{w}}, \mathscr{B}) = \sqrt{\frac{8}{29}} \left\| (\sigma^2 + \sigma + 1)\tilde{\mathbf{y}}_2 + \frac{1}{8}(\sigma^2 - 2\sigma - 3)\tilde{\mathbf{y}}_1 - \frac{1}{8}(\sigma + 5)\tilde{\mathbf{u}} \right\|_{l_2}.$$

In general, in fact, these misfits are suitable equation errors.

The error level space $\mathscr{E} = (\mathbb{R}_+)^{\mathbb{Z}_+}$, introduced in Definition 6, will be endowed with the *lexicographic total ordering*. Thus $\{\varepsilon' \geqslant \varepsilon''\} :\Leftrightarrow \{$either $\varepsilon' = \varepsilon''$ or there exists $t \in \mathbb{Z}_+$ such that $\varepsilon'_t > \varepsilon''_t$ and $\varepsilon'_{t'} = \varepsilon''_{t'}$ for $0 \leqslant t' < t\}$. The vectors ε_t being vectors $\mathbb{R}_+^n$ are themselves also endowed with the usual lexicographic ordering.

The misfit function is related, but not identical, to the widely used *prediction error criterion*. It is perhaps best to illustrate the difference for the case $q = 1$, $T = \mathbb{Z}$, and $\mathscr{B} \in \mathbf{L}$ an autonomous system. Then $\mathscr{B}$ can be described by one AR relation $p(\sigma)\mathbf{w} = (p_l \sigma^l + p_{l-1} \sigma^{l-1} + \cdots + p_0)\mathbf{w} = \mathbf{0}$, where it is assumed $p_l p_0 \neq 0$. The one step ahead prediction misfit is then given by $\dfrac{\|p(\sigma)\mathbf{w}\|_{\mathscr{K}}}{|p_l|}$ compared to $\varepsilon_l(\tilde{\mathbf{w}}, \mathscr{B}) = \dfrac{\|p(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\sqrt{\sum_{i=0}^{l} p_i^2}}$ for the misfit of Definition 6, and $\dfrac{\|p(\sigma)\mathbf{w}\|_{\mathscr{K}}}{|p_0|}$ for the backwards prediction error.

Neither the misfit function nor its total ordering are compelling choices. Indeed, there are a number of other misfit functions which are suitable for the purpose. For example, $\mathscr{E} = \mathbb{R}_+$, and as another equation error oriented misfit function:

$$\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) = \underset{t \in \mathbb{Z}_+}{\text{supremum}} \tilde{\varepsilon}_t(\tilde{\mathbf{w}}, \mathscr{B})$$
$$\text{with } \tilde{\varepsilon}_t(\tilde{\mathbf{w}}, \mathscr{B}) = \underset{a \in \mathscr{B}_t^\perp}{\text{maximum}} \frac{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\|a\|_{\mathscr{L}_t^*}}$$

could have been taken. In fact, much is to be said for using $\mathscr{E} = (\mathbb{R}_+)^{\mathbb{Z}_+}$ and the sequence $\tilde{\varepsilon}(\tilde{\mathbf{w}}, \mathscr{B})$ as misfit. Clearly this sequence is monotonically non-decreasing. When equipped with the reverse lexicographic ordering this partial ordering is in fact quite appealing. Approximate modelling in this case together with the analogous reverse lexicographic ordering for the complexity function is a topic for future research.

The same thing holds for the following behaviour misfit oriented misfit function:

$$\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) = \underset{\mathbf{w} \in \mathscr{B} \cap \mathscr{K}}{\text{infimum}} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{\mathscr{K}}.$$

Behaviour misfit oriented functions are much less localized in time than equation error oriented misfit functions and as such they have important advantages, at least in principle. However, it appears much more difficult to find the best fitting behaviour than it is to find the best fitting equations.

Note, finally, that the misfit measures used do not take into consideration the purpose for which a model will be used. As such, these models are purely descriptive in nature. Models explicitly constructed for, say, prediction, or adaptive control, will have to reflect this purpose in the choice of a suitable misfit function.

However, in the present paper the multivalued misfit function of Definition 6 will be used. It takes explicitly into consideration the lag structure of $\mathscr{B}$

and can, as shown in the next section, be computed by analyzing finite dimensional matrices only (in contrast to the other above mentioned misfit functions, which are infinite dimensional in nature, in the sense that they require the analysis of infinite dimensional operators).

24. THE COVARIANCE OPERATOR

Let $\{\tilde{\mathbf{w}}\} \in \mathscr{L}$, hence $\tilde{\mathbf{w}} = (\mathrm{col}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \ldots, \tilde{\mathbf{w}}_q)$ with $\tilde{\mathbf{w}}_i \in \mathscr{K}$ for all $i \in \mathbf{q}$. Now define for all $t, t' \in \mathbb{Z}$, the real $(q \times q)$ matrix with (i,j)th element given by $\langle \sigma^t \tilde{\mathbf{w}}_i, \sigma^{t'} \tilde{\mathbf{w}}_j \rangle_{\mathscr{K}}$. This matrix will be denoted by $|\sigma^t \tilde{\mathbf{w}}\rangle\langle \sigma^{t'} \tilde{\mathbf{w}}|$. Consider the infinite matrix

$$\Pi(\tilde{\mathbf{w}}) := \begin{bmatrix} |\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}| & |\tilde{\mathbf{w}}\rangle\langle\sigma\tilde{\mathbf{w}}| & \cdots & |\tilde{\mathbf{w}}\rangle\langle\sigma^{t'}\tilde{\mathbf{w}}| & \cdots \\ |\sigma\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}| & |\sigma\tilde{\mathbf{w}}\rangle\langle\sigma\tilde{\mathbf{w}}| & \cdots & |\sigma\tilde{\mathbf{w}}\rangle\langle\sigma^{t}\tilde{\mathbf{w}}| & \cdots \\ \vdots & \vdots & \vdots\vdots & \vdots & \vdots\vdots \\ |\sigma^t\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}| & |\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma\tilde{\mathbf{w}}| & \cdots & |\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma^{t'}\tilde{\mathbf{w}}| & \cdots \\ \vdots & \vdots & \vdots\vdots & \vdots & \vdots\vdots \end{bmatrix}$$

(case $T = \mathbb{Z}_+$).

Clearly $\Pi(\tilde{\mathbf{w}})$ is a matrix representation of the real symmetric non-negative bilinear form on $\mathscr{L}^* \times \mathscr{L}^*$ which maps (a, b) ino $\langle a(\sigma)\tilde{\mathbf{w}}, b(\sigma)\tilde{\mathbf{w}}\rangle_{\mathscr{K}}$. Call $\Pi(\tilde{\mathbf{w}})$ the *covariance operator* of $\tilde{\mathbf{w}}$. Note that if $\|\cdot\|_{\mathscr{K}}$ is shift invariant, i.e. if $\|\mathbf{k}\|_{\mathscr{K}} = \|\sigma\mathbf{k}\|_{\mathscr{K}}$, $\forall \mathbf{k} \in \mathscr{K}$ (as in Examples 1 and 3 when $T = \mathbb{Z}$, and in Example 4), then $\Pi(\tilde{\mathbf{w}})$ will be Toeplitz, i.e. $|\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma^{t'}\tilde{\mathbf{w}}| = |\sigma^{t-t'}\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}|$. This offers important computational advantages which will not be pursued here. Note finally that $\Pi(\tilde{\mathbf{w}})$ can also be viewed as a map from $\mathscr{L}^*$ into $\mathscr{L}$.

The truncation $\Pi_{t,t'}(\tilde{\mathbf{w}})$ consisting of the block rows indexed $0, 1, \ldots, t$ and the block columns indexed $0, 1, \ldots, t'$ of $\Pi(\tilde{\mathbf{w}})$ will also be considered. The truncation $\Pi_{t,t'}(\tilde{\mathbf{w}})$ with $t = t'$ will be denoted by $\Pi_t(\tilde{\mathbf{w}})$. The matrix $\Pi_t(\tilde{\mathbf{w}})$ is obviously symmetric and non-negative definite. Its kernel, viewed as a subspace of $\mathscr{L}_t^* \cong (\mathbb{R}^{1\times q})^{(t+1)}$ and its image, viewed as a subspace of $\mathscr{L}_t \cong \mathbb{R}^{q(t+1)}$, determine the most powerful unfalsified (AR) model for $\{\tilde{\mathbf{w}}\}$. It is interesting to observe that all the important invariants of the i/o and i/s/o representations of this most powerful model can be obtained directly from the rank increase of the consecutive matrices $\Pi_t(\tilde{\mathbf{w}})$.

Proposition 28. Let $\{\tilde{\mathbf{w}}\} \in \mathscr{L}$ and $\Pi(\tilde{\mathbf{w}})$ be its covariance operator. Let $R_{\tilde{\mathbf{w}}}^*$ define the most powerful unfalsified (AR) model for $\{\tilde{\mathbf{w}}\}$. Denote $\mathscr{B}^* := \mathscr{B}(R_{\tilde{\mathbf{w}}}^*)$, $\mathscr{B}_t^* := \mathscr{B}^*|_{T\cap[0,t]}$, $(\mathscr{B}^*)^\perp =: \mathscr{N}^* \subset \mathscr{L}^*$, and $(\mathscr{B}_t^*)^\perp =: \mathscr{N}_t^* \subset \mathscr{L}_t^*$. For simplicity $\tilde{\mathbf{w}}$ will be dropped in the notation for $\Pi(\tilde{\mathbf{w}})$, etc. The following hold for Π viewed as a map from $\mathscr{L}^*$ into $\mathscr{L}$.

(i) $\mathrm{im}\,\Pi = \mathscr{B}^*$, $\ker\Pi = \mathscr{N}^*$, $\mathrm{im}\,\Pi_t = \mathscr{B}_t^*$, and $\ker\Pi_t = \mathscr{N}_t^*$.

(ii) Define $\dim \mathrm{im}\,\Pi_{-1} = \dim\ker\Pi_{-1} = 0$. Then $(\dim \mathrm{im}\,\Pi_t - \dim \mathrm{im}\,\Pi_{t-1})$ is monotone non-increasing and reaches its limit in a finite number of steps. This limit equals the number of inputs in any i/o representation of $\mathscr{B}^*$. Further, $(\dim\ker\Pi_t - \dim\ker\Pi_{t-1})$ is monotone non-decreasing and reaches its limit in a finite number of steps. This limit equals the number of outputs in any i/o representation of $\mathscr{B}^*$.

(iii) Let t' be such that $(\dim \mathrm{im}\,\Pi_{t'} - \dim \mathrm{im}\,\Pi_{t'-1}) = \lim_{t\to\infty}(\dim \mathrm{im}\,\Pi_t - \dim \mathrm{im}\,\Pi_{t-1}) =: m$, or, equivalently, such that $(\dim\ker\Pi_{t'} - \dim\ker\Pi_{t'-1}) = \lim_{t\to\infty}(\dim\ker\Pi_t - \dim\ker\Pi_{t-1}) =: p$. Then the excess over these limits m and p, $\dim \mathrm{im}\,\Pi_{t'} - m(t'+1) = p(t'+1) - \dim\ker\Pi_t$, equals the dimension of the state space in any minimal state space representation of $\mathscr{B}^*$.

Proof. Clearly

$$\{a(\sigma)\tilde{\mathbf{w}} = 0\} \Leftrightarrow \{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}} = 0\} \Leftrightarrow \{a^T\Pi a = 0\}$$

with $a(\sigma) = a_0 + a_1\sigma + \cdots + a_t\sigma^t + \cdots$ identified with $a = \mathrm{col}(a_0, a_1, \ldots, a_t, \ldots)$. This shows that $\ker\Pi = \mathscr{N}^*$. The results follow from Theorem 6. □

From Proposition 28 the following algorithm for computing $R_{\tilde{\mathbf{w}}}^*$ on the basis of $\Pi(\tilde{\mathbf{w}})$ is obtained. This algorithm has the advantage over those discussed in Part II in that beyond Step 2 it requires the construction of the kernel of symmetric finite dimensional matrices only. In fact, suitably modified, it can be used to compute the most powerful t-complete model for an observed time series of finite length, by examining the $q(t+1) \times q(t+1)$ dimensional matrix $\Pi_t(\tilde{\mathbf{w}})$.

Algorithm 7

Data. $\begin{cases} \tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots, \tilde{\mathbf{w}}(t), \ldots & (\text{case } T = \mathbb{Z}_+) \\ \ldots, \tilde{\mathbf{w}}(-1), \tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots & (\text{case } T = \mathbb{Z}) \end{cases}$

with $\tilde{\mathbf{w}}_i \in \mathscr{K}, \forall i \in \mathbf{q}$.

Step 1 (*Computation of the covariance*). Compute $|\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma^{t'}\tilde{\mathbf{w}}|$ for all $t, t' \in \mathbb{Z}_+$.

Step 2 (*Determination of the lag*). Determine $t' \in \mathbb{Z}_+$ such that $(\dim \mathrm{im}\,\Pi_t(\tilde{\mathbf{w}}) - \dim \mathrm{im}\,\Pi_{t-1}(\tilde{\mathbf{w}})) = (\dim \mathrm{im}\,\Pi_{t'}(\tilde{\mathbf{w}}) - \dim \mathrm{im}\,\Pi_{t'-1}(\tilde{\mathbf{w}}))$ for $t \geqslant t'$.

Step 3. (*Determination of the system parameters*). Compute first $\mathscr{N}_0 := \ker\Pi_0(\tilde{\mathbf{w}})$, and a basis $(f_1, f_2, \ldots, f_{n_0})$ for $\mathscr{N}_0$. Now compute recursively, for $t = 1, 2, \ldots, t'$, $\mathscr{N}_{t-1}'' := \mathscr{N}_{t-1} + \sigma^*\mathscr{N}_{t-1}$, $\mathscr{N}_t := \ker\Pi_t(\tilde{\mathbf{w}})$, and a complementary basis $(f_{n_0+n_1+\cdots+n_{t-1}+1}, f_{n_0+n_1+\cdots+n_{t-1}+2}, \ldots,$

$f_{n_0+n_1+\cdots+n_{t-1}+n_t}$) for $\mathscr{N}''_{t-1}$ in $\mathscr{N}_t$.

Now consider the polynomial vectors $r_i(s)$ derived from the f_is by setting $f_i =: \text{col}(f_{i,0}, f_{i,1}, \ldots)$ with $f_{i,j} \in \mathbb{R}^q$ and defining $r_i(s) := \sum_{t \in \mathbb{Z}_+} f_{i,t}^T s^t$. Then $R(s) := \text{col}(r_1(s), r_2(s), \ldots, r_{n_0+n_1+\cdots+n_{t'}}(s))$ defines the most powerful (AR) model unfalsified by $\{\tilde{\mathbf{w}}\}$.

The approximate modelling procedure discussed in the next section is very similar in structure to Algorithm 7. Also, it is possible to refine Algorithm 7 so that it gives directly a state space representation of $R^*_{\tilde{\mathbf{w}}}$.

The covariance operator allows one to compute the misfit function (as introduced in Definition 6) between an observed time series and a model $\mathscr{B} \in \mathbf{L}$.

Proposition 29. Let $\{\tilde{\mathbf{w}}\} \in \mathscr{L}$ and $\mathscr{B} \in \mathbf{L}$. Now define, for $t \in \mathbb{Z}_+$, the $q(t+1) \times q(t+1)$ matrices P_t so that they correspond to the orthogonal projection from $\mathbb{R}^{q(t+1)}$ onto $\mathscr{N}'_t$ viewed as a subspace of $\mathbb{R}^{q(t+1)}$. Then $(\sqrt{\sigma(P_t \Pi_t(\tilde{\mathbf{w}}) P_t)}, 0, \ldots, 0)$ equals $(\eta^t_1(\tilde{\mathbf{w}}, \mathscr{B}), \eta^t_2(\tilde{\mathbf{w}}, \mathscr{B}), \ldots, \eta^t_{\tau_t}(\tilde{\mathbf{w}}, \mathscr{B}))$. Consequently, $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B})$ equals $(\sqrt{\sigma_{\max}(P_t \Pi_t(\tilde{\mathbf{w}}) P_t)} 0, \ldots, 0.)$

Proof. See Appendix P.

From the above proposition the following basic properties of the misfit function can be deduced.

Theorem 30. The misfit function introduced in Definition 6 has the following properties.

(i) $\{\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) = 0\} \Leftrightarrow \{\tilde{\mathbf{w}} \in \mathscr{B}\}$.

(ii) $0 \leqslant \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B})$ and, since σ is non-expansive on $\mathscr{K}$, $\varepsilon^k_t(\tilde{\mathbf{w}}, \mathscr{B}) \leqslant \sqrt{\sum_{i=1}^{q} \|\tilde{\mathbf{w}}_i\|^2_{\mathscr{K}}/(t+1)}$. Moreover $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B})$ is non-zero for at most p elements $t \in \mathbb{Z}_+$, where p equals the number of outputs in any i/o or i/s/o representation of $\mathscr{B}$.

(iii) Generically, there holds $\{\mathscr{B}', \mathscr{B}'' \in \mathbf{L}, \mathscr{B}' \supset \mathscr{B}''\} \Rightarrow \{\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}') \geqslant \varepsilon(\mathbf{w}, \mathscr{B}'')$ (lexicographic ordering). This genericity should be interpreted as a property of $\Pi(\tilde{\mathbf{w}})$ as explained in Appendix N.

Proof. See Appendix P.

The genericity in (iii) of the above theorem can also be interpreted in terms of $\mathscr{B}'$ and $\mathscr{B}''$. In any case, the fact that (iii) holds only generically is obviously a weakness of the misfit function used in this paper.

The assumption introduced in Section 23 that each component of the observed time series $\tilde{\mathbf{w}}_i$ belongs to the inner product space $\mathscr{K}$ is a convenient way to discuss the introduction of misfit functions. However, the assumption that $\tilde{\mathbf{w}}_i$ belongs to, say, either the space $l_2(\mathbb{R}^q; \mathbb{Z})$, or the space of almost periodic sequences, is not a particularly nice one in most applications. A much more practical idea is to assume that only the average mean square of $\tilde{\mathbf{w}}_i$ exists. In other words, it is much more reasonable to assume something like

$$\lim_{t \to \infty} \frac{1}{t+1} \sum_{t'=0}^{t} \|\tilde{\mathbf{w}}_i(t')\|^2 < \infty \text{ (case } T = \mathbb{Z}_+).$$

Unfortunately, the space of sequences for which this limit exists is not even a linear space and is hence not suitable as a $\mathscr{K}$-space.

Most of the error criteria which will be discussed, in particular the one introduced in Definition 6 and which will be used in Algorithms 8 and 9, are entirely based on the equation error $\|a(\sigma)\tilde{\mathbf{w}}\|$ as a measure for the lack of fit between the observation $\tilde{\mathbf{w}}$ and the AR relation $a \in \mathbb{R}^{1 \times q}[s]$. This equation error, however, can be defined if the correlations

$$\lim_{t' \to \infty} \frac{1}{t'+1} \sum_{t''=0}^{t'} \tilde{\mathbf{w}}_i(t''+t) \tilde{\mathbf{w}}_j(t'') \qquad (\text{case } T = \mathbb{Z}_+)$$

$$\lim_{t' \to \infty} \frac{1}{2t'+1} \sum_{t''=-t'}^{t'} \tilde{\mathbf{w}}_i(t''+t) \tilde{\mathbf{w}}_j(t'') \qquad (\text{case } T = \mathbb{Z})$$

are assumed to exist $\forall t \in \mathbb{Z}$, $i, j \in \mathbf{q}$. In that case it is clear that

$$\lim_{t \to \infty} \frac{1}{t+1} \sum_{t'=0}^{t} (a(\sigma)\tilde{\mathbf{w}})(t')(b(\sigma)\tilde{\mathbf{w}})(t') \qquad (\text{case } T = \mathbb{Z}_+)$$

$$\lim_{t \to \infty} \frac{1}{2t+1} \sum_{t'=-t}^{t} (a(\sigma)\tilde{\mathbf{w}})(t')(b(\sigma)\tilde{\mathbf{w}})(t') \qquad (\text{case } T = \mathbb{Z})$$

is well-defined for all $a, b \in \mathbb{R}^{1 \times q}[s]$. With a slight abuse of notation this limit can then be denoted as $\langle a(\sigma)\tilde{\mathbf{w}}, b(\sigma)\tilde{\mathbf{w}} \rangle_{\mathscr{K}}$, with the obvious resulting meaning for $\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}$. With this definition of $\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}$ and the resulting interpretation of the misfit in Definition 6 and the covariance operator the sequel can also be used (with a few exceptions where the most powerful unfalsified model is referred to) under the assumption that the correlations mentioned all exist. Note, in particular, that this assumption will be satisfied with probability one whenever $\tilde{\mathbf{w}}$ is a realization of an ergodic second order stochastic process.

25. AN ALGORITHM FOR APPROXIMATE MODELLING OF AN OBSERVED TIME SERIES

In this section the main results of this paper will be given. These consist of two algorithms for approximate modelling, i.e. identification of a multivariable time series. First the algorithms will be outlined. They are based on the methodology put forward in Section 20. For the complexity and misfit the definitions introduced in Sections 22 and 23 will be used. However, since the ordering introduced there on the complexity is only a partial

ordering, it is not directly employable for minimization and a variation of this ordering relation will be used. The algorithms themselves examine the covariance operator of the observed time series, as introduced and discussed in Section 24. The analysis of the covariance operator is a least squares type of analysis, along the lines of the algorithm put forward in Section 21.

The setting to be used is formally defined as follows.

- *The *model set* $\mathscr{M}$ is $\mathbf{L}$, the collection of all linear shift invariant closed subspaces of $(\mathbb{R}^q)^T$, equipped with the topology of pointwise convergence.
- *The *measurement set* is $\mathscr{Z}$, the collection of all singletons $\{\tilde{\mathbf{w}}\}$ with $\tilde{\mathbf{w}} \in (\mathbb{R}^q)^T$ such that each component $\tilde{\mathbf{w}}_i \in \mathscr{K}$ for all $i \in \mathbf{q}$.
- *The *complexity* is defined as $\mathbf{c}: \mathbf{L} \to \mathscr{C} := [0,1]^{\mathbb{Z}_+}$ with $\mathbf{c}_t(\mathscr{B}) := \dfrac{\dim \mathscr{B}_t}{q(t+1)}$. The *lexicographic ordering* on $\mathscr{C} = [0,1]^{\mathbb{Z}_+}$ will be used, i.e. $\{\mathbf{c}' \geqslant \mathbf{c}''\} \Leftrightarrow \{$either $\mathbf{c}' = \mathbf{c}''$ or there exists $t \in \mathbb{Z}_+$ such that $\mathbf{c}'_t \geqslant \mathbf{c}''_t$ and $\mathbf{c}'_{t'} = \mathbf{c}''_{t'}$ for $0 \leqslant t' < t\}$. Clearly this defines a total ordering on $\mathscr{C}$ which is isotone with regard to the partial ordering studied in Section 23.
- *The *misfit* is defined as $\varepsilon: \mathscr{Z} \times \mathbf{L} \to \mathscr{E} := (\mathbb{R}_+)^{\mathbb{Z}_+}$ with $\varepsilon_t^k(\tilde{\mathbf{w}}, \mathscr{B}) := \min_{\mathscr{N}' \supset \mathscr{N}'_t} \dim \mathscr{N}' \leqslant \tau_t - k - 1 \max_{a \in \mathscr{N}'_t} \dfrac{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\|a\|^*_{\mathscr{L}_t}}$ where $\mathscr{N}'_t := \mathscr{B}_t^{\perp} \cap (\mathscr{B}_{t-1}^{\perp} + \sigma^* \mathscr{B}_{t-1}^{\perp})^{\perp}$. The *lexicographic ordering* on $(\mathbb{R}_+^9)^{\mathbb{Z}_+}$ will be used, i.e. $\{\varepsilon' \geqslant \varepsilon''\} :\Leftrightarrow \{$either $\varepsilon' = \varepsilon''$ or there exists $t \in \mathbb{Z}_+$ such that $\varepsilon'_{t'} = \varepsilon''_{t'}$ for $0 \leqslant t' < t$ and $\varepsilon'_t > \varepsilon''_t\}$. On $\mathbb{R}^9$ itself we use the lexicographic ordering. Clearly this defines a total ordering on $\mathscr{E}$.

It is important to note that using the lexicographic ordering on the complexity and the misfit level space shows a great preference for short lag relations. Structured, non-complex systems have in this definition a relatively large number of relations with short lags, and a good fit between data and model means that the (AR) relations with short lags are put in evidence by the data with small equation error. This preference for short lag relations is reasonable but not compelling. It is reasonable because an equation involving a short lag inspires *prima facie* more confidence than one with a large lag. Indeed, in order to use the latter (for prediction purposes for example) data from the far past will have to be employed which is, in principle, undesirable. Similarly, everything else being the same, it makes sense to prefer models which fit the short lag relations well but the long lag relations poorly over those where the converse is true. However, this prejudice for short order relations is not compelling: why should an accurate (AR) relation of lag 1 and a very poor one of lag 4 be preferred above a model consisting of two moderately accurate relations, one of lag 1 and one of lag 2?

25.1. *Modelling with limited complexity*

In our first approximate modelling algorithm, assume that the *maximal admissible complexity*, $\mathbf{c}^{\text{adm}} = (\mathbf{c}_0^{\text{adm}}, \mathbf{c}_1^{\text{adm}}, \ldots, \mathbf{c}_t^{\text{adm}}, \ldots) \in [0,1]^{\mathbb{Z}_+}$ is given. It is important to realize what this means. In effect it implies that the model is required to contain at least a certain number of (independent) zeroth order lag (i.e. static) relations. If it has this minimal number of zeroth order lag relations, then it must have a certain number of first order lag relations etc. A typical choice for $\mathbf{c}^{\text{adm}}$ would be $\mathbf{c}_t^{\text{adm}} = 1$ for $t < l$ and $\mathbf{c}_t^{\text{adm}} = \dfrac{m}{q} + \left(1 - \dfrac{m}{q}\right)\dfrac{l}{t+1}$ for $t \geqslant l$. This is equivalent to requiring $p = q - m$ relations, all of maximal lag l.

Note that $\mathbf{d}_t(\mathscr{B}) = q(t+1)\mathbf{c}_t(\mathscr{B})$ will be integer valued and monotone non-decreasing for whatever model $\mathscr{B} \in \mathbf{L}$ is chosen. Consequently one may as well assume that $\mathbf{c}^{\text{adm}}$ has these properties to start with. Otherwise, simply go through the substitutions

$$\mathbf{c}_t^{\text{adm}} \to \frac{\text{ent}(q(t+1)\mathbf{c}_t^{\text{adm}})}{q(t+1)} \to \frac{\min_{t' \geqslant t} q(t'+1)\mathbf{c}_{t'}^{\text{adm}}}{q(t+1)}.$$

The algorithm examines the SVs of the consecutive truncated correlation matrices and uses their singular vectors in order to decide how many and which (AR) relations of a given lag should be honoured at each stage. The main part of the algorithm is recursive. It has been structured as follows. First define the variables involved in the algorithm, then show the initiation step, the next step computation, and finally the termination rule. The algorithm works with any data set $\tilde{\mathbf{w}}$ with $\tilde{\mathbf{w}}_i \in \mathscr{K}$ for all $i \in \mathbf{q}$. However, since considerable simplifications occur in the generic case, the resulting implications will be mentioned explicitly. Genericity should be interpreted as a property of $\Pi(\tilde{\mathbf{w}})$ in the sense explained in Appendix N.

In the algorithm

$$\mathscr{L}_t^* := \mathbb{R}_t^{1 \times q}[s] := \{a(s) \in \mathbb{R}^{1 \times q}[s] \mid \partial(a) \leqslant t\}$$

will be identified with $\mathbb{R}^{q(t+1)}$ by identifying $a_0 + a_1 s + \cdots + a_t s^t = a(s) \in \mathbb{R}_t^{1 \times q}[s]$ with $a = \text{col}\,(a_0^T, a_1^T, \ldots, a_t^T) \in \mathbb{R}^{q(t+1)}$. The notation will indicate how $a \in \mathscr{L}_t^*$ should be viewed.

Algorithm 8
Input:

$$\textit{Data.}\ \begin{cases}\tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots, \tilde{\mathbf{w}}(t), \ldots & (\text{case } T = \mathbb{Z}_+)\\ \ldots, \tilde{\mathbf{w}}(-1), \tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots & (\text{case } T = \mathbb{Z})\end{cases}$$

with $\tilde{\mathbf{w}}_i \in \mathscr{K}, \forall i \in \mathbf{q}$;
$\mathbf{c}^{\text{adm}} = (\mathbf{c}_0^{\text{adm}}, \mathbf{c}_1^{\text{adm}}, \ldots, \mathbf{c}_t^{\text{adm}}, \ldots)$, $0 \leqslant \mathbf{c}_t^{\text{adm}} \leqslant 1$, with $\mathbf{d}_t^{\text{adm}} := q(t+1)\mathbf{c}_t^{\text{adm}}$ integer valued and monotone non-decreasing in t.

Introduce also the function $\tilde{\mathbf{d}}$: $\mathbb{Z}_+ \to \mathbb{Z}$ deduced from $\mathbf{c}^{\text{adm}}$ as follows:

$$\tilde{\mathbf{d}}_t := \min(\mathbf{d}_t^{\text{adm}}, \tilde{\mathbf{d}}_{t-1} + (\tilde{\mathbf{d}}_{t-1} - \tilde{\mathbf{d}}_{t-2}))$$

(define $\tilde{\mathbf{d}}_{-1} = 0$ and $\tilde{\mathbf{d}}_{-2} = -q$). This $\tilde{\mathbf{d}}$ is non-negative, monotone non-decreasing, and concave. In fact, it is the largest (in the sense of the lexicographic partial ordering) concave function satisfying $\tilde{\mathbf{d}} \leqslant \mathbf{d}^{\text{adm}}$. From $\tilde{\mathbf{d}}$ deduce $\tilde{\mathbf{k}}$: $\mathbb{Z}_+ \to \{0, 1, \ldots, q\}$ as follows:

$$\begin{aligned}\tilde{\mathbf{k}}_t &:= \max(0, (\tilde{\mathbf{d}}_{t-1} - \tilde{\mathbf{d}}_{t-2}) - (\mathbf{d}_t^{\text{adm}} - \tilde{\mathbf{d}}_{t-1}))\\ &= \tilde{\mathbf{d}}_{t-1} - \tilde{\mathbf{d}}_{t-2} - (\tilde{\mathbf{d}}_t - \tilde{\mathbf{d}}_{t-1}).\end{aligned}$$

Preliminary step (*Examination of the trivial model*). If $\mathbf{c}_0^{\text{adm}} = 0$ then $\mathbf{w} = \mathbf{0}$ defines the optimal approximate model and the algorithm is terminated.

Recursive part. (*Computation of the* (*AR*) *relations of the approximate model*).

(1) Declaration of the variables and their significance.
$*\Pi_t(\tilde{\mathbf{w}}) \in \mathbb{R}^{q(t+1)\times q(t+1)}$: the tth order truncated correlation matrix of the data.
$*n_0^t, n_1^t, \ldots, n_t^t \in \mathbb{Z}_+$: the number of zeroth, first, …, tth order (AR) relations honoured by the approximate model at stage t. [Generically, $n_i^t = \tilde{\mathbf{k}}_t \forall 0 \leqslant i < t$.]
$*r_{i,j}^t(s) \in \mathbb{R}_i^{1\times q}[s]$, $i = 0, 1, \ldots, t$, $j \in \mathbf{n}_i^t$: the (AR) relations honoured by the approximate model at stage t.
$*\mathscr{N}_t \subset \mathscr{L}_t^* = \mathbb{R}^{q(t+1)}$: the span of these (AR) relations and their shifts.
$*m_t$: the dimension of $\mathscr{N}_t$. There holds $m_t = \sum_{i=0}^{t} (t-i+1)n_i^t$. [Generically, $m_t = \sum_{i=0}^{t} (t-i+1)\tilde{\mathbf{k}}_i$.]
$*f_1, f_2, \ldots, f_{m_t} \in \mathbb{R}^{q(t+1)}$: an orthogonal basis for $\mathscr{N}_t$.

(2) Initialization.
*Compute $\Pi_0(\tilde{\mathbf{w}}) = |\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}|$.
*Examine its SVs $\sigma_1^0 \geqslant \sigma_2^0 \geqslant \cdots \geqslant \sigma_{r_0}^0 > 0$. [Generically, $r_0 = q$.]
*Define k_0 as follows. If $q\mathbf{c}_0^{\text{adm}} \geqslant r_0$, set $k_0 = r_0$. Otherwise, choose k_0 such that $\sigma_{k_0}^0 > \sigma_{k_0+1}^0 = \sigma_{q\mathbf{c}_0^{\text{adm}}+1}^0$. [Generically, $k_0 = \tilde{k}_0 = q\mathbf{c}_0^{\text{adm}}$.]
*Compute the corresponding left singular vectors $u_{k_0+1}, u_{k_0+2}, \ldots, u_q \in \mathbb{R}^q$ of $\Pi_0(\tilde{\mathbf{w}})$.
*Define

$$\begin{aligned}n_0^0 &= q - k_0\\ r_{0,j}^0 &\cong u_{k_0+j}, r_{0,j}^0 \in \mathbb{R}_0^{1\times q}[s], j \in \mathbf{n}_0^0\\ \mathscr{N}_0 &= \operatorname{span}(r_{0,j}^0, j \in \mathbf{n}_0^0)\\ m_0 &= n_0^0\\ f_j &= r_{0,j}^0, f_j \in \mathbb{R}^q, j \in \mathbf{m}_0\end{aligned}$$

*Examine the termination rule

(3) Next step computation.
*Compute $|\tilde{\mathbf{w}}\rangle\langle\sigma^t\tilde{\mathbf{w}}|, |\sigma\tilde{\mathbf{w}}\rangle\langle\sigma^t\mathbf{w}|, \ldots, |\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma^t\tilde{\mathbf{w}}|$ and form, from these and $\Pi_{t-1}(\tilde{\mathbf{w}})$, $\Pi_t(\tilde{\mathbf{w}})$.
*Compute $\mathscr{N}_{t-1} + \sigma^*\mathscr{N}_{t-1} = \mathscr{M}_t \subset \mathscr{L}_t^*$ and extend $f_1, f_2, \ldots, f_{m_{t-1}}$ with $f_{m_{t-1}+1}, f_{m_{t-1}+2}, \ldots, f_{p_t}$ so that $(f_i \in \mathbb{R}^{q(t+1)}, i \in \mathbf{p}_t)$, forms an orthonormal basis for $\mathscr{M}_t$. Note that

$$\begin{aligned}p_t &= m_{t-1} + n_0^{t-1} + n_1^{t-1} + \cdots + n_{t-1}^{t-1}\\ &= \sum_{i=0}^{t-1} (t+1-i)n_i^{t-1}.\end{aligned}$$

[Generically, $p_t = \sum_{i=0}^{t-1} (t+1-i)\tilde{\mathbf{k}}_i$.]
*Compute $\Pi_t' = P_t\Pi_t(\tilde{\mathbf{w}})P_t$ where $P_t = I - \sum_{i\in\mathbf{p}_t} f_i f_i^T$. Note that $\mathscr{M}_t = \ker P_t \subset \ker \Pi_t'$. [Generically, $\ker P_t = \ker \Pi_t' = \mathscr{M}_t$.]
*Compute its SVs $\sigma_1^t \geqslant \sigma_2^t \geqslant \ldots \geqslant \sigma_{r_t}^t > 0$. Clearly $r_t \leqslant q(t+1) - p_t$. [Generically, $r_t = q(t+1) - p_t$.]
*Define the following selection rule for d_t:
If $q(t+1)\mathbf{c}_t^{\text{adm}} \geqslant r_t$, set $d_t = r_t$.
Otherwise, choose d_t such that $\sigma_{d_t}^t > \sigma_{d_t+1}^t = \sigma_{q(t+1)\mathbf{c}_t^{\text{adm}}+1}^t$. [Generically, $d_t \cong \tilde{\mathbf{d}}_t$.]
Set $n_t = q(t+1) - d_t - p_t$. [Generically, $n_t = \tilde{\mathbf{k}}_t$.]
*Compute the left singular vectors corresponding to the smallest SVs of Π_t' $u_{d_t+1}, u_{d_t+2}, \ldots, u_{d_t+n_t}$, $f_1, f_2, \ldots, f_{p_t} \in \mathbb{R}^{q(t+1)}$ of Π_t'.
Let $\mathscr{U}_t := \operatorname{span}(u_{d_t+j}, j \in \mathbf{n}_t)$.
**If $\mathscr{U}_t \cap \mathscr{L}_{t-1}^* = \{0\}$ [generically this will be the case] then set
$n_i^t = n_i^{t-1}$ for $i = 0, 1, \ldots, t-1$;
$n_t^t = n_t$;
$r_{i,j}^t(s) = r_{i,j}^{t-1}(s)$ for $j \in \mathbf{n}_i^{t-1}$ for $i = 0, 1, \ldots, t-1$;
$r_{t,j}^t(s) = u_{d_t+j}$, $r_{t,j}^t \in \mathbb{R}_t^{1\times q}[s]$ for $j \in \mathbf{n}_t$.
**If $\mathscr{U}_t \cap \mathscr{L}_{t-1}^* \neq 0$ to observe that, as a consequence of the fact that σ is non-expansive on $\mathscr{K}$, $s(\mathscr{U}_t \cap \mathscr{L}_{t-1}^*) \subset \mathscr{U}_t$. Now apply the procedure STRUCTURE (see 4) to $\mathscr{U}_t$ in order to find an orthonormal basis for $\mathscr{U}_t$. Note that since $\mathscr{U}_t$ is

already orthogonal to $\mathscr{M}_t$, this will yield an orthonormal basis for $\mathscr{M}_t + \mathscr{U}_t$. This returns $\bar{n}_0$, $\bar{n}_1, \ldots, \bar{n}_t \in \mathbb{Z}_+$ and $p_j^i(s) \in \mathbb{R}_i^{1\times q}[s]$, $i = 0, 1, \ldots, t$; $j \in \bar{\mathbf{n}}_i$. Set
$n_i^t = n_i^{t-1} + \bar{n}_i$ for $i = 0, 1, \ldots, t-1$;
$n_t^t = \bar{n}_t$;
$r_{i,j}^t(s) = r_{i,j}^{t-1}(s)$ for $j \in \mathbf{n}_i^{t-1}$, $i = 0, 1, \ldots, t-1$;
$r_{i,n_i^{t-1}+j}^t(s) = p_j^i(s)$ for $j \in \bar{\mathbf{n}}_i$, $i = 0, 1, \ldots, t-1$;
$r_{t,j}^t(s) = p_j^t(s)$ for $j \in \bar{\mathbf{n}}_t$.

*Further, set
$\mathscr{N}_t = \text{span}(u_{d_t+j}, f_i; j \in \mathbf{n}_t, i \in \mathbf{p}_t)$
$= \mathscr{M}_t + \mathscr{U}_t$;
$m_t = p_t + n_t$;
$f_{p_t+j} = u_{d_t+j}, j \in \mathbf{n}_t$.
*Examine the termination rule.

(4) Procedure STRUCTURE.
This procedure [which generically will never have to be invoked] streamlines and orthonormalizes a set of (AR) relations.

Let $\mathscr{N}_t \subset \mathscr{L}_t^*$ be such that $s(\mathscr{N}_t \cap \mathscr{L}_{t-1}^*) \subset \mathscr{N}_t$. It computes
$\bar{n}_i = \dim \mathscr{N}_i'$ with $\mathscr{N}_i' = (\mathscr{N}_t \cap \mathscr{L}_i^*) \cap (\mathscr{N}_t \cap \mathscr{L}_{i-1}^* + s(\mathscr{N}_t \cap \mathscr{L}_{i-1}^*))^\perp$ for $i = 0, 1, \ldots, t$ and $p_j^i(s) \in \mathbb{R}_j^{1\times q}[s]$ such that $(p_j^i, i \in \bar{n}_j)$ is an orthonormal basis for $\mathscr{N}_i'$ for $i = 0, 1, \ldots, t$.

(5) Termination rule.
The termination rule examines two things: an inequality involving the most powerful unfalsified (AR) model (of course, the approximate model will always contain the most powerful unfalsified model, since this causes no augmentation of the error) and an inequality involving $\mathbf{c}^{\text{adm}}$. Assume that a t' has already been computed as defined for example in Step 1 of Algorithm 3 of Part II. [Generically t' will be zero].
*If $t < t'$, then augment $t \to t+1$ and proceed with the recursive step.
*If $t \geqslant t'$ examine whether

$$q(t''+1) - (t''+1)n_0^t - t''n_1^t - \cdots - (t''-t+1)n_t^t \leqslant q(t''+1)\mathbf{c}_{t''}^{\text{adm}}$$

holds for all $t'' > t$.
**If it does not hold, augment $t \to t+1$ and proceed with the next recursive step.
**Otherwise terminate the algorithm.

Output:
The result of this computation is a set of polynomials $r_{i,j}^t(s) \in \mathbb{R}^{1\times q}[s]$, $i = 0, 1, \ldots, t$, $j \in \mathbf{n}_i^t$ with t the stage at which the algorithm was terminated.

Define

$$R^* := \text{col}(r_{0,1}^t, r_{0,2}^t, \ldots, r_{0,n_0^t}^t, r_{1,1}^t, r_{1,2}^t, \ldots, r_{1,n_1^t}^t, \ldots, r_{t,1}^t, r_{t,2}^t, \ldots, r_{t,n_t^t}^t)$$

and

$$\mathscr{B}^* := \mathscr{B}(R^*).$$

This ends the description of Algorithm 8.

We emphasize that $\mathscr{B}^*$ is *always* defined and that R^* will be obtained in a *finite* number of steps.

As will be seen in the next theorem, $\mathscr{B}^*$ is (only) generically the optimal approximate model. The finiteness of the algorithm should be understood in the sense that the recursive step of the computation will have to be invoked at most q times. However an *a priori* upper bound for the stage at which the algorithm can be terminated cannot be given. [Generically the algorithm will terminate at stage t if

$$\mathfrak{d}_{t+1} - \mathfrak{d}_t = \lim_{t' \to \infty} (\mathfrak{d}_{t'+1} - \mathfrak{d}_{t'}) = \min_{t'}(\mathfrak{d}_{t'+1} - \mathfrak{d}_{t'}).]$$

To see that the recursive step will have to be invoked at most q times, observe that each time it is invoked, it will lead to an augmentation of the total number of (AR) relations, $n_0^t + n_1^t + \cdots + n_t^t$, with at least one. Now if $n_0^t + n_1^t + \cdots + n_t^t = q$, then, for $t' \geqslant t$, there holds

$$q(t'+1) - (t'+1)n_0^t - t'n_1^t - \cdots - (t'-t+1)n_t^t = q(t+1) - (t+1)n_0^t - tn_1^t - \cdots - n_t^t \leqslant q(t+1)\mathbf{c}_t^{\text{adm}} \leqslant q(t'+1)\mathbf{c}_{t'}^{\text{adm}}.$$

This shows that the termination rule will automatically be satisfied.

Theorem 31. Let $\tilde{\mathbf{w}}$ and $\mathbf{c}^{\text{adm}}$ be given and $\mathscr{B}^*$ be computed as in Algorithm 8. Then there holds:

(i) $\mathbf{c}(\mathscr{B}^*) \leqslant \mathbf{c}^{\text{adm}}$;

and, generically in $\Pi(\tilde{\mathbf{w}})$,

(ii) $\{\mathscr{B} \in \mathbf{L}, \mathbf{c}(\mathscr{B}) \leqslant \mathbf{c}^{\text{adm}}\} \Rightarrow \{\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^* (\leqslant \varepsilon(\tilde{\mathbf{w}}, \mathscr{B})\}$;

(iii) $\{\mathscr{B} \in \mathbf{L}, \mathbf{c}(\mathscr{B}) \leqslant \mathbf{c}^{\text{adm}}, \varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) = \varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^*)\} \Rightarrow \{\mathscr{B} = \mathscr{B}^*\}$;

(iv) $\mathbf{c}_t(\mathscr{B}^*) = \dfrac{\mathfrak{d}_t}{q(t+1)}$ for $t \in \mathbb{Z}_+$.

Proof. See Appendix P.

Note that it follows from Theorem 31 that for a generic data sequence $\tilde{\mathbf{w}}$, Algorithm 8 will generate, with $\mathscr{B}^*$, the optimal approximate (AR) model (in

the sense of the first methodology of Section 20) with complexity limited by $\mathbf{c}^{\text{adm}}$.

25.2 *Modelling with limited misfit*

In the second approximate modelling algorithm, assume that the maximal tolerated misfit, $\varepsilon^{\text{tol}} = (\varepsilon_0^{\text{tol}}, \varepsilon_1^{\text{tol}}, \ldots, \varepsilon_t^{\text{tol}}, \ldots) \in (\mathbb{R}_+)^{\mathbb{Z}_+}$ is given. We will assume that each element $\varepsilon_t^{\text{tol}}$ is of the form $\alpha_t \operatorname{col}(1, 1, \ldots, 1)$. By slight abuse of notation we will also denote α_t by $\varepsilon_t^{\text{tol}}$. A typical choice for ε^{tol} would be $\varepsilon_t^{\text{tol}} = \varepsilon_{\max}, \forall t \in \mathbb{Z}_+$: all accepted (AR) relations are required to be corroborated by the data within a given required accuracy. Another suitable choice would be $\varepsilon_t^{\text{tol}} = \varepsilon_{\max}$ for $0 \leqslant t \leqslant l$ and $\varepsilon_t^{\text{tol}} = 0$ for $t > l$: only exact (AR) relations of large lag are then acceptable. For algorithm 9 the same notation as in Algorithm 8 will be used.

Algorithm 9

Input:

$$\textit{Data.} \begin{cases} \tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots, \tilde{\mathbf{w}}(t), \ldots & (\text{case } T = \mathbb{Z}_+) \\ \ldots, \tilde{\mathbf{w}}(-1), \tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \ldots & (\text{case } T = \mathbb{Z}) \end{cases}$$

with $\tilde{\mathbf{w}}_i \in \mathscr{K}, \forall i \in \mathbf{q}$.

$$\varepsilon^{\text{tol}} = (\varepsilon_0^{\text{tol}}, \varepsilon_1^{\text{tol}}, \ldots, \varepsilon_t^{\text{tol}}, \ldots), \varepsilon_t^{\text{tol}} \geqslant 0.$$

Recursive part (Computation of the (AR) relations of the approximate model).

(1) Declaration of the variables and their significance.

*$\Pi_t(\tilde{\mathbf{w}}) \in \mathbb{R}^{q(t+1) \times q(t+1)}$: the tth truncated correlation matrix of the data data.

*$n_0^t, n_1^t, \ldots, n_t^t \in \mathbb{Z}_+$: the number of 0th, first, ..., tth order (AR) relations honoured by the approximate model at stage t.

*$r_{i,j}^t(s) \in \mathbb{R}_i^{1 \times q}[s]$, $i = 0, 1, \ldots, t$, $j \in \mathbf{n}_i^t$: the (AR) relations honoured by the approximate model at stage t.

*$\mathscr{N}_t \subset \mathscr{L}_t^* = \mathbb{R}^{q(t+1)}$: the span of these (AR) relations and their shifts.

*m_t: the dimension of $\mathscr{N}_t$; there holds: $m_t = \sum_{i=0}^{t} (t - i + 1) n_i^t$.

*$f_1, f_2, \ldots f_{m_t} \in \mathbb{R}^{q(t+1)}$: an orthogonal basis for $\mathscr{N}_t$.

(2) Initialization.

*Compute $\Pi_0(\tilde{\mathbf{w}}) = |\tilde{\mathbf{w}}\rangle\langle\tilde{\mathbf{w}}|$.

*Examine its SVs $\sigma_1^0 \geqslant \sigma_2^0 \geqslant \cdots \geqslant \sigma_{r_0}^0 > 0$. [Generically, $r_0 = q$.]

**if $\varepsilon_0^{\text{tol}} \geqslant \sqrt{\sigma_1^0}$, set $\mathscr{B}^* = \{\mathbf{0}\}$ and the algorithm is terminated.

*Otherwise, define k_0 as follows:

**if $\varepsilon_0^{\text{tol}} < \sqrt{\sigma_{r_0}^0}$, set $k_0 = r_0$;

**otherwise, choose k_0 such that $\sqrt{\sigma_{k_0}^0} > \varepsilon_0^{\text{tol}} \geqslant \sqrt{\sigma_{k_0+1}^0}$.

*Compute the corresponding left singular vectors $u_{k_0+1}, u_{k_0+2}, \ldots, u_q \in \mathbb{R}^q$ of $\Pi_0(\tilde{\mathbf{w}})$.

*Define $n_0^0 = q - k_0$

$$r_{0,j}^0 \cong u_{k_0+j}, r_{0,j}^0 \in \mathbb{R}_0^{1 \times q}[s], j \in \mathbf{n}_0^0$$

$$\mathscr{N}_0 = \operatorname{span}(r_{0,j}^0, j \in \mathbf{n}_0^0)$$

$$m_0 = n_0^0$$

$$f_j = r_{0,j}^0, f_j \in \mathbb{R}^q, j \in \mathbf{m}_0$$

(3) Next step computation.

*Compute $|\tilde{\mathbf{w}}\rangle\langle\sigma^t\tilde{\mathbf{w}}|, |\sigma\tilde{\mathbf{w}}\rangle\langle\sigma^t\mathbf{w}|, \ldots, |\sigma^t\tilde{\mathbf{w}}\rangle\langle\sigma^t\tilde{\mathbf{w}}|$ and, from these and $\Pi_{t-2}(\tilde{\mathbf{w}})$, form $\Pi_t(\tilde{\mathbf{w}})$.

*Compute $\mathscr{N}_{t-1} + \sigma^*\mathscr{N}_{t-1} = \mathscr{M}_t \subset \mathscr{L}_t^*$ and extend $f_1, f_2, \ldots, f_{m_{t-1}}$ with $f_{m_{t-1}+1}, f_{m_{t-1}+2}, \ldots, f_{p_t}$ so that $(f_i \in \mathbb{R}^{q(t+1)}, i \in \mathbf{p}_t)$ forms an orthonormal basis for $\mathscr{M}_t$. Note that

$$p_t = m_{t-1} + n_0^{t-1} + n_1^{t-1} + \cdots + n_{t-1}^{t-1} = \sum_{i=0}^{t-1} (t + 1 - i) n_i^{t-1}.$$

*Compute $\Pi_t' = P_t \Pi_t(\tilde{\mathbf{w}}) P_t$ where $P_t = I - \sum_{i \in \mathbf{p}_t} f_i f_i^T$.

Note that $\mathscr{M}_t = \ker P_t \subset \ker \Pi_t'$. [Generically, $\ker P_t = \ker \Pi_t' = \mathscr{M}_t$.]

*Compute its SVs $\sigma_1^t \geqslant \sigma_2^t \geqslant \cdots \geqslant \sigma_{r_t}^t > 0$. Clearly $r_t \leqslant q(t+1) - p_t$. [Generically, $r_t = q(t+1) - p_t$.]

*Define the following selection rule for d_t:

**if $\varepsilon_t^{\text{tol}} \geqslant \sqrt{\sigma_1^t}$, set $d_t = 1$;

**if $\varepsilon_t^{\text{tol}} < \sqrt{\sigma_{r_t}^t}$, set $d_t = r_t$;

**otherwise, choose k_t such that $\sqrt{\sigma_{d_t}^t} > \varepsilon_t^{\text{tol}} \geqslant \sqrt{\sigma_{d_t+1}^t}$.

*Set $n_t = q(t+1) - d_t - p_t$.

**if $\sum_{i=0}^{t-1} n_i^{t-1} + n_t > q$, invoke the procedure TERMINATE;

**otherwise, proceed as follows:

*Compute the left singular vectors

$$u_{d_t+1}, u_{d_t+2}, \ldots, u_{d_t+n_t}, f_1, f_2, \ldots, f_{p_t} \in \mathbb{R}^{q(t+1)}$$

corresponding to the smallest SVs of Π_t'.

Let $\mathscr{U}_t := \operatorname{span}(u_{k_t+j}, j \in \mathbf{n}_t)$.

**If $\mathscr{U}_t \cap \mathscr{L}_{t-1}^* = \{0\}$ [generically this will be the case] then set

$$n_i^t = n_i^{t-1} \text{ for } i = 0, 1, \ldots, t-1;$$

$$n_t^t = n_t;$$

$$r_{i,j}^t(s) = r_{i,j}^{t-1}(s) \text{ for } j \in \mathbf{n}_i^{t-1}, i = 0, 1, \ldots, t-1;$$

$$r_{t,j}^t(s) = u_{d_t+j}, r_{t,j}^t \in \mathbb{R}_t^{1 \times q}[s] \text{ for } j \in \mathbf{n}_t.$$

**If $\mathscr{U}_t \cap \mathscr{L}_{t-1}^* \neq 0$ observe that, as a consequence of the fact that σ is non-expansive on $\mathscr{K}$,

$s(\mathscr{U}_t \cap \mathscr{L}^*_{t-1}) \subset \mathscr{U}_t$. Now apply the procedure STRUCTURE (see 4) to $\mathscr{U}_t$ in order to find an orthonormal basis for $\mathscr{U}_t$. Note that since $\mathscr{U}_t$ is already orthogonal to $\mathscr{M}_t$, this will yield an orthonormal basis for $\mathscr{M}_t + \mathscr{U}_t$. This returns n_0, $n_1, \ldots, n_t \in \mathbb{Z}_t$ and $p^i_j(s) \in \mathbb{R}^{1\times q}_i[s]$, $i = 0, 1, \ldots, t$; $j \in \bar{\mathbf{n}}_i$. Set

$n^t_i = n^{t-1}_i + \bar{n}_i$ for $i = 0, 1, \ldots, t-1$;

$n^t_t = \bar{n}_t$;

$r^t_{i,j}(s) = r^{t-1}_{i,j}(s)$ for $j \in \mathbf{n}^{t-1}_i$, $i = 0, 1, \ldots, t-1$;

$r^t_{i,n^{t-1}_i + j}(s) = p^i_j(s)$ for $j \in \bar{\mathbf{n}}_i$, $i = 0, 1, \ldots, t-1$;

$r^t_{t,j}(s) = p^t_j(s)$ for $j \in \bar{\mathbf{n}}_t$.

*Further set

$$\mathscr{N}_t = \operatorname{span}(u_{d_t+j}, f_i; j \in \mathbf{n}_t, i \in \mathbf{p}_t) = \mathscr{M}_t + \mathscr{U}_t;$$

$m_t = p_t + n_t$;

$f_{p_t+j} = u_{d_t+j}, j \in \mathbf{n}_t$.

*If $\sum_{i=0}^{t-1} n^{t-1}_i + n_t = q$, terminate the algorithm; otherwise augment $t \to t+1$ and proceed with the next recursive step.

(4) Procedure STRUCTURE.
This procedure [which generically will never have to be invoked] streamlines and orthonormalizes a set of (AR) relations.

Let $\mathscr{N}_t \subset \mathscr{L}^*_t$ be such that $s(\mathscr{N}_t \cap \mathscr{L}^*_{t-1}) \subset \mathscr{N}_t$. It computes $\bar{n}_i = \dim \mathscr{N}'_i$ with

$$\mathscr{N}'_i = (\mathscr{N}_t \cap \mathscr{L}^*_i) \cap (\mathscr{N}_t \cap \mathscr{L}^*_{i-1} + s(\mathscr{N}_t \cap \mathscr{L}^*_{i-1}))^\perp$$

for $i = 0, 1, \ldots, t$ and $p^i_j(s) \in \mathbb{R}^{1\times q}_j[s]$ such that $(p^i_j, i \in \bar{\mathbf{n}}_j)$ is an orthonormal basis for $\mathscr{N}'_i$ for $i = 0, 1, \ldots, t$.

(5) Procedure TERMINATE.

*Set $n'_t = q - \sum_{i=0}^{t-1} n^{t-1}_i$ and compute left singular vectors

$$u_{p_t+n'_t}, u_{p_t+n'_t-1}, \ldots, u_{p_t+1}, f_1, f_2, \ldots, f_{p_t} \in \mathbb{R}^{q(t+1)}$$

corresponding to the smallest SVs of Π'_t.

*Set $n^t_i = n^{t-1}_i$ for $i = 0, 1, \ldots, t-1$;

$n^t_t = n'_t$;

$r^t_{i,j}(s) = r^{t-1}_{i,j}(s)$ for $j \in \mathbf{n}^{t-1}_i$, $i = 0, 1, \ldots, t-1$;

$r^t_{t,j}(s) = u_{p_t+j}(s) \in \mathbb{R}^{1\times q}_t[s], j \in \mathbf{n}'_t$.

*Terminate the algorithm.

Output:

The result of this computation is a set of polynomials

$$r^t_{i,j}(s) \in \mathbb{R}^{1\times q}[s], i \in \mathbb{Z}_+; j \in \mathbf{n}^t_i.$$

Define

$$R^* := \operatorname{col}(r^t_{0,1}, r^t_{0,2}, \ldots, r^t_{0,n^t_0}, r^t_{1,1}\, r^t_{1,2}, \ldots, r^t_{1,n^t_1}, \ldots, r^t_{t,1}, r^t_{t,2}, \ldots, r^t_{t,n^t_t}, \ldots)$$

and

$$\mathscr{B}^* := \mathscr{B}(R^*).$$

This ends the description of Algorithm 9.

Note that $\mathscr{B}^*$ is *always* defined, and that R^* will be obtained in a *finite* number of steps.

As becomes apparent in the next theorem, $\mathscr{B}^*$ is (only) generically an admissible and optimal approximative model. The finiteness of the algorithm should be understood in the sense that (as in obvious from the procedure TERMINATE) there will be at most q stages where new singular vectors and new (AR) relations need to be computed. However, it is not *a priori* clear how many times SVs will have to be computed, unless, of course, one assumes $\varepsilon^{\text{tol}}_t = 0$ for t sufficiently large, in which case only a finite number of stages of the recursive part of the algorithm, augmented with an examination of the most powerful unfalsified model, need be carried out.

As will be clear from the next theorem, it is very convenient to assume that $\varepsilon^{\text{tol}}_t$ is non-increasing in t. This assumption is moreover natural in the context of the total orderings which are used on the complexity and the error level space. Under this assumption it follows that the procedure STRUCTURE will never have to be invoked, since $\varepsilon^{\text{tol}}_t \leqslant \varepsilon^{\text{tol}}_{t-1}$ implies $\mathscr{U}_t \cap \mathscr{L}^*_{t-1} = 0$. Indeed, assume that this was not so and that $0 \neq a \in \mathscr{U}_t \cap \mathscr{L}^*_{t-1}$. Then

$$\frac{\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}}{\|a\|_{\mathscr{L}^*_{t-1}}} \leqslant \varepsilon^{\text{tol}}_t < \varepsilon^{\text{tol}}_{t-1},$$

which shows that $a \in \mathscr{N}_{t-1}$. However, $\mathscr{U}_t \perp \mathscr{N}_{t-1}$, which gives a contradiction. In addition it may be shown that under this assumption the procedure TERMINATE will never have to be invoked. In order to see this, it suffices to observe that the linear span of more than q independent tth order (AR) relations will always contain non-zero elements of $\mathscr{L}^*_{t'}$ for some $t' < t$. From this it follows that, if $\varepsilon^{\text{tol}}_{t'} \geqslant \varepsilon^{\text{tol}}_t$ for $t' < t$, at most $q - \sum_{i=0}^{t-1} n^{t-1}_i$

singular values of Π_t' will be bounded by $\varepsilon_t^{\text{tol}}$. Indeed, if there were more, a suitable linear combination of the corresponding singular vectors would lie in $\mathscr{L}_{t'}^*$, for some $t' < t$, and would have been discovered and incorporated in $\mathscr{B}_t^*$ at a previous stage of the algorithm.

Theorem 32. Let $\tilde{\mathbf{w}}$ and ε^{tol} be given and $\mathscr{B}^*$ be computed as in Algorithm 9. Then if *either* we consider the situation generically in $\Pi(\tilde{\mathbf{w}})$ *or* if we assume that $\varepsilon_t^{\text{tol}}$ is non-increasing in t, there holds:

(i) $\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^*) \leqslant \varepsilon^{\text{tol}}$;

(ii) $\{\mathscr{B} \in \mathscr{L}, \varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) \leqslant \varepsilon(\mathbf{w}, \mathscr{B}^*) \leqslant \varepsilon^{\text{tol}}\}$
$\Rightarrow \{\mathbf{c}(\mathscr{B}^*) \leqslant \mathbf{c}(\mathscr{B})\}$;

(iii) $\{\mathscr{B} \in \mathbf{L}, \varepsilon(\mathbf{w}, \mathscr{B}) \leqslant \varepsilon^{\text{tol}}\} \Rightarrow$
$\{$*either* $\mathbf{c}(\mathscr{B}) > \mathbf{c}(\mathscr{B}^*)$ *or* $\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) > \varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^*)$
or $\mathbf{c}(\mathscr{B}) = \mathbf{c}(\mathscr{B}^*)$ *and* $\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}) = \varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^*)\}$.

Proof. See Appendix P.

Note that it follows from Theorem 32 that for a generic data sequence $\tilde{\mathbf{w}}$ or for non-increasing $\varepsilon_t^{\text{tol}}$s, Algorithm 9 will generate, with $\mathscr{B}^*$, an approximate (AR) model (in the spirit of the second methodology of Section 20) with misfit limited by $\varepsilon(\tilde{\mathbf{w}}, \mathscr{B}^*) \leqslant \varepsilon^{\text{tol}}$.

This section closes with a few preliminary comments on the robustness of the approximate model obtained in Algorithms 8 and 9 with respect to the observed time series $\tilde{\mathbf{w}}$. As a consequence of the sensitivity of singular vectors (as discussed after Theorem 24), this sensitivity will be related to $\dfrac{1}{\sigma_{d_t} - \sigma_{d_t+1}}$. If $\sigma_{d_t} \cong \sigma_{d_t+1}$, then it is advisable to decrease c^{adm} or increase ε^{tol} somewhat. This will lead to a robust optimal approximate model which is less complex but only slightly less accurate than the original one. However, the sensitivity will also be related to the singular values obtained at stage t and at stage $t+1$ of the algorithm. In fact, if one obtains very little improvement in the accuracy of a model by increasing the model lag by 1, then one should expect a high sensitivity of the higher order model. The exact study of the robustness of Algorithms 8 and 9 is still a matter of research. In any case, a judicious use of these algorithms will ask for a good feedback between the specifications c^{adm} and ε^{tol} and the SVs of the consecutive (reduced) correlation matrices.

26. SIMULATIONS

The approximate modelling algorithms will be now illustrated by means of two purely numerical examples. Applications to signals deduced from industrial processes and econometric time series will be considered elsewhere.

Simulation 1

In the first simulation the algorithms will be applied to a two dimensional time series generated by an autonomous system. The experiment consists of two parts. In the first part an essentially perfect (AR) signal will be analyzed, while in the second part a non-linear, noise corrupted, modification of the first signal will be considered.

Simulation 1—*First experiment.* The first component $\tilde{\mathbf{w}}_1$ of the observation vector consists of a periodic signal $\mathbf{s}$ with a period 3 added to two exponentials, yielding: $\tilde{\mathbf{w}}_1(t) = \mathbf{s}(t) - \rho_1'^{-1} + \rho_2'^{-1}$, $t = 1, 2, \ldots, 500$, with $\mathbf{s}(1) = 1$, $\mathbf{s}(2) = -0.5$, $\mathbf{s}(3) = -0.7$, $\rho_1 = 0.996$, and $\rho_2 = -0.995$. The second component $\tilde{\mathbf{w}}_2$ of the observation vector is deduced from the first by the recursion: $\tilde{\mathbf{w}}_2(t+2) = -0.81 * \tilde{\mathbf{w}}_2(t) + 0.5 * \tilde{\mathbf{w}}_1(t)$ for $t = 1, 2, \ldots, 498$, and with $\tilde{\mathbf{w}}_2(1) = \tilde{\mathbf{w}}_2(2) = 0$.

The signal flow graph generating $\tilde{\mathbf{w}} = \text{col}(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$ is shown in Diagram 3. The matrix $R(s)$ of the (AR) relations describing $\tilde{\mathbf{w}}$ are:

$$\left[\begin{array}{c|c} -0.5 & \sigma^2 + 0.81 \\ \hline (\sigma^3 - 1)(\sigma - 0.996)(\sigma + 0.995) & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \end{array}\right] = R(\sigma)\mathbf{w} = 0.$$

The resulting signals $\tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{w}}_2$ are displayed in Fig. 1.

In order to analyze the data, first compute the truncated correlation matrix $\Pi_{15}(\tilde{\mathbf{w}})$. (What was actually analyzed was the correlation matrix with a suitable exponential weighting, in order to avoid effects due to the finite data length, and with both components of $\tilde{\mathbf{w}}$ normalized to have equal norm). Figure 2 shows the square root of the SVs of $\Pi_{15}(\tilde{\mathbf{w}})$ (with σ_1 normalized to 1). From Fig. 2 and Proposition 28 it is clear that, since rank $\Pi(\tilde{\mathbf{w}}) \cong 6$, a model with zero inputs and six states should be expected. From the way the model was generated seven states would have been expected. The reason for this discrepancy is the fact that it appears impossible to distinguish the pole at $s = 1$ (in $s^3 - 1$) from the one at $s = 0.996$. (Simple and easy to explain facts such as this one already indicate that approximation should be the central idea in identification.)

Note that it is not possible to see from the SVs of $\Pi_{15}(\tilde{\mathbf{w}})$ (or $\Pi(\tilde{\mathbf{w}})$) what order the resulting (AR) relations will have. This can be done by applying the procedure. The square roots of the SVs of $\Pi_0(\tilde{\mathbf{w}})$, $\Pi_1(\tilde{\mathbf{w}})$ and $\Pi_2(\tilde{\mathbf{w}})$ (with $\|\tilde{\mathbf{w}}\|$ normalized to one) are given by:

for $\Pi_0(\tilde{\mathbf{w}})$	for $\Pi_1(\tilde{\mathbf{w}})$:	for $\Pi_2(\tilde{\mathbf{w}})$:
1.1644	1.2542	1.4586
0.5989	1.1578	1.1953
	0.6192	1.1075
	0.3795	0.5627
		0.2272
		0.0027

This indicates that there is one lag 2 (AR) relation which fits the data with an accuracy of better than 0.3%. Computation of the SVD of $\Pi_2(\tilde{\mathbf{w}})$ yields this (AR) relation:

$$\sigma^2\mathbf{w}_2 - 0.0002 * \sigma\mathbf{w}_2 + 0.8100 * \mathbf{w}_2 = 0.0001 * \sigma^2\mathbf{w}_1 + 0.0001 * \sigma\mathbf{w}_1 - 0.5001 * \mathbf{w}_1$$

which indeed fits very closely the first of the original (AR) relations producing the data.

In order to find the next (AR) relation, one must examine the correlation matrices $\Pi_3(\tilde{\mathbf{w}})$, $\Pi_4(\tilde{\mathbf{w}})$ etc., reduced by the (AR) relation which has already been accepted. This reduction works according to the projection procedure explained in the next step computation of Algorithm 9, following Proposition 29. The normalized square roots of the SV's of $\Pi_3^{\text{reduced}}(\tilde{\mathbf{w}})$ and $\Pi_4^{\text{reduced}}(\tilde{\mathbf{w}})$ are given by:

for $\Pi_3^{\text{reduced}}(\tilde{\mathbf{w}})$:	for $\Pi_4^{\text{reduced}}(\tilde{\mathbf{w}})$:
1.5328	1.6635
1.4802	1.5058
1.1705	1.4537
0.9010	1.0891
0.3319	0.3565
0.1895	0.3063
0.0000	0.0039
0.0000	0.0000
	0.0000
	0.0000

This indicates that there is a lag 4 (AR) relation which fits the data with an accuracy of better than 0.4%. Computation of the SVD of $\Pi_3^{\text{reduced}}(\tilde{\mathbf{w}})$ yields this (AR) relation:

$$\begin{aligned} &-0.2542 * \sigma^4\mathbf{w}_1 - 0.2539 * \sigma^3\mathbf{w}_1 + 0.0317 * \sigma^2\mathbf{w}_1 \\ &\quad + 0.1434 * \sigma\mathbf{w}_1 + 0.1339 * \sigma\mathbf{w}_1 \\ &= -0.0672 * \sigma^4\mathbf{w}_2 + 0.2178 * \sigma^3\mathbf{w}_2 + 0.1817 * \sigma^2\mathbf{w}_1 \\ &\quad + 0.1764 * \sigma\mathbf{w}_1 + 0.1912 * \mathbf{w}_1. \end{aligned}$$

In order to explain the resulting (AR) polynomial matrix $\tilde{R}(s)$

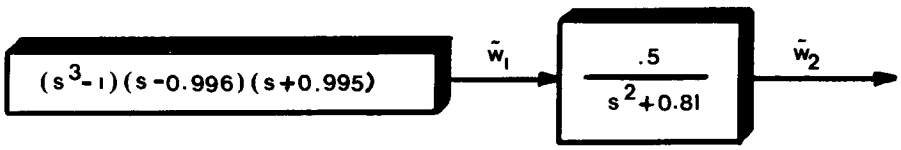
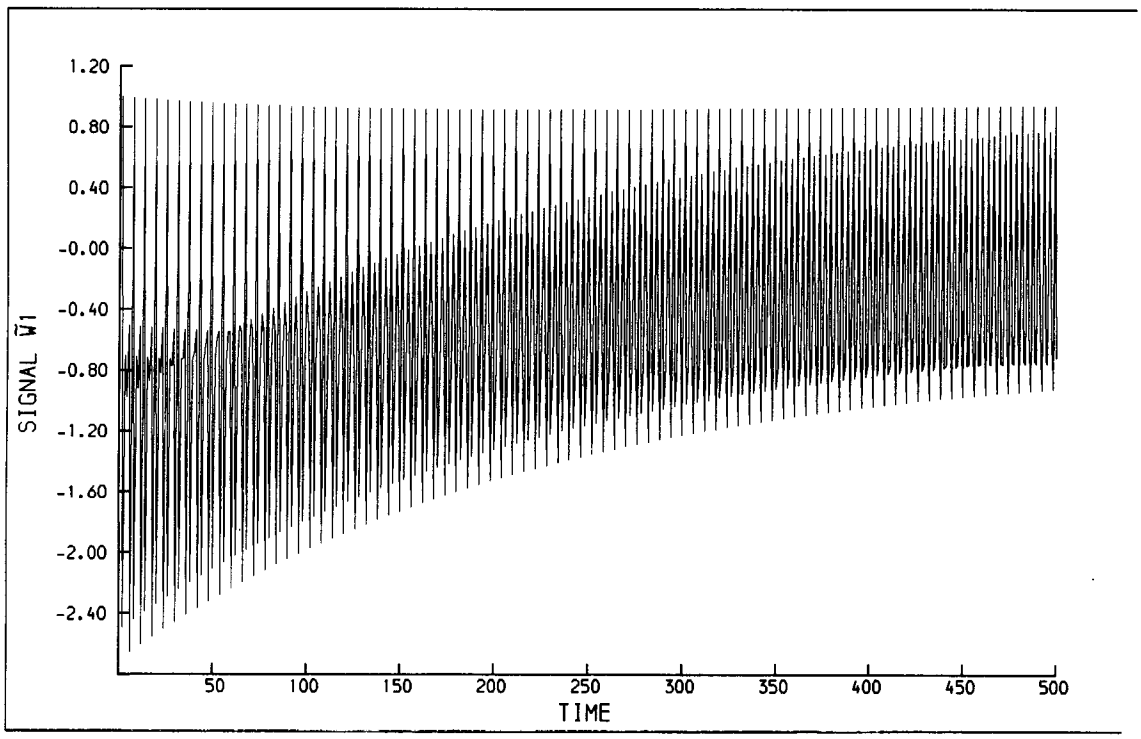


DIAGRAM 3.



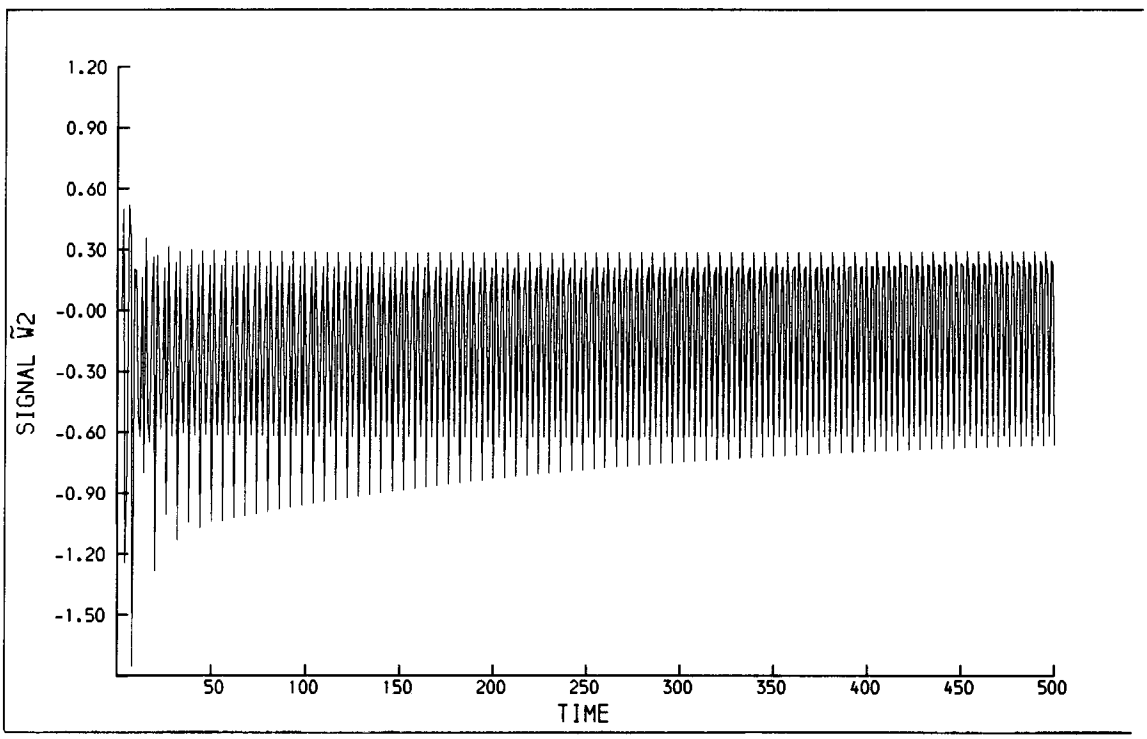


FIG. 1. Simulation 1—Experiment 1: the signals.

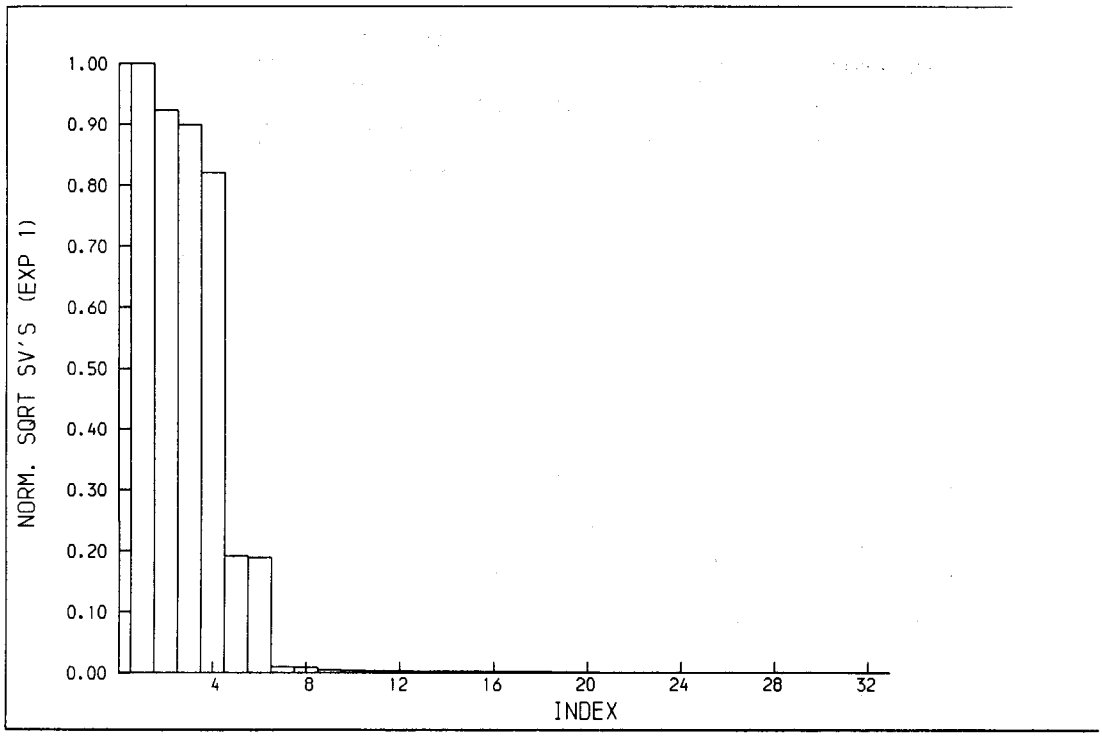


FIG. 2. Simulation 1—Experiment 1: the normalized square root of the SVs.

formed by the two (AR) relations found, compute the roots of $\det \tilde{R}(s)$ and compare them with those of $\det R(s)$. The result is shown below:

roots of $\det \tilde{R}$:	roots of $\det R$:
	1.0000
0.9963	0.9960
−0.9950	−0.9950
−0.5000 + 0.8660 i	−0.5000 + 0.8660 i
−0.5000 − 0.8660 i	−0.5000 − 0.8660 i
−0.0000 + 0.9000 i	−0.0000 + 0.9000 i
−0.0000 − 0.9000 i	−0.0000 − 0.9000 i

Observe the inability of this (or any) algorithm to bifurcate the (theoretical) roots at 1.0000 and 0.9960 present in $R(s)$.

Simulation 1—Second experiment. In the second experiment, the first component $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_1$ of the observation vector was taken to be $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_1 = \tilde{\mathbf{w}}_1 + \mathbf{n}_1$ with $\tilde{\mathbf{w}}_1$ as in the first experiment and with $\mathbf{n}_1$ zero mean gaussian white noise with variance adjusted such that the signal-to-noise ratio is 10. The second component $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_2$ of the observation vector was obtained from the first one by first computing $\mathbf{y}$ according to

$$\mathbf{y}(t+3) = 0.81 * f(\mathbf{y}(t+1) + \mathbf{y}^2(t)) + 0.5 * f(\tilde{\mathbf{w}}\tilde{\mathbf{w}}(t+1)),$$

with $\mathbf{y}(1) = \mathbf{y}(2) = 0$, $\mathbf{y}(3) = 0.5 * f(\tilde{\mathbf{w}}\tilde{\mathbf{w}}(1))$, and $f: \mathbb{R} \to \mathbb{R}$ the saturating function with characteristic shown in Diagram 4.

The time series $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_2$ was produced from $\mathbf{y}$ by $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_2 = \mathbf{y} + \mathbf{n}_2$, with $\mathbf{n}_2$ zero mean gaussian white noise with variance adapted such that the signal-to-noise ratio is 10.

The signal flow graph generating $\tilde{\mathbf{w}}\tilde{\mathbf{w}} = \text{col}(\tilde{\mathbf{w}}\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}\tilde{\mathbf{w}}_2)$ is shown in Diagram 5. The resulting signals $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{w}}\tilde{\mathbf{w}}_2$ are displayed in Fig. 3.

Computation of the normalized square roots of the SVs of $\Pi_{15}(\tilde{\mathbf{w}}\tilde{\mathbf{w}})$ leads to the results shown in Fig. 4. It is difficult to decide from there what a reasonable order of an autonomous model would be. The approximate modelling algorithm will now be applied to the data sequence $\tilde{\mathbf{w}}\tilde{\mathbf{w}}$. First assume that a maximal tolerated relative misfit of $1/6 \approx 17\%$ is imposed. In terms of the notation of Section 25, $\varepsilon^{\text{tol}} = [1/6, 1/6, \ldots, 1/6, \ldots]$.

Computation of the normalized square roots of the SVs of the relevant correlation matrices leads to the following results:

for $\Pi_{0(\tilde{\mathbf{w}}\tilde{\mathbf{w}})}$:	for $\Pi_{1(\tilde{\mathbf{w}}\tilde{\mathbf{w}})}$:	for $\Pi_{2(\tilde{\mathbf{w}}\tilde{\mathbf{w}})}$:	for $\Pi_{3(\tilde{\mathbf{w}}\tilde{\mathbf{w}}_3)}$:
1.3071	1.3584	1.6104	1.7524
0.6245	1.3250	1.3631	1.5924
	0.6586	1.1613	1.2643
	0.4183	0.5962	0.9435
		0.3198	0.3719
		0.1796	0.2919
			0.1837
			0.1585

From these results it follows that there will be one third order lag (AR) relation satisfied with an accuracy of about 16%. Computation of the SVD of $\Pi_3(\tilde{\mathbf{w}}\tilde{\mathbf{w}})$ yields this (AR) relation:

$$\begin{aligned} -0.0920 * \sigma^3\mathbf{w}_1 - 0.0399 * \sigma^2\mathbf{w}_1 + 0.1716 * \sigma\mathbf{w}_1 - 0.0130 * \mathbf{w}_1 \\ = -0.4330 * \sigma^3\mathbf{w}_2 + 0.4797 * \sigma^2\mathbf{w}_2 \\ - 0.3851 * \sigma\mathbf{w}_2 + 0.1689 * \mathbf{w}_2 \qquad (*) \end{aligned}$$

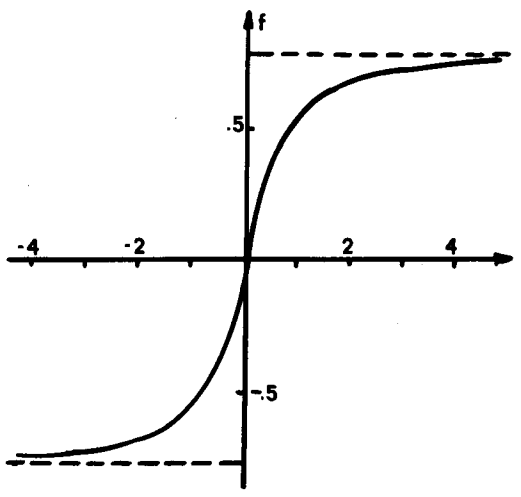


DIAGRAM 4.

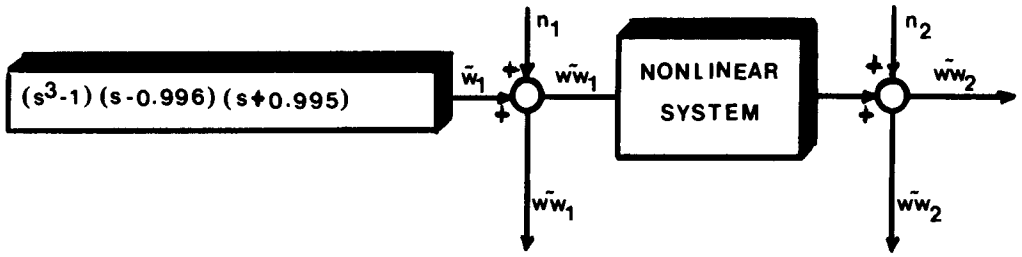


DIAGRAM 5.

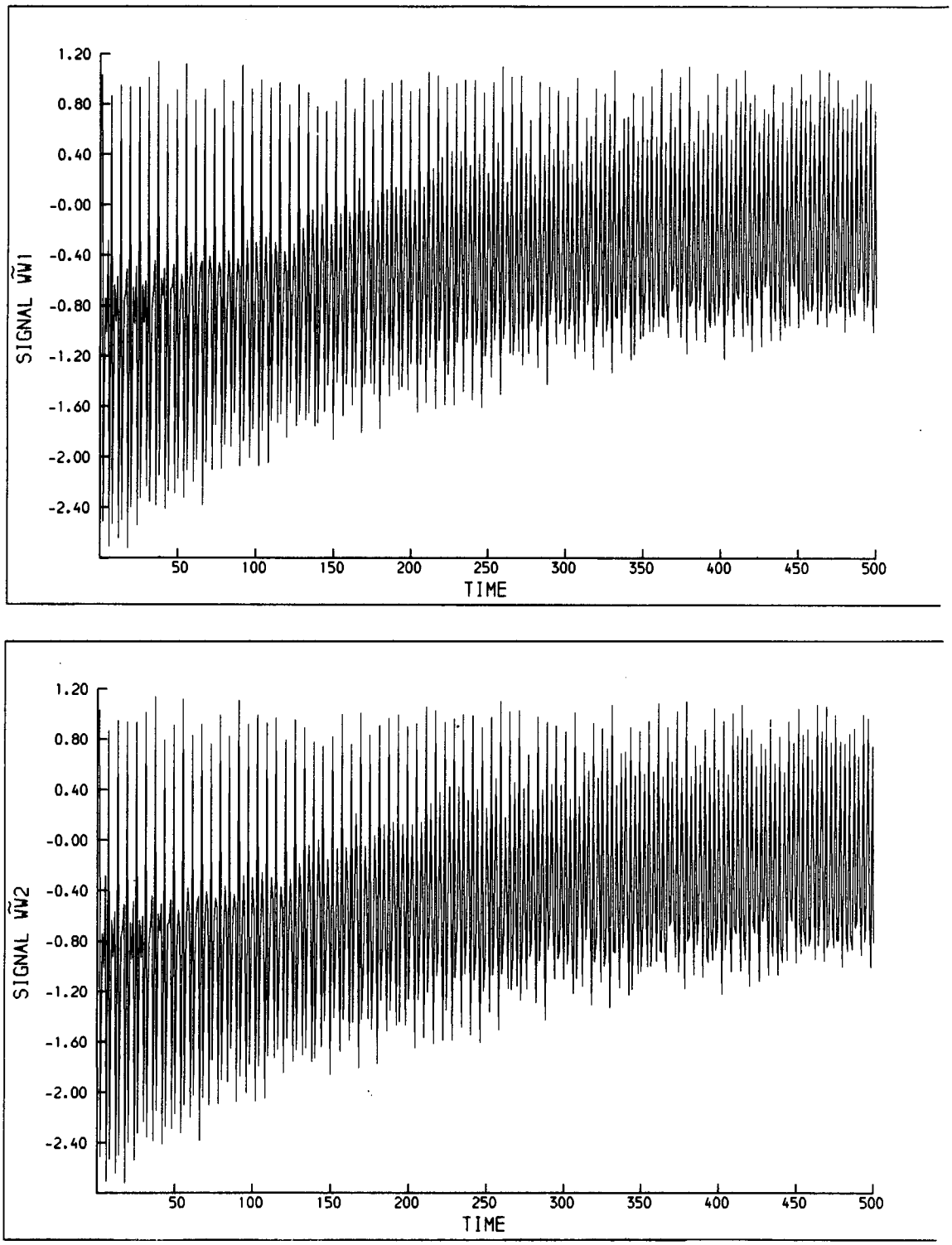


FIG. 3. Simulation 1—Experiment 2: the signals.

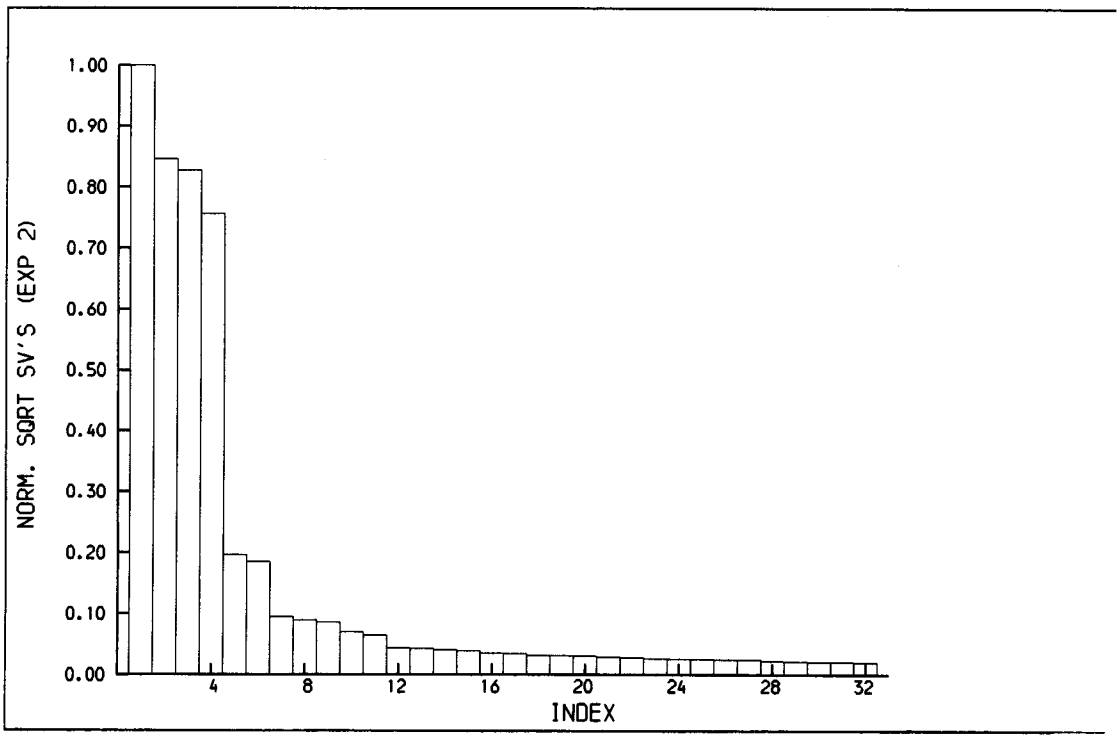


FIG. 4. Simulation 1—Experiment 2: the normalized square root of the SVs.

In order to find other (AR) relations, examine $\Pi_4(\tilde{w}\tilde{w})$ reduced by the above (AR) relation which has already been accepted. The normalized square roots of the SVs are:

for $\Pi_4^{\text{reduced}}(\tilde{w}\tilde{w})$:

1.9251
1.6211
1.5464
1.1599
0.4160
0.3293
0.2123
0.1362
0.0000
0.0000

Examination of this table results in a fourth order lag (AR) relation, which will be satisfied with a relative misfit of about 14%:

$$0.1244 * \sigma^4 w_1 + 0.0957 * \sigma^3 w_1 + 0.0921 * \sigma^2 w_1 - 0.0592 * \sigma w_1 - 0.2118 * w_1 = -0.2097 * \sigma^4 w_2 - 0.0058 * \sigma^3 w_2 + 0.1350 * \sigma^2 w_2 - 0.2926 * \sigma w_2 + 0.1384 * w_2. \quad (**)$$

The (AR) relation (*) and (**) together define the optimal approximate model obtained by requiring a maximal relative tolerated misfit of 1/6.

If a maximal admissible complexity had been imposed, then in view of the signal flow graph of the systems which generated the data $\tilde{w}\tilde{w}$, it would have been logical to impose $c^{\text{adm}} = (1, 1, 1, 7/8, 8/10, 8/12, 8/14, \ldots)$. This corresponds to requiring one third order lag and one fifth order lag (AR) equation.

The resulting (AR) relations would be (*) together with (**)', obtained form the SVD of $\Pi_5^{\text{reduced}}(\tilde{w}\tilde{w})$:

$$0.0721 * \sigma^5 w_1 - 0.1247 * \sigma^4 w_1 - 0.0884 * \sigma^3 w_1 - 0.0877 * \sigma^2 w_1 + 0.0284 * \sigma w_1 + 0.2171 * w_1 = -0.2146 * \sigma^5 w_2 + 0.1777 * \sigma^4 w_2 - 0.0328 * \sigma^3 w_2 - 0.1794 * \sigma^2 w_1 + 0.1648 * \sigma w_1 - 0.0277 * w_1. \quad (**)'$$

This last equation is satisfied with a misfit of 12.66%. In order to compare these models, consider now the poles of the various (AR) systems found so far.

for R:	for $\tilde{R}$:
1.0000	
0.9960	0.9963
−0.9950	−0.9950
−0.5000 + 0.8660 i	−0.5000 + 0.8660 i
−0.5000 + 0.8660 i	−0.5000 − 0.8660 i
−0.0000 + 0.9000 i	−0.0000 + 0.9000 i
−0.0000 − 0.9000 i	−0.0000 − 0.9000 i
for (*), (**):	for (*), (**)':
	1.6763
0.6287	0.6567
0.9963	0.9983
−0.9972	−0.9961
−0.5081 + 0.8699 i	−0.5017 + 0.8673 i
−0.5081 − 0.8699 i	−0.5017 − 0.8673 i
0.2581 + 0.8157 i	0.2796 + 0.7550 i
0.2581 − 0.8157 i	0.2796 − 0.7550 i

The discrepancy between R and $\tilde{R}$ has already been explained. The discrepancy between $\tilde{R}$ and (*), (**) is due to the non-linear term in the system which generates $\tilde{w}\tilde{w}_2$ from $\tilde{w}\tilde{w}_1$. Indeed, computation of the poles of the (AR) relation corresponding to the left hand side of (*) yields:

poles of LHS of (*):

0.6575
0.2251 + 0.7365 i
0.2251 − 0.7365 i.

Taking this into account, together with the signal generation of $\tilde{w}\tilde{w}_1$, explains the results (*), (**) very well. The additional pole (1.6783), obtained in (*), (**)', must be explained by overspecification of the order and will be very unrobust.

This can be seen by comparing $\Pi_5^{\text{reduced}}(\tilde{w}\tilde{w})$ with $\Pi_4^{\text{reduced}}(\tilde{w}\tilde{w})$. It has already been seen what $\Pi_4^{\text{reduced}}(\tilde{w}\tilde{w})$ is equal to. Computation of $\Pi_5^{\text{reduced}}(\tilde{w}\tilde{w})$ yields:

for $\Pi_5^{\text{reduced}}(\tilde{w}\tilde{w})$:

2.0106
1.6428
1.6408
1.6199
0.4175
0.3995
0.2125
0.1919
0.1266
0.0000
0.0000
0.0000

This shows that $(**)'$ give less than 1% improvement over $(**)$ and as such one should expect, as found here, one very unrobust pole in the combination of $(*)$ and $(**)'$. This demonstrates that fixing the tolerated misfit is much more appealing than fixing the admissible complexity.

Simulation 2

The second simulation will show how the approximate modelling procedure allows computation of a low dimensional model for a (non-causal) impulse response. For the observed time series $\tilde{w} = \text{col}(\tilde{w}_1, \tilde{w}_2)$, with $T = \mathbb{Z}$, $\tilde{w}_1(t) = \delta$, with δ an impulse: $\delta(0) = 1$ and $\delta(t) = 0$ *for* $t \neq 0$, and $\tilde{w}_2(t) = 1/(|t| + 1)$ were taken. The resulting impulse response $\tilde{w}_2$ is shown in Fig. 5. From this graph it is clear that it should be possible to model $\tilde{w}_2$ reasonably well by means of a sum of few exponentials: $\sum_{i=1}^{n} \alpha_i \rho_i^{|t|}$. Hence low order systems with a non-causal symmetric impulse response should give adequate approximations.

First analyze what accuracy can be expected from the best (AR) relation of order $n = 0, 1, 2\ldots$ Computation of the relevant SVs yield the following misfit vector:

$$e = \begin{bmatrix} 0.3997 \\ 0.1061 \\ 0.0185 \\ 0.0053 \\ 0.0042 \\ 0.0041 \\ 0.0040 \\ 0.0040 \\ 0.0039 \\ 0.0039 \\ 0.0039 \end{bmatrix}.$$

Here e_{n+1} equals the equation misfit of the best nth order (AR) relation (with $\|\tilde{w}\|$ normalized to one). This shows that very little improvement should be expected from using a relation of lag larger than 4.

The best fitting (AR) relations of order 0, 1, 2, 3 and 4, computed from the relevant SVs, are given below:

$$0.4816 * w_2 = 0.8764 * w_1 \qquad \text{(optmod0)}$$

$$-0.6283 * \sigma w_2 + 0.6283 * w_2 = -0.3244 * \sigma w_1 + 0.3244 * w_1 \qquad \text{(optmod1)}$$

$$0.3563 * \sigma^2 w_2 - 0.7540 * \sigma w_2 + 0.3563 * w_2 = -0.0981 * \sigma^2 w_1 - 0.3981 * \sigma w_1 + 0.0981 * w_1 \qquad \text{(optmod2)}$$

$$-0.1802 * \sigma^3 w_2 + 0.6225 * \sigma^2 w_2 - 0.6225 * \sigma w_2 + 0.1802 * w_2 = -0.314 * \sigma^3 w_1 + 0.2812 * \sigma^2 w_1 - 0.2812 * \sigma w_1 + 0.0314 * w_1 \qquad \text{(optmod3)}$$

$$0.0853 * \sigma^4 w_2 - 0.4269 * \sigma^3 w_2 + 0.6846 * \sigma^2 w_2 - 0.4269 * \sigma w_2 + 0.0853 * w_2 = 0.0104 * \sigma^4 w_1 - 0.1630 * \sigma^3 w_1 + 0.3145 * \sigma^2 w_1 - 0.1630 * \sigma w_1 + 0.0104 * w_1. \qquad \text{(optmod4)}$$

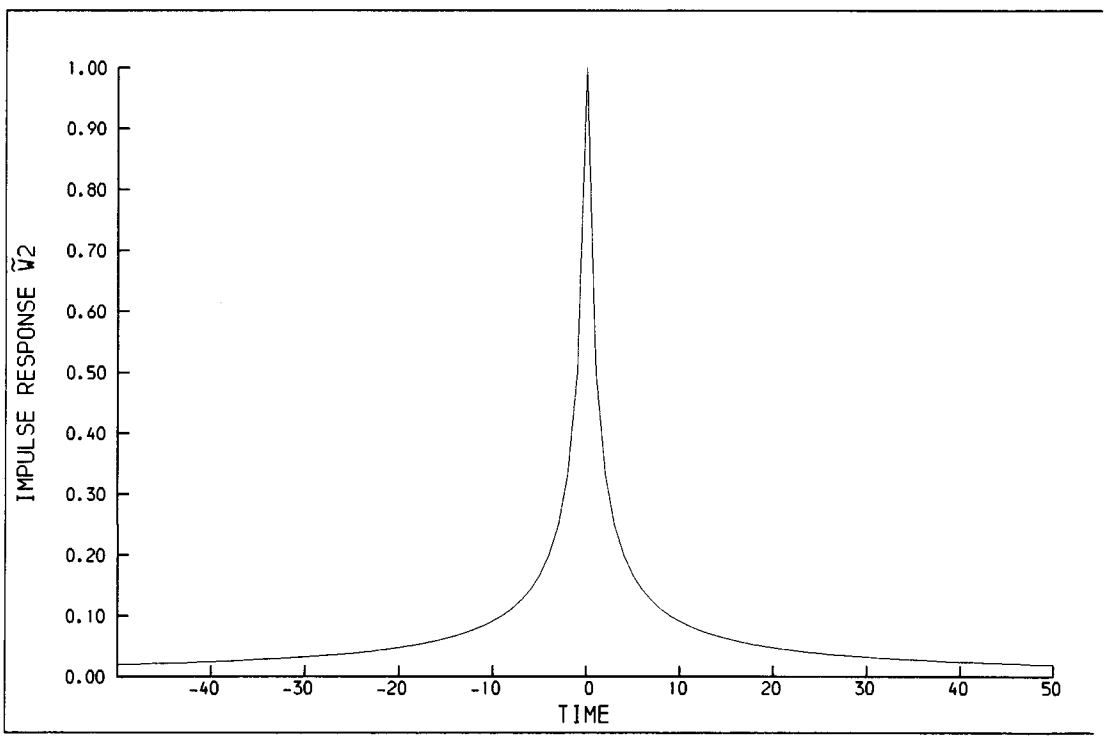


FIG. 5. Simulation 2. The impulse response.

In using these results formally in the optimal approximate modelling algorithm, assume first that the maximal admissible complexity has been specified as $\mathbf{c}^{\text{adm}} = \{1, 1, 5/6, 6/8, 7/10, 8/12, \ldots\}$. This corresponds to looking for a model described by one (AR) relation of lag 2.

This yields (optmod2) as the optimal approximate model. This model fits with an error of about 2%. If, on the other hand, the maximal tolerated relative misfit had been imposed as $\varepsilon^{\text{tol}} = \{0.5\%, 0.5\%, 0.5\%, \ldots\}$, then the optimal approximate model would have been (optmod4), which fits with a relative accuracy of about 0.42%.

Note that (optmod1) and (optmod3) contain an autonomous component with characteristic root at 1. This root is actually imposed by the time symmetry. Hence these models are not reachable and allow an arbitrary constant col $(0, a)$ to be added to any signal col$(\mathbf{w}_1, \mathbf{w}_2)$ in their behaviour.

The poles and the zeros of (optmod2) and (optmod4) are shown below:

for(optmod2):	for(optmod4):
poles: 1.4042	*poles*: 2.6029
0.7122	1.1363
	0.8801
	0.3842
zeros: 3.7933	*zeros*: 13.5329
0.2636	1.3211
	0.7570
	0.0739

We have already seen that going to models of order higher than 4 will yield very little improvement as far as the misfit is concerned. This can also be seen from examining for example the optimal 10th order model. Its poles and zeros are given by:

poles:	2.5013	*zeros*: 13.6561
	1.1085	1.2473
	0.9021	0.8018
	0.3998	0.0732
	−0.5558 + 0.8313 i	−0.5555 + 0.8315 i
	−0.5558 + 0.8313 i	−0.5555 + 0.8315 i
	0.0983 + 0.9952 i	0.0991 + 0.9951 i
	0.0983 − 0.9952 i	0.0991 − 0.9951 i
	0.8160 + 0.5780 i	0.8131 + 0.5821 i
	0.8160 − 0.5780 i	0.8131 − 0.5821 i

This shows that higher order models obtained by this method do little more than finely tune the poles and zeros of (optmod4), in addition to adding essentially non-reachable factors with characteristic roots of unity modulus.

Figure 6 shows the graphs of the impulse response of the original system compared to those of (optmod2) and (optmod4). It follows from all this that the fourth order lag approximation (optmod4) is a very adequate one.

These results have also been compared with what would have been obtained by using reduction by balancing. In order to accommodate the fact that reduction by balancing considers only strictly causal systems, the reduction of the impulse response $\tilde{\mathbf{w}}_2$ has, in fact, been computed for $t \geqq 1$ and the reduced non-causal impulse response has been taken to be equal to this reduction for $t \geqslant 1$, to be equal to 1 at $t = 0$, and to be symmetric in t. The normalized SVs of the relevant Hankel matrix give some insight into the behaviour of the achievable fit by this reduction:

$$s = \begin{bmatrix} 1.0000 \\ 0.2047 \\ 0.0884 \\ 0.0644 \\ 0.0429 \\ 0.0374 \\ 0.0289 \\ 0.0265 \\ 0.0220 \\ 0.0205 \\ 0.0178 \end{bmatrix}.$$

Here s_{n+1} equals the misfit (defined as the first neglected SV) of an nth order approximation. Note that it is much less obvious from this that the impulse response allows a good low order approximation. This is of course due to the fact that the infinite Hankel matrix defined by the sequence $\left\{\frac{1}{n}\right\}$ has a continuous spectrum.

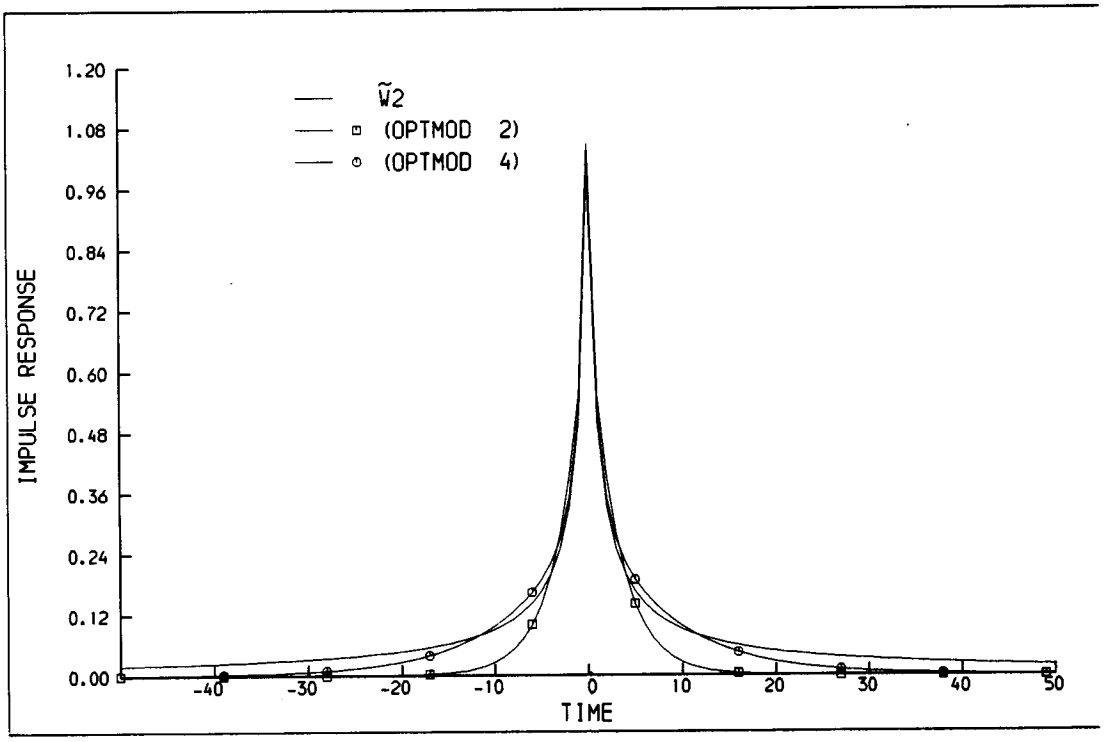


FIG. 6. Simulation 2. Impulse responses of (optmod2) and (optmod4).

The poles and zeros of the second (balred2) and fourth order (balred4) models obtained from reduction by balancing are given by:

for (balred2):	for (balred4):
poles: 1.1287	*poles*: 1.5279
0.8860	1.0477
	0.9545
	0.6545
zeros: 1.5118	*zeros*: 3.4186
0.6614	1.0806
	0.9254
	0.2925

These differ considerably from what was obtained with the approximation procedure of Section 25.

Figure 7 shows the graphs of the impulse response of the original system compared to those of (balred2) and (balred4).

In order to provide a final comparison of the various results obtained so far, Fig. 8 shows the difference between $\tilde{w}_2$ and the impulse responses of (optmod2), (optmod4), (balred2) and (balred4). The conclusion to be drawn from this simulation is that the method of approximately fitting an observed time series to a low order system also yields very satisfactory results when used for model reduction.

Note, finally, that the framework of studying systems as being defined by any set of (AR) relations works very effectively in the context of non-causal impulse responses. In fact, in the final models obtained, say in (optmod4), w_1 can be considered as an input to a causal system with output w_2. However, this system is an unstable one: its transfer function has poles both inside and outside the unit disc. The system (optmod4) can however also be viewed as a non-causal, stable system. This equivalence of causal unstable and non-causal stable systems is discussed in detail in Willems and Heij (1986).

27. CONCLUSIONS

In this sequence of papers there have been several goals. The first, to present a clean conceptual framework for discussing dynamical systems; the second, to put forward an effective language for discussing exact and approximate modelling on the basis of data and the third, to propose an algorithm for the construction of approximate multivariable dynamical models directly on the basis of an observed vector time series.

The emphasis throughout has been on linear time invariant complete systems. As shown, completeness is equivalent to finite dimensionality. However, contrary to the classical approach, no *a priori* distinction has been made between inputs and outputs. This assumption, which in any case is natural and compelling, has functioned very effectively in the modelling algorithms, both for the most powerful model constructed in Part II and for the optimal approximate model constructed in Part III.

The results presented here should be of interest both on the conceptual and on the algorithmic level. As such, the approximate modelling algorithms should be of immediate interest in applications. Particularly the fact that the input/output structure itself will be decided by the data is of much potential relevance (for example in econometrics). The problem of determining the input/output structure has been pursued earlier in the work on errors-in-variables models (Deistler, 1986).

Finally, the work presented here should be viewed as an alternative but very much related approximation philosophy as that proposed by L. Ljung in his basic work on identification theory (Ljung and Söderström, 1983; Ljung, 1987).

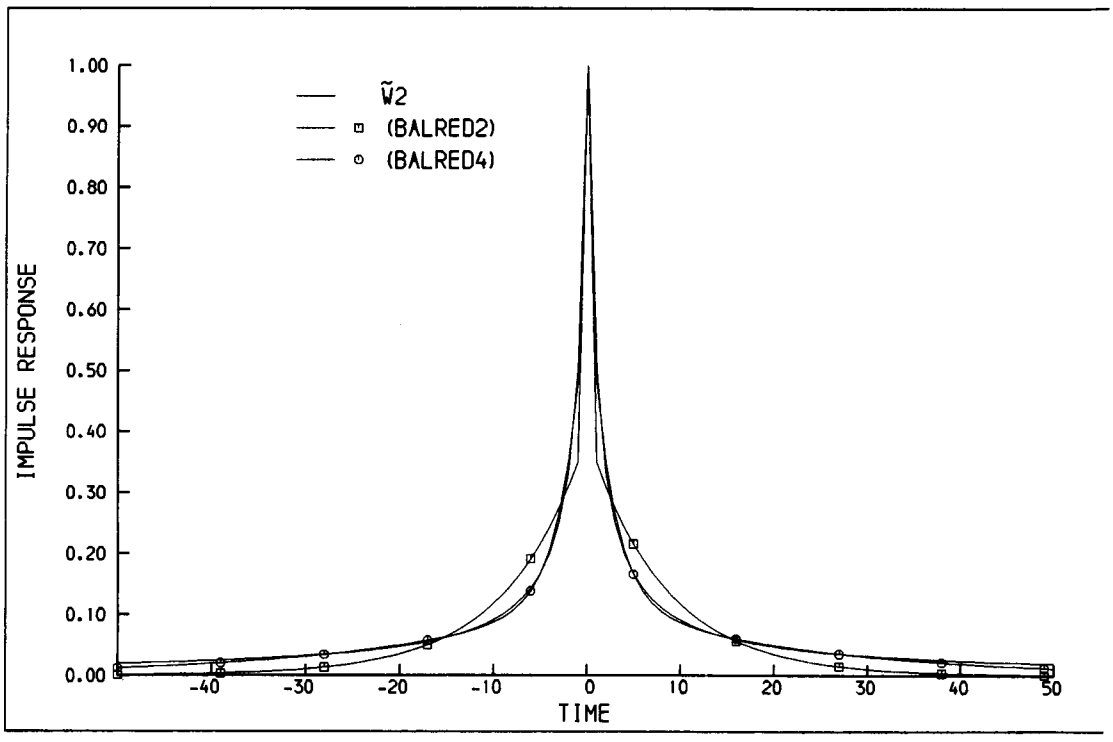


FIG. 7. Simulation 2. Impulse responses of (balred2) and (balred4).

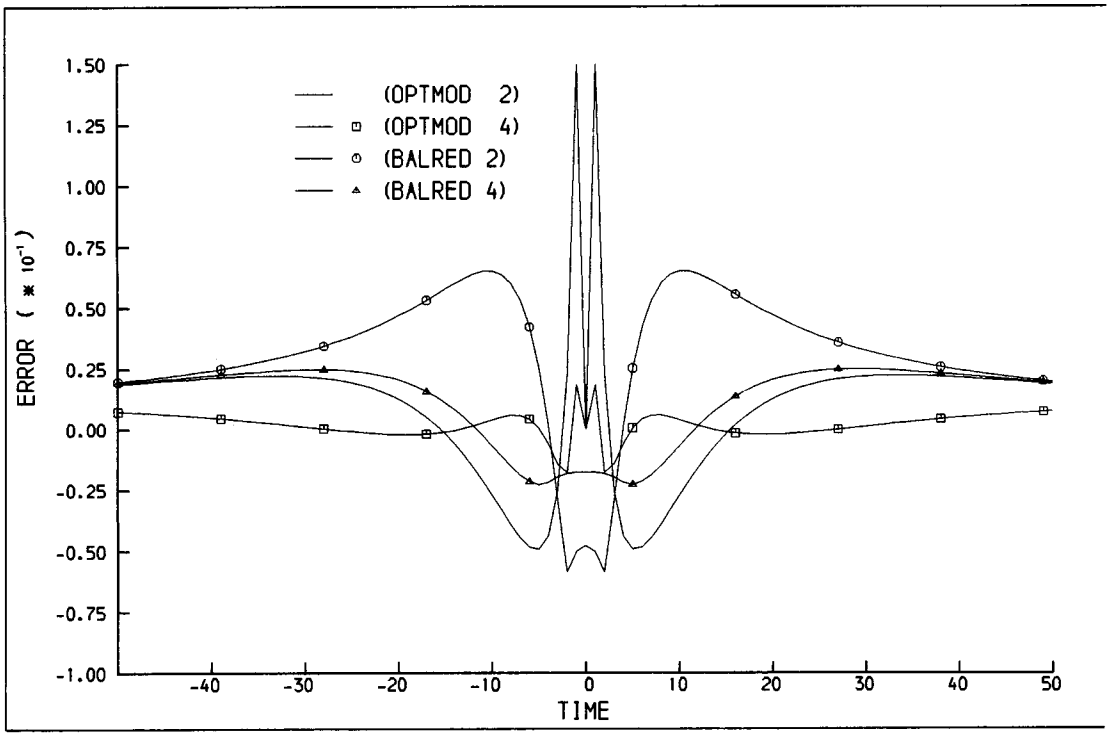


FIG. 8. Simulation 2. Approximation errors.

Optimal approximation, with a tolerated misfit as an essential part, is a much more logical approach to dynamic system identification than the more classical statistical consistency thinking. One way of arguing this goes as follows. If a robust modelling procedure is required which also yields a finite dimensional model when data produced by an infinite dimensional system are observed (which, presumably will always be the case in reality), then consistent identification methods simply cannot be used.

The ideas presented here are presently being extended in several directions:

—to l_2-systems (as defined in Willems and Heij (1986)); in this context a natural symbiosis with model reduction is aimed for, *à la* AAK and Nehari;

—to finite time systems;

—to systems with a more general "time" set, of relevance, for example, in image processing;

and finally,

—to stochastic systems; introduction of complexity considerations in stochastic systems offers a potential justification of stochasticity. Is it too much to speculate that a stochastic, chance 1/2, model for explaining the outcome of the flip of a fair coin is as accurate as a very complicated deterministic model, but preferable precisely because it is less complex?

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APPENDIX N: NOTATION (CONTINUED)

The cardinality of a finite set A equals the number of elements which it contains and is denoted by $|A|$. If $|A| = 1$, A is called a singleton. The set $Z \cap (Z')^{\text{complement}}$ is denoted by $Z - Z'$.

Let $(S, \leqslant)$ be a partially ordered space. Then it is said to be totally ordered if $\forall s_1, s_2 \in S$ there holds either $s_1 \leqslant s_2$, or $s_2 \leqslant s_1$. Let $(S_1, \underset{1}{\leqslant})$ and $(S_2, \underset{2}{\leqslant})$ be partially ordered spaces. Then f: $S_1 \to S_2$ is said to be isotone if $\{s_1' \underset{1}{\leqslant} s_1''\} \Rightarrow \{f(s_1') \underset{2}{\leqslant} f(s_1'')\}$. An

example of a partial order on $\mathbb{R}^n$ is the order of pointwise domination:

$$\{\mathrm{col}(x_1, x_2, \ldots, x_n) \leqslant \mathrm{col}(y_1, y_2, \ldots, y_n)\} :\Leftrightarrow \{x_i \leqslant y_i \, \forall i \in \mathbf{n}\}.$$

Another example is the lexicographic order:

$$\{\mathrm{col}(x_1, x_2, \ldots, x_n) \leqslant \mathrm{col}(y_1, y_2, \ldots, y_n)\} :\Leftrightarrow \{\text{either } x_i = y_i, \forall i \in \mathbf{n}, \text{ or } \exists i \in n$$

such that $x_j = y_j$ for $j < i$ and $x_i < y_i\}$. Obviously the identity map on $\mathbb{R}^n$ is isotone as a map from $\mathbb{R}^n$ endowed with the order of pointwise domination to $\mathbb{R}^n$ endowed with the lexicographic order. Analogous structure holds on $\mathbb{R}^{\mathbb{Z}_+}$, etc.

The matrix whose (i,j)th element equals $a_{i,j}$ will be denoted by $[a_{i,j}]$. If $A = [a_{i,j}] \in \mathbb{R}^{n_1 \times n_2}$ has $a_{i,j} = 0$ for $i \neq j$, then A is called diagonal and denoted as $A = \mathrm{diag}\,[a_{1,1}, a_{2,2}, \ldots, a_{n,n}]$ with $n = \min(n_1, n_2)$. For example, $I_n = \mathrm{diag}\,[1,1,\ldots,1]$ (n-times). The zero matrix in $\mathbb{R}^{n_1 \times n_2}$ will be denoted by $0_{n_1,n_2}$. A matrix $M \in \mathbb{R}^{n \times n}$ is said to be orthogonal if $MM^T = M^TM = I_n$.

A mapping $L\colon V_1 \times V_2 \to \mathbb{R}$ with V_1 and V_2 real vector spaces is said to be bilinear if $L(v_1, .)$ and $L(., v_2)$ are both linear maps for all $v_1 \in V_1$ and $v_2 \in V_2$. If $V_1 = V_2 = V$ then L is called a bilinear form on V. If, in addition, $L(v_1, v_2) = L(v_2, v_1)$ for all $v_1, v_2 \in V$ it is called symmetric and if, moreover, $L(v, v) \geqslant 0$ for all $v \in V$ it is called a real symmetric non-negative bilinear form on V.

Let $a \in \mathbb{R}_+$. Then ent(a) denotes the largest integers smaller than or equal to a and ENT(a) denotes the smallest integer larger than or equal to a or n, $m \in \mathbb{Z}_+$, $m > 0$, such that $n = dm + r$ with $d, r \in \mathbb{Z}_+$ and $0 \leqslant r \leqslant m$, define $d =: (n)\mathrm{DIV}(m)$ and $r := (n)\mathrm{MOD}(m)$.

The expected value of a random vector is denoted by $\mathscr{E}$.

An important notion in modern applied mathematics is that of genericity. It will be defined here only in the context in which it is used in this paper. Let $P_{t',t''} \in \mathbb{R}^{q \times q}$, t', $t'' \in \mathbb{Z}_+$, be a family of matrices defining the infinite matrix

$$P = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,t'} & \cdots \\ P_{1,0} & P_{1,1} & \cdots & P_{1,t'} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ P_{t,0} & P_{t,1} & \cdots & P_{t,t'} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}.$$

Assume that $P_{t',t''} = P^T_{t'',t'}$. Then obviously the infinite matrix P is also symmetric. Denote by $\mathscr{P}$ all such infinite symmetric matrices. Let P_t denote the truncation of P consisting of its first $q(t+1)$ rows and columns. In an obvious way P_t can be considered to be an element of $\mathbb{R}^{q(t+1) \times q(t+2)/2}$. Denote by Π_t the map which takes the matrix $P \in \mathscr{P}$ into its truncation $\Pi_t P := P_t \in \mathbb{R}^{q(t+1) \times q(t+2)/2}$. Let $\mathscr{G} \subset \mathscr{P}$. Elements of $\mathscr{G}$ can be thought to consist of all infinite symmetric matrices having a certain given property. This property will be called generic if it can be considered a property of typical elements of $\mathscr{P}$. Formally, if, for all $t \in \mathbb{Z}_+$, $\Pi_t \mathscr{G}$ (which is hence a subset of $\mathbb{R}^{q(t+1) \times q(t+2)/2}$) contains an open, dense, measure exhausting (meaning that its complement has zero Lebesgue measure) subset, then we will call $\mathscr{G}$ generic.

APPENDIX SV: SINGULAR VALUES

Let $M \in \mathbb{R}^{n_1 \times n_2}$. Then a *singular value decomposition* (SVD) of M is a decomposition of M into the product $M = U\Sigma V^T$ with $U \in \mathbb{R}^{n_1 \times n_1}$, $\Sigma \in \mathbb{R}^{n_1 \times n_2}$, $V \in \mathbb{R}^{n_2 \times n_2}$, U and V orthogonal ($U^TU = I_{n_1}$, $VV^T = I_{n_2}$), and Σ of the form

$$\Sigma = \left[\begin{array}{c|c} \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) & O_{r \times (n_1 - r)} \\ \hline O_{(n_2 - r) \times r} & O_{(n_2 - r) \times (n_1 - r)} \end{array}\right]$$

with $\sigma_{\max} := \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r =: \sigma_{\min} > 0$.

It is easy to see that every matrix admits a SVD. The numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the *singular values* (SV) of M, and $\sigma_{\max}$ is called the principal singular value of M. Note that $\sigma_{\max}$ is the induced norm of M. Denote $\sigma(M) = (\sigma_1, \sigma_2, \ldots, \sigma_r)$. The columns of $U = (u_1, u_2, \ldots, u_{n_1})$ are called the left singular vectors of M, while the columns of $V = (v_1, v_2, \ldots, v_{n_2})$ are called the right singular vectors of M.

Some basic properties of the SVD in the SVD are listed in the following proposition.

Proposition. The SVD of M has the following properties.

(i) $\sigma(M) = \sigma(M^T) = \sqrt{\sigma(MM^T)} = \sqrt{\sigma(M^TM)}$. If $M = M^T$ then the SVs of M are the absolute values of its eigenvalues (ordered non-increasingly).

(ii) $r = \mathrm{rank}\ M$.

(iii) $$\sigma_k = \underset{\dim L \geqslant k}{\mathrm{maximum}} \ \underset{0 \neq x \in L}{\mathrm{minimum}} \frac{\|Mx\|}{\|x\|}$$

$$= \underset{\dim L \geqslant n-k+1}{\mathrm{minimum}} \ \underset{0 \neq x \in L}{\mathrm{maximum}} \frac{\|Mx\|}{\|x\|}$$

here L denotes a linear subspace of $\mathbb{R}^{n_1}$. In particular $\sigma_1 = \max_{0 \neq x} \frac{\|Mx\|}{\|x\|} = \|M\|$ (induced norm).

(iv) Let $U_k := \mathrm{col}(u_1, u_2, \ldots, u_k)$. Then

$$\sigma_k = \underset{0 \neq x \in \mathrm{im} U_k}{\mathrm{minimum}} \frac{\|Mx\|}{\|x\|} \text{ and } \sigma_{k+1} = \underset{0 \neq x \in (\mathrm{im} U_k)^\perp}{\mathrm{maximum}} \frac{\|Mx\|}{\|x\|}.$$

(v) Define, for a k-dimensional subspace $\mathscr{L}$ of $\mathbb{R}^{n_2}$ $\varepsilon_i(\mathscr{L}) = \underset{\substack{\mathscr{L}_i \subset \mathscr{L} \\ \dim \mathscr{L}_i = k - i + 1}}{\mathrm{minimum}} \ \underset{0 \neq x \in \mathscr{L}_i}{\mathrm{maximum}} \frac{\|Mx\|}{\|x\|}$. Equip $\varepsilon(\mathscr{L}) :=$ $(\varepsilon_1(\mathscr{L}), \varepsilon_2(\mathscr{L}), \ldots, \varepsilon_k(\mathscr{L}))$ with the lexicographic ordering. Then

$$\varepsilon(\mathscr{L}) \geqslant \varepsilon((\mathrm{im}\, V_k)^\perp) = (\sigma_{k+1}, \ldots, \sigma_r, 0, \ldots, 0).$$

Moreover, if $\sigma_k > \sigma_{k+1}$, then $(\mathrm{im}\, V_k)^\perp$ is the unique subspace having $\varepsilon(\mathscr{L}) = (\sigma_{k+1}, \ldots, \sigma_r, 0, \ldots, 0)$.

APPENDIX P: PROOFS (CONTINUED)

Proof of Proposition 23.

Clearly for all $a \in M^\perp$ there holds

$$d^2(Z, a) := \frac{\frac{1}{N}\sum_{i=1}^{N} |\langle a, z_i \rangle|^2}{\|a\|^2} = \frac{\frac{1}{N}\|a^TQ_MZ\|^2}{\|a\|^2} = \frac{a^TQ_M\Pi_ZQ_Ma}{\|a\|^2}.$$

The result is an immediate consequence of statement (iii) of the proposition in Appendix SV. □

Proof of Theorem 24

This proof is based on the proposition in Appendix SV.

(i) Assume c^{adm} given. Let $k := \mathrm{ent}(n \cdot c^{\mathrm{adm}})$. If $k' \geqslant r$ then $c(\mathrm{im}\, Z) = \frac{r}{n} \leqslant c^{\mathrm{adm}}$ and $\varepsilon(Z, \mathrm{im} Z) = 0$. Further $\{\varepsilon(Z, M) = 0\} \Leftrightarrow$ $\{M \supset \mathrm{im}\, Z\}$ which shows that $M^* = Z$ is then the unique optimal approximate model. Assume now that $k' < r$ and that k is such that $\sigma_k > \sigma_{k+1} = \sigma_{k'+1}$. Clearly $c(M^*) = k$ and $\varepsilon(Z, M^*) = \sigma_{k+1}$. Moreover,

$$\varepsilon(Z, M) = \max_{a \perp M} \frac{\|a^TZ/\sqrt{N}\|}{\|a\|} \geqslant \underset{\dim \mathscr{L} \geqslant n - \dim M}{\mathrm{minimum}} \ \underset{0 \neq a \in \mathscr{L}}{\mathrm{maximum}} \frac{\|a^TZ/\sqrt{N}\|}{\|a\|}.$$

Hence $\{\dim M \leqslant k'\} \Rightarrow \{\varepsilon(Z, M) \geqslant \sigma_{k+1} = \varepsilon(Z, M^*)\}$ and

$$\{\dim M < k = \dim M^*\} \Rightarrow \{\varepsilon(Z, M) \geqslant \sigma_{\dim M + 1} \geqslant \sigma_k > \sigma_{k+1} = \varepsilon(Z, M^*)\}.$$

This shows that M^* is optimal. Moreover, since $\sigma_k > \sigma_{k+1}$, M^* is a k-dimensional subspace which achieves $\varepsilon(Z, M) = \sigma_{k+1}$.

(ii) Assume ε^{tol} given. If $\varepsilon^{\text{tol}} < \sigma_r$, then $\{\dim M < r\} \Rightarrow \{\varepsilon(Z, M) \leqslant \sigma_{\dim M+1} \leqslant \sigma_r\}$. Hence an admissible model must have $\dim M \geqslant \dim Z$. If $M^* = Z$ is taken, then $\varepsilon(Z, M^*) = 0$, while if $M \not\supset \operatorname{im} Z$ then $\varepsilon(Z, M) > 0$. This shows that $M^* = Z$ is the unique optimal approximate model in this case. Assume now that $\varepsilon^{\text{tol}} \geqslant \sigma_r$ and that k is such that $\sigma_k > \varepsilon^{\text{tol}} \geqslant \sigma_{k+1}$. Clearly $c(M^*) = \dfrac{k}{n}$ and $\varepsilon(Z, M^*) = \sigma_{k+1} \leqslant \varepsilon^{\text{tol}}$ and

$$\{\dim M < k\} \Rightarrow \{\varepsilon(Z, M) \geqslant \sigma_{\dim M+1} \geqslant \sigma_k > \varepsilon^{\text{tol}}\}.$$

This shows that M^* is optimal. Moreover, since $\sigma_k > \sigma_{k+1}$, M^* is a k-dimensional subspace such that $\varepsilon(Z, M) = \sigma_{k+1} = \varepsilon(Z, M^*)$. If $\varepsilon^{\text{tol}} \geqslant \sigma_1$, then $\sigma_1 = \varepsilon(Z, 0) \leqslant \varepsilon^{\text{tol}}$ and $M^* = 0$ is clearly optimal. □

Proof of Theorem 25

In the proof the $\mathscr{B}$ in $\mathbf{c}_t(\mathscr{B})$, etc. will be dropped when there is no possibility of confusion.

(i) Clearly $\{\mathscr{B}' \subset \mathscr{B}''\} \Rightarrow \{\mathscr{B}'_t \subset \mathscr{B}''_t\} \Rightarrow \{\dim \mathscr{B}'_t \leqslant \dim \mathscr{B}''_t\}$, which shows that $\{\mathscr{B}' \subset \mathscr{B}''\} \Rightarrow \{\mathbf{c}(\mathscr{B}') \leqslant \mathbf{c}(\mathscr{B}'')\}$.

That $0 \leqslant \mathbf{c} \leqslant 1$ is obvious from the definition of $\mathbf{c}$. That $\mathbf{c}$ is monotone non-increasing follows from the concavity of $\mathbf{d}$. Indeed, $\mathbf{d}'' \leqslant 0$ implies $\mathbf{d}'_t \leqslant \mathbf{d}'_{t-1} \leqslant \dots$. Hence $\mathbf{d}_t = \mathbf{d}'_t + \mathbf{d}'_{t-1} + \cdots + \mathbf{d}'_0 \geqslant (t+1)\mathbf{d}'_t$, which implies $\mathbf{d}_t \geqslant (t+1)\mathbf{d}'_t$. Now

$$\begin{aligned} q\mathbf{c}'_t &= \frac{\mathbf{d}_t}{t+1} - \frac{\mathbf{d}_{t-1}}{t} \\ &= \frac{(t+1)(\mathbf{d}_t - \mathbf{d}_{t-1}) - \mathbf{d}_t}{t(t+1)} \\ &= \frac{(t+1)\mathbf{d}'_t - \mathbf{d}_t}{t(t+1)} \leqslant 0. \end{aligned}$$

(ii) From the proof of Theorem 6 (v) it follows that for t such that $\mathbf{d}'_t = \lim_{t' \to \infty} \mathbf{d}'_{t'}$, $\mathbf{d}_t = (t+1)m + n$. This yields (ii).

(iii) Since $\mathbf{c}_t = \dfrac{\mathbf{d}_t}{q(t+1)}$ the partial ordering on $\mathbf{c}$ is identical to the one on $\mathbf{d}$ defined by

$$\{\mathbf{d}(\mathscr{B}') \geqslant \mathbf{d}''(\mathscr{B}'')\} :\Leftrightarrow \{\mathbf{d}_t(\mathscr{B}') \geqslant \mathbf{d}_t(\mathscr{B}'')$$

for all $t \in \mathbb{Z}_+\}$. Also $\mathbf{d}_t = \rho_0 + \rho_1 + \cdots + \rho_t$. Now examine Table 2 of Part I and relation (vi) of Theorem 6 to obtain the desired result. □

Proof of Theorem 26.

From Theorem 25, (i) and (ii) are obvious. To see (iii), observe that for

$$\begin{aligned} \mathscr{B} \in \mathbf{L}, \mathbf{d}_t(\mathscr{B}) &= \dim \mathscr{B}_t \\ &= m(t+1) + \sum_i \min(\nu_i, t+1). \end{aligned}$$

Further n, the minimal dimension of the state space, equals $\sum_i \nu_i$. Hence

$$\begin{aligned} \mathbf{d}_t(\mathscr{B}) &= \dim \mathscr{B}_t \\ &= m(t+1) + n - \sum_i \max(\nu_i - t - i, 0) \end{aligned}$$

From this, (iii) follows immediately. □

Proof of Proposition 29

Let $a \in \mathscr{L}_t^*$ and identify $a(s) = a_0 + a_1 s + \cdots + a_t s^t$ with $\operatorname{col}(a_0^T, a_1^T, \dots, a_t^T)$. Then $\|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}}^2 = a^T \Pi_t(\tilde{\mathbf{w}}) a$. Hence for $a \in \mathscr{N}'_t$, $\|a(\sigma)\tilde{\mathbf{w}}\|^2 = a^T P'_t \Pi_t(\tilde{\mathbf{w}}) P'_t a$, while for $a \perp \mathscr{N}'_t$, $a \in \ker P'_t \Pi_t(\tilde{\mathbf{w}}) P'_t$. Hence $\max_{\|a\|=1} a^T P'_t \Pi_t(\tilde{\mathbf{w}}) P'_t a$ is attained for an element $a \perp (\mathscr{N}'_t)^\perp$, i.e. for $a \in \mathscr{N}'_t$. This implies that $\eta'_1(\tilde{\mathbf{w}}, \mathscr{B}) = \sqrt{\sigma_{\max}(P'_t \Pi_t(\tilde{\mathbf{w}}) P'_t)}$, as claimed. The expression for $\eta'_2(\mathbf{w}, \mathscr{B})$, $\eta'_3(\mathbf{w}, \mathscr{B})$, ..., can be obtained in a similar way. □

Proof of Theorem 30

The $\tilde{\mathbf{w}}$ and $\mathscr{B}$ dependence will be dropped whenever this can cause no confusion.

(i) ($\Leftarrow$) is obvious. The proof of ($\Rightarrow$) goes by induction. It will be shown that $\{(\sigma^{t'}\tilde{\mathbf{w}})\big|_{T \cap [0,t-1]} \in \mathscr{B}_{t-1}, \ \forall t' \in T, \text{ and } \varepsilon_t = 0\} \Rightarrow \{(\sigma^{t'}\tilde{\mathbf{w}})\big|_{T \cap [0,t]} \in \mathscr{B}_t, \ \forall t' \in T\}$. Let $a \in \mathscr{B}_t^\perp$. Write $a = a_1 + a_2$ with $a_1 \in \mathscr{B}_{t-1}^\perp + \sigma^* \mathscr{B}_{t-1}^\perp$ and $a_2 \in \mathscr{N}'_t$. Then $a_1(\sigma)\tilde{\mathbf{w}} = \mathbf{0}$ since $\sigma^{t'}\tilde{\mathbf{w}}\big|_{T \cap [0,t-1]} \in \mathscr{B}_{t-1}$, $\forall t' \in T$, and $a_2(\sigma)\tilde{\mathbf{w}} = \mathbf{0}$ since $\varepsilon_t = 0$. The result follows.

(ii) $\varepsilon_t \geqslant 0$ is obvious. Further observe that, with $a(s) = \sum_{t \in \mathbb{Z}_+} a_t s^t$ and $a_t = (a_t^1, a_t^2, \dots, a_t^q)$, there holds

$$a(\sigma)\tilde{\mathbf{w}} = \sum_{t \in \mathbb{Z}_+} \sum_{i \in \mathbb{q}} a_t^i \sigma^t \tilde{\mathbf{w}}_i.$$

Hence

$$\begin{aligned} \|a(\sigma)\tilde{\mathbf{w}}\|_{\mathscr{K}} &\leqslant \sum_{t \in \mathbb{Z}_+} \sum_{i \in \mathbb{q}} |a_t^i| \, \|\sigma^t \tilde{\mathbf{w}}_i\|_{\mathscr{K}} \\ &\leqslant \|a\|_{\mathscr{L}^*} \sqrt{\sum_{i \in \mathbb{q}} \sum_{t=0}^{\partial(a)} \|\sigma^t \tilde{\mathbf{w}}_i\|_{\mathscr{K}}^2} \\ &\leqslant \sqrt{\partial(a)+1} \, \|a\|_{\mathscr{L}^*} \sqrt{\sum_{i \in \mathbb{q}} \|\tilde{\mathbf{w}}_i\|_{\mathscr{K}}^2}. \end{aligned}$$

The inequality follows immediately from this. The fact that ε_t is non-zero for at most q (in fact p) ts follows immediately from the observation that $\mathscr{B}_t^\perp = \mathscr{B}_{t-1}^\perp + \sigma^* \mathscr{B}_{t-1}^\perp$ whenever the shortest lag (AR) description of $\mathscr{B}$ contains no equations of lag t. Since the shortest lag (AR) description contains at most p equations, the result follows.

(iii) (*Outline of the proof*). Fix $\mathscr{B}', \mathscr{B}'' \in \mathbf{L}, \mathscr{B}' \supset \mathscr{B}''$. Let t' be such that $\mathscr{B}'_{t'-1} = \mathscr{B}''_{t'-1}$ and $\mathscr{B}'_{t'} \neq \mathscr{B}''_{t'}$. Let P' denote orthogonal projection onto $\mathscr{N}'_{t'} = (\mathscr{B}'_{t'})^\perp \cap ((\mathscr{B}'_{t'-1})^\perp + \sigma^*(\mathscr{B}'_{t'-1})^\perp)^\perp$ and let $\mathscr{N}''_{t'}$ and P'' be analogously defined. Clearly $\mathscr{N}''_{t'} \subset \mathscr{N}'_{t'} \neq 0$ and $\mathscr{N}''_{t'} \neq \mathscr{N}'_{t'}$. Observe that $\ker P' \supset \ker P''$, $\ker P' \neq \ker P''$. Now it may be shown that in the set of symmetric matrices $N = N^T \in \mathbb{R}^{q(t'+1)(t'+2)/2}$ those for which $\sigma_{\max}(P''NP'') = \sigma_{\max}(P'NP')$ is closed and has Lebesgue measure zero. The result follows when this N is allowed to play the role of $\Pi_{t'}(\tilde{\mathbf{w}})$. □

Proof of Theorem 31

(i) From the selection rule for k_t in Algorithm 8 it follows that $\dfrac{\mathbf{d}_t}{q(t+1)} \leqslant \mathbf{c}_t^{\text{adm}}$. Consequently at stage t the algorithm will already honour enough (AR) relations so as to obtain $\dim \mathscr{B}_t = k_t$. New (AR) relations which could be added at a later stage may possibly further reduce $\dim \mathscr{B}_t$ [even though generically that will not happen]. Hence

$$\frac{\dim \mathscr{B}_t^*}{q(t+1)} \leqslant \frac{\mathbf{d}_t}{q(t+1)} \leqslant \mathbf{c}_t^{\text{adm}}.$$

In order to prove (ii)–(iv), observe first the following lemma:

Lemma. Generically in $\Pi(\tilde{\mathbf{w}})$ there holds:

(i) Π'_t has $q(t+1) - p_t$ distinct positive and $p_t = \sum_{i=0}^{t-1} (t+1-i)\mathbf{k}_i$ zero eigenvalues where $p_t := \sum_{i=0}^{t-1} (t+1-i)\mathbf{k}_i$;

(ii) $\mathscr{U}_t \cap \mathscr{L}_{t-1}^* = \{0\}$.

The proof of this lemma is rather technical. Since the lemma is also intuitively reasonable and acceptable, its proof will be deleted.

Observe that this lemma yields immediately the generic statements in Algorithm 8, in particular (iv) of Theorem 31, and the fact that the procedure STRUCTURE will generically never have to be invoked. Indeed, the lemma implies that the singular values u computed at the tth stage of Algorithm 8 will generate $\mathbf{k}_t(\text{AR})$ relations of order exactly t, and hence $\mathscr{B}_t^*$ will be specified in the tth stage already.

Now turn to the proof of (ii) and (iii). It will be shown that, generically in $\Pi(\tilde{\mathbf{w}})$, there holds:

$$\{\mathscr{B} \in \mathbf{L}, \mathscr{B}_{t-1} = \mathscr{B}^*_{t-1}\}$$
$$\Rightarrow \{\textit{either } \mathbf{c}_t(\mathscr{B}) > \mathbf{c}_t(\mathscr{B}^*) = \mathbf{c}_t^{\text{adm}};$$
$$\textit{or} \quad \mathbf{c}_t(\mathscr{B}) = \mathbf{c}_t(\mathscr{B}^*) = \mathbf{c}_t^{\text{adm}},$$
$$\textit{and} \quad \varepsilon_t(\mathbf{w}, \mathscr{B}) > \varepsilon_t(\mathbf{w}, \mathscr{B}^*);$$
$$\textit{or} \quad \mathbf{c}_t(\mathscr{B}) = \mathbf{c}_t(\mathscr{B}^*) = \mathbf{c}_t^{\text{adm}},$$
$$\textit{and} \quad \varepsilon_t(\mathbf{w}, \mathscr{B}) = \varepsilon_t(\mathbf{w}, \mathscr{B}^*)\text{—}$$
$$\text{which can only be the case if } \mathscr{B}_t = \mathscr{B}^*_t\}.$$

In order to prove this implication, consider first the case $\bar{\mathbf{k}}_t = 0$, i.e. $\mathbf{d}_t = q(t+1) - \mathbf{p}_t$. Then $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) = 0$. Further, from the assumption $\mathscr{B}_{t-1} = \mathscr{B}^*_{t-1}$, we obtain

$$\mathscr{B}_t^\perp \supset \mathscr{B}_{t-1}^\perp + \sigma^* \mathscr{B}_{t-1}^\perp = (\mathscr{B}^*_{t-1})^\perp + \sigma^*(\mathscr{B}^*_{t-1})^\perp = (\mathscr{B}^*_t)^\perp,$$

which is generically equal to $\ker \Pi_t'$. It follows that $\mathscr{B}_t \subset \mathscr{B}^*_t$ and that $\mathscr{B}_t = \mathscr{B}^*_t$ iff $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) = 0$. The required implication follows when $\bar{\mathbf{k}}_t = 0$. When $\bar{\mathbf{k}}_t > 0$, then $\mathbf{d}_t < q(t+1) - \mathbf{p}_t$. In this case

$$(\sqrt{\sigma^t_{q(t+1)\mathbf{c}_t^{\text{adm}}+1}}, 0, \ldots, 0) = \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}^*)$$

Further, from the above lemma and the proposition in Appendix SV we have, in fact, that $\{\dim \mathscr{L} \geqslant q(t+1)(1 - \mathbf{c}_t^{\text{adm}})$ and

$$\mathscr{L} \neq (\mathscr{B}^*_t)^\perp\} \Rightarrow \left\{\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}^*) > (\sqrt{\sigma^t_{q(t+1)\mathbf{c}_t^{\text{adm}}+1}}, 0, \ldots, 0)\right\}.$$

Therefore

$$\{\mathscr{B}_t \neq \mathscr{B}^*_t \text{ and } \dim \mathscr{B}_t \leqslant q(t+1)\mathbf{c}_t^{\text{adm}}\} \Rightarrow \{\varepsilon_t(\mathbf{w}, \mathscr{B}) > \varepsilon_t(\mathbf{w}, \mathscr{B}^*)\}.$$

The implication follows. □

Proof of Theorem 32.

Observe first the following lemma:

Lemma. Generically in $\Pi(\tilde{\mathbf{w}})$, or if $\varepsilon_t^{\text{tol}}$ is non-increasing in t, there holds that the singular vectors u computed at the tth stage of Algorithm 9 satisfy $\mathscr{U}_t \cap \mathscr{L}^*_{t-1} = \{0\}$.

Proof. If $\tilde{\mathbf{w}}$ is generic, the result follows from the lemma in the proof of Theorem 31. If $\varepsilon_t^{\text{tol}}$ is non-increasing in t, the result follows from the comment preceding the statement of Theorem 32. □

Now return to the proof of Theorem 32. It follows from the above lemma that under the conditions of Theorem 32, the procedure STRUCTURE will never have to be invoked and that we will have $\varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}^*) = (\sqrt{\sigma^t_{\bar{\mathbf{d}}_t+1}}, 0, \ldots, 0) \leqslant \varepsilon_t^{\text{tol}}$ (suitably modified at the terminal stage). This yields (i).

Now turn to the proof of (ii) and (iii). If the procedure TERMINATE is invoked, denote by t_{fin} the stage at which it is invoked. It will be shown that

$$\{\mathscr{B} \in \mathbf{L}, \mathscr{B}_{t-1} = \mathscr{B}^*_{t-1}, t < t_{\text{fin}}\}$$
$$\Rightarrow \{\textit{either} \quad \mathbf{c}_t(\mathscr{B}) < \mathbf{c}_t(\mathscr{B}^*)$$
$$\textit{and } \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) > \varepsilon_t^{\text{tol}};$$
$$\textit{or} \quad \mathbf{c}_t(\mathscr{B}) > \mathbf{c}_t(\mathscr{B}^*);$$
$$\textit{or} \quad \mathbf{c}_t(\mathscr{B}) = \mathbf{c}_t(\mathscr{B}^*);$$
$$\textit{and } \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) > \varepsilon t(\tilde{\mathbf{w}}, \mathscr{B}^*);$$
$$\textit{or} \quad \mathbf{c}_t(\mathscr{B}) = \mathbf{c}_t(\mathscr{B}^*),$$
$$\textit{and } \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}) = \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B});$$
$$\text{—which can only be the case provided } \mathscr{B}_t = \mathscr{B}^*_t\}$$

In order to see this, observe first that as a consequence of the proposition in Appendix SV, there holds:

$$\{\dim \mathscr{B}_t < \dim \mathscr{B}^*_t\} \Rightarrow \left\{\max_{a \in \mathscr{B}_t^\perp} \sqrt{a^T \Pi_t' a}/\|a\| \right.$$
$$\left. \geqslant \sqrt{\sigma_{\bar{\mathbf{d}}_t}} > \varepsilon_t^{\text{tol}} \geqslant \sqrt{\sigma_{\bar{\mathbf{d}}_t+1}} = \varepsilon_t^1(\tilde{\mathbf{w}}, \mathscr{B}^*)\right\}$$

Also $\{\dim \mathscr{B}_t = \dim \mathscr{B}^*_t \text{ and } \mathscr{B}_t \neq \mathscr{B}^*_t\} \Rightarrow \{\varepsilon_t|\tilde{\mathbf{w}}, \mathscr{B}) > \varepsilon_t(\tilde{\mathbf{w}}, \mathscr{B}^*)\}$.

Next, it will be shown that

$$\{\mathscr{B} \in \mathbf{L}, \mathscr{B}_{t-1} = \mathscr{B}^*_{t-1}, t = t_{\text{fin}}\} \Rightarrow \{\mathbf{c}_t(\mathscr{B}) \geqslant \mathbf{c}_t(\mathscr{B}^*)$$
$$\textit{and } \varepsilon_{t'}(\tilde{\mathbf{w}}, \mathscr{B}) \geqslant \varepsilon_{t'}(\tilde{\mathbf{w}}, \mathscr{B}^*) \text{ for } t' \geqslant t\}$$

In order to see that $\dim \mathscr{B}_t \geqslant \dim \mathscr{B}^*_t$ observe that otherwise $\mathscr{B}_t$ would lead to more than q independent (AR) relations, contradicting the assumption $\mathscr{B}_{t-1} = \mathscr{B}^*_{t-1}$.

Consequently, $\dim \mathscr{B}_t \geqslant \dim \mathscr{B}^*_t$ which yields, by the proposition in Appendix SV, $\varepsilon_t(\mathbf{w}, \mathscr{B}) \geqslant \varepsilon_t(\mathbf{w}, \mathscr{B}^*)$. For $t' > t_{\text{fin}}$, finally, it is obvious that $\varepsilon_{t'}(\tilde{\mathbf{w}}, \mathscr{B}) \geqslant \varepsilon_{t'}(\tilde{\mathbf{w}}, \mathscr{B}^*) = 0$. The implication follows.

Using these implications recursively in t, yields (ii) and (iii) of Theorem 32. □