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ROBUST STABILIZATION OF UNCERTAIN SYSTEMS*

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Abstract. In this paper we consider the systems described by

$$dx = Ax\,dt + Bu\,dt + \sum_i \sigma_i F_i x\,d\beta_i \quad \text{or} \quad \dot{x} = Ax + Bu + \sum_i B_i F_i(x, t) C_i x,$$

and we will derive conditions under which there exists a feedback control law $u = Kx$ such that the closed loop system is stable for all σ_i or (smooth) nonlinearities F_i. The nonlinearities F_i and the noisy gains $\sigma_i\,d\beta_i$ are unknown uncertainties in the system, and the problem considered is to obtain a control law which is robust against these uncertainties, as far as stability is concerned.

Key words. robustness, feedback stabilization, invariant subspaces, stochastic stabilizability

1. Introduction. Robustness is a very important feature of control system design; it deals with the question whether some relevant qualitative properties, such as stability, are preserved if unknown perturbations are present in the dynamic system. This property is also often called *structural stability*. Consequently, it is of interest to incorporate this property as a feature of control system synthesis.

We consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mathbf{I}} B_i F_i(x, t) C_i x(t). \tag{1}$$

In this equation the last terms represent nonlinear and/or time-varying unknown (deterministic) perturbations. In this paper we will be concerned with the question whether there exists a linear stationary feedback control law $u(t) = Kx(t)$, such that the dynamic system described by (1) remains stable for *all* $F_i(x, t)$ satisfying only a Lipschitz or some smoothness condition. A similar question is analysed for the stochastic system described by the Ito equation

$$dx(t) = Ax(t)\,dt + Bu(t)\,dt + \sum_{i \in \mathbf{I}} \sigma_i F_i x(t)\,d\beta_i(t) \tag{2}$$

where the processes β_i are standard Wiener processes. Intuitively (2) should be regarded as the equation

$$\dot{x}(t) = \left[A + \sum_i F_i f_i(t)\right] x(t) + Bu(t)$$

where the processes $f_i(t)$ are stationary white noise stochastic processes.

There has been some previous work on these stabilizability problems. In [1] conditions have been derived in terms of the solution of an algebraic Riccati equation. In § 2 of the present paper the same question will be reexamined; it is shown that concise stabilizability criteria can be developed by means of geometrical techniques using the concepts of (A, B)-invariant subspaces [2] and almost (A, B)-invariant subspaces [3], [4]. In § 3 the robust stabilization of the deterministic system (1) is

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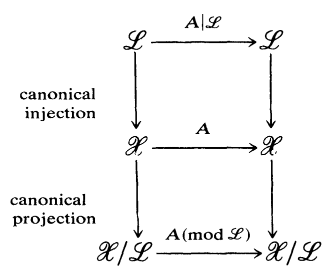
discussed; using techniques similar to those used for the Ito equation (2), criteria for robust stabilization are derived. It is shown that the model and also the results are more general than in Molander's thesis [5], which contains a rather general discussion of the robust stabilization question. A related reference is [6] where the importance of the problem considered here is argued. Finally, a recent special issue of the IEEE Transactions on Automatic Control (February (1981)) demonstrates a great deal of interest in robustness questions from the point of view of control system design.

A few words on the notation used in this paper: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ the complex plane, $\mathbb{C}_g := \{s \in \mathbb{C} | \mathrm{Re}\,(s) < 0\}$, and $\bar{\mathbb{C}}_g := \{s \in \mathbb{C} | \mathrm{Re}\,(s) \leqq 0\}$. If q is a positive integer, then $\mathbf{q} := \{1, 2, \cdots, q\}$. Script capitals are used for vector spaces and subspaces. If $A : \mathscr{X} \to \mathscr{X}$ is linear and $A\mathscr{L} \subset \mathscr{L}$, then $A|\mathscr{L}$ and $A(\mathrm{mod}\,\mathscr{L})$ are the maps defined by the commutative diagram



$\langle \mathscr{K} | A \rangle$ is the largest A-invariant subspace in a given subspace $\mathscr{K}$. $\mathscr{X}_g(A)$ denotes the A-invariant subspace spanned by the eigenspaces of A corresponding to its eigenvalues in $\mathbb{C}_g$; $\mathscr{X}_{\bar{g}}(A)$ is similarly defined with respect to $\bar{\mathbb{C}}_g$. $\sigma(A)$ is the spectrum of A and $\sigma_g(A) := \sigma(A) \cap \mathbb{C}_g$. The kernel (null space) is denoted by Ker and the image (range space) by im.

Finally, for the linear system $\dot{x} = Ax + Bu$, $y = Cx$, with state space $\mathscr{X}$, we use $\langle A | \mathrm{im}\, B \rangle$ for the reachable subspace, i.e. the smallest A-invariant subspace containing $\mathrm{im}\, B$, and $\langle \mathrm{Ker}\, C | A \rangle$ for the nonobservable subspace, i.e. the largest A-invariant subspace contained in Ker C. Finally we will be considering (almost) (A, B)-invariant and controllability subspaces [2], [3], [4] freely; the relevant facts and results are summarized in Appendix D. For a subspace $\mathscr{S}$ of $\mathscr{X}$, $\mathscr{V}^*(\mathscr{S})$, $\mathscr{V}_a^*(\mathscr{S})$, $\mathscr{V}_b^*(\mathscr{S})$ denote respectively the supremal (A, B)-invariant, $\mathscr{L}_\infty$-almost-(A, B)-invariant, and $\mathscr{L}_1$-almost-invariant subspace contained in $\mathscr{S}$, while $\mathscr{R}^*(\mathscr{S})$, $\mathscr{R}_a^*(\mathscr{S})$, $\mathscr{R}_b^*(\mathscr{S})$ denote the similarly defined (almost) controllability subspaces. The subspace $\mathscr{V}_g^*(\mathscr{S})$ is the supremal stabilizable (relative $\mathbb{C}_g$) subspace contained in $\mathscr{S}$, i.e.

$$\mathscr{V}_g^*(\mathscr{S}) = \sup \{\mathscr{V} \subset \mathscr{S} | \exists K \text{ such that } (A + BK)\mathscr{V} \subset \mathscr{V}, \sigma(A + BK) \subset \mathbb{C}_g\}.$$

$\mathscr{V}_{\bar{g}}^*(\mathscr{S})$ is similarly defined relative $\bar{\mathbb{C}}_g$.

2. Robust stabilization of stochastic systems.

2.1. Problem statement. Consider the system described by the Ito stochastic differential equation (2) where, without loss of generality, the Brownian motions β_i are assumed to be zero mean and independent:

$$E[d\beta_i(t)] = 0 \qquad \forall t, \quad \forall i \in \mathbf{l},$$

$$E[d\beta_i(t)^2] = dt \qquad \forall i \in \mathbf{l},$$

$$E[d\beta_i(t) d\beta_j(t)] = 0 \quad \forall i, j \in \mathbf{l}, \quad i \neq j.$$

In other words, β_i is a standard Wiener process. In (2) $x \in \mathscr{X} = \mathbb{R}^n$ denotes the state, $u \in \mathscr{U} = \mathbb{R}^m$ denotes the control input. The constant matrices A, B, F_i have appropriate dimensions. The positive factors σ_i indicate the intensities of the disturbances. The symbol E denotes expectation.

We will consider the stabilizability of (2) by means of a time-invariant memoryless state feedback law

$$u(t) = Kx(t) \tag{3}$$

with K a constant matrix of appropriate dimension. Then (2) reduces to

$$dx(t) = (A + BK)x(t)\,dt + \sum_{i \in \mathbf{I}} \sigma_i F_i x(t)\,d\beta_i(t). \tag{4}$$

For this closed loop system, the mean square asymptotic stability property expressed by the definition below, will be analysed:

DEFINITION 1. System (4) is said to be *mean square asymptotically stable* if for all initial states $x(0)$

$$\lim_{t \to \infty} E[x(t)x(t)^T] = 0.$$

This leads to the following stabilizability definitions:

DEFINITION 2. System (2) is said to be *perfectly robustly stabilizable* if there exists a feedback control (3) such that (4) is mean square asymptotically stable for *all* noise intensities σ_i.

DEFINITION 3. System (2) is said to be *robustly stabilizable for all noise intensities* if for all bounds $\{s_1, \cdots, s_k\}$, there exists a feedback control (3) such that (4) is mean square asymptotically stable for all noise intensities satisfying

$$\sigma_i \leqq s_i \qquad (i \in \mathbf{I}).$$

The property expressed by Definition 3 is somewhat weaker than the property expressed by Definition 2 in that the feedback matrix K may depend on the bounds s_i; some entries of K may increase without bound as some of these bounds s_i tend to infinity.

2.2. Stability of uncontrolled systems with state-dependent noise. In order to derive stabilizability conditions for (2), we first discuss criteria for mean square asymptotic stability of the stochastic system described by the Ito differential equation

$$dx(t) = Ax(t)\,dt + \sum_{i \in \mathbf{I}} \sigma_i F_i x(t)\,d\beta_i(t). \tag{5}$$

This system is autonomous (in the sense that there are no exogenous inputs), but it contains a state-dependent noise term. The second moment matrix

$$M(t) := E[x(t)x(t)^T]$$

satisfies the matrix differential equation

$$\dot{M}(t) = AM(t) + M(t)A^T + \sum_{i \in \mathbf{I}} \sigma_i^2 F_i M(t) F_i^T \tag{6}$$

which evolves in the cone of nonnegative definite symmetric $(n \times n)$ matrices. The mean square stability properties of (5) hence depend on the eigenvalues of the linear mapping L_1 on the linear space of symmetric $(n \times n)$ matrices, defined by

$$L_1(M) := AM + MA^T + \sum_{i \in \mathbf{I}} \sigma_i^2 F_i M F_i^T. \tag{7}$$

The problem considered here is the asymptotic stability of (7) for all noise intensities σ_i. This may be resolved by introducing the subspaces $\mathscr{W}_j$, defined recursively by the following algorithm:

$$\mathscr{W}_0 := \{0\},$$

$$\mathscr{W}_j := \left\langle \bigcap_{i\in\mathbf{l}} F_i^{-1}\mathscr{W}_{j-1} \middle| A \right\rangle,$$

i.e. $\mathscr{W}_j$ is the maximal A-invariant subspace contained in $\bigcap_i F_i^{-1}\mathscr{W}_{j-1}$. It is easily seen by induction that the subspaces $\mathscr{W}_j$ are nested, i.e., $\mathscr{W}_{j+1} \supset \mathscr{W}_j$. Hence

$$\mathscr{W}_\infty := \lim_{j\to\infty} \mathscr{W}_j = \mathscr{W}_n$$

exists and satisfies

$$\mathscr{W}_\infty = \left\langle \bigcap_{i\in\mathbf{l}} F_i^{-1}\mathscr{W}_\infty \middle| A \right\rangle.$$

This limit is obtained monotonically in a finite number of steps.

THEOREM 1. *The following conditions are equivalent*:

(i) $\mathscr{W}_\infty = \mathscr{X}$ *and* $\sigma(A) \subset \mathbb{C}_g$.

(ii) *System* (5) *is mean square asymptotically stable for all* $\{\sigma_i\}$, $i \in \mathbf{l}$.

(iii) *In a suitable basis the matrices* A *and* F_i ($i \in \mathbf{l}$) *take the block triangular form*:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ 0 & A_{22} & \cdots & A_{2q} \\ 0 & 0 & \cdots & A_{3q} \\ & & \ddots & \\ 0 & 0 & \cdots & A_{qq} \end{bmatrix}, \qquad F_i = \begin{bmatrix} 0 & F_{i,12} & \cdots & F_{i,1q} \\ 0 & 0 & \cdots & F_{i,2q} \\ 0 & 0 & \cdots & F_{i,3q} \\ & & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\sigma(A_{ii}) \subset \mathbb{C}_g$ *for* $i \in \mathbf{q}$.

(iv) *The Lie algebra generated by the set of matrices* $\{A, F_i; i \in \mathbf{l}\}$ (*i.e. the smallest Lie algebra containing this set*) *is solvable* [7]; *the matrices* F_i *are nilpotent, and* $\sigma(A) \subset \mathbb{C}_g$.

Proof. The equivalence of (ii), (iii), and (iv) has been shown in [1]. The elegant and computationally feasible geometrical condition (i) is proved in Appendix A. □

The geometrical criterion (i) turns out to be very well suited to attacking the stabilizability problem of system (2). This is the subject of the next section. The possibility of writing $\{A, F_i; i \in \mathbf{l}\}$ in block triangular form is related to the Jordan–Hölder theorem and has been studied in the context of constructing canonical forms for bilinear systems [8]. In fact, through condition (i) Theorem 1 yields a simple test for generalization of the question when a family of nilpotent matrices can be simultaneously triangularized. The solution of this problem is known as Engel's theorem [15]. It is concerned with a basic problem in the theory of Lie algebras, and it has implications in the theory of associative algebras and quivers.

The condition of the above theorem can be simplified if there is only one stochastic element ($l = 1$) with the corresponding F_1 of rank one: $F_1 = b_1c_1$, where b_1 is a column vector and c_1 a row vector; in this case (5) becomes:

$$dx(t) = Ax(t)\,dt + \sigma_1 b_1 c_1 x(t)\,d\beta_1(t).$$

The condition for mean square asymptotic stability for all σ_1 is that the matrix A be Hurwitz (i.e. $\sigma(A) \subset \mathbb{C}_g$) and

$$\mathcal{W}_2 = \mathcal{X},$$

or equivalently

$$\operatorname{im} b_1 \subset \langle \operatorname{Ker} c_1 | A \rangle.$$

This condition is equivalent to

$$c_1 \exp(At) b_1 = 0 \quad \forall t \in \mathbb{R},$$

or

$$c_1 (Is - A)^{-1} b_1 = 0 \quad \forall s \in \mathbb{C}.$$

This decoupling condition [2] is an obvious sufficient condition also for $F_1 = B_1 C_1$ of any rank. However, if the rank of F_1 is larger than one, then the decoupling condition is in general much too strong.

2.3. Feedback stabilizability of stochastic systems. The results of § 2.2 will now be used to analyse the perfect robust stabilizability of (2). This system is perfectly robustly stabilizable if and only if there exists a matrix K such that the matrices $\{A + BK, F_i; i \in \mathbf{I}\}$ satisfy the conditions of Theorem 1. This condition can be made explicit by means of the concept of (A, B)-invariant subspaces and stabilizability subspaces (see Appendix D). To derive the criterion the following definition is required:

DEFINITION 4. Consider the subspace $\mathcal{V}^*_{g,\infty}$ defined by the following recursive algorithm:

$$\mathcal{V}^*_{g,0} := \{0\},$$

$$\mathcal{V}^*_{g,j} := \mathcal{V}^*_g\Big(\bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{V}^*_{g,j-1}\Big),$$

$$\mathcal{V}^*_{g,\infty} := \lim_{j \to \infty} \mathcal{V}^*_{g,j} = \mathcal{V}^*_{g,n}.$$

As was the case for $\mathcal{W}_\infty$, this limit is attained monotonically in a finite number of steps.

THEOREM 2. *System* (2) *is perfectly robustly stabilizable if and only if* $\mathcal{V}^*_{g,\infty} = \mathcal{X}$.

Proof. (i) *The condition is necessary.* Suppose there exists a feedback matrix K such that the conditions of Theorem 1 are satisfied with respect to the system

$$dx(t) = (A + BK)x(t)\,dt + \sum_{i \in \mathbf{I}} \sigma_i F_i x(t)\,d\beta_i(t).$$

Then

$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{j+1} := \Big\langle \bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{W}_j \Big| A + BK \Big\rangle$$

yields $\mathcal{W}_\infty = \mathcal{X}$. Moreover, $\sigma(A + BK) \subset \mathbb{C}_g$. We claim that $\mathcal{V}^*_{g,j} \supset \mathcal{W}_j$. This is easily proved by induction. It is obviously true for $j = 0$. Moreover

$$\mathcal{V}^*_{g,j+1} = \mathcal{V}^*_g\Big(\bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{V}^*_{g,j}\Big) \supset \mathcal{V}^*_g\Big(\bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{W}_j\Big) \supset \Big\langle \bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{W}_j \Big| A + BK \Big\rangle = \mathcal{W}_{j+1}$$

yields the inductive step. Hence $\mathcal{V}^*_{g,\infty} \supset \mathcal{W}_\infty = \mathcal{X}$, which proves the necessity of the condition.

(ii) *The condition is sufficient.* To prove the sufficiency of the condition by means of Theorem 1, we need to show that there exists a single feedback matrix K such that for all j the (A, B)-invariant subspaces $\mathcal{V}^*_{g,j}$ become $(A+BK)$-invariant subspaces with the properties required by Theorem 1, in particular stabilizability and inclusion of $\mathcal{V}^*_{g,j}$ in $\bigcap_i F_i^{-1}\mathcal{V}^*_{g,j-1}$. This is not trivial, since we have no guarantee that the (A, B)-invariant subspaces can be made $(A+BK)$-invariant by means of the *same* matrix K (independent of j). This feature is called *compatibility* of the (A, B)-invariant subspaces $\mathcal{V}^*_{g,j}$ (see Appendix D). In general, compatibility is difficult to analyse. It is not hard to show that the (A, B)-invariant subspaces $\mathcal{V}^*_{g,j}$ are compatible as (A, B)-invariant subspaces, because they are nested ($\mathcal{V}^*_{g,j} \subset \mathcal{V}^*_{g,j+1}$). However, here we have to prove in addition that they are also compatible with respect to the stabilizability and inclusion properties. Let the state space be partitioned as

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_p$$

where p is the integer such that $\mathcal{V}^*_{g,p} = \mathcal{X}$, $\mathcal{V}^*_{g,p-1} \neq \mathcal{X}$, and where the subspaces $\mathcal{X}_j$ are chosen in such a way that for all $j \in \mathbf{p}$,

$$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_j = \mathcal{V}^*_{g,j}. \tag{8}$$

If the conditions of the theorem hold, then for all $j \in \mathbf{p}$ there exists a feedback matrix K_j such that $\sigma(A+BK_j) \subset \mathbb{C}_g$, and

$$\mathcal{V}^*_{g,j} = \left\langle \bigcap_{i \in \mathbf{l}} F_i^{-1}\mathcal{V}^*_{g,j-1} \middle| A+BK_j \right\rangle.$$

Let K_j^i be defined by

$$K_j x = \sum_i K_j^i x_i$$

where x_i is the component of x in $\mathcal{X}_i$. Then we check that the feedback law

$$K^* x = K_1^1 x_1 + K_2^2 x_2 + \cdots + K_p^p x_p$$

makes the subspaces $\mathcal{V}^*_{g,j}$ $(A+BK^*)$-invariant, and such that $\mathcal{V}^*_{g,j} \subset \bigcap_{i \in \mathbf{l}} F_i^{-1}\mathcal{V}^*_{g,j-1}$. In a basis compatible with the above partitioning of the state space, $A^* := A+BK^*$ and F_i have hence the form:

$$A^* = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ & & \ddots & \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \qquad F_i = \begin{bmatrix} 0 & F_{i,12} & \cdots & F_{i,1p} \\ 0 & 0 & \cdots & F_{i,2p} \\ & & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then there exists a transformation of the input and of the state, which does not change the structure of A^* and F_i (by redefining $\mathcal{X}_2, \cdots, \mathcal{X}_p$, such that (8) remains true), but which transforms the input matrix B into the form [9, pp. 543–544]:

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_p \end{bmatrix}.$$

Since $\mathcal{V}^*_{g,1}$ is a stabilizable (A, B)-invariant subspace, the pair (A_{11}, B_1) is stabilizable; there hence exists a partial feedback of the state $K_1 x_1$ such that $\sigma(A_{11}+B_1K_1) \subset \mathbb{C}_g$.

Also $\mathcal{V}^*_{g,2}$ is a stabilizable (A, B)-invariant subspace. Hence the pair

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

is stabilizable. This, however, implies the stabilizability of the pair (A_{22}, B_2). Therefore a feedback K_2x_2 exists such that $\sigma(A_{22}+B_2K_2) \subset \mathbb{C}_g$. Proceeding in this fashion all subspaces $\mathcal{V}^*_{g,j}$ are stabilized without altering the structure of A^* or F_i. □

From the definition of $\mathcal{V}^*_{g,\infty}$ it is immediately clear that an equivalent statement to the criterion of Theorem 2 is as follows:

COROLLARY 1. *System* (2) *is perfectly robustly stabilizable if and only if for some finite integer k*

$$\sum_{i \in \mathbf{I}} \operatorname{im} F_i \subset \mathcal{V}^*_{g,k}. \tag{9}$$

Proof. Condition (9) implies

$$\bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{V}^*_{g,k} = \mathcal{X}$$

and hence

$$\mathcal{V}^*_{g,k+1} = \mathcal{X}.$$ □

Here also the condition can be simplified if there is only one stochastic element and the corresponding matrix F_1 has rank one:

$$dx(t) = [Ax(t) + Bu(t)]\, dt + \sigma_1 b_1 c_1 x(t)\, d\beta_1(t). \tag{10}$$

Then the stabilizability condition becomes

$$\mathcal{V}^*_{g,2} = \mathcal{X}$$

or

$$\operatorname{im} F_1 = \operatorname{im} b_1 \subset \mathcal{V}^*_{g,1} = \mathcal{V}^*_g(\operatorname{Ker} c_1).$$

This condition implies the existence of a feedback matrix K such that $A+BK$ is Hurwitz and $c_1(Is - A - BK)^{-1}b_1$ vanishes identically. The condition of Theorem 2 is then equivalent to the criterion for disturbance decoupling with stability from the disturbance input $\operatorname{im} F_1$ or $\operatorname{im} b_1$ to the output with $\operatorname{Ker} F_1$ or $\operatorname{Ker} c_1$. In general, however, the condition of Theorem 2 is much weaker than the disturbance decoupling requirement.

We notice that the criteria of Theorem 2 or Corollary 1 are also sufficient for feedback stabilizability in cases where:

(i) the stochastic disturbances are zero-mean but not necessarily white, provided they have finite second order moments which are uniformly bounded in time,

(ii) stabilizability with respect to other moments than the second moment is considered.

2.4. High-gain stabilizability of stochastic systems. In this section it is investigated to what extent the criterion of § 2.3 can be relaxed if only stabilizability of system (4) is required for all σ_i; this means that for any $\{\sigma_i\}$ a stabilizing feedback matrix K must exist, such that

$$\dot{M} = (A+BK)M + M(A+BK)' + \sum_{i \in \mathbf{I}} \sigma_i^2 F_i M F_i' \tag{11}$$

is asymptotically stable in the cone of nonnegative definite $(n \times n)$ matrices. Since the matrix K may depend on the noise intensities σ_i, some elements may go to infinity as the noise intensities increase without bound. Then there does not exist a single feedback matrix which stabilizes (4) in the mean square for all noise intensities. Considering the criterion derived in § 2.3, one might be tempted to conjecture that for this type of stabilizability the conditions of Theorem 2 may be relaxed by replacing

- (A, B)-invariant subspaces by almost (A, B)-invariant subspaces [4],
- $\mathbb{C}_g$ by $\bar{\mathbb{C}}_g$.

The notions of almost invariant subspaces have been introduced in [3] and further worked out in [4]. The relevant facts from that reference are summarized in Appendix D.

It is unlikely that the above conjecture is correct because of the high gains involved in the transfer function which results when the gains $\sigma_i \to \infty$. The criterion of Theorem 3 below is not as strong as the above conjecture, but nevertheless it yields a useful relaxation of the conditions of Theorem 2; indeed *in the last step* (A, B)-invariance may be replaced by almost (A, B)-invariance and $\mathbb{C}_g$ by $\bar{\mathbb{C}}_g$.

THEOREM 3. *Let the subspaces $\mathcal{V}^*_{g,j}$ be as defined in* § 2.3. *Let $\mathcal{R}^*_b(\mathcal{S})$ be as defined above. Then system* (2) *is robustly stabilizable for all noise intensities if the pair (A, B) is stabilizable and*

$$\sum_{i \in \mathbf{I}} \operatorname{im} F_i \subset \mathcal{V}^*_{\bar{g}}\Big(\bigcap_{i \in \mathbf{I}} F_i^{-1}\mathcal{V}^*_{g,\infty}\Big) + \mathcal{R}^*_b\Big(\bigcap_{i \in \mathbf{I}} F_i^{-1}\mathcal{V}^*_{g,\infty}\Big). \tag{12}$$

Proof. From the definition of the subspaces $\mathcal{V}^*_{g,j}$, it follows that there exists a constant feedback matrix K such that in an appropriate basis and with the control

$$u(t) = Kx(t) + v(t)$$

the system representation (2) takes the form

$$dx(t) = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,q+1} \\ 0 & A_{2,2} & \cdots & A_{2,q+1} \\ & & \ddots & \\ 0 & 0 & \cdots & A_{q+1,q+1} \end{bmatrix} x(t)\,dt + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{q+1} \end{bmatrix} v(t)\,dt$$

$$+\sum_{i \in \mathbf{I}} \sigma_i \begin{bmatrix} 0 & F_{i,1,2} & \cdots & F_{i,1,q} & F_{i,1,q+1} \\ 0 & 0 & \cdots & F_{i,2,q} & F_{i,2,q+1} \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & F_{i,q,q+1} \\ 0 & 0 & \cdots & 0 & F_{i,q+1,q+1} \end{bmatrix} x(t)\,d\beta_i(t)$$

with $\sigma(A_{i,i}) \subset \mathbb{C}_g$ for $i \in \mathbf{q}$, and with $\mathcal{V}^*_{g,\infty} = \mathcal{V}^*_{g,q}$. The conditions of the theorem imply that the pair $(A_{q+1,q+1}, B_{q+1})$ is stabilizable and that

$$\sum_{i \in \mathbf{I}} \operatorname{im} F_{i,q+1,q+1} \subset \mathcal{V}^*_{\bar{g}}\Big(\bigcap_{i \in \mathbf{I}} \operatorname{Ker} F_{i,q+1,q+1}\Big) + \mathcal{R}^*_b\Big(\bigcap_{i \in \mathbf{I}} \operatorname{Ker} F_{i,q+1,q+1}\Big)$$

where $\mathcal{R}^*_b$ and $\mathcal{V}^*_{\bar{g}}$ are taken relative to $(A_{q+1,q+1}, B_{q+1})$. Corollary B.2 and Lemma A.3 from the appendices then show that the reduced order system

$$dx_{q+1}(t) = A_{q+1,q+1}x_{q+1}(t)\,dt + B_{q+1}v(t)\,dt + \sum_{i \in \mathbf{I}} \sigma_i F_{i,q+1,q+1}x_{q+1}(t)\,d\beta_i(t) \tag{13}$$

is robustly stabilizable for all noise intensities. The remainder of the proof of the theorem now easily follows from the triangular structure of the system equation. □

Theorem 3 is particularly interesting in the special case considered in (10), where there is only one stochastic element and the corresponding matrix F_1 has rank one. Then the criterion for robust stabilizability can be derived from the criterion for perfect robust stabilizability from § 2.3 by just replacing (A, B)-invariance by almost (A, B)-invariance and $\mathbb{C}_g$ by $\bar{\mathbb{C}}_g$: system (10) is robustly stabilizable for all noise intensities if and only if (A, B) is stabilizable and

$$\operatorname{im} b_1 \subset \mathcal{V}_{\bar{g}}^*(\operatorname{Ker} c_1) + \mathcal{R}_b^*(\operatorname{Ker} c_1).$$

Suppose in addition that there is only one input: B is a column vector which is denoted by b. Then the perfect robust stabilizability and the high-gain stabilizability conditions for system (10) can be expressed in terms of the transfer function

$$F(s) := \frac{c_1(Is - A)^{-1} b_1}{c_1(Is - A)^{-1} b}.$$

We assume that (A, c_1) is a detectable pair [2]; this entails no loss of generality, since the stabilizability of the pair (A, b) implies that there exists a feedback vector k such that $(A + bk, c_1)$ is detectable. System (10) is perfectly robustly stabilizable if and only if (A, b) is stabilizable, $F(s)$ is strictly proper, and, after cancellation of common factors, $F(s)$ has no poles with nonnegative real parts. System (10) is robustly stabilizable for all noise intensities if and only if (A, b) is stabilizable and, after cancellation of common factors, $F(s)$ has no poles with positive real parts.

3. Robust stabilization of uncertain deterministic systems.

3.1. Problem formulation. In this section we consider the deterministic counterpart of the problem analysed in § 2. Here the question is: When can a system with an unknown nonlinear and/or time-varying element can be stabilized by means of a linear state feedback regulator? In § 1 we introduced the class of systems (1) which we have in mind. However, this equation may be written in the following form, which makes it more alike to the system considered in § 2:

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mathbf{l}} f_i(x(t), t) F_i x(t). \tag{14}$$

This formulation has (1) as a special case. To see this write the nonlinear term in (1) as

$$B_i F_i(x, t) C_i = \sum_{r,s} [F_i(x, t)]_{r,s} [B_i]_r [C_i]_s$$

where $[B_i]_r$ denotes the rth column, $[C_i]_s$ denotes the sth row, and $[F_i(x, t)]_{r,s}$ denotes the (r, s) entry of B_i, C_i and $F_i(x, t)$, respectively. The system formulation (14) also has as a special case the system

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mathbf{k}} \sum_{j \in \mathbf{l}} B_{ij} F_i(x(t), t) C_{ij} x(t), \tag{15}$$

which is perhaps the most logical starting point for the class of robustness problems considered here.

DEFINITION 5. We say that (14) is *perfectly robustly stabilizable* if there exists a feedback law (3) such that the null solution of

$$\dot{x}(t) = (A + BK)x(t) + \sum_{i \in \mathbf{l}} f_i(x(t), t) F_i x(t) \tag{16}$$

is asymptotically stable in the large for all bounded nonlinear and/or time-varying gains $f_i(x, t)$. We assume throughout that the gains $f_i(x, t)$ are sufficiently smooth, e.g. Lipschitz, such that the existence and the uniqueness of the solution of (1) is ensured. We say that (14) is *robustly stabilizable for all uncertain gains* if for any set $\{m_i; i \in \mathbf{I}\}$, there exists a control law (3) such that the null solution of (16) is asymptotically stable in the large for all $f_i(x, t)$ satisfying $|f_i(x, t)| < m_i$.

Note again that in the second formulation K may depend on the bounds m_i, while in the first formulation this is not possible. The results which will be obtained, actually imply the stabilizability of the system with the structure of (14) in which the nonlinearity f_i is any $\mathcal{L}_2$-input/output stable operator. The robust stabilizability problem for the related but more restricted class of systems

$$\dot{x}(t) = Ax(t) + Bu(t) + Gf(Hx, t) \tag{15'}$$

has been studied previously by Molander [5] in essentially the same setting. Without actually introducing almost invariant subspaces he does obtain results which are important special cases of ours. Specifically he shows that (15′) is robustly stabilizable if the system (A, B, G, H) may be stably almost disturbance decoupled in the $\mathcal{L}_1$-sense. This result is a special case of our Theorem 6; it requires that

$$\operatorname{im} G \subset \mathcal{V}_g^*(\operatorname{Ker} H) + \mathcal{R}_b^*(\operatorname{Ker} H).$$

A similar result has been obtained independently in [4, Thm. 17].

3.2. Criterion for perfect robustness. In order to derive a criterion for robust stabilizability we first consider the uncontrolled system

$$\dot{x}(t) = Ax(t) + \sum_{i \in \mathbf{I}} f_i(x(t), t) F_i x(t) \tag{17}$$

and investigate when the null solution of this system is asymptotically stable in the large for all bounded functions $f_i(x, t)$. Our sufficient conditions are:

(i) the matrix A is Hurwitz;

(ii) the matrices F_i are nilpotent;

(iii) the matrices $\{A, F_i, i \in \mathbf{I}\}$ can be transformed to upper block triangular form by means of the same similarity transformation.

Expressed geometrically this yields:

THEOREM 4. *The null solution of* (17) *is asymptotically stable in the large for all bounded gains* $f_i(x, t)$, *if the matrix* A *is Hurwitz, and* $\mathcal{W}_\infty = \mathcal{X}$, *with* $\mathcal{W}_\infty$ *defined preceding Theorem* 1.

This result follows immediately from Theorem 1. By means of Theorem 4 and the ideas used in proving Theorem 2 from Theorem 1, we obtain:

THEOREM 5. *Let* $\mathcal{V}_{g,\infty}^*$ *be as in Definition* 4. *Then* (14) *is perfectly robustly stabilizable if* $\mathcal{V}_{g,\infty}^* = \mathcal{X}$.

The condition of Theorem 5 is of course equivalent to $\sum_{i \in \mathbf{I}} \operatorname{im} F_i \subset \mathcal{V}_{g,q}^*$ for some integer q. The criteria of Theorems 4 and 5 can be simplified in the cases

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 F_1(x(t), t) C_1 x(t), \tag{18}$$

in which case the criterion requires that the system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 d(t), \qquad z(t) = C_1 x(t)$$

should be disturbance decouplable with internal stability by state feedback [2]. In the more general situation

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mathbf{I}} B_i F_i(x(t), t) C_i x(t),$$

the criterion requires that the system

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i \in \mathbf{I}} B_i d_i(t), \qquad z_i(t) = C_i x(t), \qquad i \in \mathbf{I},$$

should be strictly triangularly disturbance decouplable with internal stability in the sense that there should exist a feedback K with $\sigma(A+BK) \subset \mathbb{C}_g$ such that in the closed loop system

$$\dot{x}(t) = (A+BK)x(t) + \sum_{i \in \mathbf{I}} B_i d_i(t), \qquad z_i(t) = C_i x(t), \qquad i \in \mathbf{I},$$

there should exist a permutation of $\mathbf{I}$ such that the resulting transfer function $(d_1, d_2, \cdots, d_k) \mapsto (z_1, z_2, \cdots, z_k)$ is strictly upper block triangular.

3.3. Criterion for robustness for all uncertain gains. As in § 2.4, it is tempting to conjecture from Theorem 5 that robustness for all uncertain gains would be achievable under the conditions of Theorem 5, but with almost (A, B)-invariance replacing (A, B)-invariance and $\bar{\mathbb{C}}_g$ replacing $\mathbb{C}_g$. However, since the stabilizability condition in this case comes down to impulse response quenching in the $\mathscr{L}_1$-sense, it is not possible to replace $\mathbb{C}_g$ by $\bar{\mathbb{C}}_g$ (see the example at the end of Appendix C). Nevertheless it is possible to use almost (A, B)-invariant subspaces *in the last step* of the algorithm of Theorem 5.

THEOREM 6. *Let the subspace* $\mathscr{V}^*_{g,\infty}$ *be as defined in* § 2.3 *and let* $\mathscr{R}^*_b(\bigcap_{i\in\mathbf{I}} F_i^{-1}\mathscr{V}^*_{g,\infty})$ *be as defined preceding Theorem* 3. *Then* (14) *is robustly stabilizable for all uncertain gains if* (A, B) *is stabilizable and*

$$\sum_{i\in\mathbf{I}} \operatorname{im} F_i \subset \mathscr{V}^*_{g,\infty} + \mathscr{R}^*_b\Big(\bigcap_{i\in\mathbf{I}} F_i^{-1}\mathscr{V}^*_{g,\infty}\Big). \tag{19}$$

Proof. The proof of this theorem follows exactly the same route as the proof of Theorem 3 except where Lemma A.3 of Appendix A was used. Here instead Proposition C.1 of Appendix C yields the result. □

Note that condition (19) could equivalently be expressed as

$$\sum_{i\in\mathbf{I}} \operatorname{im} F_i \subset \mathscr{V}^*_g\Big(\bigcap_{i\in\mathbf{I}} F_i^{-1}\mathscr{V}^*_{g,\infty}\Big) + \mathscr{R}^*_b\Big(\bigcap_{i\in\mathbf{I}} F_i^{-1}\mathscr{V}^*_{g,\infty}\Big). \tag{20}$$

This shows more clearly the relationship between Theorems 3 and 6.

It is straightforward to specialize the result of Theorem 6 to the case where, as in equation (10) for the stochastic case, there is only one nonlinear term; the corresponding matrix F_1 has rank one, and there is only one input:

$$\dot{x}(t) = Ax(t) + bu(t) + f_1(x(t), t) b_1 c_1 x(t) \tag{21}$$

where the same notation as in (10) is used, and b is a column vector. Suppose (A, c_1) detectable and (A, b) stabilizable. This system is perfectly robustly stabilizable if the transfer function $F(s)$, defined in § 2.4, (i) is strictly proper and (ii), after cancellation of common factors, has only poles with negative real parts. It is robustly stabilizable for all uncertain gains if (ii) holds.

The conditions of Theorem 3 are in general not sufficient to guarantee robust stabilizability for all uncertain gains in the deterministic case. This distinction is an intrinsic one and may be illustrated by means of (21) and

$$dx(t) = Ax(t)\,dt + bu(t)\,dt + \sigma_1 b_1 c_1 x(t)\,d\beta_1(t). \tag{22}$$

Robust stabilizability of (22) for all noise intensities requires for any $\varepsilon > 0$ the existence of a feedback vector k such that $\sigma(A+bk) \subset \mathbb{C}_g$ and

$$\int_0^\infty w(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\,d\omega \leqq \varepsilon$$

where

$$w : t \mapsto c_1 \exp\,[(A+bk)t]b_1,$$
$$H : s \mapsto c_1(Is - A - bk)^{-1}b_1.$$

On the other hand, robust stabilizability of (21) for all uncertain gains requires

$$\int_0^\infty |w(t)|\,dt \leqq \varepsilon \quad \text{or} \quad \sup_{\omega\in\mathbb{R}} |H(j\omega)| \leqq \varepsilon.$$

Take for example

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_1 = [0 \quad 1].$$

Then system (22) is stabilizable for any noise intensity. However, (21) is not perfectly robustly stabilizable. Even for linear time-invariant gains $f(x, t) = k$, there does not exist a feedback strategy which stabilizes the system at the same time for all gains satisfying $|k| < k_{\max}$ if $k_{\max} > 1$. This is in agreement with the above reasoning. Indeed, taking

$$u(t) = kx(t) = [-\alpha \quad -\beta]x(t)$$

yields the closed loop transfer function

$$H(s) = \frac{-\alpha}{s^2 + (1+\alpha+\beta)s + \alpha}.$$

The condition $\sigma(A+bk) \subset \mathbb{C}_g$ requires $\alpha > 0$, $1+\alpha+\beta > 0$. Now $|H(0)| = 1$ cannot be influenced by α and β, whereas

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\,d\omega = \frac{\alpha}{2(1+\alpha+\beta)}$$

can indeed be made arbitrarily small.

4. Discrete-time systems. A similar analysis can be performed on the stabilizability of the discrete-time stochastic system

$$x_{t+1} = Ax_t + Bu_t + \sum_{i\in\mathbf{I}} \sigma_i F_i x_t f_{it} \tag{23}$$

where the scalar processes f_{it} are zero mean uncorrelated normalized white noise processes, and with respect to the robustness of the nonlinear discrete-time deterministic system

$$x_{t+1} = Ax_t + Bu_t + \sum_{i\in\mathbf{I}} f_i(x_t, t)F_i x_t. \tag{24}$$

It follows from Appendix A that the criteria for perfect robust stabilizability of (23) and of (24) are exactly the same as in the continuous-time case, provided of course $\operatorname{Re}(s) < 0$ is replaced by $|z| < 1$. However for robust stabilizability of (23) for all noise intensities or robust stabilizability of (24) for all uncertain gains, it is not possible to relax the conditions as much as in the continuous-time cases. For (24) in fact no relaxation has been obtained. For robust stabilizability for all noise intensities of (23) it is possible to replace the condition $|z|<1$ by $|z|\leqq 1$ in the last step.

The distinction between discrete-time and continuous-time systems can be seen as follows. The feedback strategy $u(t)=Kx(t)$ stabilizes the stochastic continuous-time system (2) if and only if the linear mapping

$$M \mapsto L_c(M) := (A+BK)M + M(A+BK)^T + \sum_{i\in \mathbf{I}} \sigma_i^2 F_i M F_i^T \tag{25}$$

has only eigenvalues with negative real parts. The feedback strategy $u_t = Kx_t$ stabilizes the stochastic discrete-time system (23) if and only if the linear mapping

$$M \mapsto L_d(M) := (A+BK)M(A+BK)^T + \sum_{i\in \mathbf{I}} \sigma_i^2 F_i M F_i^T \tag{26}$$

has only eigenvalues with magnitude smaller than 1. The eigenvalues of $L_d(M)$ are larger than the eigenvalues of the mappings

$$M \mapsto L_i(M) := \sigma_i^2 F_i M F_i^T.$$

The eigenvalues of $L_i(M)$ are $\sigma_i^2 \lambda_a(F_i)\lambda_b(F_i)$, where $\lambda_a(F_i)$ and $\lambda_b(F_i)$ are arbitrary eigenvalues of F_i. Hence the existence of a stabilizing feedback for all noise intensities requires that the matrices F_i have only zero eigenvalues. This is also true if the feedback matrix K is allowed to depend on the noise intensities $\{\sigma_i\}$, hence for the property of *robust stabilizability for all noise intensities.* A similar conclusion is not valid however for continuous-time systems.

5. Example. In this section the application of the criteria developed in §§ 2, 3, and 4, is illustrated on the example [2] of a second-order system with the data:

$$A=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix},\quad b=\begin{bmatrix}0\\1\end{bmatrix},\quad F_1=b_1c_1,\quad b_1=\begin{bmatrix}2\\-1\end{bmatrix},\quad c_1=[a \quad 1].$$

The continuous-time and discrete-time, stochastic and deterministic, cases will be examined:

$$dx(t)=Ax(t)\,dt+bu(t)\,dt+\sigma_1F_1x(t)\,d\beta_1(t), \tag{27}$$

$$\dot{x}(t)=Ax(t)+bu(t)+f_1(x(t),t)F_1x(t), \tag{28}$$

$$x_{t+1}=Ax_t+bu_t+\sigma_1F_1x_tf_{1t}, \tag{29}$$

$$x_{t+1}=Ax_t+bu_t+f_1(x_t,t)F_1x_t. \tag{30}$$

For the continuous-time case we obtain:

(i) $a<0$: $\mathcal{V}^*_{g,\infty}=\{0\}$, $\mathcal{V}^*_{\bar{g}}(F_1^{-1}\mathcal{V}^*_{g,\infty})=\{0\}$, $\mathcal{R}^*_b(F_1^{-1}\mathcal{V}^*_{g,\infty})=\operatorname{im} b$,

(ii) $a=0$: $\mathcal{V}^*_{g,\infty}=\{0\}$, $\mathcal{V}^*_{\bar{g}}(F_1^{-1}\mathcal{V}^*_{g,\infty})=\operatorname{im}\begin{bmatrix}1\\0\end{bmatrix}$, $\mathcal{R}^*_b(F_1^{-1}\mathcal{V}_{g,\infty})=\operatorname{im} b$,

(iii) $a>0$, $a\neq .5$: $\mathcal{V}^*_{g,\infty}=\operatorname{im}\begin{bmatrix}1\\-a\end{bmatrix}$, $\mathcal{R}^*_b(F_1^{-1}\mathcal{V}^*_{g,\infty})=\operatorname{im} b$,

(iv) $a=.5$: $\mathcal{V}^*_{g,\infty}=\mathcal{X}$.

For the discrete-time case the results are:

(i) $a=.5$: $\mathcal{V}^*_{g,\infty}=\mathcal{X}$,

(ii) $|a|<1$, $a\neq .5$: $\mathcal{V}^*_{g,\infty}=\operatorname{im}\begin{bmatrix}1\\-a\end{bmatrix}$, $\mathcal{V}^*_{\bar{g}}(F_1^{-1}\mathcal{V}^*_{g,\infty})=\mathcal{V}^*_{g,\infty}$,

(iii) $|a|=1$: $\mathcal{V}^*_{g,\infty}=\{0\}$, $\mathcal{V}^*_{\bar{g}}(F_1^{-1}\mathcal{V}^*_{g,\infty})=\operatorname{im}\begin{bmatrix}1\\-a\end{bmatrix}$,

(iv) $|a|>1$: $\mathcal{V}^*_{g,\infty}=\{0\}$, $\mathcal{V}^*_{\bar{g}}(F_1^{-1}\mathcal{V}^*_{g,\infty})=\{0\}$.

Hence the stabilizability criteria are derived:

(i) The stochastic continuous-time system (27) is perfectly robustly stabilizable if $a=.5$. It is robustly stabilizable for all noise intensities if $a\geqq 0$.

(ii) The deterministic continuous-time system (28) is perfectly robustly stabilizable if $a=.5$. It is robustly stabilizable for all uncertain gains if $a>0$.

(iii) The stochastic discrete-time system (29) is perfectly robustly stabilizable if $a=.5$. The same condition holds for robust stabilizability for all noise intensities.

(iv) The deterministic discrete-time system (30) is perfectly robustly stabilizable if $a=.5$. No relaxation of this condition is obtained for robust stabilizability for all uncertain gains.

Appendix A. The first part of this appendix is relevant to the proof of Theorem 1. We consider the following linear mappings in the space of $(n\times n)$ symmetric matrices:

(A.1) $$M\mapsto L_1(M):=AM+MA^T+\sum_{i=1}^{l}\sigma_i^2F_iMF_i^T,$$

(A.2) $$M\mapsto L_2(M):=\sum_{i=1}^{l}\sigma_i^2F_i\int_0^\infty \exp(A\tau)M\exp(A^T\tau)\,d\tau\,F_i^T,$$

(A.3) $$M\mapsto L_3(M):=AMA^T+\sum_{i=1}^{l}\sigma_i^2F_iMF_i^T,$$

(A.4) $$M\mapsto L_4(M):=\sum_{i=1}^{l}\sigma_i^2F_i\sum_{j=0}^{\infty}A^jMA^{Tj}F_i^T,$$

where L_2 is only defined if A is a Hurwitz matrix, i.e. $\sigma(A)\subset\mathbb{C}_g$, and where L_4 is only defined if A has only eigenvalues smaller than 1 in modulus. Since L_2 and L_4 map the cone of nonnegative definite matrices into itself, it follows that the largest eigenvalue of L_2 and L_4 is real and positive, and that it increases with increasing $\sigma_1,\sigma_2,\cdots,\sigma_l$.

LEMMA A.1. (i) *The linear mapping L_1 has all its eigenvalues in $\mathbb{C}_g$ if and only if the matrix A is a Hurwitz matrix and the mapping L_2 has only eigenvalues with modulus smaller than* 1.

(ii) *The linear mapping L_3 has all its eigenvalues inside the open unit disk of the complex plane if and only if all eigenvalues of the matrix A and of the mapping L_4 are smaller than* 1 *in modulus.*

Part (ii) follows from an earlier paper [10]; part (i) is proved in a similar fashion and is left to the reader. The lemma can also be obtained using the analysis of [11]. Lemma A.1 yields the following theorem:

THEOREM A.1. (i) *The mapping L_1 has all its eigenvalues in $\mathbb{C}_g$ for all $\{\sigma_i; i \in \mathbf{I}\}$ if and only if the eigenvalues of the mapping L_2 vanish for some nonzero values of $\sigma_1, \cdots, \sigma_l$. In this case all eigenvalues of L_2 vanish for all $\{\sigma_i; i \in \mathbf{I}\}$, i.e. L_2 is nilpotent; moreover, the eigenvalues of L_1 are independent of $\{\sigma_i; i \in \mathbf{I}\}$.*

(ii) *The mapping L_3 has all its eigenvalues in the open unit disk $\mathcal{D}_g := \{z \in \mathbb{C} \,|\, |z| < 1\}$ for all $\{\sigma_i; i \in \mathbf{I}\}$ if and only if the eigenvalues of the mapping L_4 vanish for some nonzero values of $\sigma_1, \cdots, \sigma_l$. In this case all eigenvalues of L_4 vanish for all $\{\sigma_i; i \in \mathbf{I}\}$, i.e., L_4 is nilpotent; moreover, the eigenvalues of L_3 are independent of $\{\sigma_i; i \in \mathbf{I}\}$.*

Let the subspaces $\mathcal{W}_j$ and $\mathcal{W}_\infty$ be defined as in § 2.2, preceding Theorem 1.

LEMMA A.2. *The following statements are equivalent:*

(i) $\mathcal{W}_\infty = \mathcal{X}$,

(ii) *L_2 is nilpotent,*

(iii) *L_4 is nilpotent.*

Proof. Only (i)$\Leftrightarrow$(ii) is proven; (i)$\Leftrightarrow$(iii) is completely similar. We need to prove that $L_2^m(M) = 0$ for all $M = M^T$ and $m \geqq n(n+1)/2$.

(i) *The condition is sufficient.* Let $x \in \mathcal{W}_j$. Compute $L_2(xx^T)$. Because of the definition of $\mathcal{W}_j$, we have

$$\exp(At)x \in \mathcal{W}_j$$

and

$$F_i \exp(At)x \in \mathcal{W}_{j-1} \qquad (\forall i \in \mathbf{I}).$$

Hence

$$L_2(xx^T) = \sum_k y_k y_k^T$$

with all $y_k \in \mathcal{W}_{j-1}$. Repeatedly applying L_2 yields

$$L_2^\alpha(xx^T) = 0$$

for $\alpha \geqq j$. This proves the sufficiency of the condition since any symmetric matrix can be expressed as the linear combination of dyads of the form xx^T.

(ii) *The condition is necessary.* If $\mathcal{W}_\infty \neq \mathcal{X}$, then

$$\mathcal{W}_\infty = \left\langle \bigcap_{i \in \mathbf{I}} F_i^{-1} \mathcal{W}_\infty \middle| A \right\rangle.$$

Let $x^* \in \mathcal{X}$ and $x^* \notin \mathcal{W}_\infty$. Consider $L_2(x^* x^{*T})$; from the above property of $\mathcal{W}_\infty$ it follows that $F_i \exp(At)x^* \in \mathcal{W}_\infty$ cannot be true for all i and all t. Hence

$$L_2(x^* x^{*T}) = \sum_k y_k y_k^T$$

where at least one of the vectors $y_k \notin \mathcal{W}_\infty$. Repeatedly applying L_2 yields that $L_2(x^* x^{*T})$ cannot vanish for any integer α. □

The second part of this appendix is relevant to the proof of Theorem 3.

LEMMA A.3. *For all $\{F_i; i \in \mathbf{I}\}$ and $\{K_i < \infty; i \in \mathbf{I}\}$ there exist bounds $\{\alpha_{ij} > 0; i, j \in \mathbf{I}\}$ such that system (5) is mean square asymptotically stable for all noise intensities $\{\sigma_i | \sigma_i \leqq K_i\}$ and all system matrices A such that $\sigma(A) \subset \mathbb{C}_g$ and*

$$\int_0^\infty \|F_i \exp(At) F_j\|^2 \, dt < \alpha_{ij} \qquad (i, j \in \mathbf{I}).$$

Proof. The largest eigenvector λ^* of the mapping L_2, defined by (A.2), corresponds to a nonnegative definite eigenvector M^* which, for nonzero λ^*, is of the form

$$M^* = \sum_{j \in \mathbf{I}} F_j N_j F_j^T.$$

Let M_j denote $F_j N_j F_j^T$. We have $N_j \geqq 0$, $M_j \geqq 0$, $j \in \mathbf{I}$. The eigenvalue equation

$$\lambda^* M^* = \sum_{\substack{i \in \mathbf{I} \\ j \in \mathbf{I}}} \sigma_i^2 F_i \int_0^\infty \exp(A\tau) F_j N_j F_j^T \exp(A^T \tau)\, d\tau\, F_i^T$$

leads to

$$\lambda^* M^* = \sum_{\substack{i \in \mathbf{I} \\ j \in \mathbf{I}}} \sigma_i^2 F_i \int_0^\infty \exp(A\tau) F_j F_j^+ F_j N_j F_j^T F_j^{+T} F_j^T \exp(A^T \tau)\, d\tau\, F_i^T$$

where F_j^+ denotes the generalized inverse [12, pp. 142–144] of the matrix F_j. This yields

$$\lambda^* M^* = \sum_{\substack{i \in \mathbf{I} \\ j \in \mathbf{I}}} \sigma_i^2 F_i \int_0^\infty \exp(A\tau) F_j F_j^+ M_j F_j^{+T} F_j^T \exp(A^T \tau)\, d\tau\, F_i^T$$

and

$$|\lambda| \leqq \|M^*\|^{-1} \sum_{\substack{i \in \mathbf{I} \\ j \in \mathbf{I}}} \sigma_i^2 F_j^{+2} \int_0^\infty \|F_i \exp(A\tau) F_j\|^2\, d\tau\, \|M_j\|.$$

The matrix M^* can be taken to be of unit norm; since $M^* = \sum M_j$ and since the matrices M_j are symmetric and nonnegative definite, then $\|M_j\| \leqq 1$, $j \in \mathbf{I}$. Hence the eigenvalues of the mapping L_2 are smaller than 1 in modulus if the constants α_{ij} are sufficiently small. □

Appendix B. In this appendix the following problem is investigated: let A, B, G, H, respectively, be $(n \times n)$, $(n \times m)$, $(n \times q)$, $(p \times n)$ matrices; we want to state conditions on these matrices such that for all $\varepsilon > 0$ there exists an $(m \times n)$ feedback matrix K such that

(i) $$\sigma(A + BK) \subset \mathbb{C}_g,$$

(ii) $$\int_0^\infty \|W_K(t)\|^2\, dt \leqq \varepsilon$$

where

$$W_K : t \in R^+ \mapsto H \exp[(A + BK)t] G.$$

This property is called *impulse response quenching in the $\mathscr{L}_2$-sense with internal asymptotic stability*. It is well known that a constant K exists such that (i) is true and $W_K = 0$ if and only if

$$\operatorname{im} G \subset \mathscr{V}_g^*(\operatorname{Ker} H).$$

If it is only required that (ii) hold, i.e., that W_K can be made arbitrarily small in the $\mathscr{L}_2$-sense, then one could expect two refinements:

(i) $\mathscr{V}_g^*(\operatorname{Ker} H)$ may be replaced by $\mathscr{V}_{\bar{g}}^*(\operatorname{Ker} H)$ since by a small feedback the eigenvalues can be shifted from the imaginary axis into the left half plane.

(ii) im G may be allowed to have a component in $\mathcal{R}_b^*(\text{Ker } H)$, since in that space it is possible to make W_K arbitrarily small by high gain feedback [4].

The following result is indeed obtained.

THEOREM B.1. *Impulse response quenching in the $\mathcal{L}_2$-sense with internal asymptotic stability is possible if*

(i) (A, B) *is stabilizable (relative* $\mathbb{C}_g$),

(ii) im $G \subset \mathcal{R}_b^*(\text{Ker } H) + \mathcal{V}_g^*(\text{Ker } H)$.

The proof of this theorem proceeds via a number of propositions and lemmas:

PROPOSITION B.1. *Assume that* (A, B) *is stabilizable (relative* $\mathbb{C}_g$) *and that* $x_0 \in \mathcal{V}_g^*(\text{Ker } H)$. *Consider now*

$$J(x_0) := \inf \int_0^\infty \|y(t)\|^2 \, dt$$

subject to: $\dot{x} = Ax + Bu$, $y = Hx$, $x(0) = x_0$, $u \in \mathcal{L}_2(0, \infty)$, $x \in \mathcal{L}_2(0, \infty)$. *Then* $J(x_0) = 0$.

In order to prove this proposition, we start with a lemma.

LEMMA B.1. *Assume* (A, B) *controllable and* $\sigma(A) \subset \{s \in \mathbb{C} | \text{Re}(s) = 0\}$. *Then*

$$\lim_{t_f \to \infty} W^{-1}(0, t_f) = 0$$

where

$$W(0, t_f) := \int_0^{t_f} \exp(-A\sigma) BB^T \exp(-A^T\sigma) \, d\sigma.$$

Proof. It suffices to prove that $a^T W(0, t_f) a = M_{t_f} \|a\|^2$ with $\lim_{t_f \to \infty} M_{t_f} = \infty$. By controllability of (A, B) there is a $\delta > 0$ such that

$$\int_0^1 \|B^T \exp(-A^T\sigma) a\|^2 \, d\sigma \geqq \delta \|a\|^2.$$

Now, since $\sigma(A) \subset \{s \in \mathbb{C} | \text{Re}(s) = 0\}$, the solutions of $\dot{x} = -A^T x$ have the property that there exists $T > 1$ such that $\|x(T)\|^2 \geqq \|x(0)\|^2$ (to see this, assume A in Jordan form: if A is semisimple, it is immediate, otherwise it follows from some simple estimates). This yields

$$\int_0^{NT} \|B^T \exp(-A^T\sigma) a\|^2 \, d\sigma \geqq N\delta \|a\|^2.$$

This yields the desired growth of $W(0, t_f)$. □

LEMMA B.2. *Assume* (A, B) *controllable,* x_0 *given and* $\sigma(A) \subset \{s \in \mathbb{C} | \text{Re}(s) = 0\}$. *Then, for all* $\varepsilon > 0$, *there exist* $T > 0$ *and* $u \in \mathcal{L}_2(0, \infty)$ *such that the solution of* $\dot{x} = Ax + Bu$, $x(0) = x_0$, *satisfies* $x(T) = 0$ *and* $\|u\|_{\mathcal{L}_2(0,\infty)} \leqq \varepsilon$.

Proof. Consider, for t_f fixed, $J(x_0) := \min \int_0^{t_f} \|u(t)\|^2 \, dt$ subject to $\dot{x} = Ax + Bu$, $x(0) = x_0$, $x(t_f) = 0$. It is well known (see [13, p. 137]) that $J(x_0) = x_0^T W^{-1}(0, t_f) x_0$. The result follows then from Lemma B.1. □

Proof of Proposition B.1. Since (A, B) is stabilizable (relative $\mathbb{C}_g$), there exists K such that

$$(A + BK)\mathcal{V}_g^*(\text{Ker } H) \subset \mathcal{V}_g^*(\text{Ker } H),$$

$$\sigma[(A + BK) | \mathcal{V}_g^*(\text{Ker } H)] \subset \mathbb{C}_g,$$

$$\sigma[(A + BK)(\text{mod } \mathcal{V}_g^*(\text{Ker } H))] \subset \mathbb{C}_g.$$

By suitably choosing the basis, this yields

$$\mathscr{X}=\mathscr{X}_1\oplus\mathscr{X}_2, \qquad A+BK=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

with

$$\sigma(A_1)\subset\{s\in\mathbb{C}|\mathrm{Re}\,(s)=0\}, \quad \sigma(A_2)\subset\mathbb{C}_g, \quad B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad H=[0 \quad H_2].$$

Furthermore, since (A,B) is stabilizable, (A_1,B_1) will be controllable. From $x_0\in\mathscr{V}_g^*(\mathrm{Ker}\,H)$ it follows that $H\exp[(A+BK)t]x_0$ vanishes for all t. Lemma B.2 implies that for all $\varepsilon>0$, there exists u and $T>0$ such that $\|u\|_{\mathscr{L}_2(0,\infty)}\leqq\varepsilon$, $x(0)=x_0$, and $x(T)\in\mathscr{X}_2$. This yields

$$y(t)=H_2\int_0^t \exp[A_2(t-\tau)]B_2u(\tau)\,d\tau$$

which, since $\sigma(A_2)\subset\mathbb{C}_g$, is the convolution of an $\mathscr{L}_1$-kernel $\{t\mapsto\exp(A_2t)B\}$ with an arbitrary small $u\in\mathscr{L}_2(0,\infty)$. Hence y is arbitrarily small in the $\mathscr{L}_2$-norm. It is also immediate that the corresponding $x\in\mathscr{L}_2(0,\infty)$. This yields $J(x_0)=0$, as desired. □

PROPOSITION B.2. *Assume* $x_0\in\mathscr{R}_b^*(\mathrm{Ker}\,H)$. *Consider now* $J(x_0):=\inf\int_0^\infty\|y(t)\|^2\,dt$, *subject to* $\dot{x}=Ax+Bu$; $y=Hx$; $x(0)=x_0$, $u\in\mathscr{L}_2(0,\infty)$, $x\in\mathscr{L}_2(0,\infty)$. *Then* $J(x_0)=0$.

Proof. That $\int_0^\infty\|y(t)\|^2\,dt$ vanishes without the constraints $u\in\mathscr{L}_2(0,\infty)$ and $x\in\mathscr{L}_2(0,\infty)$ follows immediately from [4, Thm. 10]. However, it is easily seen by examining the proof that the u and x used for showing that this infimum is zero are indeed $\mathscr{L}_2$-functions. This yields the proposition. □

Proof of Theorem B.1. Consider the least squares control for the system $\dot{x}=Ax+Bu$ with cost functional, with $\varepsilon>0$

$$\int_0^\infty[\varepsilon(\|u\|^2+\|x\|^2)+\|Hx\|^2]\,dt.$$

Let $J_\varepsilon(x_0)$ be the optimal cost with $x_0=x(0)$ and $u=K_\varepsilon x$ the optimal control law. From Proposition B.1 it follows that $\lim_{\varepsilon\to0}J_\varepsilon(x_0)=0$ for $x_0\in\mathscr{V}_g^*(\mathrm{Ker}\,H)$ and from Proposition B.2 this follows for $x_0\in\mathscr{R}_b^*(\mathrm{Ker}\,H)$. Since (A,B) is stabilizable (relative $\mathbb{C}_g$), $u=K_\varepsilon x$ is an asymptotically stabilizing control law with $J_\varepsilon(x_0)\geqq\int_0^\infty\|Hx\|^2\,dt$ arbitrarily small for $\varepsilon\downarrow0$ and $x_0\in\mathscr{R}_b^*(\mathrm{Ker}\,H)+\mathscr{V}_g^*(\mathrm{Ker}\,H)$. This yields the theorem. □

COROLLARY B.1. *Simultaneous quenching of the impulse responses*

$$H_i\exp[(A+BK)t]G_j \qquad (i\in\mathbf{k}, j\in\mathbf{l})$$

in the $\mathscr{L}_2$*-sense, with internal asymptotic stability* (*by means of a common feedback matrix* K) *is possible if*

(i) (A,B) *is stabilizable* (*relative* $\mathbb{C}_g$),

(ii) $\sum_{j\in\mathbf{l}}\mathrm{im}\,G_j\subset\mathscr{R}_b^*(\bigcap_{i\in\mathbf{k}}\mathrm{Ker}\,H_i)+\mathscr{V}_g^*(\bigcap_{i\in\mathbf{k}}\mathrm{Ker}\,H_i)$.

This corollary is an immediate consequence of Theorem B.1. The next result follows directly from Corollary B.1 and Lemma A.3 in Appendix A.

COROLLARY B.2. *Consider the linear mapping*

$$M\to L_K(M):=\sum_{i\in\mathbf{l}}\sigma_i^2F_i\int_0^\infty\exp[(A+BK)\tau]M\exp[(A+BK)^T\tau]\,d\tau\,F_i^T$$

in the space of $(n \times n)$ *symmetric matrices. Then for all* $\{\sigma_i\}$ *there exists a matrix K such that the eigenvalues of* $L_K(M)$ *are smaller than* 1 *in modulus if*

$$\sum_{i \in \mathbf{I}} \operatorname{im} F_i \subset \mathcal{R}_b^*\Big(\bigcap_{i \in \mathbf{I}} \operatorname{Ker} F_i\Big) + \mathcal{V}_g^*\Big(\bigcap_{i \in \mathbf{I}} \operatorname{Ker} F_i\Big).$$

Appendix C. In this appendix a question similar to that in Appendix B is considered, but now with respect to the $\mathcal{L}_1$-norm. With the same notations we say that *impulse response quenching in the* $\mathcal{L}_1$*-sense with internal asymptotic stability* is possible if for all $\varepsilon > 0$ there exists a feedback matrix K such that

(i) $$\sigma(A+BK) \subset \mathbb{C}_g,$$

(ii) $$\int_0^\infty \|W_K(t)\| dt \leqq \varepsilon.$$

The obtained condition is slightly stronger than the criterion of Theorem B.1; it is expressed by the following result:

THEOREM C.1. *Impulse response quenching in the* $\mathcal{L}_1$*-sense with internal asymptotic stability is possible if*

(i) (A, B) *is stabilizable* (*relative* $\mathbb{C}_g$),

(ii) $\operatorname{im} G \subset \mathcal{R}_b^*(\operatorname{Ker} H) + \mathcal{V}_g^*(\operatorname{Ker} H)$.

Proof. (i) It may be shown that there exists an (A, B)-invariant subspace $\mathcal{V}_1$ and a matrix K_1 such that $(A+BK_1)\mathcal{V}_1 \subset \mathcal{V}_1$, $\sigma((A+BK_1)|\mathcal{V}_1) \subset \mathbb{C}_g$, and

$$\mathcal{R}_b^*(\operatorname{Ker} H) + \mathcal{V}_g^*(\operatorname{Ker} H) = \mathcal{R}_b^*(\operatorname{Ker} H) \oplus \mathcal{V}_1.$$

(ii) By the results of [4, Thm. 12] there exists an (A, B)-invariant $\mathcal{R}_\varepsilon$ and a matrix K_ε such that $\mathcal{R}_\varepsilon \to_{\varepsilon \to 0} \mathcal{R}_b^*(\operatorname{Ker} H)$, $(A+BK_\varepsilon)\mathcal{R}_\varepsilon \subset \mathcal{R}_\varepsilon$, $\sigma((A+BK_\varepsilon)|\mathcal{R}_\varepsilon) \subset \mathbb{C}_g$, and

$$\int_0^\infty \|H \exp[(A+BK_\varepsilon)t]G'\| \, dt \leqq \varepsilon$$

where G': $\operatorname{im} G \cap \mathcal{R}_\varepsilon \to \mathcal{X}$ is the canonical injection.

(iii) Let $K: \mathcal{X} \to \mathcal{U}$ be defined by $K|\mathcal{V}_1 = K_1|\mathcal{V}_1$, $K|\mathcal{R}_\varepsilon = K_\varepsilon|\mathcal{R}_\varepsilon$ and $\sigma(A+BK) \subset \mathbb{C}_g$. The stabilizability of (A, B) guarantees the existence of such a K. Also

$$\int_0^\infty \|H \exp[(A+BK)t]G\| \, dt = \int_0^\infty \|H \exp[(A+BK_\varepsilon)t]G'\| \, dt \leqq \varepsilon$$

which yields Theorem C.1. □

Notice the difference between the conditions (ii) in Theorems B.1 and C.1. In the former case im G should lie in the *almost stabilizable almost* (A, B)*-invariant* subspace "contained" in Ker H; in the latter case im G should be part of the *stabilizable almost* (A, B)*-invariant* subspace "contained" in Ker H. It is not possible to replace condition (ii) of Theorem C.1 by the slightly weaker condition (ii) of Theorem B.1. This is illustrated by the following example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H = [0 \quad 1].$$

Then

$$\mathscr{V}_{\bar{g}}^{*}(\operatorname{Ker} H)=\operatorname{im}\begin{bmatrix}1\\0\end{bmatrix},\quad \mathscr{V}_{g}^{*}(\operatorname{Ker} H)=\{0\},\quad \mathscr{R}_{b}^{*}(\operatorname{Ker} H)=\begin{bmatrix}1\\1\end{bmatrix}.$$

This shows that condition (ii) of Theorem C.1 is not satisfied; it is shown in § 3.3 that impulse response quenching in the $\mathscr{L}_1$-sense with internal asymptotic stability is not possible. On the other hand, condition (ii) of Theorem B.1 holds, and impulse response quenching in the $\mathscr{L}_2$-sense is possible.

COROLLARY C.1. *Simultaneous quenching of the impulse responses*

$$H_i \exp[(A+BK)t]G_j \qquad (i\in \mathbf{k}, j\in \mathbf{l})$$

in the $\mathscr{L}_1$-sense with internal asymptotic stability (by means of a common feedback matrix K) is possible if

(i) *(A, B) is stabilizable (relative $\mathbb{C}_g$),*

(ii) $\sum_{j\in \mathbf{l}} \operatorname{im} G_j \subset \mathscr{R}_b^*(\bigcap_{i\in\mathbf{k}} \operatorname{Ker} H_i)+\mathscr{V}_{\bar{g}}^*(\bigcap_{i\in\mathbf{k}} \operatorname{Ker} H_i)$.

The result of Corollary C.1 can be used to derive a condition for stabilizability of the nonlinear time-varying system

$$\dot{x}(t)=Ax(t)+Bu(t)+[F_1 \quad F_2 \quad \cdots \quad F_l]M(x(t),t)\begin{bmatrix}F_1\\F_2\\\vdots\\F_l\end{bmatrix}x(t) \tag{C.1}$$

with $x\in\mathbb{R}^n$. The matrices F_i are square $(n\times n)$ matrices. The gain matrix $M(x,t)$ is of dimension $(ln\times ln)$. Let a linear time-invariant feedback $u(t)=Kx(t)$ be applied to this system. Then, according to the small loop theorem [14], the system is $\mathscr{L}_p$-input-output-stable if

(i) $\sigma(A+BK)\subset\mathbb{C}_g$,

(ii) $\|M(x,t)\|<\alpha \quad \forall x, \quad \forall t$,

(iii) $\max_{i,j\in\mathbf{l}}\int_0^\infty \|F_i \exp[(A+BK)t]F_j\|\,dt<1/\alpha l^2$.

Hence $F_i x(t)\in\mathscr{L}_2(0,\infty)$; the Hurwitz character of $A+BK$ then shows that the solution of (C.1) tends to zero as $t\to\infty$ for all initial conditions. Hence the null solution of (C.1) is asymptotically stable in the large. Sufficient conditions for the existence of a feedback matrix K satisfying (i) and (ii) can be derived from Corollary C.1. Consider now the special case that $M(x,t)$ is a block diagonal matrix

$$M(x)=\operatorname{diag}[f_1(x,t)F_1^+ \quad f_2(x,t)F_2^+ \quad \cdots \quad f_l(x,t)F_l^+]$$

when the functions f_i are scalar and F_i^+ denotes the generalized inverse [12, pp. 142–144] of the matrix F_i. Then (C.1) reduces to (14); the following result is hence obtained:

PROPOSITION C.1. *For any $\alpha>0$ there exists a constant feedback matrix K such that the null solution of*

$$\dot{x}(t)=(A+BK)x(t)+\sum_{i\in\mathbf{l}} f_i(x(t),t)F_i x(t) \tag{C.2}$$

is asymptotically stable in the large for all nonlinear gains satisfying

$$|f_i(x,t)|<\alpha \qquad (i\in\mathbf{l}, \forall x, \forall t)$$

if (A, B) is stabilizable (relative $\mathbb{C}_g$), and if

$$\sum_{i\in\mathbf{l}} \operatorname{im} F_i \subset \mathscr{R}_b^*\Big(\bigcap_{i\in\mathbf{l}} \operatorname{Ker} F_i\Big)+\mathscr{V}_{\bar{g}}^*\Big(\bigcap_{i\in\mathbf{l}} \operatorname{Ker} F_i\Big).$$

Appendix D. Following the suggestion of one of the reviewers we have collected in this appendix the relevant facts on (A, B)-invariant and almost (A, B)-invariant subspaces used in this paper. More details may be found in references [2], [3], [4].

Consider the system $\dot{x} = Ax + Bu$ with $x \in \mathscr{X} := \mathbb{R}^n$. A subspace $\mathscr{V} \subset \mathscr{X}$ is said to be an *(A, B)-invariant subspace* if there exists a matrix K such that $\mathscr{V}$ is $(A+BK)$-invariant (i.e. such that $(A+BK)\mathscr{V} \subset \mathscr{V}$). An equivalent property is that $\mathscr{V}$ satisfies

$$A\mathscr{V} \subset \mathscr{V} + \mathscr{B}$$

where $\mathscr{B} := \operatorname{im} B$.

Let $\mathbf{V}(\mathscr{S})$ denote the set of all (A, B)-invariant subspaces contained in a given subspace $\mathscr{S}$. Then this set is closed under subspace addition, i.e. $\mathscr{V}_1, \mathscr{V}_2 \in \mathbf{V}(\mathscr{S}) \Rightarrow \mathscr{V}_1 + \mathscr{V}_2 \in \mathbf{V}(\mathscr{S})$. Hence there exists a largest (A, B)-invariant subspace in $\mathscr{S}$, which is denoted by $\mathscr{V}^*(\mathscr{S})$. Systematic finite and linear algorithms are available [2, 3] to compute $\mathscr{V}^*(\mathscr{S})$. A related concept, denoted by $\mathscr{V}_g^*(\mathscr{S})$, is defined as follows

$$\mathscr{V}_g^*(\mathscr{S}) := \sup \{\mathscr{V} \in \mathbf{V}(\mathscr{S}) | \exists K \text{ such that } (A+BK)\mathscr{V} \subset \mathscr{V} \text{ and } \sigma(A+BK) \subset \mathbb{C}_g\}.$$

It is easily proven that this subspace is well defined. It is called the largest *stabilizability* subspace contained in $\mathscr{S}$ and is readily computed from $\mathscr{V}^*(\mathscr{S})$ [2]. Finally $\mathscr{V}_{\bar{g}}^*$ is similarly defined with $\bar{\mathbb{C}}_g$ replacing $\mathbb{C}_g$ in the definition.

Let $\mathscr{V}_i$, $i \in \mathbf{k}$, be a family of (A, B)-invariant subspaces. Then, by definition, there exist matrices K_i such that $(A+BK_i)\mathscr{V}_i \subset \mathscr{V}_i$. However, there is no guarantee that there exists a single K such that $(A+BK)\mathscr{V}_i \subset \mathscr{V}_i$ for all $i \in \mathbf{k}$. If this is the case, then the subspaces $\mathscr{V}_i$ are said to be *compatible* (A, B)-invariant subspaces. It is easy to prove that the subspaces $\mathscr{V}_i$ are compatible, for example, if they are nested ($\mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_k$), but in general compatibility is a difficult matter to verify.

A further generalization leads to controllability subspaces. Thus

$$\mathscr{R}^*(\mathscr{S}) := \sup \{\mathscr{V} \in \mathbf{V}(\mathscr{S}) | \text{for } K \text{ such that } (A+BK)\mathscr{V} \subset \mathscr{V}, \text{ there holds } \langle A+BK | \mathscr{S} \cap \operatorname{im} B\rangle = \mathscr{V}\}.$$

Again, $\mathscr{R}^*(\mathscr{S})$ is well defined. For equivalent definitions and algorithms for computing $\mathscr{R}^*(\mathscr{S})$ we refer the reader to [2].

An interesting generalization of (A, B)-invariance is *almost (A, B)-invariance*. These notions have been introduced in [3] and further worked out in [4]. The largest almost (A, B)-invariant subspace contained in a given subspace $\mathscr{S}$ is the subspace of initial states in $\mathscr{S}$ for which there exists an input such that the resulting state trajectory is *almost* contained in $\mathscr{S}$. However, this depends on the topology chosen. In particular, we obtain a somewhat larger subspace if we measure "*almost being contained in*" in the $\mathscr{L}_p$-sense $(1 \leqq p < \infty)$ rather than in the $\mathscr{L}_\infty$-sense. Similarly, the largest almost controllability subspace contained in $\mathscr{S}$ is the subspace of initial states in $\mathscr{S}$ which, by means of an input, may be transferred to any terminal state in that subspace, such that the resulting state trajectory is *almost* contained in $\mathscr{S}$. Let $\mathscr{V}_a^*(\mathscr{S})$ and $\mathscr{V}_b^*(\mathscr{S})$ denote respectively the supremal $\mathscr{L}_\infty$-almost-controllability and the $\mathscr{L}_p$-$(1 \leqq p < \infty)$-almost-controllability subspace contained in $\mathscr{S}$. Similarly $\mathscr{R}_a^*(\mathscr{S})$ and $\mathscr{R}_b^*(\mathscr{S})$ denote respectively the supremal $\mathscr{L}_\infty$-almost-(A, B)-invariant and the $\mathscr{L}_p$-$(1 \leqq p < \infty)$-almost-(A, B)-invariant subspace contained in $\mathscr{S}$.

In the present paper we use primarily $\mathscr{R}_b^*(\mathscr{S})$. We therefore define it formally:

$$\mathscr{R}_b^*(\mathscr{S}) := \sup \mathbf{R}_b(\mathscr{S})$$

where

$$\mathbf{R}_b(\mathcal{S}) := \{\mathcal{R}_b \subset \mathcal{X} \mid \forall x_0, x_1 \in \mathcal{R}_b \exists T > 0, \text{ such that for all } \varepsilon > 0, \text{ there exists } x \in \Sigma_x \text{ with the properties: (i) } x(0) = x_0, \text{ (ii) } x(T) = x_1, \text{ and (iii)} \|d(x(t), \mathcal{S})\|_{\mathcal{L}_1(0,T)} \leqq \varepsilon\}.$$

Here

$$d(x(t), \mathcal{S}) := \inf_{s \in \mathcal{S}} \|x(t) - s\|$$

and

$$\Sigma_x := \{x : \mathbb{R} \to \mathcal{X} \mid x \text{ is absolutely continuous and } \exists u : \mathbb{R} \to \mathcal{U}, \text{ such that } \dot{x}(t) = Ax(t) + Bu(t) \text{ almost everywhere}\}.$$

This definition merely says that $\mathcal{R}_b^*(\mathcal{S})$ is the largest subspace of $\mathcal{X}$ in which any two states can be transferred to one another while keeping the $\mathcal{L}_1$-norm of the distance of the state trajectory to $\mathcal{S}$ arbitrarily small. This has an obvious interpretation in terms of $\mathcal{L}_1$-(almost) output nulling for the system $\dot{x} = Ax + Bu$, $z = Hx$, with $\mathcal{S} = \operatorname{Ker} H$. The subspace $\mathcal{R}_b^*(\mathcal{S})$ is readily computed by means of the following finite linear recursive algorithm:

$$\mathcal{S}_{k+1} = \operatorname{im} B + A(\mathcal{S} \cap \mathcal{S}_k),$$
$$\mathcal{S}_0 = \{0\}.$$

Then

$$\mathcal{R}_b^*(\mathcal{S}) = \mathcal{S}_\infty := \lim_{k \to \infty} \mathcal{S}_k$$

where this limit is obtained monotonically in at most Min [codim (im B), $1 + \dim(\mathcal{S})$] steps.

The subspace $\mathcal{V}_b^*(\mathcal{S})$ may be defined completely analogously as

$$\mathcal{V}_b^*(\mathcal{S}) := \sup \mathbf{V}_b(\mathcal{S})$$

where

$$\mathbf{V}_b(\mathcal{S}) = \{\mathcal{V}_b \subset \mathcal{X} \mid \forall x_0 \in \mathcal{V}_b \text{ and } \varepsilon > 0 \exists x \in \Sigma_x \text{ with the properties: (i) } x(0) = x_0 \text{ and (ii) } \|d(x(t), \mathcal{S})\|_{\mathcal{L}_1(0,\infty)} \leqq \varepsilon\}.$$

In [4, Thm. 10] it is proven that $\mathcal{V}_b^*(\mathcal{S}) = \mathcal{V}^*(\mathcal{S}) + \mathcal{R}_b^*(\mathcal{S})$. Its main use in feedback system synthesis stems from the following result [4, Thm. 12].

THEOREM D.1. *Consider the finite dimensional linear system* $\dot{x} = Ax + Bu$, $z = Hx$. *Let* $1 \leqq p < \infty$. *Then for all* $\varepsilon > 0$ *there exists a matrix* K *such that*

$$\int_0^\infty \|H e^{(A+BK)t} G\| \, dt \leqq \varepsilon$$

if and only if

$$\operatorname{im} G \subset \mathcal{V}_b^*(\operatorname{Ker} H).$$

This theorem is also the basic tool for our results on robust stabilizability. However, a number of refinements were needed (Theorems B.1 and C.1). We note in closing that, because $1 \leqq p < \infty$, $\mathcal{R}_b^*(\mathcal{S})$ and $\mathcal{V}_b^*(\mathcal{S})$ need not be contained in $\mathcal{S}$. This fact is amply discussed in [4].

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