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Hiroyuki Imai (M' 79) was born in Kyoto, Japan, on March 21, 1939. He received the B.S. and M.S. degrees in mechanical engineering from Doshisha University, Kyoto, Japan, in 1962 and 1964, respectively and the Doctor of Engineering degree from Kyoto University, Kyoto, Japan, in 1978.

From 1964 to 1966 he was employed by Toshiba Electric Company, Ltd. as a Design Engineer. From 1966 to 1968 he was a graduate student at Doshisha University. Since 1968 he has been a Research Associate in the Department of Precision Mechanics, Kyoto University. During the academic year of 1974–1975 he was with the Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY. His fields of interests include multivariable system theory and system identification.

Hajime Akashi (M' 78) received the B.S. degree in engineering from the Department of Aeronautics Engineering, Kyoto University, Kyoto, Japan, in 1944, and the Ph.D. degree in engineering from Kyoto University in 1960.

He spent one year from 1953 to 1954 at Brown University, Providence, RI, in the Graduate School of Applied Mathematics as a Fulbright exchange student. He taught at Ritsumeikan University, Kyoto, Japan, as a Lecturer and Professor from 1955 to 1961, and at Doshisha University, Japan, as a Professor from 1961 to 1968. He was a Visiting Professor at the University of Kansas, Lawrence, from 1963 to 1965. Since 1968 he has been a Professor with Kyoto University. His main field of interest has been the theory of control, especially nonlinear control, optimal control, identification, and estimation.

Dr. Akashi is a member of the Japan Association of Automatic Control Engineers, the Japan Society of Mechanical Engineers, and other societies. He is the author of a book on control engineering, and is involved in the administrative work of IFAC, for which he is now a Vice Chairman of the Advisory Committee and the Chairman of the IPC for IFAC 81 Congress.

Almost Invariant Subspaces: An Approach to High Gain Feedback Design—Part I: Almost Controlled Invariant Subspaces

JAN C. WILLEMS

***Abstract*—In a previous paper [1] we have introduced the notion of "almost controlled invariant subspaces" which are subspaces to which one can steer the state of a linear system arbitrarily close. In the present paper we will show how these subspaces may be viewed as ordinary controlled invariant subspaces when one allows distributional inputs, or as those subspaces which can be approximated by controlled invariant subspaces. The results are applied to a number of control synthesis problems, i.e., disturbance decoupling, robustness, noisy gain stabilization, and cheap control. Part II of the paper will treat the dual theory of almost conditionally invariant subspaces.**

I. Introduction

ONE of the most important new developments in linear system theory in the last decade has been, without any doubt, the introduction of $(A-B)$ and $(A-C)$ invariant subspaces [2], [3]. From a linear algebra point of view it is most logical to call these subspaces $A(\text{mod im}\,B)$ and $A|\ker C$ invariant subspaces; we will call them *controlled invariant* and *conditionally invariant* subspaces, respectively. This nomenclature is suggested by their system theoretic interpretation and was also used in [3]. The "geometric" approach of Wonham [2] provides an elegant and, in applications, very effective approach to this problem area. In addition, this approach admits a very effective separation of the structural questions (linear operators and subspace operations) from the numerical and computational ones (matrices and linear equations), which leads to a very clear, transparent, and sophisticated theoretical picture. Not only have these concepts been of crucial importance for providing a deep understanding of the "fine structure" of linear systems but they have also served as an excellent framework for solving a number of

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The author is with the Mathematics Institute, University of Groningen, Groningen, The Netherlands.

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very convincing control system synthesis problems as, for example, the disturbance decoupling problem, tracking, regulation, the synthesis of noninteracting controllers, etc.

In a recent paper [1] we have introduced a new concept which fits very nicely into this geometric circle of ideas. The concept is that of "almost" $A(\mathrm{mod}\,\mathsf{B})$-invariant subspaces which in a sense (see Section IV) is the high gain feedback generalization of an $A(\mathrm{mod}\,\mathsf{B})$-invariant subspace. Here we will call these subspaces *almost controlled invariant subspaces*. In [1] the basic definition of these subspaces has been given together with an explicit feedback representation for them. In the present paper further properties of these subspaces will be studied together with some extensions of these ideas and explicit applications to a number of control theoretic synthesis problems. In Part II of the paper we will treat the duals, i.e., *almost conditionally invariant subspaces* or, as one may like to call them, *almost* $A|\ker C$-invariant subspaces. In there the synthesis problems will be mainly concerned with observers. We finally mention a recent manuscript [4] where a more restricted but somewhat analogous idea called asymptotic holdability has been introduced. However, since all asymptotically holdable subspaces as introduced there need to be subspaces of $\mathsf{B}:=\mathrm{im}\,B$, these constitute a small subset of our almost controlled invariant subspaces.

We will denote throughout vectors by lower case italic letters, time functions and distributions by boldface italic lower case letters, matrices (linear operators) by italic capitals, and subspaces by sans serif capitals. As usual, $\mathbb{R}$ will denote the real line, $\mathbb{R}^+:=[0,\infty)$, $\mathbb{R}^-:=(-\infty,0]$, $\mathbb{C}$ denotes the complex plane, and, for a positive integer n, $\boldsymbol{n}:=\{1,2,\cdots,n\}$. The spectrum (i.e., the set of eigenvalues, counting multiplicity) of the square matrix M will be denoted by $\sigma(M)$. $\mathrm{Re}\,\sigma(M)<K$ means that all points of $\sigma(M)$ have real part less than K.

We will sometimes use the notation $\mathsf{L}_1\subset\mathsf{L}$ to indicate that L_1 is a *linear* subspace of L when its linear structure is apparent from the context. We will often consider families of subspaces of a given vector space. These will be denoted by underlined sans serif capitals. Let $\underline{\mathsf{L}}$ be such a family of subspaces of X. We will say that $\underline{\mathsf{L}}$ is closed under addition if $\{\mathsf{L}_1,\mathsf{L}_2\in\underline{\mathsf{L}}\}\Rightarrow\{\mathsf{L}_1+\mathsf{L}_2\in\underline{\mathsf{L}}\}$. We will denote by $\mathsf{L}^*:=\sup_{\mathsf{L}\in\underline{\mathsf{L}}}\mathsf{L}$ the smallest subspace of X such that 1) $\mathsf{L}^*\supset\mathsf{L}\ \forall\mathsf{L}\in\underline{\mathsf{L}}$. Thus L^* is defined by 1) and 2): $\{\mathsf{L}_1\supset\mathsf{L}\ \forall\mathsf{L}\in\underline{\mathsf{L}}\}\Rightarrow\{\mathsf{L}^*\subset\mathsf{L}_1\}$. It is easily seen that this uniquely defines the subspace L^*. Of course, L^* may or may not belong to $\underline{\mathsf{L}}$. It obviously does if $\underline{\mathsf{L}}$ is closed under addition. In fact, $\mathsf{L}^*=\Sigma_{\mathsf{L}\in\underline{\mathsf{L}}}\mathsf{L}$. This trivial fact is rather important to us: we record it as follows.

Lemma 0: If $\underline{\mathsf{L}}$ is closed under addition, then $\sup_{\mathsf{L}\in\underline{\mathsf{L}}}\mathsf{L}=:\mathsf{L}^*\in\underline{\mathsf{L}}$.

If $\underline{\mathsf{L}}_1,\underline{\mathsf{L}}_2$ are families of subspaces of X, then $\underline{\mathsf{L}}_1+\underline{\mathsf{L}}_2:=\{\mathsf{L}\subset\mathsf{X}|\mathsf{L}=\mathsf{L}_1+\mathsf{L}_2,\ \mathsf{L}_1\in\underline{\mathsf{L}}_1,\ \mathsf{L}_2\in\underline{\mathsf{L}}_2\}$. A sequence of subspaces $\{\mathsf{L}_i\}$, $i\in\boldsymbol{n}$, will be called a *chain* in L if $\mathsf{L}\supset\mathsf{L}_1\supset\mathsf{L}_2\supset\cdots\supset\mathsf{L}_n$. If L is a subspace of X then $\mathsf{X}(\mathrm{mod}\,\mathsf{L})$ is the vector space $\{x+\mathsf{L}|x\in\mathsf{X}\}$. In a suitable basis this corresponds to ignoring some of the coordinates of x.

Let $x\in\mathsf{X}$, $\mathsf{L}\subset\mathsf{X}$, with X a normed vector space. Then $d(x,\mathsf{L}):=\inf_{x'\in\mathsf{L}}\|x-x'\|$. Let $\boldsymbol{l}$: $\mathbb{R}\to\mathsf{X}$ be measurable then we will say that $\boldsymbol{l}\in\mathfrak{L}_p$ if $\|\boldsymbol{l}\|_{\mathfrak{L}_p}<\infty$ where $\|\boldsymbol{l}\|_{\mathfrak{L}_p}$ is defined by

$$\|\boldsymbol{l}\|_{\mathfrak{L}_p}:=\begin{cases}\left(\int_{-\infty}^{+\infty}\|\boldsymbol{l}(t)\|^p\,dt\right)^{1/p} & \text{for } 1\leqslant p<\infty\\ \operatorname*{ess\,sup}_t\|\boldsymbol{l}(t)\| & \text{for } p=\infty.\end{cases}$$

This notation will also be used for functions defined on an interval. The codomain will always be obvious from the context. We will use the notation $\boldsymbol{l}\in C^\infty$ if $\boldsymbol{l}$ is infinitely differentiable, and $\boldsymbol{l}\in\mathfrak{L}_1^{\mathrm{loc}}$ if $\boldsymbol{l}$ restricted to any finite interval belongs to $\mathfrak{L}_1$. The abbreviation a.c. stands for 'absolutely continuous' and a.e. stands for 'almost everywhere' and is here always used in connection with Lebesgue measure.

Most of the paper deals with the system Σ: $\dot{x}=Ax+Bu$ with $x\in\mathsf{X}:=\mathbb{R}^n$, $u\in\mathsf{U}:=\mathbb{R}^m$ and (A,B) matrices of appropriate dimensions. We will use as standard notation $\mathsf{B}:=\mathrm{im}\,B$ and $A_F:=A+BF$. Thus $\dot{x}=A_Fx$ is the flow obtained by using the feedback law $u=Fx$ on Σ. Finally, $\langle A|\mathsf{B}\rangle$ denotes the reachable subspace of Σ given by $\mathsf{B}+A\mathsf{B}+\cdots+A^{n-1}\mathsf{B}$, i.e., the smallest A-invariant subspace containing B.

II. Almost Controlled Invariant Subspaces

In this section we will introduce the "almost" versions of controlled invariant subspaces. In Part II we will dualize the ideas to almost conditionally invariant subspaces. However, since the present paper deals with controlled invariant subspaces we will most of the time delete the adjective "controlled."

We will take as a starting point that an almost invariant subspace is a subspace of the state space of a linear system to which one can stay arbitrarily close by choosing the input properly. However, we will see that there exist a number of appealing equivalent defining properties. Among others, they are those subspaces inside which one can remain by using impulsive type controls or those subspaces which can be approximated arbitrarily closely by ordinary controlled invariant subspaces.

Section II of the paper is mainly a reexposition, without proofs, of the results in [1].

A. Basic Definitions

Consider the linear system

$$\boxed{\Sigma:\ \dot{x}=Ax+Bu.}$$

We will denote by $\Sigma_x(A,B)$ all possible state trajectories generated by Σ. Formally, $\Sigma_x(A,B):=\{\boldsymbol{x}:\ \mathbb{R}\to\mathsf{X}|\boldsymbol{x}$ is a.c. and $\exists\boldsymbol{u}$: $\mathbb{R}\to\mathsf{U}$ such that $\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B\boldsymbol{u}(t)$ a.e.$\}$. Equivalently, $\Sigma_x(A,B)=\{\boldsymbol{x}:\ \mathbb{R}\to\mathsf{X}|\boldsymbol{x}$ is a.c. and $\dot{\boldsymbol{x}}(t)-A\boldsymbol{x}(t)\in\mathsf{B}$

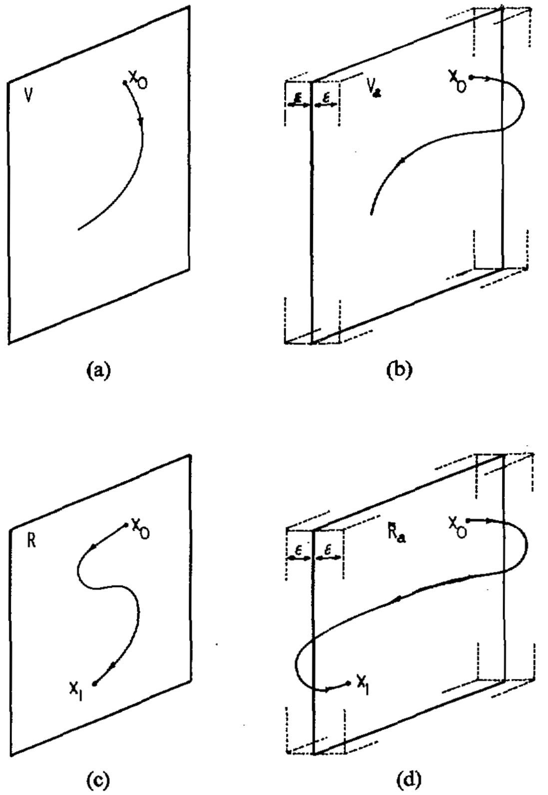

Fig. 1. (a) A controlled invariant subspace. (b) An almost controlled invariant subspace. (c) A controllability subspace. (d) An almost controllability subspace.

a.e.}. If there is no chance for confusion we will denote $\Sigma_x(A, B)$ by Σ_x. Furthermore, since $\Sigma_x(A, B)$ depends only on A and B, we will often denote it by $\Sigma_x(A, \mathrm{B})$.

Definition 1: A subspace $V \subset X$ is said to be a *(controlled) invariant subspace* if $\forall x_0 \in V$, $\exists x \in \Sigma_x$ such that $x(0) = x_0$ and $x(t) \in V$, $\forall t$. A subspace $V_a \subset X$ is said to be an *almost (controlled) invariant subspace* if $\forall x_0 \in V_a$ and $\epsilon > 0$, $\exists x \in \Sigma_x$ such that $x(0) = x_0$ and $d(x(t), V_a) \leqslant \epsilon$, $\forall t$.

In the controllability version of these notions one actually requires to steer between any two given points of the subspace while staying in or arbitrarily close to it. Formally, we have the following.

Definition 2: A subspace $R \subset X$ is said to be a *controllability subspace* if $\forall x_0, x_1 \in R$, $\exists T > 0$ and $x \in \Sigma_x$ such that $x(0) = x_0$, $x(T) = x_1$, and $x(t) \in R$, $\forall t$. A subspace $R_a \subset X$ is said to be an *almost controllability subspace* if $\forall x_0, x_1 \in R_a$, $\exists T > 0$ such that $\forall \epsilon > 0$, $\exists x \in \Sigma_x$ with the properties that $x(0) = x_0$, $x(T) = x_1$, and $d(x(t), R) \leqslant \epsilon$, $\forall t$.

These notions are visualized in Fig. 1.

Let $\underline{V}$, $\underline{R}$, $\underline{V}_a$, $\underline{R}_a$ denote the sets of all invariant, etc. subspaces, and $\underline{V}(K)$, $\underline{R}(K)$, $\underline{V}_a(K)$, and $\underline{R}_a(K)$ those contained in a given subspace K of X. Note that $\underline{R} \subset \underline{V} \subset \underline{V}_a$ and $\underline{R} \subset \underline{R}_a \subset \underline{V}_a$.

The following property, essentially trivial, is crucial in applications.

Theorem 1: $\underline{V}$, $\underline{R}$, $\underline{V}_a$, *and* $\underline{R}_a$ *are closed under subspace addition (i.e.,* $V_1, V_2 \in \underline{V} \Rightarrow V_1 + V_2 \in \underline{V}$, *etc.). Consequently,*

$$\sup \underline{V}(K) =: V_K^* \in \underline{V}, \quad \sup \underline{R}(K) =: R_K^* \in \underline{R},$$
$$\sup \underline{V}_a(K) =: V_{a,K}^* \in \underline{V}_a, \quad \textit{and} \quad \sup \underline{R}_a(K) =: R_{a,K}^* \in \underline{R}_a.$$

Proof: The first part of the theorem is an easy consequence of the fact that Σ_x is a linear subspace of the space of all a.c. maps from $\mathbb{R}$ into X. Indeed, let $x_0 \in V_a = V_{a,1} + V_{a,2}$. Hence, it may be written as $x_0 = a_1 + a_2$ with $a_i \in V_{a,i}$. Consequently, given any $\epsilon > 0$, $\exists x_i \in \Sigma_x$ such that $x_i(0) = a_i$ and $d(x_i(t), V_{a,i}) \leqslant \epsilon/2$. Now, $x := x_1 + x_2 \in \Sigma_x$ and satisfies $x(0) = x_0$ and

$$d(x(t), V_a) \leqslant d(x_1(t), V_a) + d(x_2(t), V_a)$$
$$\leqslant d(x_1(t), V_{a,1}) + d(x_2(t), V_{a,2}) \leqslant \epsilon.$$

The same sort of proof applies to $\underline{V}$, $\underline{R}$, and $\underline{R}_a$. For the last two it is convenient to proceed by first showing that it suffices to consider only the situation with x_0 or x_1 equal 0. This in order to be able to choose the T appearing in the definition of $\underline{R}_a$ the same for the two trajectories. The second part of the theorem follows from the closure under addition and Lemma 0 of the introduction. □

B. Feedback Characterizations

The following "feedback" characterizations of invariant and controllability subspaces are well known and are often taken as the definitions [2].

Proposition 1:

1) $\{V \in \underline{V}\} \Leftrightarrow \{\exists F$ such that $A_F V \subset V\} \Leftrightarrow \{AV \subset V + B\}$. (This last characterization justifies the "$A(\mathrm{mod}\, B)$-invariant" nomenclature.)

2) $\{R \in \underline{R}\} \Leftrightarrow \{\exists F$ and $B_1 \subset B$ such that $R = \langle A_F | B_1 \rangle\}$.

Analogous feedback characterizations may be found for almost invariant and almost controllability subspaces.

Theorem 2:

1) $\underline{V}_a = \underline{V} + \underline{R}_a$, i.e., $\{V_a \in \underline{V}_a\} \Leftrightarrow \{\exists V \in \underline{V}$ *and* $R_a \in \underline{R}_a$ *such that* $V_a = V + R_a\}$;

2) $\{R_a \in \underline{R}_a\} \Leftrightarrow \{\exists F$ *and a chain* $\{B_i\}$ *in* B *such that* $R_a = B_1 + A_F B_2 + \cdots + A_F^{n-1} B_n\}$

The above theorem is the main result of [1]: consequently, it need not be reproven here. Note that it follows immediately from the above theorem that for scalar input systems there are a finite number of almost controllability subspaces given by $R_i := B + AB + \cdots + A^{i-1}B$; $i \in \boldsymbol{n}$. It also follows from this theorem that $\{V_a \in \underline{V}_a$ and $V_a \cap B = \{0\}\} \Leftrightarrow \{V_a \in \underline{V}\}$.

C. Almost Controllability Subspace Algorithm

In applications it is very important to have algorithms for computing V_K^*, $V_{a,K}^*$, R_K^*, $R_{a,K}^*$. It turns out, however, that very little new needs to be done in order to set these up. Indeed, such algorithms may be obtained by suitably combining the invariant subspace algorithm (ISA) [2, p. 91] and the almost controllability subspace algorithm (ACSA) [2, p. 108]. (In [2] this algorithm is called the controllability subspace algorithm (CSA); in view of the theorem which follows, we propose the more appropriate name (ACSA).)

Proposition 2: Consider the algorithms

$\mathsf{V}_\mathsf{K}^{k+1}=\mathsf{K}\cap A^{-1}(\mathsf{V}_\mathsf{K}^k+\mathsf{B});$	$\mathsf{V}_\mathsf{K}^0=\mathsf{X}$	(ISA)
$\mathsf{R}_\mathsf{K}^{k+1}=\mathsf{K}\cap(A\mathsf{R}_\mathsf{K}^k+\mathsf{B});$	$\mathsf{R}_\mathsf{K}^0=\{0\}$	(ACSA)

Then V_K^k is monotone nonincreasing; moreover, $\mathsf{V}_\mathsf{K}^{\dim \mathsf{K}+1}=\mathsf{V}_\mathsf{K}^\infty:=\lim_{k\to\infty}\mathsf{V}_\mathsf{K}^k$, and $\{\mathsf{V}_\mathsf{K}^{k+1}=\mathsf{V}_\mathsf{K}^k\}\Rightarrow\{\mathsf{V}_\mathsf{K}^k=\mathsf{V}_\mathsf{K}^\infty\}$. Similarly R_K^k is monotone nondecreasing; moreover, $\mathsf{R}_\mathsf{K}^{\dim \mathsf{K}}=\mathsf{R}_\mathsf{K}^\infty:=\lim_{k\to\infty}\mathsf{R}_\mathsf{K}^k$, and $\{\mathsf{R}_\mathsf{K}^{k+1}=\mathsf{R}_\mathsf{K}^k\}\Rightarrow\{\mathsf{R}_\mathsf{K}^k=\mathsf{R}_\mathsf{K}^\infty\}$.

In terms of these algorithmic definitions we obtain the following.

Theorem 3:

$$\mathsf{V}_\mathsf{K}^*=\mathsf{V}_\mathsf{K}^\infty,$$
$$\mathsf{R}_{a,\mathsf{K}}^*=\mathsf{R}_\mathsf{K}^\infty,$$
$$\mathsf{V}_{a,\mathsf{K}}^*=\mathsf{V}_\mathsf{K}^\infty+\mathsf{R}_\mathsf{K}^\infty, \quad \textit{and}$$
$$\mathsf{R}_\mathsf{K}^*=\mathsf{V}_\mathsf{K}^\infty\cap\mathsf{R}_\mathsf{K}^\infty=\mathsf{V}_{\mathsf{R}_\mathsf{K}^\infty}^\infty=\mathsf{R}_{\mathsf{V}_\mathsf{K}^\infty}^\infty.$$

These results are proven in [1] and we refer to there for the proof.

The above algorithms are basically conceptual ones and it would be naive to suggest that they can simply be implemented on a digital computer, even though they involve, in principle, solving only linear equations. There are issues of numerical stability and the robustness against the "soft" values in the matrices A and B. In [5] some of the difficulties and considerations which enter into the numerical computation of V_K^* and R_K^* are discussed. The algorithms proposed there are not a direct implementation of those in Theorem 3 but involve generalized eigenvalue/eigenvector computations and, hence, leave the realm of finite arithmetic.

We note that the intermediate steps V_K^k and R_K^k in (ISA) and (ACSA) have some significance of their own (see [1]). Finally, the symmetry that is apparent from comparing (ISA) and (ACSA) may be viewed in terms of time reversal in discrete-time systems where, if $\Sigma(A,B)$ describes the forward time evolution, then $\Sigma(A^{-1},A^{-1}\mathsf{B})$ describes the backward time evolution. This shows that (ACSA) becomes (ISA) under time reversal. Unfortunately, because of space limitations we cannot elaborate these points further at this time.

III. Systems with Distributions as Inputs

In this section we will prove the equivalence of almost invariant or controllability subspaces with what would be "ordinary" invariant or controllable subspaces when distributions are also allowed as inputs. However natural this point of view may be, it is a bit awkward to set up a completely satisfactory mathematical framework to treat this. We will take a somewhat pragmatic approach and comment later on some of the mathematical difficulties involved.

We will denote the space of (finite dimensional valued) distributions by D', by D'_+ those with support on $\mathbb{R}^+$, and by D'_T those with support on $[0,T]$ (the dimension of the vector space involved will always be clear from the context); $\boldsymbol{\delta}^{(-1)}$ denotes the Heaviside step, $\boldsymbol{\delta}$ Dirac's delta, $\boldsymbol{\delta}^{(i)}$ its ith (distributional) derivative, etc.

Take again the systems $\dot{x}=Ax+Bu$, but now with $\boldsymbol{u}\in\mathsf{D}'$, and consider the state "trajectories" then obtained. Formally, $\Sigma_\mathsf{D}:=\{\boldsymbol{x}\in\mathsf{D}'|\exists\boldsymbol{u}\in\mathsf{D}'$ such that $\dot{\boldsymbol{x}}=A\boldsymbol{x}+B\boldsymbol{u}\}$ with derivatives, of course, to be understood in the sense of distributions. In trying to generalize the concept of invariant subspaces to the situation at hand we need to give a meaning to the condition $x(0)=x_0$ which is not possible as such in the context of distributions. We therefore have to introduce the following subclasses of Σ_D:

1) $\Sigma_\mathsf{D}^+:=\{\boldsymbol{x}\in\Sigma_\mathsf{D}|\boldsymbol{x}=\boldsymbol{f}^-+\boldsymbol{x}^+$ with $\boldsymbol{x}^+\in\mathsf{D}'_+$ and $\boldsymbol{f}^-$ a map $\mathbb{R}\to\mathsf{X}$ with support on $\mathbb{R}^-$ and a.c. there$\}$; and

2) for $T\geqslant 0$, $\Sigma_\mathsf{D}^T:=\{\boldsymbol{x}\in\Sigma_\mathsf{D}|\boldsymbol{x}=\boldsymbol{f}^-+\boldsymbol{x}^{[0,T]}+\boldsymbol{f}^+$ with $\boldsymbol{x}^{[0,T]}\in\mathsf{D}'_T$ and $\boldsymbol{f}^-$, $\boldsymbol{f}^+$ maps $\mathbb{R}\to\mathsf{X}$ with support on $\mathbb{R}^-$ and $[T,\infty)$ respectively, and a.c. there$\}$.

For elements of Σ_D^+ we may now define $x(0^-):=\lim_{t\uparrow 0}x(t)$ while for elements of Σ_D^T we may also define $x(T^+):=\lim_{t\downarrow T}x(t)$. Thus, in effect, an element of Σ_D^+ is generated by an input which is a regular function for $t<0$ and a distribution for $t\geqslant 0$, while elements of Σ_D^T have inputs which are in addition also regular functions for $t>T$. For elements of Σ_D^+ and Σ_D^T we can speak about their restrictions to $\mathbb{R}^+$ and $[0,T]$, respectively, as the distributions $\boldsymbol{x}^+$ and $\boldsymbol{x}^{[0,T]}$ appearing in their definitions. We will denote these restrictions accordingly.

Let L be a subspace of X. We will say that an X-valued distribution $\boldsymbol{f}$ lies in L if, for all scalar test functions ϕ, $\int_{-\infty}^{+\infty}\phi(t)\boldsymbol{f}(t)\,dt=:\langle\phi,\boldsymbol{f}\rangle\in\mathsf{L}$. This allows us, using the notation introduced above, to define distributionally invariant and controllability subspaces:

Definition 3: A subspace $\mathsf{V}_\mathsf{D}\subset\mathsf{X}$ is said to be a *distributionally invariant* subspace if $\forall x_0\in\mathsf{V}_\mathsf{D}$ there exists $\boldsymbol{x}\in\Sigma_\mathsf{D}^+$ such that $\boldsymbol{x}(0^-)=x_0$ and $\boldsymbol{x}^+$ lies in V_D. A subspace $\mathsf{R}_\mathsf{D}\subset\mathsf{X}$ is said to be a *distributionally controllability* subspace if in addition $\forall x_0,x_1\in\mathsf{R}_\mathsf{D}$ there exists $T\geqslant 0$ and $\boldsymbol{x}\in\Sigma_\mathsf{D}^T$ such that $\boldsymbol{x}(0^-)=x_0$, $\boldsymbol{x}(T^+)=x_1$, and $\boldsymbol{x}^{[0,T]}$ lies in R_D.

Let $\underline{\mathsf{V}}_\mathsf{D},\underline{\mathsf{R}}_\mathsf{D}$ denote the sets of all distributionally invariant and controllability subspaces, and $\underline{\mathsf{V}}_\mathsf{D}(\mathsf{K}),\underline{\mathsf{R}}_\mathsf{D}(\mathsf{K})$ those contained in K.

The following theorem is an immediate consequence of the definitions.

Theorem 4: $\underline{\mathsf{V}}_\mathsf{D}$ and $\underline{\mathsf{R}}_\mathsf{D}$ are closed under subspace addition. Consequently

$$\sup\underline{\mathsf{V}}_\mathsf{D}(\mathsf{K})=:\mathsf{V}_{\mathsf{D},\mathsf{K}}^*\in\underline{\mathsf{V}}_\mathsf{D}$$

and

$$\sup\underline{\mathsf{R}}_\mathsf{D}(\mathsf{K})=:\mathsf{R}_{\mathsf{D},\mathsf{K}}^*\in\underline{\mathsf{R}}_\mathsf{D}.$$

Note also that $\underline{\mathsf{R}}_\mathsf{D}\subset\underline{\mathsf{V}}_\mathsf{D}$. (This may be seen by taking $x_1=0$ on the definition of distributionally controllability subspaces.) Our goal is to prove that $\underline{V}_D = \underline{V}_a$ and $\underline{R}_D = \underline{R}_a$. In order to do this it is convenient to introduce a special class of distributions.

Definition 4: A distribution $f \in D'_+$ is said to be of *Bohl type* if there exists vectors f_i and matrices F, G, and H such that $f = \Sigma_{i=0}^{N} f_i \delta^{(i)} + f_{-1}$ with $f_{-1}: t \mapsto He^{Ft}G$. Equivalently, iff the Laplace transform of f is rational.

Bohl type distributions have an important property. Indeed, while the scalar distributions in D'_+ form a convolution algebra without zero divisors, they do not form a field. However, the subalgebra of Bohl type distributions in $\mathcal{D}'_+$ does form a field. This is easily seen by considering Laplace transforms: the inverse of a nonzero scalar rational function exists and is rational. This field is actually isomorphic to the field of rational functions over $\mathbb{R}$. It is easily seen that Bohl type inputs have the special property in linear systems that they lead to Bohl type state trajectories when restricted to $\mathbb{R}^+$. What may be more surprising, however, is that for motions in distributionally invariant subspaces it suffices to consider this special class of inputs.

Lemma 1: Assume $V_D \in \underline{V}_D$ and $x_0 \in V_D$. Then for all $x_0 \in V_D$ there exists an "input" $\boldsymbol{u} \in D'_+$ of Bohl type $\boldsymbol{u} = \Sigma_{i=0}^{N} u_i \boldsymbol{\delta}^{(i)} + \boldsymbol{u}_{-1}$ such that the corresponding $\boldsymbol{x} \in \Sigma_D^+$ with $\boldsymbol{x}(0^-) = x_0$ has restriction $\boldsymbol{x}^+$ which lies in V_D and is Bohl. In fact, $\boldsymbol{x}^+ = \Sigma_{i=1}^{N} \Sigma_{k=1}^{i} A^{i-k} B u_i \boldsymbol{\delta}^{(k-1)} + \boldsymbol{x}_{-1}$ with $\boldsymbol{x}_{-1} = \boldsymbol{r} + \boldsymbol{h} * \boldsymbol{u}_{-1}$ where $\boldsymbol{r}$: $t \in \mathbb{R}^+ \mapsto e^{At}\boldsymbol{x}(0^+)$, $\boldsymbol{x}(0^+) = \boldsymbol{x}(0^-) + \Sigma_{i=0}^{N} A^i B u_i$, $\boldsymbol{h}$: $t \in \mathbb{R}^+ \mapsto e^{At}B$, and $*$ denotes convolution.

Proof: Let C be a matrix such that $V_D = \ker C$. Since $V_D \in \underline{V}_D$, there exists $\boldsymbol{u} \in D'_+$ such that

$$\boldsymbol{f} + \boldsymbol{g} * \boldsymbol{u} = \boldsymbol{0}$$

where $\boldsymbol{f}$: $t \in \mathbb{R}^+ \mapsto Ce^{At}x_0$, $\boldsymbol{g}$: $t \in \mathbb{R}^+ \mapsto Ce^{At}B$, and $*$ denotes convolution. Consider now this equation as a vector-matrix equation in the convolution algebra D'_+ of scalar distributions with $\boldsymbol{f}$, $\boldsymbol{g}$ known (respectively, vector and matrix-valued) and $\boldsymbol{u}$ the unknown (vector valued). This equation may be viewed as a finite system of linear equations. These equations are by assumption solvable for $\boldsymbol{u}$ in the algebra D'_+ and we wish to demonstrate its solvability in the class of Bohl type elements of D'_+. Now, since the elements of $\boldsymbol{f}$ and $\boldsymbol{g}$ are obviously Bohl and since the equation is solvable over the algebra D'_+, it is actually solvable over the field of Bohl distributions in D'_+, which yields the first part of the lemma. The expression for $\boldsymbol{x}^+$ may be found in most books on distribution theory (see, e.g., [6]). □

The above lemma provides one of the key steps in the following.

Theorem 5: $\underline{V}_D = \underline{V}_a$ *and* $\underline{R}_D = \underline{R}_a$.

Proof:

1) We will first prove that $\underline{R}_D \supset \underline{R}_a$. Assume that $R_a = B_1 + A_F B_2 + \cdots + A_F^{n-1} B_n$ with $\{B_i\}$ a chain in B: by Theorem 2 every $R_a \in \underline{R}_a$ can be written this way. Since $\Sigma_x(A_F, B) = \Sigma_x(A, B)$ we may as well think $F = 0$. Hence, using the result of Theorem 4, it suffices to show that for any $Br =: b \in B$ and $j \in \boldsymbol{n}$, $R_j := \operatorname{span}\{b, Ab, \cdots, A^{j-1}b\}$ belongs to $\underline{R}_D$. Let $x_0, x_1 \in R_j$. Hence, $x_1 - x_0$ may be written as $\Sigma_{i=0}^{j-1} \alpha_i A^i b$. Apply now the "input" $\Sigma_{i=0}^{j-1} \alpha_i r \boldsymbol{\delta}^{(i)}$. Then, with $T = 0$, we obtain $\boldsymbol{x}(0^+) = \boldsymbol{x}(0^-) + \Sigma_{i=0}^{j-1} \alpha_i A^i Br$. Hence, with $\boldsymbol{x}(0^-) = x_0$, $\boldsymbol{x}(0^+)$ equals x_1. Moreover, the restriction $\boldsymbol{x}^{[0,T]}$ is given by $\Sigma_{i=1}^{j-1} \Sigma_{k=1}^{i} \alpha_i A^{i-k} Br \boldsymbol{\delta}^{(k-1)}$, which clearly lies in R_j.

2) Next, we prove that $\underline{V}_D \supset \underline{V}_a$. This is easy. Indeed, by Theorem 2, $\underline{V}_a = \underline{V} + \underline{R}_a$ which, since $\underline{V} \subset \underline{V}_D$ and $\underline{R}_a \subset \underline{R}_D \subset \underline{V}_D$, establishes the desired inclusion.

3) We will now show the inclusion $\underline{V}_a \supset \underline{V}_D$. Here we use Lemma 1 in a crucial way. Let $V_D \in \underline{V}_D$ and $x_0 \in V_D$. We need to show that $x_0 \in V_{V_D}^* + R_{a,V_D}^*$, since this implies $V_D = V_{a,V_D}^* \in \underline{V}_a$. By Lemma 1 there exists a Bohl type distribution $\boldsymbol{u} \in D'_+$ such that the corresponding solution $\boldsymbol{x}$ with $\boldsymbol{x}(0^-) = x_0$ has a restriction $\boldsymbol{x}^+$, as given in Lemma 1, which lies in V_D. This implies that $\boldsymbol{x}(0^+) \in V_{V_D}^*$ and it suffices to show that $\boldsymbol{x}(0^+) - \boldsymbol{x}(0^-) \in R_{a,V_D}^*$. Since $\boldsymbol{x}^+$ and $\boldsymbol{x}(0^+) - \boldsymbol{x}(0^-)$ lie in V_D it follows that

$$\sum_{i=k}^{N} A^{i-k} B u_i \in V_D, \qquad \forall k = 0, 1, \cdots, N \qquad (*)$$

We will show that this implies that $\Sigma_{i=k}^{N} A^{i-k} B u_i \in R_{a,V_D}^*$ and hence that $\Sigma_{i=0}^{N} A^i B u_i = \boldsymbol{x}(0^+) - \boldsymbol{x}(0^-) \in R_{a,V_D}^*$. The proof goes by induction. For $k = N$, $(*)$ implies $Bu_N \in V_D \cap B =: R_{V_D}^1$, as defined by (ACSA). Assume now that $\Sigma_{i=k}^{N} A^{i-k} B u_i \in R_{V_D}^{N-k+1}$. Then, $\Sigma_{i=k-1}^{N} A^{i-k+1} B u_i = A(\Sigma_{i=k}^{N} A^{i-k} B u_i) + Bu_{k-1} \in V_D \cap (AR_{V_D}^{N-k+1} + B) =: R_{V_D}^{N-k+2}$ which yields the inductive step and finally, $\Sigma_{i=0}^{N} A^i B u_i \in R_{a,V_D}^*$, as desired.

This conclusion could also have been obtained by approximating the distribution $\boldsymbol{u}$ by smooth functions, but the above proof requires less analysis.

4) A slight adaptation of the proof of 3). (Take $V_{V_D}^* = \{0\}$); this yields $\underline{R}_a \supset \underline{R}_D$. □

Comments

1) It is a consequence of Lemma 1 and the proof of the above theorem that in a controllability subspace it is possible to transfer x_0 at 0^- to x_1 at T^+ for any $T \geq 0$ with an input that is the sum of a Bohl distribution plus the translate to T of a Bohl distribution.

2) The definitions of $\underline{V}_D$ and $\underline{R}_D$ are formulated in terms of elements of Σ_D whose restrictions to $\mathbb{R}^+$ or $[0, T]$ were only to lie in a given subspace. This may seem awkward, but it is not possible, at least as far as we see things now, to reformulate this condition in terms of elements of Σ_D which lie in the subspaces on all of $\mathbb{R}$. For example, consider the almost controllability subspace B. For an initial state $x_0 \in B$, the obvious candidate for a trajectory $\boldsymbol{x}$ which lies totally in B would be $\boldsymbol{x}(t) = 0$ for $t \neq 0$ and $\boldsymbol{x}(0) = x_0$. However, even in the sense of distributions, this is equivalent to the zero trajectory and the condition $\boldsymbol{x}(0) = x_0$ does not mean anything. What we need is to be able to distinguish between $\boldsymbol{\delta}$-functions at $t = 0^+$ and $t = 0^-$ such that the input $\boldsymbol{u} = (\boldsymbol{\delta}^+ - \boldsymbol{\delta}^-)x_0$ would be admissible and unequal to zero. However, a

theory of distributions which achieves this distinction does not seem to be available at present.

3) It is possible in principle to generate the distributional inputs which hold $x(0^-)$ in a subspace $\mathsf{V}_D \in \underline{\mathsf{V}}_D$ by a feedback law. In order to see this, consider the system Σ: $\dot{x}=Ax+Bu$ with the feedback law u: $x \mapsto Fx+\Sigma_{i=0}^{N}\delta^{(i)}F_i x$. Intuitively, one should interpret this expression as stating that if the system is in state x_0 then the input "value" $Fx_0+\Sigma_{i=0}^{N}\delta^{(i)}F_i x_o$ should be applied, whatever be the instant of time at which this occurs (hence, $\delta^{(i)}$ does not signify a distribution at $t=0$ since it is assumed that this feedback acts always). Plugging this feedback law into the system equations yields the autonomous (it contains no external inputs) time-invariant (the right-hand side does not depend explicitly on t) linear system: $\dot{x}=(A_F+\Sigma_{i=0}^{N}BF_i\delta^{(i)})x$ with $x(0)=a$. What would we mean by the solution of this? One possible approach is to identify the solution with that of the system $\dot{x}=A_F x+\Sigma_{i=0}^{N}BF_i\delta^{(i)}a$, $x(0^-)=a$, with distributional inputs: we know how to solve this. In particular, this leads to $x(0^+)=(I+\Sigma_{i=0}^{N}A_F^iBF_i)a=:La$ (it is logical to require $L^2=L$ in this setting; otherwise the distributional input should act again at $t=0^+$, etc.), and $x(t)=e^{A_F t}La$ for $t>0$.

Applying now these ideas to the situation at hand, assume that $\mathsf{V}_D=\mathsf{V}\oplus\mathsf{B}_1\oplus\cdots\oplus A_F^{n-1}\mathsf{B}_n=\mathsf{V}\oplus\mathsf{R}_a$ with $\mathsf{V}\in\underline{\mathsf{V}}$ (as is easily seen, V_D may always be written as such a direct sum). Because of the assumed independence F can be chosen such that $A_F\mathsf{V}\subset\mathsf{V}$. Consider now the feedback law $x\mapsto(F+\Sigma_{i=0}^{n-1}BF_i\delta^{(i)})x$ where the F_i's are such that $\operatorname{im} BF_i=\mathsf{B}_i$, $BF_0+A_FBF_1+\cdots+A_F^{n-1}BF_{n-1}=-I$ on R_a, and $F_i=0$ on $\mathsf{V}\oplus\mathsf{L}$ where L is any complement of V_D. The closed-loop system is then defined by $\dot{x}=(A_F+\Sigma_{i=0}^{n-1}BF_i\delta^{(i)})x$, which yields, as explained above ($L^2=L$ in this case), a well-defined solution process x having the property $\{x(0^-)\in\mathsf{V}_D\}\Rightarrow\{x^+$ lies in V_D and $x(0^+)=x(0^-)+\Sigma_{i=0}^{n-1}A_F^iBF_ix(0^-)\in\mathsf{V}\}$. Hence, $x(t)=e^{A_Ft}x(0^+)\in\mathsf{V}$ for $t>0$.

From a mathematical point of view this procedure works fine. However, what one would like to set up is some type of convergence when the $F_i\delta^{(i)}$'s are approximated by high gain feedback elements. This remains to be worked out.

The mathematical problems addressed in Comments 2) and 3) are, in our opinion, worthwhile questions and relevant to the understanding of high gain feedback design problems. Just as we have a theory of modeling distribution like time functions as distributions, we need a theory in which high gain feedback signals could be modeled as distribution-valued feedback.

IV. Almost Invariant Subspaces as Limits of Invariant Subspaces

In this section we will prove that arbitrarily close to any almost invariant subspace there exists an ordinary invariant subspace. In fact, the feedback gain which makes this invariant subspace a closed-loop invariant subspace goes to infinity in this approximation process. Hence, the direct connection with high gain feedback. Conversely, we will show that if a given subspace can be approximated arbitrarily closely by an invariant subspace, then it is actually an almost invariant subspace. Before stating the relevant theorem, let us recall the topological structure which one usually puts on the space of subspaces of a finite dimensional vector space. Let $\underline{\mathsf{G}}_q^n(\mathbb{R})$ denote, as usual, the set of all q-dimensional subspaces of $\mathbb{R}^n$. It is possible to provide $\underline{\mathsf{G}}_q^n(\mathbb{R})$ with a topology by taking the ϵ-neighborhood of the element $\mathsf{P}=\operatorname{Im}\begin{bmatrix} I_q \\ 0\end{bmatrix}$ to be $\{\operatorname{im}\begin{bmatrix} I_q \\ Z\end{bmatrix} \mid \|Z\|\leq\epsilon\}$ (by suitably choosing the basis, any element $\mathsf{P}\in\underline{\mathsf{G}}_q^n(\mathbb{R})$ may thus be expressed). Hence, $\mathsf{P}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{P}$ in $\underline{\mathsf{G}}_q^n(\mathbb{R})$ signifies that if $\mathsf{P}=\operatorname{span}[r_1,\cdots,r_q]$ then there exist vectors r_i^ϵ, $i\in\underline{q}$, such that $\mathsf{P}_\epsilon=\operatorname{span}[r_1^\epsilon,\cdots,r_q^\epsilon]$ and $r_i^\epsilon \underset{\epsilon\to 0}{\to} r_i$. This way $\underline{\mathsf{G}}_q^n$ actually becomes a $q(n-q)$-dimensional analytic manifold called a *Grassmannian.*

The primary purpose of this section is to prove the following.

Theorem 6: $\underline{\mathsf{V}}_a=\underline{\mathsf{V}}^{\text{closure}}$. Explicitly, $\{\mathsf{V}_a\in\underline{\mathsf{V}}_a\}\Leftrightarrow\{\exists\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$ such that $\mathsf{V}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{V}_a\}$, where this convergence should be understood in the Grassmannian sense.

Before proving the theorem we note that, in general, $\underline{\mathsf{R}}_a\neq\underline{\mathsf{R}}^{\text{closure}}$.

Proof: ($\Rightarrow$): We need to construct $\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$ such that $\mathsf{V}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{V}_a$. Since $\underline{\mathsf{V}}_a=\underline{\mathsf{V}}+\underline{\mathsf{R}}_a$ and since every $\mathsf{R}\in\underline{\mathsf{R}}_a$ may be written as $\mathsf{R}_a=\mathsf{B}_1+A_F\mathsf{B}_2+\cdots+A_F^{n-1}\mathsf{B}_n$ with $\{\mathsf{B}_i\}$ a chain in B, it suffices to construct such a sequence for a subspace of the form $b+A_Fb+\cdots+A_F^{i-1}b$ with $b\subset\mathsf{B}$ one-dimensional. This reduces the problem to the scalar input case and we may further obviously restrict our attention to reachable systems. Now, using feedback invariance of $\underline{\mathsf{V}}$ we may as well assume that the scalar input system consists of a bank of integrators and is in control canonical form. We thus have to show that $\mathsf{R}_i:=\operatorname{span}[e_{n-i+1},\cdots,e_n]$, with e_j the jth standard basis vector, may be approximated by invariant subspaces of $\dot{x}_1=x_2,\cdots,\dot{x}_{n-1}=x_n$, $\dot{x}_n=u$.

Every one-dimensional subspace of the form $\operatorname{span}[1,\lambda,\cdots,\lambda^{n-1}]^T$ is an invariant subspace; we will now show that by letting $\lambda_1,\cdots,\lambda_i\to\infty$ in a suitable way $\mathsf{L}_i:=\operatorname{span}\{[1,\lambda_k,\cdots,\lambda_k^{n-1}]^T, k\in\underline{i}\}$ approaches R_i.

We proceed recursively. Obviously, for $i=1$ and $\lambda\to\infty$, $\mathsf{L}_1=\operatorname{span}[\lambda^{-n+1},\cdots,\lambda^{-1},1]^T \underset{\lambda\to\infty}{\to} \mathsf{R}_1$. Assume now that $\mathsf{L}_{k-1}\to\mathsf{R}_{k-1}$. Hence,

$$\mathsf{L}_{k-1}=\operatorname{im}\begin{bmatrix} Z_{k-1} \\ \hdashline I_{k-1}\end{bmatrix}$$

with $\|Z_{k-1}\|\to 0$. Now, for $\lambda_k\to\infty$,

$$\mathsf{L}_k:=\operatorname{im}\left[\begin{array}{c|c} Z_{k-1} & \begin{matrix}1\\ \lambda_k\\ \vdots\\ \lambda_k^{n-1}\end{matrix} \\ \hline I_{k-1} & \end{array}\right]$$

$$=\operatorname{im}\left[\begin{array}{c|c} Z_{k-1} & \begin{matrix}1\\ \lambda_k\\ \vdots\\ \lambda_k^{n-k}\end{matrix} \\ \hline I_{k-1}-\begin{bmatrix}\lambda_k^{n-k+1}\\ \vdots\\ \lambda_k^{n-1}\end{bmatrix} Z_{k-1}^1 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \end{array}\right]$$

$$=\operatorname{im}\left[\begin{array}{c|c} Z_{k-1} & \begin{matrix}\lambda_k^{-n+k}\\ \vdots\\ \lambda_k\\ 1\end{matrix} \\ \hline I_{k-1}-\begin{bmatrix}\lambda_k^{n-k+1}\\ \vdots\\ \lambda_k^{n-1}\end{bmatrix} Z_{k-1}^1 & 0 \end{array}\right].$$

Here Z_{k-1}^1 denotes the first row of Z_{k-1}. Now, since $Z_{k-1}\to 0$, it is clearly possible to let $\lambda_k\to\infty$ such that this subspace approaches

$$\operatorname{im}\left[\begin{array}{c|c} 0 & \begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix} \\ \hline I_{k-1} & 0 \end{array}\right]=\mathsf{R}_k$$

which yields the inductive step.

($\Leftarrow$): Assume $\mathsf{V}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{L}$, $\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$. We wish to show that $\mathsf{L}\in\underline{\mathsf{V}}_a$. It is convenient to proceed via the following.

Lemma 2: Assume $\mathsf{V}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{L}$, $\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$ and $\mathsf{L}\cap\mathsf{B}=\{0\}$. Then $\mathsf{L}\in\underline{\mathsf{V}}$.

Proof: Let $x\in\mathsf{L}$. We need to show that there exists $u\in\mathsf{U}$ such that $Ax+Bu\in\mathsf{L}$. Now, there exist $x_\epsilon\in\mathsf{V}_\epsilon$ such that $\lim_{\epsilon\to 0}x_\epsilon=x$ and u_ϵ such that $Ax_\epsilon+Bu_\epsilon\in\mathsf{V}_\epsilon$. Assume, without loss of generality, that B is injective. Now, B injective, $x_\epsilon\to x$, $\mathsf{V}_\epsilon\to\mathsf{L}$, $Ax_\epsilon+Bu_\epsilon\in\mathsf{V}_\epsilon$, $\mathsf{B}\cap\mathsf{L}=\{0\}$ may be seen to imply that there exists ϵ such that $\|u_{\epsilon'}\|$ is bounded for $0\leqslant\epsilon'\leqslant\epsilon$. Hence, u_ϵ has a condensation point u for $\epsilon\to 0$. This yields $Ax_\epsilon+Bu_\epsilon\to Ax+Bu$, which, since $Ax_\epsilon+Bu_\epsilon\in\mathsf{V}_\epsilon$ and $\mathsf{V}_\epsilon\to\mathsf{V}$ implies $Ax+Bu\in\mathsf{V}$, as desired. □

In order to complete the proof of the theorem we need to develop a considerable machinery which we have delegated to the Appendix.

With these results we now proceed with the proof of ($\Leftarrow$) of Theorem 6. Assume that $\mathsf{V}_\epsilon \underset{\epsilon\to 0}{\to} \mathsf{L}$. Consider now $\mathsf{R}^*_{a,\mathsf{L}}$ and $\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})$. By Theorem $\tilde{A}$, $\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})$ is of the form $\tilde{\Sigma}_x(A',\mathsf{B}')$ with $\mathsf{B}'=(A\mathsf{R}^*_{a,\mathsf{L}}+\mathsf{B})(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})$. Moreover, by the algorithm (ACSA) of Proposition 2

$$\begin{aligned}\mathsf{B}'\cap(\mathsf{L}(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}}))&\\ &=((A\mathsf{R}^*_{a,\mathsf{L}}+\mathsf{B})(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}}))\cap(\mathsf{L}(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}}))\\ &=((A\mathsf{R}^*_{a,\mathsf{L}}+\mathsf{B})\cap\mathsf{L})(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})=\{0\}\end{aligned}$$

and hence by the above lemma $\mathsf{L}(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})\in\underline{\mathsf{V}}^{\tilde{\Sigma}(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})}=\underline{\mathsf{V}}^{\Sigma(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{L}})}$ which implies that $\mathsf{L}\in\underline{\mathsf{V}}$ as claimed. □

V. Decomposition and Behavior Under Approximation of Almost Invariant Subspaces

In this section we will show that every almost invariant subspace can be written as the sum of 1) a controllability subspace; the closed-loop eigenvalues corresponding to eigenvectors in this subspace are arbitrarily assignable; 2) a controlled invariant subspace in which the control is uniquely defined by the requirement that the trajectory should lie in that subspace (we call such subspaces *coasting* subspaces); the closed-loop eigenvalues corresponding to eigenvectors in this subspace are fixed; and 3) an almost invariant subspace which can be arbitrarily approximated by an invariant subspace but which contains no trajectories other than the null-trajectory (we will call such subspaces *sliding* subspaces). All trajectories in that subspace must be generated by singularity inputs. In approximating such subspaces by invariant subspaces, the closed-loop eigenvalues of the eigenvectors of this approximating invariant subspace will have to go to infinity but may be chosen to be arbitrarily stable.

Definition 5: A subspace $\mathsf{C}\in\underline{\mathsf{V}}$ is said to be a *coasting subspace* if $x_1,x_2\in\Sigma_x$, $x_1(0)=x_2(0)$, and $x_1(t),x_2(t)\in\mathsf{C}\,\forall t$ imply $x_1=x_2$, i.e., if $\mathsf{R}^*_{\mathsf{C}}=\{0\}$. A subspace $\mathsf{S}\in\underline{\mathsf{V}}_a$ is said to be a *sliding subspace* if $x\in\Sigma_x$ and $x(t)\in\mathsf{S}\,\forall t$ imply $x(t)=0\,\forall t$, i.e., if $\mathsf{V}^*_{\mathsf{S}}=\{0\}$.

The family of all coasting, respectively, sliding, subspaces are denoted by $\underline{\mathsf{C}}$ and $\underline{\mathsf{S}}$. Note that $\underline{\mathsf{S}}\subset\underline{\mathsf{R}}_a$. The following easy theorem gives the decomposition which we are after.

Theorem 7: Let $\mathsf{V}_a\in\underline{\mathsf{V}}_a$. *Then there exist* $\mathsf{C}\in\underline{\mathsf{C}}$ *and* $\mathsf{S}\in\underline{\mathsf{S}}$ *such that* $\mathsf{R}^*_{\mathsf{V}_a}\oplus\mathsf{C}=\mathsf{V}^*_{a,\mathsf{V}_a}$ and $\mathsf{V}^*_{\mathsf{V}_a}\oplus\mathsf{S}=\mathsf{R}^*_{a,\mathsf{V}_a}$. *This yields a decomposition* $\mathsf{V}_a=\mathsf{R}^*_{\mathsf{V}_a}\oplus\mathsf{C}\oplus\mathsf{S}$.

Proof: We need to show only two decompositions. 1) for $\mathsf{V}\in\underline{\mathsf{V}}$ there exists $\mathsf{C}\in\underline{\mathsf{C}}$ such that $\mathsf{V}=\mathsf{R}^*_{\mathsf{V}}\oplus\mathsf{C}$; and 2) for $\mathsf{R}_a\in\underline{\mathsf{R}}_a$ there exists $\mathsf{S}\in\underline{\mathsf{S}}$ such that $\mathsf{R}_a=\mathsf{R}^*_{\mathsf{R}_a}\oplus\mathsf{S}$. Fact 1) is well known and follows by taking an F such that $A_F\mathsf{V}\subset\mathsf{V}$ and $\sigma(A_F|\mathsf{R}^*_{\mathsf{V}})\cap\sigma((A_F|\mathsf{V})(\mathrm{mod}\,\mathsf{R}^*_{\mathsf{V}}))=\varnothing$ and examining a spectral decomposition of V. To see 2) consider

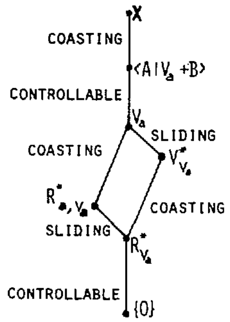

Fig. 2. Decomposition of V_a.

the representation $\mathsf{R}_a=\mathsf{B}_1\oplus A_F\mathsf{B}_2\oplus\cdots\oplus A_F^{n-1}\mathsf{B}_n$. We may obviously choose this F such that $\mathsf{R}^*_{\mathsf{R}_a}=\langle A_F|\mathsf{B}\cap\mathsf{R}_a\rangle$. Letting now B'_i be a chain in B such that $\mathsf{B}\cap\mathsf{R}_a\oplus\mathsf{B}'_i=\mathsf{B}_i$ yields $\mathsf{R}_a=\mathsf{R}^*_{\mathsf{R}_a}\oplus(\mathsf{B}'_1\oplus A_F\mathsf{B}'_2\oplus\cdots\oplus A_F^{n-1}\mathsf{B}'_n)$ as desired. □

The lattice diagram suggested by the above theorem is shown in Fig. 2.

Note that $\mathsf{V}^*_{\mathsf{V}_a}(\mathrm{mod}\,\mathsf{R}^*_{\mathsf{V}_a})\cong\mathsf{V}_a(\mathrm{mod}\,\mathsf{R}^*_{a,\mathsf{V}_a})$ and $\mathsf{V}_a(\mathrm{mod}\,\mathsf{V}^*_{\mathsf{V}_a})\cong\mathsf{R}^*_{a,\mathsf{V}_a}(\mathrm{mod}\,\mathsf{R}^*_{\mathsf{V}_a})$.

The next results concern the behavior of the approximations of almost invariant subspaces by invariant subspaces. According to Theorem 6, given any $\mathsf{V}_a\in\underline{\mathsf{V}}_a$ there exists $\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$ and a feedback gain F_ϵ such that $A_{F_\epsilon}\mathsf{V}_\epsilon\subset\mathsf{V}_\epsilon$ and $\mathsf{V}_\epsilon\underset{\epsilon\to0}{\to}\mathsf{V}_a$. Denote by $\sigma_\epsilon:=\sigma(A_{F_\epsilon}|\mathsf{V}_\epsilon)$. An interesting question concerns the behavior of F_ϵ, σ_ϵ, A_{F_ϵ}, and its invariant subspaces as $\epsilon\to0$. In the next theorem we will prove an expected result in this direction: it illustrates the high gain behavior of F_ϵ as $\epsilon\to0$. A complete theory of this limit behavior is not yet available. In particular, if $\mathsf{V}_a=\mathsf{R}\oplus\mathsf{C}\oplus\mathsf{S}$ with $\mathsf{S}=\mathsf{B}_1\oplus A_F\mathsf{B}_2\oplus\cdots\oplus A_F^{n-1}\mathsf{B}_n$ then the limit behavior of F_ϵ, σ_ϵ and A_{F_ϵ} is surely related to this decomposition. Undoubtly this limit behavior is also closely related to singular perturbations, infinite zeros, and the like. However, the situation appears to be rather involved, and we limit ourselves to the following two theorems.

Theorem 8: Assume $\mathsf{V}_a\in\underline{\mathsf{V}}_a$, $\mathsf{V}_\epsilon\in\underline{\mathsf{V}}$, $\mathsf{V}_\epsilon\underset{\epsilon\to0}{\to}\mathsf{V}_a$, *and* $A_{F_\epsilon}\mathsf{V}_\epsilon\subset\mathsf{V}_\epsilon$. *Then* $\{F_\epsilon\underset{\epsilon\to0}{\to}F\}\Rightarrow\{\mathsf{V}_a\in\underline{\mathsf{V}}$ *and* $A_F\mathsf{V}_a\subset\mathsf{V}_a\}$ *and* $\{\mathsf{V}_a\notin\underline{\mathsf{V}}\}\Rightarrow\{F_\epsilon\underset{\epsilon\to0}{\to}\infty\}$.

Proof: The first part is proven as follows. Let $\{x^i\}$ be a basis for V_a and $\{x^i_\epsilon\}$ for V_ϵ, such that $x^i_\epsilon\underset{\epsilon\to0}{\to}x^i$. Then $A_{F_\epsilon}x^i_\epsilon\underset{\epsilon\to0}{\to}A_Fx^i$, hence, $A_Fx^i\in\mathsf{V}_a$, and thus $\mathsf{V}_a\in\underline{\mathsf{V}}$, as claimed. Consider now the second part of the theorem. If $F_\epsilon\underset{\epsilon\to0}{\not\to}\infty$, then it has a convergent subsequence which, by the first part of the theorem, implies $\mathsf{V}_a\in\underline{\mathsf{V}}$ and yields the desired contradiction. □

The following theorem illustrates the behavior of the spectrum σ_ϵ under a particular choice of the V_ϵ's.

Theorem 9: Let $\mathsf{V}_a\in\underline{\mathsf{V}}_a$ *be given. Write* $\mathsf{V}_a=\mathsf{R}^*_{\mathsf{V}_a}\oplus\mathsf{C}\oplus\mathsf{S}$ *with* $\mathsf{C}\in\underline{\mathsf{C}}$ *and* $\mathsf{S}\in\underline{\mathsf{S}}$. *Then there exist* $\mathsf{S}_\epsilon\in\underline{\mathsf{C}}$ *such that* $\mathsf{S}_\epsilon\underset{\epsilon\to0}{\to}\mathsf{S}$. *Moreover, one may choose the* S_ϵ*'s such that* $\sigma_{\mathsf{S},\epsilon}:=\sigma(A_{F_\epsilon}|\mathsf{S}_\epsilon)$ *is real and* $\max\sigma_{\mathsf{S},\epsilon}\underset{\epsilon\to0}{\to}-\infty$. *In any case,* $\min_{\lambda\in\sigma_{\mathsf{S},\epsilon}}|\lambda|=:\min|\sigma_{\mathsf{S},\epsilon}|\underset{\epsilon\to0}{\to}\infty$. *Correspondingly, there exists* $\mathsf{V}_\epsilon\underset{\epsilon\to0}{\to}\mathsf{V}_a$ *with* $\mathsf{V}_\epsilon=\mathsf{R}^*_{\mathsf{V}_a}\oplus\mathsf{C}\oplus\mathsf{S}_\epsilon$ *and* $\sigma_\epsilon=\sigma_{\mathsf{R},\epsilon}\cup\sigma_{\mathsf{C}}\cup\sigma_{\mathsf{S},\epsilon}$ *with* $\sigma_{\mathsf{R},\epsilon}$ *arbitrary for all* ϵ *(modulo, of course, the symmetry with respect to the real axis),* $\sigma_{\mathsf{C}}=\sigma(A_F|\mathsf{C})$ *with* F *any matrix such that* $A_F\mathsf{C}\subset\mathsf{C}$ *(it is well known that* σ_{C} *is then fixed by* V_a *itself and is given by* $\sigma((A_F)\mathsf{V}^*_{\mathsf{V}_a})(\mathrm{mod}\,\mathsf{R}^*_{\mathsf{V}_a}))$*—these eigenvalues are, in one possible definition of zeros, the zeros of* $\dot{x}=Ax+Bu$, $y=x(\mathrm{mod}\,V_a)$*), and* $\sigma_{\mathsf{S},\epsilon}$ *as above.*

Proof: The desired subspaces S_ϵ have been constructed in the proof of Theorem 6; this construction shows as a side result the claim about the spectrum $\sigma_{\mathsf{S},\epsilon}$ as $\epsilon\to0$. We only need to prove that $\min|\sigma_{\mathsf{S},\epsilon}|\underset{\epsilon\to0}{\to}\infty$ always holds. In order to see this, assume the contrary. Then there exists a subsequence ϵ' and $\lambda_{\epsilon'}\in\sigma_{\mathsf{S},\epsilon'}$ such that $\lambda_{\epsilon'}\underset{\epsilon'\to0}{\to}\lambda$. Let $v_{\epsilon'}$ be a (unit magnitude) eigenvector corresponding to $\lambda_{\epsilon'}$. The $v_{\epsilon'}$'s have an accumulation point $v\neq0$. We claim that span $v\in\underline{\mathsf{V}}$ which, since obviously also $v\in\underline{\mathsf{S}}$, will yield the desired contradiction. To see this, observe that $Av_{\epsilon'}-\lambda_{\epsilon'}v_{\epsilon'}\in\mathsf{B}$. Taking limits along the accumulation subsequence yields $Av-\lambda v\in\mathsf{B}$, hence span $v\in\underline{\mathsf{V}}$. □

Note finally that the construction in the proof of Theorem 6 actually shows that one may let $\sigma_{\mathsf{S},\epsilon}$ go to infinity along any *a priori* given asymptotic root locus directions (but not at arbitrary independent rates).

VI. The Σ-Distance to a Subspace and L_p-Almost Invariant Subspaces

We have seen how the system Σ: $\dot{x}=Ax+Bu$ defines a family of trajectories Σ_x. This family of trajectories provides in a natural way a notion of distance from a point to a subspace by defining this to be the smallest (integrated) pointwise distance to the subspace while traveling along a trajectory through the point. This is illustrated in the Fig. 3. This natural and appealing concept will turn out to provide a very direct approach to the almost disturbance decoupling problem which will be treated in Section VIII. This notion does not define a distance function in the mathematical sense of the term. It is not unlike the traveling distance between two points in a city taking into consideration permitted traffic patterns as one-way streets. Moreover, since there are many ways to measure (integrated) trajectories, we arrive at a number of such measures.

Definition 6: The Σ-*distance in the* $\mathcal{L}_p$-*sense* from a point $x_0\in\mathsf{X}$ to a subspace $\mathsf{K}\subset\mathsf{X}$ is defined by

$$d_p(x_0,\mathsf{K}):=\inf_{\substack{x\in\Sigma_x\\ x(0)=x_0}}\|d(x,\mathsf{K})\|_{\mathcal{L}_p},\qquad 1\leqslant p\leqslant\infty,$$

where

$$\|d(x,\mathsf{K})\|_{\mathcal{L}_p}:=\left(\int_{-\infty}^{+\infty}d^p(x(t),\mathsf{K})\,dt\right)^{1/p},\qquad \text{for } 1\leqslant p<\infty$$

and

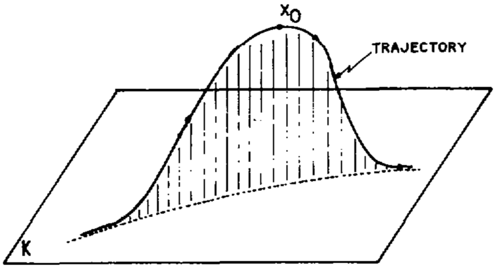

Fig. 3. The Σ-distance from x_0 to K.

$$\|d(x,\mathsf{K})\|_{\mathcal{L}_\infty} := \sup_t \|d(x(t),\mathsf{K})\|.$$

Note that in principle the norm on X will influence d_p. However, since all norms on X are equivalent, the specific norm chosen will not influence the fact that x_0 is zero distance away from K. Those points play a special role in the sequel.

Definition 7: $\mathsf{V}^*_{p,\mathsf{K}} := \{x_0 \in \mathsf{X} \mid d_p(x_0,\mathsf{K})=0\}$ will be called the *supremal* $\mathcal{L}_p$*-almost invariant subspace "contained" in* K and

$$\mathsf{R}^*_{p,\mathsf{K}} := \mathsf{R}^*_{a,\mathsf{V}^*_{p,\mathsf{K}}}$$

will be called the *supremal* $\mathcal{L}_p$*-almost controllability subspace "contained" in* K. Note (see Theorem 10) that $\mathsf{V}^*_{\infty,\mathsf{K}} \subset \mathsf{K}$ but that $\mathsf{V}^*_{p,\mathsf{K}} \subset \mathsf{K}$ need *not* be the case if $p<\infty$!

It is easily seen from the linearity of Σ_x that $\mathsf{V}^*_{p,\mathsf{K}}$ is a linear subspace. Also, $\mathsf{R}^*_{p,\mathsf{K}}$ admits the interpretation that it is the subspace of X with the property that one can travel between any two points of that subspace while remaining arbitrarily close (in the $\mathcal{L}_p$-sense) to K.

Whereas the Σ-distance will in general be hard to compute, it turns out that evaluation of zero distance conditions is feasible in terms of almost invariant subspaces. We record this below.

Theorem 10:

1) $\mathsf{R}^*_{\infty,\mathsf{K}} = \mathsf{R}^*_{a,\mathsf{K}}$ *and* $\mathsf{V}^*_{\infty,\mathsf{K}} = \mathsf{V}^*_{a,\mathsf{K}}$; *and*

2) *for* $1 \leqslant p < \infty$: $\mathsf{R}^*_{p,\mathsf{K}} = A\mathsf{R}^*_{a,\mathsf{K}} + \mathsf{B}$ *and* $\mathsf{V}^*_{p,\mathsf{K}} = \mathsf{R}^*_{p,\mathsf{K}} + \mathsf{V}^*_{\mathsf{K}} = A\mathsf{V}^*_{a,\mathsf{K}} + \mathsf{B} + \mathsf{V}^*_{a,\mathsf{K}}$.

Notation: This result justifies introducing the notation

$$\mathsf{R}^*_{b,\mathsf{K}} := A\mathsf{R}^*_{a,\mathsf{K}} + \mathsf{B}$$

and

$$\mathsf{V}^*_{b,\mathsf{K}} := \mathsf{R}^*_{b,\mathsf{K}} + \mathsf{V}^*_{\mathsf{K}} = A\mathsf{V}^*_{a,\mathsf{K}} + \mathsf{B} + \mathsf{V}^*_{a,\mathsf{K}} = A\mathsf{V}^*_{a,\mathsf{K}} + \mathsf{B} + \mathsf{V}^*_{\mathsf{K}}.$$

We will prove Theorem 10 later. Theorems 3 and 10 provide, via (ISA) and (ACSA), algorithms for computing $\mathsf{R}^*_{p,\mathsf{K}}$ and $\mathsf{V}^*_{p,\mathsf{K}}$. A more direct algorithm may be based on the following.

Proposition 3: Consider the algorithm

$$\boxed{\mathsf{S}^{k+1}_{\mathsf{K}} = \mathsf{B} + A(\mathsf{K} \cap \mathsf{S}^k_{\mathsf{K}}); \qquad \mathsf{S}^0_{\mathsf{K}} = \{0\}.} \qquad \text{(ACSA)}'$$

Then $\mathsf{S}^k_{\mathsf{K}}$ is monotone nondecreasing; moreover, $\mathsf{S}^{\dim \mathsf{K}+1}_{\mathsf{K}} = \mathsf{S}^\infty_{\mathsf{K}} := \lim_{k\to\infty} \mathsf{S}^k_{\mathsf{K}}$, and $\{\mathsf{S}^{k+1}_{\mathsf{K}} = \mathsf{S}^k_{\mathsf{K}}\} \Rightarrow \{\mathsf{S}^k_{\mathsf{K}} = \mathsf{S}^\infty_{\mathsf{K}}\}$.

Proof: This follows immediately from Lemma 3.

Lemma 3: Let $\mathsf{R}^k_{\mathsf{K}}$ be as defined in (ACSA) (see Proposition 2). Then $\mathsf{S}^{k+1}_{\mathsf{K}} = A\mathsf{R}^k_{\mathsf{K}} + \mathsf{B}$.

Proof: The equality obviously holds for $k=0$. Assume now that it holds for $0,1,\cdots,k$. Then

$$\begin{aligned}\mathsf{S}^{k+2}_{\mathsf{K}} &= \mathsf{S}^{k+1}_{\mathsf{K}} + A(\mathsf{K} \cap \mathsf{S}^{k+1}_{\mathsf{K}})\\ &= A\mathsf{R}^k_{\mathsf{K}} + \mathsf{B} + A(\mathsf{K} \cap (A\mathsf{R}^k_{\mathsf{K}} + \mathsf{B}))\\ &= A\mathsf{R}^k_{\mathsf{K}} + \mathsf{B} + A\mathsf{R}^{k+1}_{\mathsf{K}} + \mathsf{B}\end{aligned}$$

which yields the inductive step. □

Theorem 10 and the above proposition result in the following.

Theorem 11: Consider (ACSA)′. Then $\mathsf{R}^*_{b,\mathsf{K}} = \mathsf{S}^\infty_{\mathsf{K}}$ *and* $\mathsf{V}^*_{b,\mathsf{K}} = \mathsf{S}^\infty_{\mathsf{K}} + \mathsf{V}^\infty_{\mathsf{K}}$ *(as defined in (ACSA) in Proposition 2).*

Consequently, $\mathsf{R}^*_{a,\mathsf{K}}$, $\mathsf{R}^*_{b,\mathsf{K}}$ and $\mathsf{V}^*_{a,\mathsf{K}}$, $\mathsf{V}^*_{b,\mathsf{K}}$ are well-defined subspaces which are in principle computable through linear algorithms, and they deliver $\mathsf{R}^*_{p,\mathsf{K}}$ and $\mathsf{V}^*_{p,\mathsf{K}}$ for $1 \leqslant p \leqslant \infty$. We now return to the proof.

Proof of Theorem 10:

1) This claim is an immediate consequence of the definitions: almost invariance and zero $\mathcal{L}_\infty$-distance are identical.

2) We now consider the case $1 \leqslant p < \infty$. We will first prove that $\{x_0 \in \mathsf{V}^*_{p,\mathsf{K}} = A\mathsf{R}^*_{a,\mathsf{K}} + \mathsf{B} + \mathsf{V}^*_{\mathsf{K}}\} \Rightarrow \{d_p(x_0,\mathsf{K}) = 0\}$. To see this observe that by the linearity of $\mathsf{V}^*_{p,\mathsf{K}}$ and the reasoning used in the proof of Theorem 6 it suffices to show for the single input system $\dot{x} = Ax + Bu$, that $\{x_0 \in \mathsf{R}_{i+1}\} \Rightarrow \{d_p(x_0,\mathsf{R}_i) = 0\}$, where $\mathsf{R}_0 = \{0\}$, $\mathsf{R}_i := \mathsf{B} + A\mathsf{B} + \cdots + A^{i-1}\mathsf{B} = A\mathsf{R}_{i-1} + \mathsf{B}$. By feedback invariance we may as well assume that the system consists of a bank of integrators and that it is in control canonical form. The question then reduces to showing $d_p(x_0,\mathsf{R}_i) = 0$ for $x_0 = e_{n-i}$, where $\{e_i\}$, $i \in \boldsymbol{n}$, denotes the standard basis for X: this will immediately imply the same for $x_0 = e_j$ with $j \geqslant n-i$, since $d_p(x_0,\mathsf{R}_i) = 0 \Rightarrow d_p(x_0,\mathsf{R}_j) = 0$ for $j \geqslant i$. Let $\boldsymbol{x} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]^T \in \Sigma_x$ with $\boldsymbol{x}(0) = x_0$ and $\boldsymbol{x}(t) = 0$ for $t \geqslant 1$. A simple calculation shows that

$$\begin{aligned} t \mapsto \big[\, &\alpha^{-n+i+1}\boldsymbol{a}_1(\alpha t), \cdots, \alpha^{-1}\boldsymbol{a}_{n-i-1}(\alpha t),\\ &\boldsymbol{a}_{n-i}(\alpha t), \alpha\boldsymbol{a}_{n-i+1}(\alpha t),\\ &\cdots, \alpha^i\boldsymbol{a}_n(\alpha t)\big]^T\end{aligned}$$

is also a trajectory through x_0. By taking α sufficiently large this yields $d_p(x_0,\mathsf{R}_i) = 0$ as desired.

We still need to show that $\{d_p(x_0,\mathsf{K}) = 0\} \Rightarrow \{x_0 \in A\mathsf{V}^*_{a,\mathsf{K}} + \mathsf{V}^*_{a,\mathsf{K}} + \mathsf{B}\}$. Consider, therefore, $\Sigma_x(\mathrm{mod}\, \mathsf{V}^*_{a,\mathsf{K}}) = \Sigma_x(A', \mathsf{B}')$ as introduced in the Appendix. Define $x_0' := x_0(\mathrm{mod}\, \mathsf{V}^*_{a,\mathsf{K}})$ and $\mathsf{K}' = \mathsf{K}(\mathrm{mod}\, \mathsf{V}^*_{a,\mathsf{K}})$. Clearly $d_p(x_0', \mathsf{K}) = 0$ and $\mathsf{K}' \cap \mathsf{B}' = \{0\}$. Hence, it suffices to show that $\{d_p(x_0,\mathsf{K}) = 0,\ \mathsf{B} \cap \mathsf{K} = \{0\}$, and $\mathsf{V}^*_{\mathsf{K}} = \{0\}\} \Rightarrow \{x_0 \in \mathsf{B}\}$. Use feedback to write $\dot{x} = Ax + Bu$ in the form $\dot{z} = A'z + Bb$, $\dot{b} = u$ with $\mathsf{X} = \mathsf{Z} + \mathsf{B}$ and $\mathsf{Z} \supset \mathsf{K}$. Since $d_p(x_0, \mathsf{K}) = 0$ there exists a trajectory $(z, \boldsymbol{b})$ through $x_0 = (z_0, b_0)$ with $\|\boldsymbol{b}\|_{\mathcal{L}_p}$ arbitrarily small. Now since

$$z(t) = e^{A't}z_0 + \int_0^t e^{A'(t-\tau)}B'\boldsymbol{b}(\tau)\, d\tau$$

this implies $e^{A't}z_0 \in \mathsf{K}$ for t sufficiently small and hence $z_0 \in \mathsf{V}_{\mathsf{K}}^* = \{0\}$, and $x_0 \in \mathsf{B}$, as claimed. The expression for $\mathsf{R}_{p,K}^*$ follows immediately from its definition and the expression of $\mathsf{V}_{p,\mathsf{K}}^*$ which we have just calculated. □

Theorem 10 was concerned with open-loop control laws in order to generate $d_p(x_0, \mathsf{K}) = 0$. In analogy with the analysis in Section IV, this may also be stated in feedback form. In the next theorem we formulate such a result and we will phrase it in a form which will make it immediately applicable to the disturbance decoupling problem which will be discussed in Section VIII.

Theorem 12: Assume that in addition to Σ: $\dot{x} = Ax + Bu$ *there are matrices* H *and* G *given. Consider the closed-loop impulse response* $W: t \in \mathbb{R}^+ \to He^{A_F t}G$. *Denote* $\mathsf{K} := \ker H$. *Then*

1) $\{\forall \epsilon > 0 \exists F$ *such that* $\|W\|_{\mathcal{L}_p} \leq \epsilon\} \Leftrightarrow \{\operatorname{im} G \subset \mathsf{V}_{p,\mathsf{K}}^*\}$; *and*

2) $(\forall \epsilon > 0,\ M \in \mathbb{R},\ \exists F$ *such that* $\|W\|\mathcal{L}_p \leq \epsilon$ *and* $Re\, \sigma(A_F) \geq M\} \Leftrightarrow \{\operatorname{im} G \subset \mathsf{R}_{p,\mathsf{K}}^*$ *and* Σ *is controllable*$\}$

In fact, 1) *and* 2) *hold uniformly in p in the sense that*

$\{\forall \epsilon > 0 \exists F$ *such that* $\|W\|_{\mathcal{L}_p} \leq \epsilon,\ 1 \leq p \leq \infty\} \Leftrightarrow \{\operatorname{im} G \subset \mathsf{V}_{a,\mathsf{K}}^*\}$; *and*

$\{\forall \epsilon > 0,\ 1 \leq p_0 < \infty,\ \exists F$ *such that* $\|W\|_{\mathcal{L}_p} \leq \epsilon,\ 1 \leq p \leq p_0\} \Leftrightarrow \{\operatorname{im} G \subset \mathsf{V}_{b,\mathsf{K}}^*\}$.

Similar refinements hold for 2).

Proof:

1) ($\Rightarrow$) In a suitable norm, there holds $\|Hx\| = d(x, \mathsf{K})$ and we may as well assume this to be the case. Considering now the solution of $\dot{x} = Ax + BFx$ with $x(0) \in \operatorname{im} G$, it is seen that there exists a feedback—and thus an open-loop control for Σ such that $\|Hx\|_{\mathcal{L}_p(0,\infty)} \leq \epsilon$. The same obviously holds for $t \leq 0$ which yields the inclusion $\operatorname{im} G \subset \mathsf{V}_{p,\mathsf{K}}^*$.

2) ($\Leftarrow$) is the most involved proof of the paper. In this part we are required to construct a feedback control law. Consider the case $1 \leq p < \infty$. Now $\mathsf{V}_{b,\mathsf{K}}^* = A\mathsf{R}_{a,\mathsf{K}}^* + \mathsf{B} + \mathsf{V}_{\mathsf{K}}^*$. This may be written, for some F and some chain $\{\mathsf{B}_i\}$ in B, as $\mathsf{V}_{\mathsf{K}} \oplus \mathsf{B} \oplus A_F \mathsf{B}_1 \oplus \cdots \oplus A_F^n \mathsf{B}_{n-1}$ with $\mathsf{B}_1 \oplus A_F \mathsf{B}_1 \oplus \cdots \oplus A_F^{n-1} \mathsf{B}_{n-1} \subset \mathsf{K}$. Using this direct sum decomposition and feedback invariance of Σ_x it follows that it suffices to consider the scalar input case $\dot{x}_1 = x_2, \cdots, \dot{x}_{n-1} = x_n,\ \dot{x}_n = u$ where we are, in effect required to show that for all $\epsilon > 0$ there exists F such that

$$[\, I_{n-i+1} \mid 0 \,]\, e^{A_F t} \begin{bmatrix} 0 \\ \hline I_i \end{bmatrix}$$

has $\mathcal{L}_p(0,\infty)$ norm less than ϵ. The proof of this fact is rather indirect: we will first show that the claim is valid for $i = 1$. This obviously also yields the result for $n = 1$. We will then proceed by induction on n.

Consider the case $i = 1$. We then need to show that for the scalar input controllable system $\dot{x} = Ax + Bu$ there exists, $\forall \epsilon > 0$, F such that the $\mathcal{L}_p$-norm of $t \in \mathbb{R}^+ \mapsto e^{A_F t}B$ is less than ϵ. By [2, Lemma 3.5] there exists $\mathsf{V} \in \underline{\mathsf{V}}$ and F with $A_F \mathsf{V} \subset \mathsf{V}$ such that $\mathsf{V} \oplus \mathsf{B} = \mathsf{X}$ and $\operatorname{Re} \sigma(A_F | \mathsf{V}) < 0$. Now, with (A, B) in control canonical form

$$\mathsf{V}_\lambda := \operatorname{im}[\lambda^{-n+1}, \lambda^{-n+2}, \cdots, \lambda^{-1}, 1]^T \in \underline{\mathsf{V}} \quad \text{for all } \lambda \neq 0.$$

Hence, with λ sufficiently large, B may be written as $\mathsf{B} = v_\lambda + v'_\lambda$ with $v_\lambda \in \mathsf{V}_\lambda$ and $v'_\lambda \in \mathsf{V}$. Also, $v_\lambda \underset{\lambda \to \infty}{\to} \mathsf{B}$ and $v'_\lambda \underset{\lambda \to \infty}{\to} 0$. Let F_λ be such that $F_\lambda | \mathsf{V} = F | \mathsf{V}$ and $A_{F_\lambda} v = \lambda v_\lambda$. Then

$$e^{A_{F_\lambda} t}B = e^{A_F t} v'_\lambda + e^{A_{F_\lambda} t} v_\lambda = e^{A_F t} v'_\lambda + e^{\lambda t} v_\lambda.$$

Now letting $\lambda \to \infty$ and estimating the $\mathcal{L}_p$-norms in this decomposition yields the desired result, including the uniformity in p.

We now return to the case of arbitrary n and i. In terms of higher order differential equations, we start with the system $y^{(n)} = u$ and $1 \leq i \leq n$ given, and the problem is to show that $\forall \epsilon > 0$ there exist a_i's (defining the feedback control law) such that the differential equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$ has the property that the solution with a fixed initial condition satisfying $y(k)(0) = 0$ for $k < n - i$, is such that $\|y^{(k)}\|\mathcal{L}_p(0,\infty) \leq \epsilon$ for $k \leq n - i$. We proceed by induction on n. We have already shown that the claim is true for $n = 1$. Assume that the claim is true up to $n - 1$. In the previous paragraph we have proven the case $i = 1$ for all n. Assume now $i > 1$, define $z := y^{(n-i+1)} + b_1 y^{(n-i)} + \cdots + b_{n-i+1} y$ and consider the differential equation $z^{(i-1)} + \beta_1 z^{(i-2)} + \cdots + \beta_{i-1} z = 0$. Together these differential equations define one of the form $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$.

We will consider the first of these differential equations as a system with input z and output $y, \dot{y}, \cdots, y^{(n-i)}$. Its initial conditions are $y(0), \dot{y}(0), \cdots, y^{(n-i)}(0)$. We will show that the b_i's may be chosen in such a way that this system has an impulse response with arbitrarily small $\mathcal{L}_1$-norm and an initial condition response with arbitrarily small $\mathcal{L}_p$-norm. This claim actually follows from the case $i = 1$, because for the initial condition only $y^{(n-i)}(0) \neq 0$ and the impulse response also corresponds to such an initial condition response.

Consider now the differential equation governing z. Its initial conditions $z^{(i-2)}(0), \cdots, z(0)$ are computable from $y^{(n-1)}(0), \cdots, y(0)$ in terms of the b_i's. Now, by the induction hypothesis (applied for $n' = i' = i - 1$), one may choose, once one has the b_i's, the β_i's such that the $\mathcal{L}_p$-norm of z is arbitrarily small.

Together this yields the claim. Indeed, assume that we want the $\mathcal{L}_p$-norm of $y, \dot{y}, \cdots, y^{(n-i)}$ to be $\leq \epsilon$. Choose the b_i's such that for the first differential equation the initial condition response is $\leq \epsilon/2$ in $\mathcal{L}_p$-norm, and the impulse response is $\leq \epsilon/2$ in $\mathcal{L}_1$-norm. Adjust now the β_i's such that z has $\mathcal{L}_p$-norm ≤ 1. Together this yields an $\mathcal{L}_p$-norm $y, \dot{y}, \cdots, y^{(n-i)}$ which is $\leq \epsilon$, as desired.

This ends the proof for $1 \leq p < \infty$. The case $p = \infty$ is entirely analogous. To prove 2) it suffices to keep also Theorem 9 in mind while repeating the above arguments. □

Comment 4: It is perhaps worthwhile to point out that the problem treated in Theorem 12 promised to be a great deal harder in the "almost" case than we knew it to be true in the "exact" case. Indeed that $\{\mathsf{V}_{\ker H}^* \supset \operatorname{im} G\} \Leftrightarrow \{\exists F: He^{A}F^{t}G = 0,\ t \in \mathbb{R}\}$ is a direct consequence of the definitions. However, if $\operatorname{im} G \subset V^*_{a,\ker H}$, then we know that $\exists V_\epsilon$ in the neighborhood of $V^*_{a,\ker H}$ and F_ϵ such that $A_{F_\epsilon} V_\epsilon \subset V_\epsilon$. Hence $\operatorname{im} G$ is almost contained in the A_{F_ϵ}-invariant subspace V_ϵ which, in turn, is almost contained in $\ker H$. However, this does not yet guarantee that $He^{A_{F_\epsilon}t}G$ will be small because of the unboundedness of F_ϵ as $\epsilon \to 0$! The above theorem thus makes sure that things can be set up in such a way that $He^{A_{F_\epsilon}t}G$ remains indeed small.

As demonstrated in Theorem 10 there are only two types of $\mathcal{L}_p$-almost invariant subspaces, i.e., those with $p=\infty$ and those with $1 \leqslant p < \infty$. We think of V_a as almost invariant subspaces of the first kind, i.e., in the $\mathcal{L}_\infty$-sense and of $R_b = AR_a + B$ and $V_b = R_b + V$ as almost invariant subspaces of the second kind, i.e., in the $\mathcal{L}_1$-sense.

As we will see in Section VIII, Theorem 12 has an immediate application in control theoretic terms in the sense of making the $\mathcal{L}_p$-norm of a closed-loop impulse response arbitrarily small. Of course, there are other measures for the "size" of an impulse response which are of interest in applications. One such result which is relevant in this context is the following.

Theorem 13: Assume that in addition to $\Sigma: \dot{x} = Ax + Bu$ *there are matrices H and G given. Consider the closed-loop transfer function* $\hat{W}(s) := H(Is + A_F)^{-1}G$. *Denote* $K := \ker H$. *Then*

1) $\{\forall \epsilon > 0 \exists F$ *such that* $\hat{W}(s)$ *has all its poles in* $\operatorname{Re} s < 0$ *and* $\|\hat{W}(j\omega)\| \leqslant \epsilon \, \forall \omega \in \mathbb{R}\} \Leftrightarrow \{\operatorname{im} G \subset V^*_{b,K}\}$; *and*

2) $\{\forall \epsilon > 0,\ M \in \mathbb{R}, \exists F$ *such that* $\operatorname{Re} \sigma(A_F) \leqslant M$ *and* $\|\hat{W}(i\omega)\| \leqslant \epsilon,\ \forall \omega \in \mathbb{R}\} \Leftrightarrow \{\operatorname{im} G \subset R^*_{b,K}\}$.

Proof: We will only outline the proof of 1); 2) is proven analogously. To see 1)($\Leftarrow$) notice that, since $\hat{W}(s) = \int_0^\infty W(t) e^{-st}\, dt$ (with W as defined in Theorem 12), there exists, $\forall \epsilon > 0$, an F such that $\|W\|_{L_1} \leqslant \epsilon$, whence $\hat{W}$ is analytic in $\operatorname{Re} s \geqslant 0$ and $\|\hat{W}(i\omega)\| \leqslant \int_0^\infty \|W(t)\|\, dt \leqslant \epsilon$. The proof of 1)($\Rightarrow$) may be transformed, by an argument identical to the analogous step in the proof of Theorem 12, from a closed-loop to an open-loop question, which requires showing that if for a given x_0 there exists, $\forall \epsilon > 0$, $x \in \Sigma_x$ such that its Fourier $\hat{x}(i\omega)$ satisfies $\|H\hat{x}(i\omega)\| \leqslant \epsilon$, $\forall \omega \in \mathbb{R}$, then $x_0 \in V^*_{b,K}$. By a reasoning analogous to the last part of the proof of Theorem 10, it suffices to show this for the case $V^*_{a,K} = \{0\}$ and $K \cap B = \{0\}$ for which we need to prove that this implies $x_0 \in B$. Using the same notation as in the relevant part of the proof of Theorem 10, we obtain

$$z(t) = e^{A't} z_0 + \int_0^t e^{A'(t-\tau)} B' b(\tau)\, d\tau$$

with $\|\hat{b}(i\omega)\| \leqslant \epsilon$, $\forall \omega \in \mathbb{R}$. This obviously implies $(Ii\omega - A')^{-1} z_0 \in V^*_K = \{0\}$, hence $z_0 = 0$ and $x_0 \in B$, which yields the result. □

Comment 5: The spaces $R^*_{a,\ker H}$ and $R^*_{b,\ker H}$ have an interesting interpretation of their own. Indeed, $R^*_{a,\ker H}$ consists precisely of those points $x(0^-)$ in $\ker H$ which can be driven to 0 by distributional inputs while remaining in $\ker H$. For all such $x(0^-)$ one can make sure that $x(0^+) = 0$, but in any case one needs to have $x(0^+) \in R^*_{\ker H}$. The points in $R^*_{\ker H}$ are precisely those for which one can achieve this with smooth inputs. The subspace $R^*_{b,\ker H}$ on the other hand consists of those points in X (not necessarily in $\ker H$!) which can be driven to 0 by distributional inputs while remaining in $\ker H$. (Do not let the fact that this may be possible with $x(0^-) \notin \ker H$ worry you; an impulse will generate the "solution" $x = \mathbf{0}$ for $t > 0$ for $\dot{x} = u$, $x(0^-) = -1$ and shows, for example, that $B \subset R^*_{b,\{0\}}$).

Comment 6: There is another possible and particularly instructive approach to many of the issues discussed in our paper, i.e., what most people would call the "frequency domain" approach. Consider the system $\dot{x} = Ax + Bu$; $y = Hx$, and assume that we ask ourselves whether all initial conditions in $\operatorname{im} G$ can be held in $\ker H$ by choosing the control properly. Taking Laplace transforms yields the equation

$$H(Is - A)^{-1} BU(s) = H(Is - A)^{-1} G \qquad (**)$$

which one should solve for $U(s)$.

In the class of transforms of functions with exponential growth, this equation is solvable iff $\operatorname{im} G \subset V^*_{\ker H}$ in which case it is always solvable in the class of strictly proper rational functions of McMillan degree $\leqslant \operatorname{codim} V^*_{\ker H}$.

One can also consider this equation in the class of Laplace transformable distributions. In this case $(**)$ is solvable iff $\operatorname{im} G \subset V^*_{b,\ker H}$, which actually implies solvability in the class of rational functions. The reason why $V^*_{b,\ker H}$ and not $V^*_{a,\ker H}$ is relevant here is connected with what has been mentioned in the last sentence on Comment 5. The condition $\operatorname{im} G \supset V^*_{a,\ker H}$ is the solvability condition for $(**)$&$HG = 0$ simultaneously. The point is that the solvability of $(**)$ implies approximate solvability (in the time domain) only in the $\mathcal{L}_1$-norm, while $(**)$&$HG = 0$ implies approximate solvability of $(**)$ in the $\mathcal{L}_\infty$-sense as well.

It is also interesting to remark that solvability of $(**)$ with regular inputs leads to an equation over the ring of strictly proper rational functions while the solvability of $(**)$ with distributional inputs leads to an equation over the field of rational functions. From a pure mathematical point of view, the latter should be easier to treat but, as we have seen, this was certainly not the case in the time domain.

VII. Supremal Almost Invariant Subspaces

Most applications of the ideas of our paper involve at one stage or another the computation of the supremal elements contained in a given subspace. (ISA), (ACSA) and (ACSA)′ provide (see Theorems 4 and 11) conceptual algorithms that compute for a given subspace $K \subset X$, V^*_K, R^*_K, $R^*_{a,K}$, $V^*_{a,K}$ and $R^*_{b,K}$, $V^*_{b,K}$.

In the present section we will give some additional related results. In particular we study the generic case and the situation $\dim B \geqslant \operatorname{codim} K$ where some very nice results may be obtained which avoid these computations.

The concept of genericity which we will use is given next.

Definition 8: A subset $S \subset \mathbb{R}^N$ is said to be *generic* if there exists $S' \subset S$ with S' open, dense, and measure exhausting (i.e., $\mu((S')^{\text{compl}})=0$ with μ Lebesgue measure). A subset $Z \subset \mathbb{R}^N$ is said to be an *algebraic variety* (or a "*Zariski closed*" set) if there exist polynomials $p_1, p_2, \cdots, p_q$ in N indeterminates such that

$$Z=\left\{(s_1,s_2,\cdots,s_N)^T \in \mathbb{R}^N \mid p_i(s_1,s_2,\cdots,s_N)=0,\ \forall i \in \boldsymbol{q}\right\}.$$

If, for at least one i, $p_i \neq 0$, then Z is said to be *nontrivial.*

Our genericity proofs are based on the following well-known [2, 0.16] lemma.

Lemma 4: If $S=Z^{\text{compl}}$ with Z a nontrivial algebraic variety, then S is generic.

We denote equalities with a subscript (g) if they hold for a generic set. We think of a generic equality as a strong formalization of the intuitive concept of "almost certainly true" or "essentially always true."

Consider Σ: $\dot{x}=Ax+Bu$ with $u \in \mathsf{U}=\mathbb{R}^m$, $x \in \mathsf{X}=\mathbb{R}^n$, and let H be an $(l \times n)$-matrix. Then (A,B,H) may be considered as an element of $\mathbb{R}^{n^2+nm+ln}$. It is in this setting that the genericity statements in the following theorem should be understood.

Theorem 14: Let $\mathsf{K}:=\ker H$.

1) *If* $m>l$, *then* $\mathsf{R}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{R}_{a,\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{a,\mathsf{K}}^* \underset{(g)}{=} \mathsf{K}$ *and* $\mathsf{R}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{X}$;

2) *If* $m=l$, *then* $\mathsf{R}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{R}_{a,\mathsf{K}}^* \underset{(g)}{=} \{0\}$, $\mathsf{V}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{a,\mathsf{K}}^* \underset{(g)}{=} \mathsf{K}, \mathsf{R}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{B}$, *and* $\mathsf{V}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{X}$; *and*

3) *If* $m<l$, *then* $\mathsf{R}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{R}_{a,\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{a,\mathsf{K}}^* \underset{(g)}{=} \{0\}$ *and* $\mathsf{R}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{V}_{b,\mathsf{K}}^* \underset{(g)}{=} \mathsf{B}$.

Proof: The results about $\mathsf{R}_{\mathsf{K}}^*$, $\mathsf{V}_{\mathsf{K}}^*$ may be found in [2]. This immediately yields 1) since $\mathsf{V}_{a,\mathsf{K}}^*$, $\mathsf{R}_{a,\mathsf{K}}^*$ contain $\mathsf{R}_{\mathsf{K}}^*$. Using these inclusions it suffices to show in 2) that $\mathsf{R}_{a,\mathsf{K}}^* \underset{(g)}{=} \{0\}$. Since $m=l$ implies, by Lemma 4, that $\mathsf{B} \cap \mathsf{K} \underset{(g)}{=} \{0\}$, we immediately obtain the result by the algorithm in Theorem 4. In 3) we only need to show that $\mathsf{V}_{a,\mathsf{K}}^* \underset{(g)}{=} \{0\}$. This, however, follows since $\mathsf{V}_{\mathsf{K}}^* \underset{(g)}{=} \{0\}$ and $\mathsf{R}_{a,\mathsf{K}}^* \underset{(g)}{=} \{0\}$ [by 2)]. □

The above theorem is disappointing in the sense that it shows that generically nothing is gained by the "almost" qualification in the extremely restricted genericity setting considered here where all the parameters of (A,B,H) are considered "random." It may be expected that in more realistic situations, e.g., starting from system parameters belonging to an algebraic variety, there will be many situations where the "almost" solvability conditions hold generically, while the "exact" conditions would generically be unachievable.

If we ask for exact conditions then we are able to show the following very concrete result.

Theorem 15: Assume $\dim \mathsf{B} \geqslant \operatorname{codim} \mathsf{K}$ (*It is assumed that effectively there are at least as many inputs as outputs in the system* $\dot{x}=Ax+Bu$, $z=Hx$ *with* $\mathsf{K}=\ker H$). *Then*

1) $\mathsf{V}_{a,\mathsf{K}}^*=\mathsf{K}$; *and*

2) $\mathsf{V}_{b,\mathsf{K}}^*=A\mathsf{K}+\mathsf{K}+\mathsf{B}$.

Proof: It suffices to prove 1). This is, in fact, an easy consequence of Theorem 6. Indeed, if $\dim \mathsf{B} \geqslant \operatorname{codim} \mathsf{K}$, then $\mathsf{V} \cap \underline{\mathsf{G}}^n_{\dim \mathsf{K}}$ will be dense in $\underline{\mathsf{G}}^n_{\dim \mathsf{K}}$ since $\{\mathsf{L} \in \underline{\mathsf{G}}^n_{\dim \mathsf{K}} \mid \mathsf{L}+\mathsf{B}=\mathsf{X}\}$ is obviously dense in $\underline{\mathsf{G}}^n_{\dim \mathsf{K}}$. Hence $\mathsf{K} \in \underline{\mathsf{G}}^n_{\dim \mathsf{K}} = (\mathsf{V} \cap \underline{\mathsf{G}}^n_{\dim \mathsf{K}})$ closure $\subset \mathsf{V}_a$, as claimed. □

We mention the following corollary which is inspired by a similar result in [7] where it is proven in a much less direct way.

Corollary 1: Assume that $\dim \mathsf{B} \geqslant \operatorname{codim} \mathsf{K}$ *and* $\mathsf{R}_{\mathsf{K}}^*=\{0\}$ (*the system* $\dot{x}=Ax+Bu$, $z=Hx$ with $x(0)=0$ *is then invertible in the sense that corresponding to any output trajectory* y *there is exactly one* $x \in \Sigma_x$ *and hence, if* B *is injective, exactly one input trajectory; it is the case, e.g., if* $\mathsf{K} \cap \mathsf{B}=\{0\}$). *Then* $\mathsf{V}_{b,\mathsf{K}}^*=\mathsf{X}$.

Proof: By Theorem 15, it suffices to show that $A\mathsf{K}+\mathsf{K}+\mathsf{B}=\mathsf{X}$. Let V_c be such that $\mathsf{V}_{\mathsf{K}}^* \oplus \mathsf{V}_c=\mathsf{K}$. Then $\dim A\mathsf{V}_c=\dim \mathsf{V}_c$ (since $\{x \in \mathsf{V}_c,\ Ax=0\} \Rightarrow \{x \in \mathsf{V}_{\mathsf{K}}^*\}$). Furthermore, $A\mathsf{V}_c \cap (\mathsf{B}+\mathsf{V}_{\mathsf{K}}^*)=\{0\}$ (since $\{x \in \mathsf{V}_c,\ Ax \in \mathsf{B}+\mathsf{V}_{\mathsf{K}}^*\} \Rightarrow \{x \in \mathsf{V}_{\mathsf{K}}^*\}$). Finally $\{\mathsf{R}_{\mathsf{K}}^*=\{0\}\} \Leftrightarrow \{\mathsf{V}_{\mathsf{K}}^* \cap \mathsf{B}=\{0\}\}$. Hence $A\mathsf{V}_c$, B, and $\mathsf{V}_{\mathsf{K}}^*$ are independent, and $\dim(A\mathsf{K}+\mathsf{K}+\mathsf{B}) \geqslant \dim(A\mathsf{V}_c \oplus \mathsf{V}_{\mathsf{K}} \oplus \mathsf{B}) = \dim \mathsf{V}_c + \dim \mathsf{V}_{\mathsf{K}}^* + \dim \mathsf{B} = \dim \mathsf{K} + \dim \mathsf{B} \geqslant n$. Conclusion: $\dim(A\mathsf{K}+\mathsf{K}+\mathsf{B}) \geqslant n$, hence, $A\mathsf{K}+\mathsf{K}+\mathsf{B}=\mathsf{X}$. □

Note that the proof of the above corollary also shows that if $\dim \mathsf{B} > \operatorname{codim} \mathsf{K}$, then $\mathsf{R}_{\mathsf{K}}^* \neq \{0\}$ which is of interest in its own right. Moreover, the same method of proof shows that $\{\dim \mathsf{B} - \dim(\mathsf{B} \cap \mathsf{V}_{\mathsf{K}}^*) \geqslant \operatorname{codim} \mathsf{K}\} \Rightarrow \{\mathsf{V}_{b,\mathsf{K}}^*=\mathsf{X}\}$ which, since it has no interpretations in terms of system invertibility, we leave for what it is worth.

VIII. Applications

In this section we will illustrate the use of the concepts of almost invariant subspaces in a number of control synthesis applications.

A. Disturbance Decoupling

Consider the signal flow graph diagram shown in Fig. 4, where the plant Σ is controlled by the feedback processor Σ_f which yields a closed loop system Σ_{cl} with the disturbances as inputs and the controlled variables as outputs. One of the most easily motivated control synthesis questions is the problem of designing a feedback processor such that in the closed-loop system the controlled variables are insensitive to the disturbances.

Here, we will assume that all the states are measured and hence that we may use state feedback (output feedback will be considered in Part II of this paper). The plant is assumed to be a linear time-invariant finite dimensional system given by Σ: $\dot{x}=Ax+Bu+Gd$, $z=Hx$ with $x \in \mathsf{X}=\mathbb{R}^n$, $u \in \mathsf{U}=\mathbb{R}^m$, $d \in \mathsf{D}=\mathbb{R}^q$, and $z \in \mathsf{Z}=\mathbb{R}^l$. The feedback processor is taken to be linear time-invariant and memoryless (it may be shown that in the disturbance decoupling problems considered here and in [1, Ch. 4 and 5],

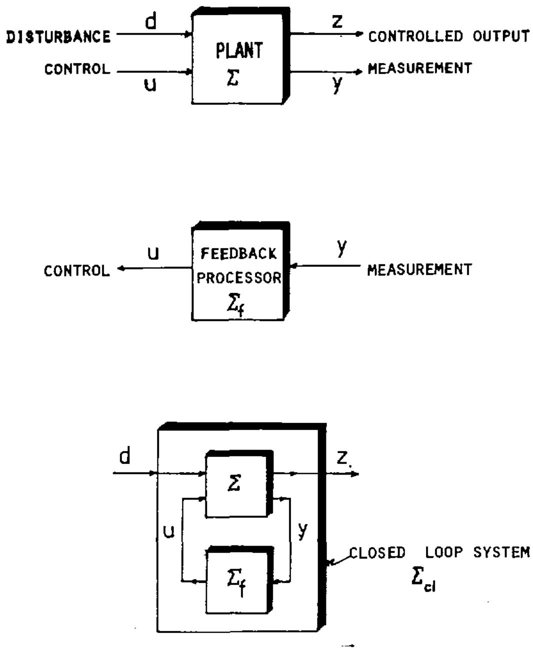

Fig. 4. Disturbance decoupling.

this constitutes no loss of generality) and is given in terms of the gain matrix F by Σ_f: $u=Fx$. The closed-loop system is then described by $\dot{x}=A_F x+G_d$, $z=Hx$.

The question of when there exists F such that in the closed-loop system the disturbances do not influence the controlled variables at all, i.e., such that $H(Is-A_F)^{-1}G=0$ has been studied by many authors (see e.g. [2]) and requires $\operatorname{im} G\subset \mathcal{V}^*_{\ker H}$. It is called the disturbance decoupling problem (DDP). If in addition one requires pole placement of A_F then we arrive at the disturbance decoupling problem with pole placement (DDPPP) for which the condition becomes $\operatorname{im} G\subset \mathcal{R}^*_{\ker H}$ and controllability of (A, B).

Recently the disturbance decoupling problem with measurement feedback and pole placement has been solved (see [8] where other relevant references may be found). Here we will consider an interesting variation on this question, namely whether disturbance decoupling is "almost" possible. An early paper where approximate disturbance decoupling has been studied, but which did not give definitive results, is [9]. Of course, there are a number of ways to quantify this almost disturbance decoupling. The most logical approach is to work in terms of induced norms.

Definition 9: $(\text{ADDP})_p$, $1\leq p\leq\infty$ (the *almost disturbance decoupling problem in the $\mathcal{L}_p$-sense*) is said to be solvable if $\forall\epsilon>0$ there exists F such that in Σ_{cl}, with $x(0)=0$, there holds $\|z\|_{\mathcal{L}_p(0,\infty)}\leq\epsilon\|d\|_{\mathcal{L}_p(0,\infty)}$. Another, possibly somewhat more artificial version is (ADDP)′ (*the almost disturbance decoupling problem in the $\mathcal{L}_p-\mathcal{L}_q$-sense*) which is said to be solvable if $\forall\epsilon>0$ there exists F such that $\|z\|_{\mathcal{L}_q(0,\infty)}\leq\epsilon\|d\|_{\mathcal{L}_p(0,\infty)}$ for all $1\leq p\leq q\leq\infty$. A particularly interesting case is to take $p=1$ or $q=\infty$. (Actually the solvability condition for this special case is exactly the same as the one for the general case involving all (p,q) with $1\leq p\leq q\leq\infty$.) If in either of the above problems one adds the condition that for any given M one should have in addition $\operatorname{Re}\sigma(A_F)\leq M$, then we speak about $(\text{ADDPSS})_p$ and (ADDPSS)′ (*almost disturbance decoupling with strong stabilization*).

We have the following.

Theorem 16:

1) *Let* $1\leq p\leq\infty$. *Then* $\{(\text{ADDP})_p$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset \mathcal{V}^*_{b,\ker H}\}$, *and* $\{(\text{ADDPSS})_p$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset\mathcal{R}^*_{b,\ker H}$ *and* (A,B) *is controllable*$\}$;

2) $\{(\text{ADDP})'$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset\mathcal{V}^*_{a,\ker H}\}$, *and* $\{(\text{ADDPSS})'$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset\mathcal{R}^*_{a,\ker H}$ *and* (A,B) *is controllable*$\}$.

Proof: Let $W: t\in\mathbb{R}^+\mapsto He^{A_F t}G$ and $\hat{W}(s):=H(Is-A_F)^{-1}G$. In order to show 1)($\Leftarrow$) it suffices to observe that the $\mathcal{L}_p$-induced norm of a convolution operator is bounded by the $\mathcal{L}_1$-norm of its kernel, in our case $\|W\|_{\mathcal{L}_1}$. Now apply Theorem 12. To show 1)($\Rightarrow$) we consider only the most interesting cases $p=1,2,\infty$ (the other cases require a slight modification of Theorem 12 or 13). For $p=1,\infty$, observe that $\|W\|_{\mathcal{L}_1}$ is exactly the $\mathcal{L}_p$-induced norm, while for $p=2$ the $\mathcal{L}_2$-induced norm is $\sup_{\omega\in\mathbb{R}}\|\hat{W}(i\omega)\|$ provided $\|W\|_{\mathcal{L}_1(0,\infty)}<\infty$. The result then follows from Theorems 12 and 13, respectively. 2)($\Leftarrow$): by Theorem 12, there exists, $\forall\epsilon>0$, F such that $\|W\|_{\mathcal{L}_1}, \|W\|_{\mathcal{L}_\infty}\leq\epsilon$. Now by a standard inequality for convolution operators (the so-called Young's inequality—see [10, p. 150]) it follows that $\|z\|_{\mathcal{L}_q}\leq\|W\|_{\mathcal{L}_\infty}^{1/p-1/q}\|W\|_{\mathcal{L}_1}^{1+1/q-1/p}\|d\|_{\mathcal{L}_p}$ which yields the result. To show 2)($\Rightarrow$) observe that the $\mathcal{L}_1\to\mathcal{L}_\infty$ induced norm is exactly $\|W\|_{\mathcal{L}_\infty}$. The conclusion follows from Theorem 12. Finally, it should be clear how Theorem 12 or 13 also give the strong stability conclusion. □

Using Theorems 14–16 we obtain the following useful corollary.

Corollary 2:

1) $\{(\text{ADDP})_p$ *is generically solvable*$\}\Leftrightarrow\{m\geq l\}$, *and* $\{(\text{ADDPSS})_p$ *is generically solvable*$\}\Leftrightarrow\{m>l\}$;

2) *if* $\dim\mathcal{B}\geq\operatorname{codim}\ker H$ *then* $\{(\text{ADDP})_p$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset A\ker H+\ker H+\mathcal{B}\}$, *and* $\{(\text{ADDP})'$ *is solvable*$\}\Leftrightarrow\{\operatorname{im} G\subset\ker H\}$;

3) if $\operatorname{im} G\subset\mathcal{B}$ (a frequently occurring situation in technological applications: the disturbances enter the system through the same input channels as the control) and (A,B) is controllable then (ADDPSS) is solvable;

4) if $\dim\mathcal{B}\geq\operatorname{codim}\ker H$ and $\mathcal{R}^*_{\ker H}=\{0\}$ then $(\text{ADDP})_p$ is solvable.

Comment 7: In the above results, as in many other places in our paper, it is of interest to add to almost disturbance decoupling stability requirements of the type $\sigma(A_F)\subset\mathbb{C}_g\subset\mathbb{C}$. This is easy to do: the subspace $\mathcal{V}^*_{\mathcal{K}}$ in $\mathcal{V}^*_{a,\mathcal{K}}$ or $\mathcal{V}^*_{b,\mathcal{K}}$ needs to be replaced by $\mathcal{V}^*_{g,\mathcal{K}}$ with $\mathcal{V}^*_{g,\mathcal{K}}:=\sup\{\mathcal{V}\in\underline{\mathcal{V}}(\mathcal{K})\mid\exists F$ such that $A_F\mathcal{V}\subset\mathcal{V}$ and $\sigma(A_F)\subset\mathbb{C}_g\}$. The relevant subspaces are then

$$\mathcal{V}^g_{a,\mathcal{K}}:=\mathcal{R}^*_{a,\mathcal{K}}+\mathcal{V}^*_{\mathcal{K}}\quad\text{and}\quad\mathcal{V}^g_{b,\mathcal{K}}:=\mathcal{R}^*_{b,\mathcal{K}}+\mathcal{V}^*_{\mathcal{K}}.$$

Comment 8: The almost disturbance decoupling results imply the quenching of the disturbances in a number of

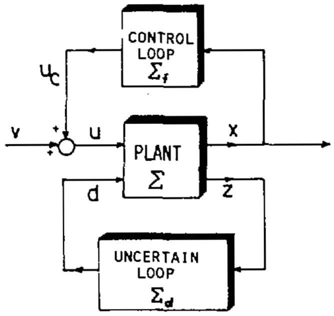

Fig. 5. Robust controller synthesis.

other norms than those considered here. In particular, it implies that if $\lim_{T\to\infty}1/T\int_0^T\|d\|^2\,dt<\infty$, then $\lim_{T\to\infty}1/T\int_0^T\|z\|^2\,dt\leqslant\epsilon\lim_{T\to\infty}1/T\int_0^T\|d\|^2\,dt$, of obvious interest in the context of stochastic disturbances.

Comment 9: Note that when (DDP) is not solvable, but (ADDP) is, then by the results of Theorem 8 the feedback gain $F\to\infty$ as $\epsilon\to 0$. The design requires then a high gain feedback synthesis approach.

B. Robustness[1]

Consider the feedback system shown in Fig. 5, where the plant $\Sigma: \dot{x}=Ax+Bu+Gd,\ z=Hx$ is as in Section VIII-A and is influenced by an uncertain loop Σ_d which we model as an unknown $\mathcal{L}_p$-input/output stable system with bounded gain operator M with known gain. Formally, $z\in\mathcal{L}_p(0,\infty)\Rightarrow d=Mz\in\mathcal{L}_p(0,\infty)$ and $\exists K<\infty$ such that $\|d\|_{\mathcal{L}_p}\leqslant K\|z\|_{\mathcal{L}_p}$. Let M_k denote all such systems. The plant is controlled by the control loop Σ_f which is to be designed. For Σ_f we take $u_c=Fx$ and with $u=u_c+v$ we then obtain a closed-loop system Σ_{cl} mapping $v\mapsto x$. The idea is to choose F such that Σ_{cl} with $x(0)=0$ is $\mathcal{L}_p$-input/output stable for all $\Sigma_d\in\mathsf{M}_K$.

Definition 10: Let $1\leqslant p\leqslant\infty$. Then (RRP) (the *robust regulator problem*) is said to be solvable if, $\forall K$, $\exists F$ such that Σ_{cl} is $\mathcal{L}_p$-input/output stable for all $\Sigma_d\in\mathsf{M}_K$ in the sense that $\{v\in\mathcal{L}_p\}\Rightarrow\{x\in\mathcal{L}_p\}$ (and hence $z, d, u_c, u\in\mathcal{L}_p$).

Note that the feedback F is allowed to depend on K. If one would need an F which works for all K, then the question requires disturbance decoupling from d to z and may hence be solved by the results in [2].

Robustness design questions as those in Definition 10 occur for example in situations where, due to failures in a system (the uncertain system Σ_d), it is necessary to introduce an override control (Σ_f) which will maintain stability. A very tight, high gain type of control could very well be desired in such situations.

From standard results on $\mathcal{L}_p$-input/output stability theory (see, e.g., [11, 12]) it is easy to see that (RRP) will be solvable iff it is possible to choose the $\mathcal{L}_p$-gain of the internal controlled loop $d\mapsto z$, i.e., of $\dot{x}=A_Fx+Gd$, $z=Hx$, smaller than $1/K$. This leads to the following.

Theorem 17: {(RRP) *is solvable*}$\Leftrightarrow$$\{\operatorname{im}G\subset\mathsf{V}^*_{b,\ker C}\}$.

The proof of this theorem is identical to that of Theorem 16. The relevant parts of Corollary 2 and the above comments (e.g., regarding stability) apply here also.

C. Stabilization of Systems with Noisy Parameters

In this section we will consider an application of the ideas in this paper when it is the $\mathcal{L}_2$-norm of an impulse response that needs to be made arbitrarily small. Consider the system described by the *Itô* equation $dx=(Ax+Bu)\,dt+G(dK)Hx$, with $x(0)$ a random vector with finite second moments and K and independent (matrix valued) Wiener process on $t\geqslant 0$. Let

$$\mathsf{E}\{\underline{\mathsf{K}}_{ij}(t_1)\underline{\mathsf{K}}_{rs}(t_2)\}=\sigma^2_{ijrs}\min(t_1,t_2)$$

and $\Sigma:=(\sigma^2_{ijrs})$. The problem is to decide whether there exists for any given $M<\infty$ a feedback matrix F such that $u=Fx$ stabilizes in the mean square sense (i.e., such that $dx=A_Fx\,dt+G(dK)Hx$ yields $\lim_{t\to\infty}\mathsf{E}\{\|x(t)\|^2\}=0$) for all $\|\Sigma\|\leqslant M$. We call this problem the *stabilization for arbitrary noise intensities*. This problem has been studied before (see [7],[13],[14]). It may be shown [15] that the solvability of this problem is equivalent to the existence for all $\epsilon>0$ of an F such that $\operatorname{Re}\sigma(A_F)<0$ and

$$\int_0^\infty\|Ge^{A_Ft}H\|^2\,dt\leqslant\epsilon.$$

Applying Theorem 12, suitably modified as indicated in Comment 4, yields the following theorem.

Theorem 18: Assume (A,B) stabilizable. Then the stabilization problem with arbitrary noise intensities is solvable iff

$$\operatorname{im}G\subset\mathsf{V}^*_{b,\ker H}:=\mathsf{R}^*_{b,\ker H}+\mathsf{V}^*_{\ker H}$$

where

$$\mathsf{V}^+_{\ker H}=\sup\{\mathsf{V}\in\underline{\mathsf{V}}(\ker H)\,|\,\exists F\colon A_F\mathsf{V}\subset\mathsf{V}\ \textit{and}\ \operatorname{Re}\sigma(A_F)<0\}.$$

D. Cheap Control

Our last application is concerned with linear quadratic control. Consider the stabilizable system $\dot{x}=Ax+Bu$ with $x(0)=x_0$ and the cost criterion $\int_0^\infty(\rho u^TRu+x^TQx)\,dt$ with $R=R^T>0$ and $Q=Q^T\geqslant 0$. We will study the behavior for $\rho\downarrow 0$. Let $J_\rho(x_0)$ be the optimal performance. Clearly $\hat{J}(x_0):=\lim_{\rho\downarrow 0}J_\rho(x_0)$ exists and is given by a quadratic form in x_0, $\hat{J}(x_0)=:x_0^TPx_0$ with $P=P^T\geqslant 0$. We say that *cheap control* is possible if $\hat{J}(x_0)=0$, i.e., if $x_0\in\ker P$. It is of interest to evaluate the *cheap control set* $\ker P$ without having to carry out the defining limiting process. (Particularly interesting are the situations with $\hat{J}=0$.)

[1]These results were presented at the ONR/MIT-LIDS Workshop on Recent Robustness Theory of Multivariable Systems, April 25–27, 1979, MIT, Cambridge, MA. Very much related results have been obtained independently by Molander [7] in his thesis. His results, however, give only the sufficiency parts and are obtained from an entirely different point of view.

Theorem 19: Assume (A, B) stabilizable. Then

$$\{\hat{J}(x_0)=0\} \Leftrightarrow \{x_0 \in \mathsf{V}^+_{b,\ker Q} := \mathsf{R}^*_{b,\ker Q} + \mathsf{V}^+_{\ker Q}\}$$

with $\mathsf{V}^+_{\ker Q}$ as defined in Theorem 18.

Proof: ($\Leftarrow$) By Theorem 12, suitably modified as indicated in Comment 4, there exists, for any given $\epsilon > 0$ an F such that, for $\dot{x}=Ax+Bu, u=Fx, x(0)=x_0$, we have $\int_0^\infty x^T Q x\, dt \leqslant \epsilon$ and A_F asymptotically stable. Hence $\int_0^\infty u^T R u\, dt < \infty$. Letting $\rho \downarrow 0$ yields a cost $\leqslant \epsilon$ and hence $\hat{J}(x_0)=0$.

($\Rightarrow$) If $\hat{J}(x_0)=0$ there exists F (the optimal feedback control law for ρ sufficiently small) such that $\operatorname{Re} \sigma(A_F)<0$ and $\int_0^\infty x^T Q x\, dt$ arbitrarily small. This implies by Theorem 12, suitably modified as indicated in Comment 4, that $x_0 \in \mathsf{V}^*_{b,\ker Q}$ as claimed. □

Necessary and sufficient conditions for cheap control have been derived before (see, e.g., [7], [16], [17]) but it would seem that the natural interpretation of the cheap control set in terms of almost invariant subspaces given in Theorem 19 yields more insight and may very well provide an excellent starting point for studying more general singular control type situations. Note in particular that the above theorem involves $\mathsf{V}^+_{b\ker Q}$ instead of $\mathsf{V}^*_{b,\ker Q}$. This brings a "*minimum phase*" condition into the picture: indeed, the cheap control set will be all of X if, e.g., $\dim \mathsf{B} \geqslant \operatorname{codim} \ker \mathsf{Q}$, $\mathsf{R}^*_{\ker Q} = \{0\}$, and the system is minimum phase in the sense that $\sigma(A_F | \mathsf{V}^*_{\ker Q})$ is in the left-half plane for F such that $A_F \mathsf{V}^*_{\ker Q} \subset \mathsf{V}^*_{\ker Q}$.

IX. Example

The following example aims to illustrate several points, among them the evaluation of the supremal almost invariant subspace in the kernel of some output map and the conditions for almost disturbance decoupling for scalar systems.

Let $\mathbb{R}[s]$ denote the polynomials with real coefficients, and $\mathbb{R}_i[s]$ those of degree $\leqslant i$. For $f \in \mathbb{R}[s]$, we will denote by $|f|$ its degree.

Consider the system described by the higher order scalar differential equation

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u + r\left(\frac{d}{dt}\right)d$$

where $p(s) \in \mathbb{R}_n[s]$ is monic and $q(s), r(s) \in \mathbb{R}_{n-1}[s]$. We will assume for simplicity that p and q are relatively prime, which translates into minimality of the state space models which will be used. This system may be written in state space form, for example by taking the standard observable realization with $\mathsf{X} = \mathbb{R}^n$ and

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \quad H = [1 \quad 0 \quad \cdots \quad 0]$$

with the p_i's defined by $p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_0$ and the b_i's and the g_i's by the Laurent series expansions

$$\frac{q(s)}{p(s)} = \frac{b_1}{s} + \frac{b_2}{s^2} + \cdots + \frac{b_n}{s^n} + \cdots,$$

$$\frac{r(s)}{p(s)} = \frac{g_1}{s} + \frac{g_2}{s^2} + \cdots + \frac{g_n}{s^n} + \cdots$$

However, for the purposes at hand, it is convenient to identify $\mathfrak{X}$ with $\mathbb{R}_{n-1}[s]$ with typical element $x(s) = x_{n-1}s^{n-1} + \cdots + x_0$ and define the realization by $A: x(s) \mapsto s x(s) - x_{n-1} p(s)$, $B = q(s)$ (more precisely $B: \alpha \mapsto q(s)\alpha$), $G = d(s)$, and $H: x(s) \mapsto x_{n-1}$. (Actually, if one coordinatizes X by $x(s) = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$ with the α_i's defined by the Laurent series expansion

$$\frac{x(s)}{p(s)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \cdots + \frac{\alpha_n}{s^n} + \cdots$$

then this realization becomes precisely the standard observable realization.)

Let $\mathsf{K} := \ker H$. Clearly $\mathsf{K} = \mathbb{R}_{n-2}[s]$. By straightforward calculations it is now possible to compute $\mathsf{V}^*_{\mathsf{K}}, \mathsf{V}^*_{a,\mathsf{K}}$, etc. We have also computed for a given symmetric set $\mathbb{C}^+ \subset \mathbb{C}$, $\mathsf{V}^+_{\mathsf{K}} := \sup\{\mathsf{V} \in \underline{\mathsf{V}}(\mathsf{K}) | \exists F \text{ such that } A_F\mathsf{V} \subset \mathsf{V} \;\&\; \sigma(A_F|\mathsf{V}) \subseteq \mathbb{C}^+\}$. This involves factoring q into $q = q_+ q_-$ with q_+ having its roots in $\mathbb{C}^+$ and q_- having its roots in $(\mathbb{C}^+)^{\text{complement}}$.

The results are shown in Table I.

TABLE I

* = *		Solvable	⇔ *
$\mathsf{V}^*_{\mathsf{K}}$	$\mathbb{R}_{\vert q\vert-1}[s]$	(DDP)	$\vert r\vert < \vert q\vert$
$\mathsf{R}^*_{\mathsf{K}}$	$\{0\}$	(DDPPP)	$r=0$
$\mathsf{R}^*_{a,\mathsf{K}}$	$q(s)\mathbb{R}_{n-\vert q\vert-2}[s]$	(ADDPSS)′	$\vert r\vert < n-1$ & q divides r
$\mathsf{R}^*_{b,\mathsf{K}}$	$q(s)\mathbb{R}_{n-\vert q\vert-1}[s]$	$(\text{ADDPSS})_p$	q divides r
$\mathsf{V}^*_{a,\mathsf{K}}$	$\mathbb{R}_{n-2}[s]\,(=\mathsf{K})$	(ADDP)′	$\vert r\vert < n-1$
$\mathsf{V}^*_{b,\mathsf{K}}$	$\mathbb{R}_{n-1}[s]\,(=\mathfrak{X})$	$(\text{ADDP})_p$	always
$\mathsf{V}^+_{\mathsf{K}}$	$q_-(s)\mathbb{R}_{\vert q_+\vert-1}[s]$	$(\text{ADDP})_p$ + closed loop poles in $\mathbb{C}_g$	q_- divides r

The above example also illustrates an important point: the contrast between setting up a theory such as ours in the time versus the frequency domain. For the scalar example at hand, deriving the solvability conditions for (DDP), $(\text{ADDP})_p$, etc. are utterly trivial in the frequency domain (see Comment 6). However, our identification of the state space with $\mathbb{R}[s]$ allowed us to recover this economy also in the state space realization. Altogether we feel that setting up this theory in a state space context provides a more conceptual insight and better potential for generalization to nonlinear or infinite dimensional systems than a frequency domain theory, i.e., the theory seems to come better to grips with the essential ideas. However, from the computational point of view the frequency domain approach is likely to have some advantages because the relevant algorithms (which for a great deal still need to be set up) start with systems which are more parsimoniously parameterized since they are not encumbered with the intrinsic freedom of the choice of the state space which every state space model always has. Nevertheless, we feel that *it will be possible to develop*—as we have done in the above example—*a dictionary which allows* step for step *to translate any frequency domain calculation into an algorithmic identical calculation in the state space*, starting from a suitably adapted realization, and hence it will be possible to view a frequency domain theory as a special case of a state space theory.

X. Conclusions

We start this conclusions section by lining up the various (equivalent) definitions of (almost) invariant subspaces which have been obtained so far. For ordinary invariance we have:

1) open loop invariance (Definition 1);

2) feedback invariance (Proposition 1). (If it would not sound so awkward, "invariantable" would be a suitable terminology to describe (1) & (2).)

3) $A(\text{mod}\,\mathsf{B})$-invariance: $A\mathsf{V}\subset\mathsf{V}+\mathsf{B}$, which is an appealing notion from a linear algebra or a functional analysis point of view. Actually one is led to wonder whether this concept and its applications in multivariable control theory may not find some nice applications outside of control theory.

4) $\Sigma|\mathsf{V}$ is a system, in the sense that $\exists A', B'$ such that $\dot{v}=A'v+B'w$ generates the elements of Σ_x contained in the subspace V.

5) $\Sigma(\text{mod}\,\mathsf{V})$ is a system (Theorem A).

6) Consider the linear *foliation* $\{x+\mathsf{V}|x\in\mathsf{X}\}$ of X. This clearly induces a partition of (and hence an equivalence relation on) X. We will say that "leaves may be followed" by trajectories of Σ if for every leaf L there exists a family of trajectories $\Sigma'\subset\Sigma_x$ covering L at $t=0$ (i.e., $\Sigma'(0)=\mathsf{L}$) and such that $\{x_1', x_2'\in\Sigma'\}\Rightarrow\{x_1'(t)=x_2'(t)\ (\text{mod}\,\mathsf{V})$ for all t, i.e., $x_1'(t)$ and $x_2'(t)$ belong to the same leaf$\}$. In the foliation $\{x+\mathsf{V}|x\in\mathsf{X}\}$ leaves may be followed iff $\mathsf{V}\in\underline{\mathsf{V}}$. Equivalent conditions (in the linear case) are $\{\forall x\in\Sigma_x$ and $x_1=x(0)\ (\text{mod}\,\mathsf{V})$, $\exists x'\in\Sigma_x$ with $x'(0)=x_1$ and $x'(t)=x(t)\ (\text{mod}\,\mathsf{V})\forall t\}$ or $\{\forall x_0, x_1\in\mathsf{X}$, $x_0=x_1(\text{mod}\,\mathsf{V})$, $\exists x, x'\in\Sigma_x$ with $x(0)=x_0$, $x'(0)=x_1$, and $x(t)=x'(t)\ (\text{mod}\,\mathsf{V})\forall t\}$.

7) There exist F such that in $\dot{x}=A_Fx+Bu$ leaves of this foliation are *input insensitive*, i.e., $\{x_1(0)=x_2(0)\ (\text{mod}\,\mathsf{V})\}\Rightarrow\{x_1(t)=x_2(t)\ (\text{mod}\,\mathsf{V})\forall t$ and $\forall u\}$

For almost invariance we have

1′) approximate open loop invariance (Definition 1);

2′) approximate feedback invariance (Theorem 6);

3′) invariance under distributional inputs (Theorem 5).

Actually all of the above other characterizations of invariant subspaces may be given a distributional analog.

4′) $\Sigma(\text{mod}\,\mathsf{V}_a)$ is a smooth system (Theorem A).

5′) The feedback representation of Theorem 2.

All these characterizations have analogs for (almost) controllability subspaces. Other characterizations may be deduced from Comment 6.

The concept of *almost controlled invariant subspace* provides, in our opinion, a very natural and direct approach to many high gain feedback synthesis questions. In many ways it would seem that this notion has been the missing link towards a clean geometric approach to such problems. When all the facts are in, the theory of almost invariant subspaces is equally tight and "algebraic" than that of ordinary invariant subspaces, but the details of the proofs may be seen to require much harder analysis because of the approximations and the limiting processes involved. Perhaps some streamlining of the proofs is still called for, however.

We have concentrated in this paper on developing the basic ideas and the theory. In order to obtain a practical design tool from these ideas, it remains to develop algorithms which are sound from the numerical analysis point of view, in the spirit of [5], taking into consideration numerical stability and robustness and interactive computer aided design packages. It should be emphasized that we are not suggesting that it is possible to separate the question of how useful a theory is from its computational feasibility. What we mean to say is the following. In our paper we have posed certain design problems [e.g., $(\text{ADDP})_p$] and we have shown when a solution to the design specifications exists [$\text{im}\,G\subset\mathsf{V}^*_{b,\ker H}$] where for the entities entering the conditions [$\mathsf{V}^*_{b,\ker H}$] we have given conceptual algorithms in the form of a finite set of linear equations [(ACSA)′] which would in principle lead to a verification of the solvability issue. The design itself would now involve further computation of approximation of some almost invariant subspace [$\mathsf{V}^*_{b,\ker H}$] by an invariant one [$\mathsf{V}_\epsilon\to\mathsf{V}^*_{b,\ker H}$] and finding suitable feedback gains [$A_{F_\epsilon}\mathsf{V}_\epsilon\subset\mathsf{V}_\epsilon$]. Following the constructions in the proofs of our theorems this can in principle be carried out by setting up the familiar decomposition $\mathsf{V}^*_{b,\ker H}=\mathsf{V}\oplus\mathsf{B}_1\oplus A_{F_\epsilon}\mathsf{B}_2\oplus\cdots\oplus A_{F_\epsilon}^{n-1}\mathsf{B}_n$. In other words, where we have demonstrated the structure of the solution and the feasibility of the computation (the algorithms are "rational:" they involve linear equations and no algebraic equations or finite iterations are required!) the theory still needs improvement in order to translate this into adequate numerical algorithms and to bring this all together into a computer-aided design tool.

As already shown in Section VIII many well-known control questions may be interpreted in terms of almost invariant subspaces. Many aspects of the results and ideas developed here have clear connections with other problems which has been studied in the control theory literature. Among those we like to mention the root-locus for nearly singular LQ problems (see, e.g., [2, Ch. 13], [16], [17], and [18] and for general feedback systems [19], [20]); the related work on cheap control [17], [21] and singular perturbations [22], [23], [24]; definitions and properties of infinite zeros [25]–[28] and, finally, the work on limit behavior and degeneracy of systems depending on parameters [29]. Many of the connections with these problems seem worthwhile studying in more detail and the fact that the work presented here seems to have so many points of contact with other research areas is obviously one of the appealing aspects of it.

One obvious question is what the discrete time, the nonlinear, and the dual analogs are of the concept of almost controlled invariant subspace. The discrete time analog does not exist as such although these may be connections with the synthesis of dead-beat controllers [30], [31] together with some type of time-reversal. Another point of contact is the limit behavior of sampled data systems with fast sampling and dead-beat control action. Nonlinear generalizations appear possible in principle through suitable generalizations of invariant distributions and such things [32], [33] and it would be very nice to obtain this way a connection with sliding modes as studied in [24], [34]. The dual of the notion of an almost controlled invariant subspace is that of an almost conditionally invariant subspace which will be the subject of Part II of this paper.

Appendix

In this Appendix we will discuss some auxiliary results which are used in text. In the interest of brevity we will only give the main ideas of the proofs of the facts which are actually used.

We have had to worry about smoothness issues in connection with the concepts introduced in this paper more than was pleasing. That it makes a difference at all is shown in Theorem A.

Consider $\Sigma: \dot{x}=Ax+Bu$ and the induced state trajectory space Σ_x. We have assumed that the elements of Σ_x need only be a.c. This is certainly a natural starting point but we could also have assumed more smoothness, say, infinite differentiability C^∞-smoothness. Denote by $\tilde{\Sigma}_x := \Sigma_x \cap C^\infty$, i.e., $\tilde{\Sigma}_x := \{x: \mathbb{R}\to \mathsf{X} \mid x \in C^\infty \text{ and } \dot{x}(t)-Ax(t)\in \mathsf{B}\,\forall t\}$. We will call $\tilde{\Sigma}$ the smooth version of Σ. We can now consider $\tilde{\underline{V}}$, defined relative to $\tilde{\Sigma}$ in the same way as $\underline{V}$ was relative to Σ. Formally $\tilde{\underline{V}} := \{\mathsf{V}\subset \mathsf{X} \mid \forall x_0\in \mathsf{V}\, \exists x \in \tilde{\Sigma}_x$ such that $x(0)=x_0$ and $x(t)\in \mathsf{V}\,\forall t\}$; $\tilde{\underline{V}}_a$, $\tilde{\underline{R}}$, and $\tilde{\underline{R}}_a$ are defined analogously. Using some obvious smoothness arguments, it is easily seen that

Proposition A: $\tilde{\underline{V}}=\underline{V}$, $\tilde{\underline{V}}_a=\underline{V}_a$, $\tilde{\underline{R}}=\underline{R}$, and $\tilde{\underline{R}}_a=\underline{R}_a$.

Let L be a subspace of X and consider $\Sigma_x(\mathrm{mod}\,\mathsf{L})$ (i.e., $\{x'\in\Sigma_x(\mathrm{mod}\,\mathsf{L})\}\Leftrightarrow\{\exists x\in\Sigma_x \mid x'(t)=x(t)\ (\mathrm{mod}\,\mathsf{L})\forall t\}$) and $\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{L})$, similarly defined. Now, $\Sigma_x(\mathrm{mod}\,\mathsf{L})$ may or may not be a "system," i.e., there may or may not exist (A', B') such that $\Sigma_x(\mathrm{mod}\,\mathsf{L})=\Sigma_x(A',\mathsf{B}')$. Similarly there may or may not exist (A',B') such that $\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{L})=\tilde{\Sigma}_x(A',\mathsf{B}')$. The conditions under which this is the case are very nice indeed.

Theorem A:

1) $\{\exists(A',\mathsf{B}')$ *such that* $\Sigma_x(A',\mathsf{B}')=\Sigma_x(\mathrm{mod}\,\mathsf{L})\}\Leftrightarrow\{\mathsf{L}\in\underline{V}\}$;

2) $\{\exists(A',\mathsf{B}')$ *such that* $\tilde{\Sigma}_x(A',\mathsf{B}')=\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{L})\}\Leftrightarrow\{\mathsf{L}\in\underline{V}_a\}$.

This theorem gives one more equivalent and interesting characterization of $A(\mathrm{mod}\,\mathsf{B})$-invariant subspaces. The surprising part is that in this case the smoothness required of the trajectories does play a role here.

Proof of Theorem A (Outline): 1) is easy to prove. The proof of 2)($\Rightarrow$) is deleted: the result is not used in the paper; 2)($\Leftarrow$) follows immediately from 1) and the following.

Lemma A: Let $\mathsf{L}=\mathsf{B}_1+A_F\mathsf{B}_2+\cdots+A_F^{n-1}\mathsf{B}_n$ for some F and some chain $\{\mathsf{B}_i\}$ in B. Then there exists (A',B') such that $\tilde{\Sigma}_x(A',\mathsf{B}')=\tilde{\Sigma}_x(\mathrm{mod}\,\mathsf{L})$. In fact, $\mathsf{B}'=(A\mathsf{L}+\mathsf{B})(\mathrm{mod}\,\mathsf{L})$.

Proof: See the ideas in [1], Proposition 10, and [35], part 3) of the proof of Lemma 7. □

A rather immediate consequence of the above theorem and the representations for $\underline{R}_a$ and $\underline{V}_a$ described in Section II is the following.

Corollary A:

$$\{\mathsf{L}\in\underline{V}\}\Rightarrow\{\underline{V}^{\Sigma(\mathrm{mod}\,\mathsf{L})}=\underline{V}^{\Sigma}(\mathrm{mod}\,\mathsf{L})\},$$

$$\{\mathsf{L}\in\underline{V}_a\}\Rightarrow\{\underline{V}_a^{\tilde{\Sigma}(\mathrm{mod}\,\mathsf{L})}=\underline{V}_a^{\tilde{\Sigma}}(\mathrm{mod}\,\mathsf{L})\},$$

$$\{\mathsf{L}\in\underline{R}\}\Rightarrow\{\underline{R}^{\Sigma(\mathrm{mod}\,\mathsf{L})}=\underline{R}^{\Sigma}(\mathrm{mod}\,\mathsf{L})\}$$

$$\{\mathsf{L}\in\underline{R}_a\}\Rightarrow\{\underline{R}_a^{\tilde{\Sigma}(\mathrm{mod}\,\mathsf{L})}=\underline{R}_a^{\tilde{\Sigma}}(\mathrm{mod}\,\mathsf{L})\}.$$

The superscripts at the right-hand sides simply denote with respect to which system (almost) invariance or (almost) controllability has to be taken.

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Jan C. Willems (S'66–M'68–SM'79–F'80) was born in Brugge, Belgium in 1939. He studied at the University of Gent in Belgium, the University of Rhode Island, and the Massachusetts Institute of Technology, where he received his Ph.D. degree in electrical engineering in 1968.

From 1968 he was an Assistant Professor of electrical engineering at MIT, Cambridge, MA until in 1973 when he was appointed to his present position of Professor of Systems and Control at the Mathematics Institute of the University of Groningen in The Netherlands. He has held brief visiting appointments at the University of Cambridge, Cambridge, England, Harvard University, Cambridge, MA, the University of Rome, the ETH, and Ben Gurion University. His research interests lie in systems theory and automatic control. His recent work is concerned with the geometric theory of linear systems, the synthesis of multivariable controllers, and problems of system representation.

Dr. Willems serves as a consulting advisor to the Mathematical Centre in Amsterdam and is member of the editorial board of several journals in the field of Systems and Control. He is a member of SIAM, the Dutch General Systems Society, and the Dutch "Wiskundig Genootschap."