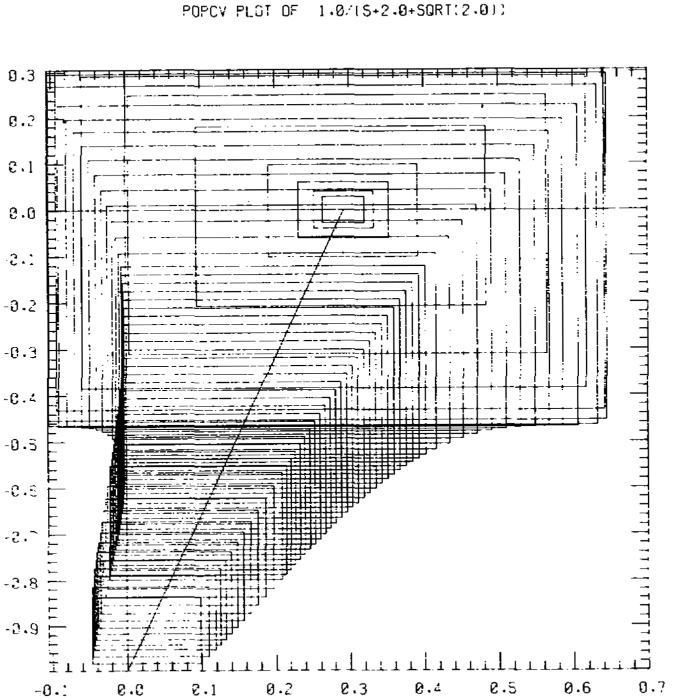
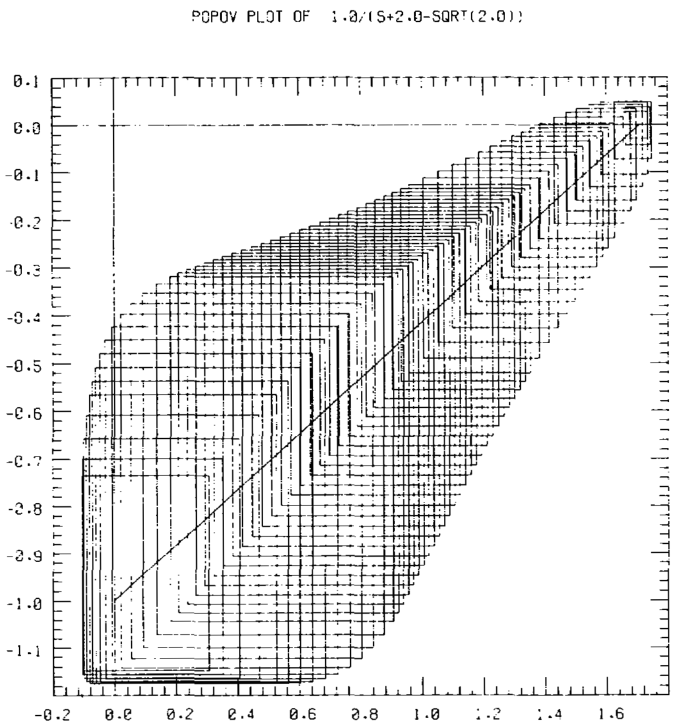


(a)



(b)

Fig. 2. Theorem 5 applied to Example 2. With $q=0$, we obtain $-1/k<\min(-0.105,-0.11)$. This gives $k<9$.

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Synthesis of State Feedback Control Laws with a Specified Gain and Phase Margin

PER MOLANDER AND JAN C. WILLEMS

Abstract—**It is shown how one may use a "circle criterion" philosophy to design a state feedback control law which yields a closed-loop system with specified robustness characteristics. This robustness is most immediately given in terms of preservation of stability when a cone-bounded nonlinearity is introduced in the loop, but may also be interpreted in terms of gain and phase margin and gain reduction tolerance.**

I. Introduction

In a seminal paper, Kalman [1] demonstrated that linear-quadratic regulators satisfy a certain frequency domain inequality which yields a degree of robustness for these control laws. In terms of classical control concepts, Kalman's inequality implies that these controllers possess a 60° phase margin, infinite gain margin, and 50 percent gain reduction tolerance. In terms of feedback stability concepts [2], [3], it implies that the closed-loop system will remain stable if an arbitrary nonlinear system contained in the sector $(1/2,\infty)$ is introduced in the loop. This result holds under the sole assumption that the integrand in the performance criterion is the usual sum of a positive definite term in the control and a nonnegative definite term in the state.

In the last few years, there has been renewed interest in this sort of results (see, e.g., [4]–[6]), motivated principally by robustness questions. Indeed, where classical control synthesis techniques were very directly concerned with robustness considerations, it appeared that modern control theory had somewhat neglected to formulate robustness questions as such. In particular, the question has recently been raised whether one can design a state feedback control law such that the closed-loop system has a specified gain and phase margin. In this paper it will be shown how this can be achieved.

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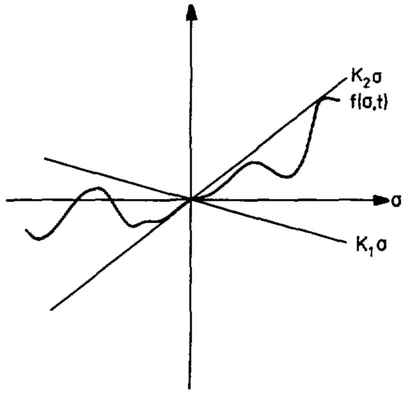

Fig. 1. An illustration of the sector condition.

II. Problem Formulation

Consider the linear time-invariant finite-dimensional system

$$\dot{x}=Ax+Bu \tag{1}$$

with $x\in\mathbf{R}^n$ and $u\in\mathbf{R}^m$. It will be assumed throughout that (A,B) is a controllable pair, although this is sometimes irrelevant and stabilizability often suffices. Assume that the desired input is given by the linear time-invariant control law

$$u=-L^Tx \tag{2}$$

and that it is necessary to choose L such that the closed-loop system is stable when the actual input is a perturbation of (2). Specifically, (2) will be called a *robust control law with robustness sector* (K_1,K_2) $(-\infty<K_1<1<K_2<\infty)$ if the closed-loop system is globally asymptotically stable when the actual input is given by

$$u=-f(L^Tx,t) \tag{3}$$

with $f\colon \mathbf{R}^m\times\mathbf{R}\to\mathbf{R}^m$ any nonlinear function satisfying

$$\sup_{\sigma,t}(f(\sigma,t)-K_1\sigma)^T(f(\sigma,t)-K_2\sigma)<0 \tag{4}$$

or, equivalently,

$$\sup_{\sigma,t}\frac{|f(\sigma,t)-\frac{1}{2}(K_1+K_2)\sigma|}{|\sigma|}<\frac{1}{2}(K_2-K_1) \tag{4$'$}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^m$. The sector condition is illustrated in Fig. 1 for the single-input case.

Moreover, the feedback system

$$\dot{x}=Ax+Bu+Gv,\quad y=L^Tx,\quad z=C^Tx \tag{5}$$

with $u(\cdot)$ determined from $y(\cdot)$ by a nonlinear input/output system with nonanticipating system function $F\colon y(\cdot)\to -u(\cdot)$ satisfying

$$\sup_{y(\cdot)\in L_2(0,\infty)}\frac{|u(\cdot)-\frac{1}{2}(K_1+K_2)y(\cdot)|_{L_2(0,\infty)}}{|y(\cdot)|_{L_2(0,\infty)}}<\frac{1}{2}(K_2-K_1) \tag{6}$$

should be $L_2(0,\infty)$ input/output stable as a system from $v(\cdot)$ to $z(\cdot)$. When $K_2=\infty$, (4) and (6) should be interpreted as holding for some $K_2<\infty$, and a similar modification is required when $K_1=-\infty$.

The robustness requirements also allow for an interpretation in terms of gain and phase margin and gain reduction tolerance. To be precise, the control law (2) has a *gain margin* g $(g>1)$ if the closed-loop system is globally asymptotically stable for all control laws

$$u=-\Lambda L^Tx \tag{7}$$

with Λ any symmetric matrix with eigenvalues $1<\lambda_i<g$. It is said to have *phase margin* ϕ $(0<\phi<\pi)$ if it is globally asymptotically stable for all control laws (7) with Λ any unitary matrix $(\Lambda^*\Lambda=\Lambda\Lambda^*=I)$ possessing eigenvalues $\lambda_i=\exp(j\cdot\phi_i)$ with $|\phi_i|<\phi$. It is said to have *gain reduction tolerance* ρ in percent if it is globally asymptotically stable for all control laws (7) with Λ any symmetric matrix with eigenvalues $1-\rho/100<\lambda_i<1$. $(\rho>0$, and usually $\rho<100$, but $\rho>100$ is also possible and corresponds to gain reversal.) These are straightforward generalizations of the classical single-input concepts.

From (4) the following proposition follows (the proof is given in the Appendix).

Proposition 1: Assume that (2) has robustness sector (K_1,K_2). Then it has

$$\text{gain margin } g=K_2$$

$$\text{phase margin } \phi \text{ with } \cos(\phi)=\frac{K_1K_2+1}{K_1+K_2}$$

$$\text{gain reduction tolerance } \rho=(1-K_1)\cdot 100.$$ □

Consequently, in order to design a feedback law (2) with gain margin g and phase margin ϕ it suffices to take for robustness sector (K_1,K_2) with

$$K_1=\frac{g\cos(\phi)-1}{g-\cos(\phi)}$$

and

$$K_2=g.$$

Notice that if $\cos(\phi)<1/g$, then $K_1<0<1<K_2$ and hence A will have to be asymptotically stable in order for this design to be possible.

The circle criterion inequalities will be used to design control laws with a given robustness sector. Of course, there may be other techniques which yield a prescribed gain and phase margin, but such results do not seem to be part of the linear-quadratic ideas on which, after all, also the nonlinear feedback stability results (Propositions 2 and 3) are based. The main facts from stability theory to be used are given in the following section.

III. Preliminaries

Let $\{A,B,C\}$ denote the system

$$\Sigma\colon \dot{x}=Ax+Bu,\qquad y=C^Tx.$$

The minimal system Σ is said to be *inside the sector* $[\alpha,\beta]$ $(-\infty<\alpha<0<\beta<\infty)$ if A is asymptotically stable and if, with $x(0)=0$, and for all $u(\cdot)\in L_2(0,\infty)$, the response $y(\cdot)$ satisfies

$$\langle y(\cdot)-\alpha u(\cdot),y(\cdot)-\beta u(\cdot)\rangle_{L_2(0,\infty)}\le 0.$$

It is said to be *outside the sector* $[\alpha,\beta]$ $(0<\alpha<\beta<\infty)$ if $A+\beta^{-1}BC^T$ is asymptotically stable and

$$\langle y(\cdot)-\alpha u(\cdot),y(\cdot)-\beta u(\cdot)\rangle_{L_2(0,\infty)}\ge 0.$$

(If $\beta=\infty$, the first inequality should read

$$\langle y(\cdot)-\alpha u(\cdot),u(\cdot)\rangle_{L_2(0,\infty)}\ge 0,$$

with a similar modification for $\alpha=-\infty$.)

The following results follow from standard calculations [2], [3] and the Kalman–Yacubovich–Popov lemma or its extensions [7]–[9].

Proposition 2: Let $\Sigma=\{A,B,C\}$ be minimal and $G(s)=C^T(sI-A)^{-1}B$ be its transfer function. Then the following conditions are equivalent.

i) Σ is inside the sector $[\alpha,\beta]$.

ii) A is asymptotically stable and for all ω,

$$\mathrm{Re}\langle G(j\omega)-\alpha,G(j\omega)-\beta\rangle\le 0.$$

iii) There exists a $P=P^T>0$ such that

$$\begin{bmatrix} A^TP+PA+CC^T & PB-\frac{\alpha+\beta}{2}C \\ \left(PB-\frac{\alpha+\beta}{2}C\right)^T & \alpha\beta \end{bmatrix} \leq 0$$

or, equivalently, if $\alpha\beta \neq 0$,

$$A^TP+PA+CC^T-\left(PB-\frac{\alpha+\beta}{2}C\right)\frac{1}{\alpha\beta}\left(PB-\frac{\alpha+\beta}{2}C\right)^T \leq 0.$$

Similarly, the following conditions are equivalent.

i) Σ is outside the sector $[\alpha,\beta]$.

ii) $A+\beta^{-1}BC^T$ is asymptotically stable and for all ω

$$\mathrm{Re}\langle G(j\omega)-\alpha, G(j\omega)-\beta\rangle \geq 0.$$

iii) There exists a $P=P^T>0$ such that

$$\begin{bmatrix} A^TP+PA-CC^T & PB+\frac{\alpha+\beta}{2}C \\ \left(PB+\frac{\alpha+\beta}{2}C\right)^T & \alpha\beta \end{bmatrix} \leq 0$$

or, equivalently, if $\alpha\beta \neq 0$,

$$A^TP+PA-CC^T+\left(PB+\frac{\alpha+\beta}{2}C\right)\frac{1}{\alpha\beta}\left(PB+\frac{\alpha+\beta}{2}C\right)^T \leq 0.$$

Obvious modifications are necessary for $\alpha=-\infty$ or $\beta=\infty$. □

The following proposition gives the basic stability result in which much of the work on feedback stability has culminated.

Proposition 3: Assume that $\{A,B,L\}$ is minimal. Then the control law (2) has robustness sector (K_1,K_2) if either

i) $0 \leq K_1 \leq K_2$ and $\{A,B,L\}$ is outside the sector $[-1/K_1,-1/K_2]$, or

ii) $K_1 \leq 0 \leq K_2$ and $\{A,B,L\}$ is inside the sector $[-1/K_2,-1/K_1]$. □

In fact, x^TPx with P as in the above proposition will be a Lyapunov function for the control law (3).

Various conditions (conicity, etc.) equivalent to those of the above theorems or specific cases in which the conditions hold may be found in the literature [2], [3], [10], [11].

IV. Robustness Synthesis

Theorem 1: Assume $0 \leq K_1 \leq K_2 < \infty$. Then the following procedure yields a feedback law (2) with robustness sector (K_1,K_2).

1) Pick any $(n\times m)$ matrix D such that (A,D) is an observable pair and any $(n\times n)$ matrix $Q=Q^T \geq 0$.

2) Solve the algebraic Riccati equation

$$A^TP+PA-\frac{4K_1K_2}{(K_1+K_2)^2}(PB+D)(PB+D)^T+DD^T+Q=0$$

for its (unique) solution $P=P^T>0$. (This solution automatically exists when $K_1>0$ and iff A is asymptotically stable when $K_1=0$).

3) Take $L=2/(K_1+K_2)(PB+D)$.

Proof: By Proposition 3, one must show that P is well-defined in 2 and that $\{A,B,L\}$ satisfies condition iii) of Proposition 2 with $\alpha=-1/K_1$ and $\beta=-1/K_2$. Part 1) follows from the standard linear-quadratic results since the constant term of the Riccati equation equals

$$\left(\frac{K_1-K_2}{K_1+K_2}\right)^2 DD^T+Q>0$$

and the required controllability and observability conditions are satisfied. To see part 2), solve for D in the algorithm getting $D=(K_1+K_2)L/2-PB$. Inserting this into the Riccati equation yields the inequality in iii) of Proposition 2, modulo a factor K_1K_2. □

Theorem 2: Assume $-\infty \leq K_1 \leq 0 \leq K_2 \leq \infty$. Then the following procedure yields a feedback law (2) with robustness sector (K_1,K_2).

1) Pick any $(n\times m)$ matrix D and any $(n\times n)$ matrix $Q=Q^T \geq 0$ such that (A,D) is observable and the algebraic Riccati equation

$$A^TP+PA-\frac{4K_1K_2}{(K_1+K_2)^2}(PB+D)(PB+D)^T+DD^T+Q=0$$

has a symmetric solution (such D and Q will exist iff A is asymptotically stable).

2) Pick any symmetric solution of this Riccati equation (which will automatically be >0).

3) Take $L=2/(K_1+K_2)(PB+D)$.

Proof: The properties of the Riccati equation follow from the results in [9] and the claimed robustness from the fact that $\{A,B,L\}$ satisfies condition iii) of Proposition 2 with $\alpha=-1/K_2$ and $\beta=-1/K_1$. □

Remark: The above procedure collapses if $K_1+K_2=0$. For a treatment of this singular case, see [12]. □

The case $K_2=\infty$ is treated in the following theorem which is proved in a similar fashion.

Theorem 3: The following procedure yields a feedback law (2) with robustness sector (K_1,∞).

Case 1—$K_1>0$:

1) Pick any $(n\times n)$ matrix $Q=Q^T \geq 0$ such that (A,Q) is observable.

2) Solve the algebraic Riccati equation

$$A^TP+PA-2K_1PBB^TP+Q=0$$

for its (unique) solution $P=P^T>0$ (this solution automatically exists when $K_1>0$ and iff A is asymptotically stable when $K_1=0$ in which case this equation becomes a Lyapunov equation).

3) Take $L=PB$.

Case 2—$K_1 \leq 0$:

1) Pick any $(n\times n)$ matrix $Q=Q^T \geq 0$ such that (A,Q) is observable and the algebraic Riccati equation

$$A^TP+PA-2K_1PBB^TP+Q=0$$

has a symmetric solution (such a Q will exist iff A is asymptotically stable).

2) Pick any symmetric solution of this Riccati equation (which will automatically be >0).

3) Take $L=PB$.

V. Discussion

1) The robustness problem considered here is not well-posed. If A is asymptotically stable, $L=0$ will give a robustness sector $(-\infty,\infty)$, and if A is not, then a robustness sector (ϵ,∞) $(\epsilon>0)$ is obtainable from the linear-quadratic theory of 1 by using $1/2\epsilon$ times the optimal gain for L. Nevertheless, the theorems serve to yield techniques for generating other control laws. In these theorems one may regard the choices of D and Q as in some sense equivalent (albeit much less intuitively interpretable) to choosing the performance criterion of a linear-quadratic design.

2) If $K_1>0$, the robustness design is always possible and one is free in choosing D and Q in the procedure of Theorems 1 and 3. In case $K_1 \leq 0 \leq K_2$, A must be asymptotically stable. In this case, the matrices D and Q of Theorems 2 and 3 should be chosen such that the Riccati equation has a solution. In fact, they should be such that for

$$\dot{x}=Ax+Bu, \quad y=D^Tx$$

with $x(0)=0$ and for all $u(\cdot)$, one has

$$(K_1-K_2)^2|u|^2_{L_2(0,\infty)}+4K_1K_2\left(|u+y|^2_{L_2(0,\infty)}+\int_0^\infty x^TQx\,dt\right) \geq 0$$

in Theorem 2, and

$$|u|^2_{L_2(0,\infty)}+K_1\int_0^\infty x^TQx\,dt \geq 0$$

in Theorem 3.

3) Theorem 3 with $K_1=1/2$ yields Kalman's original result. In fact, the other cases of Theorem 3 with $K_1>0$ may also be interpreted from this result since the procedure used actually corresponds to taking an ordinary linear-quadratic optimal design, thus obtaining the robustness sector $(1/2,\infty)$, and then using $2K_1$ times the optimal gain to obtain the robustness sector (K_1,∞).

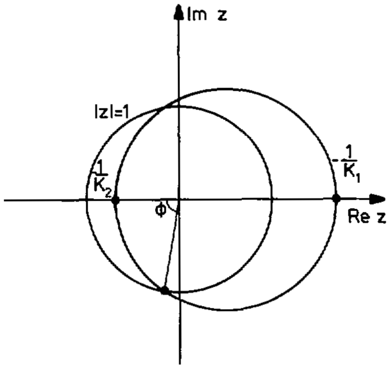

Fig. 2. Deduction of the phase-margin equation in the single-input case.

An interesting special case of Theorem 3 is $K_1=0$, in which situation one obtains the robustness sector $[0,\infty)$ by picking (A,Q) observable and solving

$$A^TP+PA=-Q.$$

$L=PB$ then yields an infinite gain margin, a 90° phase margin, and 100 percent gain reduction tolerance. These are so-called Lyapunov designs. It is possible to show that all the L's thus obtained are also linear-quadratic optimal designs, but by considering them as such, it would not be possible to guarantee this robustness sector.

4) All of the results of Theorems 1 to 3 admit an optimal control interpretation. It is easily verified using the by now standard frequency domain manipulations, which yield the results of [1] and its multivariable extensions, that using

$$J=\int_0^\infty(|u|^2+2u^TC^Tx+x^TMx)\,dt$$

on $\dot{x}=Ax+Bu$ with C and M such that

$$|u|^2+2u^TC^Tx+x^TMx\geqslant\gamma^2|u|^2(|\gamma|\leqslant 1)$$

will yield a robustness sector $(1/(1+|\gamma|),1/(1-|\gamma|))$. Similarly, if

$$|u|^2+2u^TC^Tx+x^TMx\leqslant\gamma^2|u|^2(|\gamma|\geqslant 1),$$

and if the optimal control problem has a solution, which is then not guaranteed, one obtains a robustness sector $(1/(1-|\gamma|),1/(1+|\gamma|))$. By multiplying the optimal gain by an appropriate factor, it is possible to obtain an arbitrary preassigned robustness sector. Theorems 1 to 3 are easily interpreted in this vein. The performance criterion is

$$\int_0^\infty\left(|u|^2+\frac{4K_1K_2}{(K_1+K_2)^2}(2u^TD^Tx+|D^Tx|^2+x^TQx)\right)dt$$

and the gain used is $(K_1+K_2)/2K_1K_2$ times the optimal gain.

The idea of including a cross-product term $2u^TC^Tx$ in the cost functional deserves in our opinion some attention in the linear-quadratic theory, where often only the case (u^TRu+x^TMx) with $R=R^T>0$ and $M=M^T\geqslant 0$ is treated. This extra flexibility makes it possible to generate optimal control laws that may be superior from the sensitivity or accuracy point of view. For a treatment of such optimal control problems, see [9].

5) The theory is easily generalized to the case that K_1 and K_2 become arbitrary diagonal matrices. This admits robustness design with unequal gain and phase margin requirements in the different loops.

Appendix

Proof of Proposition 1: Consider first the single-input case. The gain-margin and gain reduction tolerance assertions are immediate from the definitions. To deduce the phase-margin result, consider Fig. 2, which has been drawn for the case $K_1<0<K_2$.

It follows from the circle criterion that the condition on the Nyquist locus of $G(s)=L^T(sI-A)^{-1}B$ pertaining to condition (4) is that it be contained in a circle through $(-1/K_2,0)$ and $(-1/K_1,0)$, symmetric with respect to the real axis. This implies that the Nyquist locus intersects the unit circle at an angle at least ϕ degrees away from the negative real axis. Inserting the coordinates $(-\cos(\phi),-\sin(\phi))$ into the equation of the circle bounding the Nyquist locus yields

$$\left(-\cos(\phi)-\frac{1/K_1+1/K_2}{2}\right)^2+(-\sin(\phi))^2=\left(\frac{-1/K_1+1/K_2}{2}\right)^2$$

whence

$$\cos(\phi)=\frac{K_1K_2+1}{K_1+K_2}.$$

The multivariable case may be reduced to the above situation by using the fact that a unitary matrix is unitarily equivalent to a diagonal one. □

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Connections Between Finite-Gain and Asymptotic Stability

DAVID J. HILL AND PETER J. MOYLAN

***Abstract*—The relationship between input-output and Lyapunov stability properties for nonlinear systems is studied. Well-known definitions for the input-output properties of finite-gain and passivity, even with quite reasonable minimality assumptions on a state-space representation, do not necessarily imply any form of stability for the state. Attention is given to the precise versions of input-output and observability properties which guarantee asymptotic stability. Particular emphasis is given to the possibility of multiple equilibria for the dynamical system.**

I. Introduction

For causal linear time-invariant systems, there are well-known strong equivalences among a variety of definitions of stability [1]. In particular, $\mathcal{L}_2$ finite-gain stability implies, under minimality assumptions, global

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