Hierarchic Recursive Image Enhancement

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Abstract—**The problem that is solved in this paper can be formulated as: given an observation of an image against the background of additive noise and given the statistics of the image and the noise, find an optimal estimate of the image such that the computer-time and storage requirements of the estimator are modest for images of, say 250 × 250 points or more. A discrete-time vector-scanning model is derived that describes the statistics of a large class of images. The optimal linear smoother—with regard to the least-squares criterion—is formulated in a recursive manner as a combination of two Kalman filters. It is observed that in the model the covariance matrices are Toeplitz matrices. It is shown that the z transform defines a one-to-one relation between Toeplitz matrices and functions of a complex variable. This reduces the Riccati equation to a scalar equation in the z domain. It is further shown that multiplication by a Toeplitz matrix can be performed recursively by two linear dynamical systems. This leads to an algorithm which is not only recursive in the "time" parameter of the state space model but also in the index of the elements of the state vector. This so-called hierarchic recursive method has modest computational requirements.**

I. Introduction

IMAGE ENHANCEMENT is a process for improving a degraded image. The degradation may be due to reflections from spurious objects, inaccuracies in the sensing mechanism, or imperfect transmission. When the degradation cannot be avoided or the object cannot be observed again, image enhancement has proved to be very useful [1]. Fields of application include astronomy, electron-microscopy, and X-ray photography.

Image processing deals with data which are two dimensional in nature. A monochromatic image may be represented by its brightness $\beta(t,s)$ at every point (t,s). The coordinates t and s may be continuous, discrete, or a combination of the two. In this paper $\beta(t,s)$ is considered as a sample of a two-dimensional random process. Characteristic features of the image are then described by its first- and second-order statistics [2]. Images are thus divided into classes characterized by their mean $E\{\beta(t,s)\}$ and their covariance $E\{\beta(t,s)\beta(t',s')\}$, where $E\{\cdot\}$ stands for expected value. In many cases the disturbances may also be described by a random (noise) process with known statistics.

Image enhancement thus becomes a statistical estimation problem where one attempts to separate the observations from the noise. The most powerful and computationally efficient procedure currently available for making this estimation is the recursive Kalman-filtering technique. In the Kalman filter, however, the observation must be written as the output of a linear dynamical system with one independent variable (the time). Application of this technique (in image enhancement) thus requires the design of a model and a filter which are adopted to the special two-dimensional structure of an image, where a great deal of attention must be paid to the computer-time and storage requirements. Since images with, say, 250 × 250 points (or more) must be processed, the computational requirements tend to become unmanageably large.

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Several procedures have been proposed to convert the two-dimensional information into a form with only one variable.

Nahi and Assefi [2] consider a sort of TV scanning of an image. The periodic nature of the scanning procedure results in an output which will be a nonstationary random process. This introduces complexities in the design of a model and approximations are needed even before a model can be constructed.

Powell and Silvermann [3] found an exact, but not strictly linear, model of the TV signal. The linear approximation of the model, however, needs a further approximation as the dimensions of the image grow larger than, say, 50 × 50 as is usually the case.

Nahi and Franco [4] describe a continuous vector process in which a column of intensities is taken as the state vector of the model. But straightforward implementation of the Kalman filter for this model, even in the steady-state case, overloads most computers when images are larger than about 50 × 50.

Habibi [5] derives a two-dimensional Kalman filter which is scalar, in the sense that the dimension of the state of the filter is one. Unfortunately, the assumption that the optimal estimate can be expressed as the combination of two optimal estimates, based on partially overlapping data sets, is unfounded and thus the estimator derived will not be optimal.

Jain and Angel [6] also derive a scalar filter. However, before filtering the image has to be transformed and afterwards it has to be retransformed. This makes their method equivalent to frequency-domain filtering with $O(N^2 \log N)$ operations, where N is the image dimension.

In this paper, a new approach to image enhancement is proposed wherein an exact representation of the statistics of the image is given by a discrete linear vector model. An image enhancer is described which is hierarchical and recursive in the two indices of the image. The first index locates the state vector of the model which is estimated by the recursive Kalman technique, while along the vector itself the elements are estimated by a recursive subsystem. The computations in the subsystem involve an approximation, which, however, by selecting the dimension of the subsystem may be made as accurate as desired. In practical cases this dimension need not be larger than three for an accuracy of about 0.5 percent. The number of operations is $O(N^2)$ and the computer storage requirements are also considerably lower than the previously mentioned methods.

Following the descriptions of the statistical assumptions and the mathematical formulation of the problem given in Section II, the recursive vector scanning model is presented in Section III. In the next section, the Kalman filter is given for the resulting model, which is used twice —one sweep from left to right and one reversed. The two estimates are combined to find the optimal smoothed estimate. The efficient implementation of the smoother is due to a number of interesting properties of Toeplitz matrices and their relations with the z transform and linear dynamical systems, which are investigated in Section V. The resulting algorithm and a numerical example are given in Section VI. Finally, an application is discussed in Section VII.

II. Mathematical Problem Formulation

A monochromatic picture can be represented by a matrix of intensities B of dimension $K \times L$, with elements $\beta_{k,l}(k,l) \in S \triangleq \{1,\cdots,K\} \times \{1,\cdots,L\}$. The characteristic features of an image will be expressed, here, by the first- and second-order statistics of the (stationary) random variable $\beta_{k,l}$. Without any loss of generality, the reference level and the brightness scale may be chosen such that

$$E\{\beta_{k,l}\}=0 \tag{2.1a}$$

$$E\{\beta_{k,l}^2\}=1 \tag{2.1b}$$

where $(k,l) \in S$. We will assume that the covariance is exponentially decaying in both directions. Thus

$$E\{\beta_{k,l}\beta_{k+p,l+q}\}=\rho_1^{|p|}\rho_2^{|q|}; \qquad (k,l),(k+p,l+q) \in S \tag{2.1c}$$

with $0<\rho_1<1$ and $0<\rho_2<1$. A large number of "real-world" images can indeed be approximated by this type of autocorrelation function [1]–[4]. The observed image, represented by $\eta_{k,l}$, is supposed to be the original image $\beta_{k,l}$ contaminated by additive noise $\omega_{k,l}$; i.e.,

$$\eta_{k,l}=\beta_{k,l}+\omega_{k,l}. \tag{2.2a}$$

Assuming the noise $\omega_{k,l}$ to be white and uncorrelated with the intensity;

$$E\{\beta_{k,l}\omega_{k+p,l+q}\}=0, \qquad \forall(k,l),(k+p,l+q) \in S \tag{2.2b}$$

$$E\{\omega_{k,l}\omega_{k+p,l+q}\}=\theta\cdot\delta_p\cdot\delta_q, \qquad \forall(k,l),(k+p,l+q) \in S \tag{2.2c}$$

where $\theta>0$ is the intensity of the noise and δ denotes the Kronecker δ:

$$\delta_p=\begin{cases}0, & \text{if } p \neq 0\\ 1, & \text{if } p=0.\end{cases}$$

Since we can always subtract the average of the noise intensity from the observations, we may assume, without loss of generality, that

$$E\{\omega_{k,l}\}=0. \tag{2.2d}$$

We can now formulate the problem solved in this paper.

Problem: Let the matrix of intensities Y with elements $\eta_{k,l}$ represent the observed image of an original $B=(\beta_{k,l})$ with additive noise $W=(\omega_{k,l})$. Thus $Y=B+W$. Let the mean and the covariances of $\beta_{k,l}$ and $\omega_{k,l}$ be given by (2.1) and (2.2). We want to find an efficient algorithm to compute the optimal linear least-squares estimation $\hat{B}$ of B based on the observation Y, i.e., a linear map L in $\hat{B}=L(Y)$ such that $E\{\sum_{k,l \in S}(\hat{\beta}_{k,l}-\beta_{k,l})^2\}$ is minimized, and an efficient way to compute L and $L(Y)$.

III. The Vector-Scanning Model

To make use of a vector-scanning model we now need to consider the statistics in terms of the vectors defined below.

Definition: Let $\beta_{k,l}$, $\omega_{k,l}$, $\eta_{k,l}$ represent the brightness, observation noise, and observation at a point k, l of the image, respectively. The corresponding vectors b_l, w_l, and y_l are defined by

$$b_l \triangleq \begin{bmatrix}\beta_{1,l}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \beta_{K,l}\end{bmatrix} \qquad w_l \triangleq \begin{bmatrix}\omega_{1,l}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \omega_{K,l}\end{bmatrix} \qquad y_l \triangleq \begin{bmatrix}\eta_{1,l}\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \eta_{K,l}\end{bmatrix}. \tag{3.1}$$

The vectors are columns in the matrices representing the image. For the sake of simplicity, we take in the following $\rho_1=\rho_2=\rho (0<\rho<1)$ in (2.1c). It is a straightforward matter to extend the results to the case $\rho_1 \neq \rho_2$.

Property: When the means and (co)variances of $\beta_{k,l}$ and $\omega_{k,l}$ are given by (2.1) and (2.2) then the vectors b_l, w_l, and y_l of the preceding definition have the following first- and second-order statistics: the mean is

$$E\{b_l\}=E\{w_l\}=E\{y_l\}=0 \tag{3.2}$$

the variance is

$$E\{b_l b_l'\}=R \triangleq \begin{bmatrix}1 & \rho & \cdots & \rho^{K-1}\\ \rho & \ddots & & \vdots\\ \vdots & & \ddots & \rho\\ \rho^{K-1} & \cdots & \rho & 1\end{bmatrix}$$

$$E\{w_l w_l'\} = \Theta \triangleq \begin{bmatrix} \theta & & 0 \\ & \ddots & \\ 0 & & \theta \end{bmatrix}$$

$$E\{y_l y_l'\} = R + \Theta \tag{3.3}$$

and the covariance is

$$E\{b_l b_{l+q}'\} = \rho^{|q|} R$$
$$E\{w_l w_{l+q}'\} = 0$$
$$E\{y_l y_{l+q}'\} = \rho^{|q|} R. \tag{3.4}$$

For all l and $l+q \in \{1, \cdots, L\}$, $q \neq 0$.

Remark: The variance matrix R has a special structure in that each (sub- and super-) diagonal has identical entries. These matrices are finite Toeplitz matrices and this special structure is fully exploited later.

We now want to design a discrete dynamical system the output of which is a vector process $\{y_l\}$ with the parameters chosen such that the statistics of the process $\{y_l\}$ are the same as those assumed for the successive columns of the observation matrix Y. The observed image can then be interpreted as the output of a discrete dynamic system to which Kalman filtering can be applied.

Problem: Given R and Θ as defined in (3.3) and taking the discrete dynamical system to be

$$x_{l+1} = Ax_l + Dv_l$$
$$y_l = Cx_l + Ew_l$$

we wish to determine the dimension of x_l, the parameters A, D, C, and E, the mean and variances of $\{v_l\}$ and $\{w_l\}$, and the correlation between $\{v_l\}$, $\{w_l\}$, and $\{x_l\}$, such that the statistics of $\{y_l\}$ are as given in (3.2)–(3.4), where it is understood that the dimension of y_l is K and that $\{v_l\}$ and $\{w_l\}$ are white noise processes. The problem is a special case of the inverse problem of stationary covariance generation (Anderson [7] and Anderson and Moore [8]). It has more than one solution, a convenient one is given in the following proposition. The proof, which consists of straightforward calculation, is omitted.

Proposition: A solution to the problem stated above is given by the system

$$x_{l+1} = \rho x_l + \sqrt{1-\rho^2}\, R^{1/2} v_l, \qquad l = 0, \cdots, L-1$$
$$y_l = x_l + w_l, \qquad l = 1, \cdots, L$$
$$x_0 = 0 \tag{3.5}$$

where v_l and w_l ($l = 1, \cdots, L$) are uncorrelated zero-mean random vectors with variances

$$E\{v_l v_l'\} = I$$
$$E\{w_l w_l'\} = \Theta. \tag{3.6}$$

The matrix $R^{1/2}$ is well defined because $R = R' > 0$.

Remark: The state vector x_l in the model (3.5) is identical to the vector b_l as defined in (3.1) and thus x_l represents the intensities of a column of the image. For later convenience, we present here a second solution to the problem.

Proposition: Another solution is given by the anticausal system

$$x_{l-1} = \rho x_l + \sqrt{1-\rho^2}\, R^{1/2} v_l, \qquad l = L+1, \cdots, 2$$
$$y_l = x_l + w_l, \qquad l = L, \cdots, 1$$
$$x_{L+1} = 0 \tag{3.7}$$

where v_l and w_l ($l = L, \cdots, 1$) are uncorrelated zero-mean random vectors with variances given by (3.6).

IV. Recursive Filter and Smoother

A. The Recursive Filter

The Kalman filter of the model described in Section III is given by the following proposition [9].

Proposition: Let $\hat{x}_l^+$ denote the optimal linear least-squares estimate of the state $\hat{x}_l$ of the system (3.5) based on observation $y_1, y_2, \cdots, y_{l-1}, y_l$. Then $\hat{x}_l^+$ is recursively given by

$$\hat{x}_{l+1}^+ = \rho \hat{x}_l^+ + M_{l+1}(y_{l+1} - \rho \hat{x}_l^+)$$
$$\hat{x}_0 = 0 \tag{4.1a}$$

with

$$M_{l+1} = \tilde{\Sigma}_{l+1}(\tilde{\Sigma}_{l+1} + \Theta)^{-1}$$
$$\tilde{\Sigma}_{l+1} = \rho^2 \Sigma_{l+1}^+ + (1-\rho^2) R$$
$$\Sigma_{l+1}^+ = (I - M_{l+1})\tilde{\Sigma}_{l+1} = M_{l+1}\Theta$$
$$\tilde{\Sigma}_0 = R. \tag{4.1b}$$

The matrices Σ_l^+ and $\tilde{\Sigma}_l$ are the variance matrices of the error of the filter estimate $(x_l - \hat{x}_l^+)$ and the one-step predictor estimate $(x_{l+1} - \rho \hat{x}_l^+)$, respectively. Furthermore, it can be shown that the steady state $M = \lim_{l \to \infty}\{M_l\}$ exists and is the (unique) symmetric positive-definite solution of the Riccati equation:

$$\rho^2 M^2 \Theta + (1-\rho^2) M(R+\Theta) - (1-\rho^2) R = 0. \tag{4.2}$$

The steady-state value $\Sigma^+ = \lim_{l\to\infty} \Sigma_l^+$ is similarly given by

$$\Sigma^+ = M \cdot \Theta. \tag{4.3}$$

Solution for $L \to \infty$: To implement the filter in the form given by (4.1) the number of operations in each step is proportional to K^4, where K is the dimension of the state. This is due to (4.1b) where a $K \times K$ matrix is inverted. A considerable reduction is obtained if we take for M_{l+1} the steady-state value M. In practical applications ($\rho \sim 0.9$) M_l reaches the steady-state value in about 30 iterations, while L is usually 100 or larger. The approximations result in a slight degradation of the performance of the filter at the left border of the image where l is small.

The filter equations then become

$$\hat{x}_{l+1}^{+}=\rho\hat{x}_{l}^{+}+M(y_{l+1}-\rho x_{l}^{+}),\qquad l\in \boldsymbol{Z} \tag{4.4}$$

with M as the solution of (4.2). For later convenience, we define the innovations vector $\tilde{y}_{l+1}=y_{l+1}-\rho\hat{x}_{l}$ and rewrite (4.4) as

$$\hat{x}_{l+1}^{+}=\rho\hat{x}_{l}^{+}+M\cdot\tilde{y}_{l+1}. \tag{4.5}$$

B. The Recursive Smoother

As stated in the problem formulation, we want to find an estimate of the original matrix $X=[x_1,\cdots,x_L]$ based on observations $Y=[y_1,\cdots,y_L]$. Thus we are, in effect, dealing with a so-called fixed-interval smoothing problem. There are two main approaches to the linear-recursive fixed-interval smoothing problem. In one approach, the correction for the estimate of the Kalman filter is calculated, see, e.g., Meditch [10]. In the other, the smoothed estimate is given by a properly weighted combination of two filtered estimates—one based on observations prior to the estimated point and the other based on observations after that point, according to Fraser and Potter [11].

Using the idea of the two-filter approach, we developed a special formulation for which we need the following lemma and definition.

Lemma: Let x and y have a joint distribution with the given mean $\begin{bmatrix} m_x \\ m_y \end{bmatrix}$ and covariance $\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$ and suppose that Σ_{xx} is nonsingular. Then there always exists a linear model

$$y=Lx+v \tag{4.6}$$

where v is a random vector uncorrelated with x, such that x and y have the given mean and covariance.

Proof: The proof follows from a direct evaluation by taking

$$L=\Sigma_{yx}\Sigma_{xx}^{-1}$$
$$m_v=m_y-\Sigma_{yx}\Sigma_{xx}^{-1}m_x$$
$$\Sigma_{vv}=\Sigma_{yy}-\Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}. \tag{4.7}$$

Definition: Let x, y_1, y_2 have a joint distribution with the given mean and covariance, then y_1, y_2 are called independent observations of x if, in the underlying linear models,

$$y_1=L_1x+v_1$$
$$y_2=L_2x+v_2.$$

The observation noise vectors are uncorrelated, i.e., cov $\{v_1,v_2\}=0$.

Theorem 4.1: Let $\hat{x}_1$ and $\hat{x}_2$ be two linear least-squares estimates of a random vector x, based on independent observations y_1, y_2 with error covariances $\Sigma_1=\boldsymbol{E}\{(\hat{x}_1-x)(\hat{x}_1-x)'\}$ and $\Sigma_2=\boldsymbol{E}\{(x_2-x)(x_2-x)'\}$, respectively. Let x have mean m_x, and covariance Σ_{xx}. Then the linear least-squares estimate of x based on y_1 and y_2 is given by

$$\hat{x}=\Sigma(\Sigma_1^{-1}\hat{x}_1+\Sigma_2^{-1}\hat{x}_2-\Sigma_{xx}^{-1}m_x) \tag{4.8}$$

where $\Sigma=(\Sigma_1^{-1}+\Sigma_2^{-1}-\Sigma_{xx}^{-1})^{-1}$ is the error covariance.

Proof: The linear least-squares estimate of x based on y, when the underlying linear model is $y=Lx+v$, is given by

$$\hat{x}=\Sigma_0(L\Sigma_{vv}^{-1}y+\Sigma_{xx}^{-1}m_x) \tag{4.9}$$

where $\Sigma_0=(L'\Sigma_{vv}^{-1}L+\Sigma_{xx}^{-1})^{-1}$ is the error covariance (c.f., Schweppe [12, ch. 5]). By applying (4.9) three times, namely, with $y=y_1$, $y=y_2$, and $y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ the result (4.8) follows by straightforward manipulation.

The steady-state Kalman filter gives the linear least-squares estimate of x_l, denoted by $\hat{x}_l^{+}$, based on observations $\cdots$, y_{l-1},y_l. The smoother gives the linear least-squares estimate of x_l, denoted by $\hat{x}_l$, by combining $\hat{x}_l^{+}$ with $\hat{x}_l^{-}$, where the latter is the linear least-squares estimate of x_l based on observations y_{l+1},y_{l+2},$\cdots$, according to Theorem 4.1.

Remark: We cannot use $\hat{x}_{l|l}^{-}$, based on y_l,y_{l+1},$\cdots$, although this would give a nice symmetry, because the two estimates $\hat{x}_l^{+}$ and $\hat{x}_{l|l}^{-}$ would then no longer be independent, both being partially based on y_l. We want to find a recursive formulation for $\hat{x}_l^{-}$. We therefore consider the anticausal system, which also generates the statistics of the image as we found in Section III.

Proposition: The optimal-estimate $\hat{x}_l^{-}$, based on y_{l+1},y_{l+2},$\cdots$, state vector of the system (3.7) is recursively given by the anticausal Kalman reconstructor:

$$\hat{x}_{l-1}^{-}=\rho\hat{x}_l^{-}+\rho M(y_l-\hat{x}_l^{-}) \tag{4.10}$$

with error covariance $\Sigma^{-}=(I-M)^{-1}\Sigma^{+}$, where the steady-state matrices M and Σ^{+} are given by (4.2) and (4.3), respectively.

Remark: It is not difficult to verify that the Kalman *filter* of the anticausal system has the same M and Σ as the filter of the causal system. Equation (4.10) results from the fact that the Kalman predictor propagates the estimate of the filter. The reconstructor of the preceding proposition is simply a predictor backwards in time. We can now give the results for the optimal smoothed estimate $\hat{x}_l$.

Proposition: The smoothed estimate of the vector x_l based on observations $\cdots$, y_{l-1},y_l,y_{l+1},$\cdots$, with statistics as in (3.2)–(3.4), is given by

$$\hat{x}_l=\left[2I+M(\Theta R^{-1}-I)\right]^{-1}(\hat{x}_l^{+}+(I-M)\hat{x}_l^{-}) \tag{4.11a}$$

where $\hat{x}_l^{+}$ and $\hat{x}_l^{-}$ are obtained from

$$\hat{x}_{l+1}^{+}=\rho\hat{x}_l^{+}+M(y_{l+1}-\rho\hat{x}_l^{+}) \tag{4.11b}$$
$$\hat{x}_{l-1}^{-}=\rho\hat{x}_l^{-}+\rho M(y_l-\hat{x}_l^{-}) \tag{4.11c}$$

where $l\in\boldsymbol{Z}$ and M is given by (4.2).

Remark: In the preceding proposition the steady-state formulations of the filter and reconstructor are given. Since in any practical algorithm l will run over a finite range $\{1,\cdots,L\}$, a slight suboptimality is introduced at the borders (i.e., at $l\approx 1$ where the filter is suboptimal and at $l\approx L$ where the reconstructor is suboptimal).

Solution for $K\to\infty$: In practice K and L are large (>100). Straightforward implementation of the filter and especially the smoother, as given in the preceding proposition, then becomes impossible because difficulties arise in

1) The (efficient) solution of the quadratic matrix equation (4.2);
2) the (efficient) implementation of the matrix–vector products in (4.11) occuring at each of the L steps of the filter, the reconstructor, and the weighting.

By letting $K\to\infty$ the finite Toeplitz matrices R and Θ (c.f., (3.5)) become infinite Toeplitz matrices. By using a number of special properties of such matrices the two problems have elegant solutions.

V. Toeplitz Matrices

A. Unique Representation in the z Domain

The general form of a Toeplitz matrix is given by

$$A=\begin{bmatrix} \ddots & \ddots & \ddots \\ \cdots\alpha_{-1} & \alpha_0 & \alpha_1\cdots \\ \ddots & \ddots & \ddots \end{bmatrix} \leftarrow \text{“zeroth row”}$$

= defining sequence.

A is characterized by $(A)_{ij}=\alpha_{j-i}$. Thus on each diagonal the entries are identical and so, once its “zeroth row” $\{\alpha_k\}$ is known, the Toeplitz matrix A is defined. The sequence $\{\alpha_k\}$ is called the defining sequence.

The matrices R and Θ occurring as coefficient matrices in the recurrence equation (4.1b) for the gain matrix of the Kalman filter and in the Ricatti equation (4.2) that gives the steady-state gain are of the Toeplitz type. The gain matrix is also of the Toeplitz type. This will be shown by first establishing a one-to-one correspondence between a class of Toeplitz matrices and a class of functions defined on the unit circle $\boldsymbol{U}$ in the complex plane (the z domain).

Definition: The Toeplitz operator $A:\{\xi_k\}\to\{\eta_k\}$ induced by the Toeplitz matrix A is defined by

$$\eta_k=\sum_{j=-\infty}^{\infty}\alpha_{j-k}\xi_j \tag{5.1}$$

where $\{\alpha_k\}$ is the defining sequence of A and $\{\xi_k\}$, $\{\alpha_k\}$ are two-sided infinite sequences. In this definition the operator and the matrix are denoted by the same letter A, but no confusion arises if we identify (as is usual) the matrix and the operator defined by the matrix. We can write, equivalently,

$$\{\eta_k\}=A\{\xi_k\} \qquad \text{and} \qquad y=Ax$$

where x, y are two-sided infinite vectors of which successive elements form the sequences $\{\xi_k\}$, $\{\eta_k\}$, respectively, and

$$\{\eta_k\}=\{\alpha_k^-\}*\{\xi_k\} \tag{5.2}$$

where $\{\alpha_k^-\}$ is defined by $\alpha_k^-=\alpha_{-k}$ $\forall k\in\boldsymbol{Z}$ and $*$ denotes convolution.

Definition: The Toeplitz function $a:\boldsymbol{U}\to\boldsymbol{C}$ of a Toeplitz matrix A with defining sequence $\{\alpha_k\}$ is defined by

$$a(z)=\sum_{k=-\infty}^{\infty}\alpha_k z^{-k}, \qquad \forall z\in\boldsymbol{U}$$

such that the sequence converges, i.e., $a(\cdot)=Z(\{\alpha_k\})$, is the z transform of the defining sequence of A.

Theorem 5.1: Let A be a Toeplitz matrix with defining sequence $\{\alpha_k\}$ and let $a(\cdot)$ be the Toeplitz function of A; then

$$\|A\|=\operatorname*{ess\,sup}_{z\in\boldsymbol{U}}|A(z)| \tag{5.3}$$

here

$$\|A\|\triangleq\sup_{\|\{\xi_k\}\|=1}\|A\{\xi_k\}\|_2$$

and

$$\|\{\xi_k\}\|_2\triangleq\left(\sum_{k=-\infty}^{\infty}\xi_k^2\right)^{1/2}$$

i.e., $\|A\|$ is the l^2 induced operator norm.

Proof: See Ter Horst and Schoute [13]. An immediate consequence of (5.3) is the following theorem.

Theorem 5.2: Let A be a Toeplitz matrix; $\{\alpha_k\}$, its defining sequence; $a(\cdot)$ its Toeplitz function: then A is a bounded linear operator from l^2 in l^2 if, and only if, $a(\cdot)$ is essentially bounded on $\boldsymbol{U}$, i,e., $A\in\mathfrak{L}(l^2,l^2)\Leftrightarrow a\in L^\infty(\boldsymbol{U})$.

Notation: $\mathcal{Q}$ is the {set of Toeplitz operators} $\cap\,\mathfrak{L}(l^2,l^2)$ and $\mathfrak{B}=L^\infty(\boldsymbol{U})$. Using this notation, Theorem 5.2 can be stated as $A\in\mathcal{Q}\Leftrightarrow a(\cdot)\in\mathfrak{B}$, where $a(\cdot)$ is the Toeplitz function of A. The correspondence between $\mathcal{Q}$ and $\mathfrak{B}$ is bijective; namely, given an element of $\mathcal{Q}$, the z transform of its defining sequence exists, is unique, and is, by definition, the corresponding Toeplitz function. On the other hand, given an element of $\mathfrak{B}$, its inverse z transform exists, is unique, and is the defining sequence of a Toeplitz matrix.

An important property of the z transform is the following. Let $a(\cdot)=Z(\{\alpha_k\})$, $b(\cdot)=Z(\{\beta_k\})$, then $Z(\{\alpha_k\}*\{\beta_k\})=a(\cdot)\cdot b(\cdot)$, where $\cdot$ denotes pointwise multiplication and $*$ denotes convolution.

We know that Toeplitz operators are convolution operators (c.f., [26]) and it follows that, for $A,B\in\mathcal{Q}$ with defining sequences $\{\alpha_k\}$, $\{\beta_k\}$ and corresponding Toeplitz functions $a(\cdot)$, $b(\cdot)$; AB has a defining sequence $\{\alpha_k\}*\{\beta_k\}$ and thus a corresponding Toeplitz function

$a(\cdot)\cdot b(\cdot)$. If we define multiplication in $\mathcal{A}$ as matrix multiplication, then it is clear that $\mathcal{A}$ is an algebra. It is also easy to verify that $\mathcal{B}$ is an algebra under pointwise multiplication. Since the z transform is linear and because of the property just mentioned, it defines an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. This isomorphism is isometric according to Theorem 5.1. For the sake of completeness, we mention that $\mathcal{B}=L^\infty(U)$ is a complete normed space (Banach space), because of the isometry the same applies to $\mathcal{A}$. More details are given by, for example, Luenberger [14] and Yoshida [15]. The above can be summarized as follows.

Theorem 5.3: $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. The z transform defines an isometric isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Corollary 5.1: Let $A\in\mathcal{A}$. Define the spectrum of $A:\sigma(A)$ by $\{\lambda|(A-\lambda I)$ is not invertable$\}$. We then have, with Theorem 5.1,

$$\sigma(A)=\left\{\lambda|\operatorname*{ess\,sup}_{z\in U}|(a(z)-\lambda)^{-1}|=\infty\right\}\triangleq \text{ess range}\,\{a(\cdot)\}.$$

Again $a(\cdot)$ is the Toeplitz function corresponding to the Toeplitz matrix A (see also Widom [16, p. 186]).

B. The Riccati Equation in the z domain

In Section IV, we found that the gain of the steady-state Kalman filter was the positive root of the algebraic Riccati equation in M:

$$\rho^2M^2\Theta+(1-\rho^2)M(R+\Theta)-(1-\rho^2)R=0 \quad (5.4)$$

(cf. (4.2)). Here the matrices are of dimension $\infty\times\infty$ and the coefficient matrices R and Θ are Toeplitz matrices with defining sequences $\{\cdots,\rho^2,\rho,1,\rho,\rho^2,\cdots\}$ and $\{\cdots,0,0,\theta,0,0,\cdots\}$, respectively. The corresponding Toeplitz function for R is given by

$$r(z)=\frac{(1-\rho^2)}{(1-\rho z^{-1})(1-\rho z)},\qquad z\in U \quad (5.5)$$

and for Θ it is the constant function, equal to θ. These functions are clearly bounded and have bounded inverses for $\theta\neq 0, 0<\rho<1$. It follows that (5.4) is an equation with coefficients in the algebra $\mathcal{A}$ of bounded Toeplitz operators and has a solution in that algebra. Making use of the isomorphism between $\mathcal{A}$ and the algebra $\mathcal{B}$ of bounded functions on U we can write the corresponding equation with coefficients in $\mathcal{B}$; namely,

$$\rho^2\theta m^2(z)+(1-\rho^2)(r(z)+\theta)m(z)-(1-\rho^2)r(z)=0,\qquad z\in U \quad (5.6)$$

which has as a solution

$$m(z)=\frac{-(1-\rho^2)(\theta+r(z))+\sqrt{w(z)}}{2\theta\rho^2} \quad (5.7)$$

where

$$w(z)=(1-\rho^2)^2(\theta+r(z))^2+4(1-\rho^2)\theta\rho^2r(z),\qquad z\in U.$$

We have taken the positive root of the quadratic equation since we know that $M>0$ (see Section IV-A). Hence, the spectrum $\sigma(M)>0$, which implies by Corollary 5.1 that $m(z)>0\ \forall z\in U$.

Remark: It can be verified that for $\theta>0$ and $0<\rho<1$ the function $w(\cdot)$ cannot be written as $w(z)=v^2(z)\ \forall z\in U$, where $v(\cdot)$ is a rational function in z. Hence, the function $m(\cdot)$ is an irrational function in z.

C. Recursive Realization of Toeplitz Operators

In Section V-A, we saw that the action of a Toeplitz operator can be written as a convolution (c.f., (5.3)). On the other hand we know, from linear system theory, that the output sequence of a discrete dynamical system is the convolution of the input sequence with the impulse response of that system. Here these two facts are combined to give a recursive realization of a Toeplitz operator.

Theorem 5.4: Let A be a Toeplitz matrix with defining sequence $\{\alpha_k\}=\{\cdots\alpha_{-2},\alpha_{-1},\alpha_0,\alpha_1,\alpha_2,\cdots\}$ and suppose that the causal dynamical system

$$S^+:\begin{cases} s_{k+1}=F^+s_k+G^+\nu_k\\ \zeta_k=H^+s_k+J^+\nu_k\end{cases} \quad (5.8a)$$

and the anticausal dynamical system

$$S^-:\begin{cases} s_{k-1}=F^-s_k+G^-\nu_k\\ \zeta_k=H^-s_k+J^-\nu_k\end{cases} \quad (5.8b)$$

have impulse responses

$$\{\alpha_k^-\}^+\triangleq\left\{\cdots,0,0,\frac{1}{2}\alpha_0,\alpha_{-1},\alpha_{-2},\cdots\right\} \quad (5.9a)$$

and

$$\{\alpha_k^-\}^-\triangleq\left\{\cdots,\alpha_2,\alpha_1,\frac{1}{2}\alpha_0,0,0,\cdots\right\} \quad (5.9b)$$

respectively. Then the product $A\{\eta_k\}$ is given by $\{\zeta_k\}^++\{\zeta_k\}^-$, where $\{\zeta_k\}^+$ is the output sequence of S^+ when the input sequence is $\{\eta_k\}$ and $\{\zeta_k\}^-$ is the output sequence of S^- when the input sequence is also $\{\eta_k\}$.

Proof: The output of a discrete dynamical system is the convolution of the input sequence with the impulse response, thus

$$\begin{aligned}\{\zeta_k\}^-+\{\zeta_k\}^+&=\{\alpha_k^-\}^-*\{\eta_k\}+\{\alpha_k^-\}^+*\{\eta_k\}\\ &=(\{\alpha_k^-\}^-+\{\alpha_k^-\}^+)*\{\eta_k\}=\{\alpha_k^-\}*\{\eta_k\}=A\{\eta_k\}\end{aligned}$$

where by definition $\alpha_k^-=\alpha_{-k}$.

We will now consider the particular case where $M\tilde{y}$ has to be computed, M being the gain of the steady-state Kalman filter and $\tilde{y}$ the innovations vector.

Since M has a symmetric defining sequence $\{\mu_k\}$, we will have the same matrices F, G, H, J in the system S^+

(c.f., (5.8a)) as in the system S^-. We then need to determine F, G, H, and J such that the impulse response of, say, S^+ is $\{\mu_k\}^+ \triangleq \{\cdots,0,0,\frac{1}{2}\mu_0,\mu_1,\mu_2,\cdots\}$. This problem is solved in realization theory (see, for example, Kalman, Falb, and Arbib [17, sec. 10.6]). Unfortunately the sequence $\{\mu_k\}$ does not have a finite dimensional realization. If it had a finite dimensional realization, this would imply that the z transform of its impulse response and thus also $m(\cdot)$ is rational in z which would contradict the remark made in Section V-B. Consequently, in order to work with a finite dimensional system, some approximation $\tilde{M}\tilde{y}$ has to be realized instead of $M\tilde{y}$. A convenient approximation to make is that of partial realization, i.e., the matrices F, G, H, and J are determined such that the first $2n+1$ elements of the impulse response of the approximating system are $\{\frac{1}{2}\mu_0,\mu_1,\cdots,\mu_{2n}\}$ and the impulse response is not specified further. A relatively simple recipe for determining F (dimension $n\times n$), G, H, and J is given in the next proposition.

Proposition: Suppose the impulse response $\{\mu_k\}^+$ has no finite dimensional realization, then a partial realization of the first $2n+1$ terms $\{\frac{1}{2}\mu_0,\cdots,\mu_{2n}\}$ is found as follows.

a) Form the $(n+1)\times n$ Hankel matrix W:

$$W \triangleq \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & & & \vdots \\ \mu_n & & \cdots & \mu_{2n-1} \\ \mu_{n+1} & & \cdots & \mu_{2n} \end{bmatrix}.$$

b) By LR decomposition find the matrices P and Q satisfying

$$W = PQ$$

where P has the lower triangular form

$$P = \begin{bmatrix} 1 & & & 0 \\ * & \ddots & & \\ \vdots & & \ddots & \\ * & \cdots & * & 1 \\ * & \cdots & & * \end{bmatrix}.$$

c) Form the $n\times n$ submatrices $P_{n,n}$ consisting of the first n rows of P and $P^*_{n,n}$ consisting of the last n rows of P.

d) Let

$$F = P_{n,n}^{-1}P^*_{n,n}$$
$$G = \text{first column of } Q$$
$$H = [1,0,\cdots,0]$$
$$J = \left[\tfrac{1}{2}\mu_0\right].$$

Now the impulse response of the system S^+ (c.f., (5.8a)) is $\{\cdots 0,0,\frac{1}{2}\mu_0,\mu_1,\mu_2,\cdots,\mu_{2n},*,*,\cdots\}$.

For a more detailed treatment see, for example, Rissanen [18]. The dimension of the system n is called the order of the approximation. As a measure of the "goodness of fit" of the approximation we have the following definition.

Definition: Let $\tilde{M}$ be the Toeplitz operator that is realized by the approximating system for M. Then the relative approximation error is defined as

$$\epsilon_M = \frac{\|M-\tilde{M}\|}{\|M\|}$$

where $\|\ \|$ is the l^2 induced operator norm. Using (5.3), ϵ_M can easily be computed from

$$\epsilon_M = \frac{\sup_{z\in U}|m(z)-\tilde{m}(z)|}{\sup_{z\in U}|m(z)|} \tag{5.10}$$

where $\tilde{m}(\cdot)$ is the Toeplitz function corresponding to $\tilde{M}$.

VI. The Hierarchic Recursive Smoother

A. The Algorithm

The steady-state recursive smoother of Section IV-B and the recursive realization of Toeplitz operators described in Section V-C assume $L=\infty$ and $K=\infty$, respectively, i.e., the image is infinitely large horizontally and vertically. For a practical algorithm we have to assume the finite image to be extended with zeros on all four sides.

The algorithm presented here is hierarchic recursive; i.e., the smoother consists of a filter and a reconstructor that are recursive in l. For fixed l there is the recursive (in k) realization of the multiplication of a Toeplitz matrix with a vector.

The initial conditions follow from the assumption of extension by zeros. The global structure of the smoother is given in Fig. 1.

The smoothing is executed in three stages.

1) Filter:

$$\hat{x}_0^+ = 0$$
$$\hat{x}_{l+1}^+ = \rho\hat{x}_l^+ + M(y_{l+1} - \rho\hat{x}_l^+), \qquad l=0,1,\cdots,L-1. \tag{6.1}$$

2) Reconstructor:

$$\hat{x}_{L+1}^- = 0$$
$$\hat{x}_{l-1}^- = \rho\hat{x}_l^- + \rho M(y_l - \hat{x}_l^-), \qquad l=L+1,L,\cdots,2. \tag{6.2}$$

3) Weighter:

$$\hat{x}_l = \left[2I + M(\Theta R^{-1} - I)\right]^{-1}(\hat{x}_l^+ + \hat{x}_l^- - M\hat{x}_l^-), \qquad l=1,2,\cdots,L. \tag{6.3}$$

At each stage we have to perform a multiplication with a Toeplitz matrix. As pointed out in the preceding section this multiplication is a convolution and can be realized by

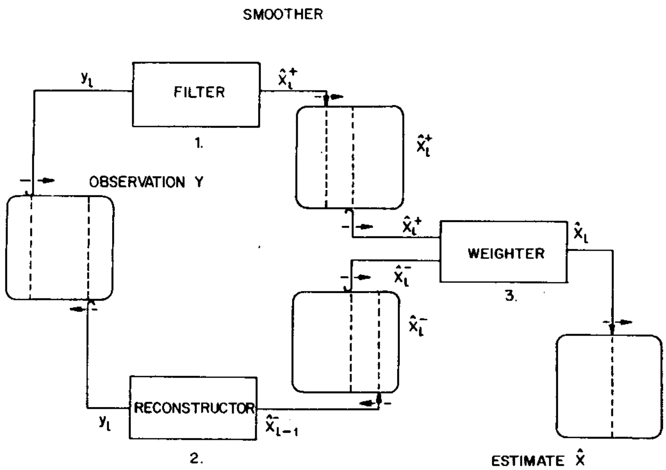

Fig. 1. The global structure of the smoother. The lth column of the estimate of the image $\hat{x}_l$ is a weighted combination of $\hat{x}_l^+$ and $\hat{x}_l^-$, where $\hat{x}_l^+$ is the estimate based on the observations $y_1, y_2, \cdots, y_l$ and $\hat{x}_l^-$ is the estimate based on the observations $y_{l+1}, y_{l+2}, \cdots, y_N$.

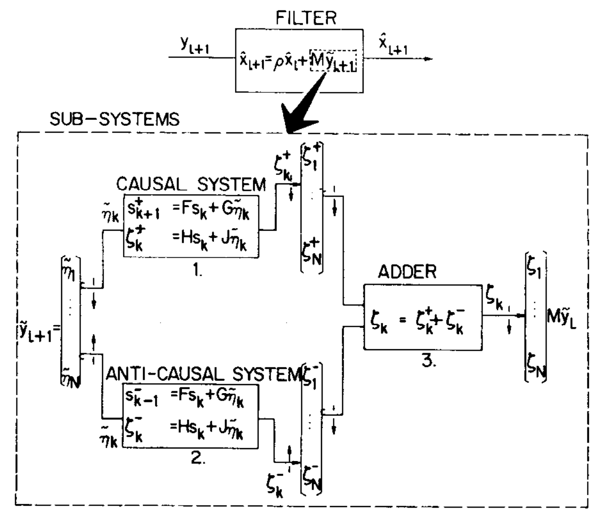

Fig. 2. The filter and its internal subsystems. Inside the filter the multiplication $M\tilde{y}_{l+1}$ is realized by two subsystems that both have as an input sequence the successive elements of the vector $\tilde{y}_{l+1}$ (but in reverse order for the second system). The output sequences form the vectors ζ_{l+1}^+ and ζ_{l+1}^-, respectively, and, by addition, we get $M\tilde{y}_{l+1} = \zeta_{l+1}^+ + \zeta_{l+1}^-$.

two discrete dynamic systems—one being causal and the other, anticausal. These subsystems appear in the filter, reconstructor, and weighter.

We will now take a look inside the filter. The reconstructor and weighter are not treated explicitly, but they can be realized in an analogous manner.

The structure of the filter is given in Fig. 2. Within the filter, for each l there are three substages, recursive in k, to compute $M\tilde{y}_{l+1}$. The result is added to the predicted value of $x_{l+1}: \rho\hat{x}_l$.

1) Causal Subsystem:

$$\begin{aligned} s_0^+ &= 0 \\ s_{k+1}^+ &= Fs_k^+ + G\eta_k, \qquad k=0,1,\cdots,K-1 \\ \zeta_k^+ &= Hs_k + J\eta_k, \qquad k=1,2,\cdots,K. \end{aligned} \tag{6.4}$$

2). Anticausal Subsystem:

$$\begin{aligned} s_{K+1}^- &= 0 \\ s_{k-1}^- &= Fs_k^- + G\eta_k, \qquad k=K+1,K,\cdots,2 \\ \zeta_k^- &= Hs_k + J\eta_k, \qquad k=K,K-1,\cdots,1. \end{aligned} \tag{6.5}$$

3) Adder:

$$\zeta_k = \zeta_k^+ + \zeta_k^-, \qquad k=1,2,\cdots,K. \tag{6.6}$$

The dimension of the subsystem n is chosen such that the relative approximation error ϵ_M is satisfactorily low. (Generally for $n=3$ a very good approximation is already obtained.) The matrices F, G, H, and J can be computed off line according to the proposition in Section V-C. They satisfy the following equations:

$$\begin{aligned} \frac{1}{2}\mu_0 &= J \\ \mu_k &= HF^{k-1}G, \qquad k=1,\cdots,2n \end{aligned} \tag{6.7}$$

where $\{\mu_k\}$ is the defining sequence of the Toeplitz matrix M.

It can be seen in (6.1), (6.2), and (6.3) that we have three places where the product with M needs to be computed. Of course the corresponding F, G, H, and J need to be computed only once. Furthermore, in (6.3) we have the multiplication with the Toeplitz matrix $P=[2I+M(\Theta R^{-1}-I)]^{-1}$. The inverse is computed in the z domain (c.f., Section V-B). The inverse z transform then gives the defining sequence $\{\pi_k\}$ of P. The causal and anticausal parts $\{\pi_k\}^+$ and $\{\pi_k\}^-$, respectively, are then partially realized, in the same way as $\{\mu_k\}^+$ and $\{\mu_k\}^-$.

As a rough estimate of the computational effort we consider the number of multiplications needed for processing one image with the hierarchic recursive smoother. Let the dimension of the image be $N \times N$, and let n be the dimension of the subsystem. Then in each point of the image for each multiplication with a Toeplitz matrix for one subsystem we have $\frac{1}{2}n(n+1)+2$ multiplications (this is the number of elements that differ from 0 or 1 in F, G, H, and J). Since we have a Toeplitz matrix multiplication at four places, this gives the number of multiplications as

$$(4n^2+4n+16)N^2. \tag{6.8}$$

Usually n will be 2 or 3.

We can compare this to smoothing, using a two-dimensional fast Fourier transform (FFT). For one transform we have $2N^2 \log_2 N$ complex multiplications (Cochman *et al.* [19] or Cooley [20]). Next we have the pointwise multiplication of the smoother with the transfer function and then the inverse transformation. If we count one complex multiplication as three real multiplications we have as the number of multiplications

$$(12 \cdot \log_2 N + 1)N^2. \tag{6.9}$$

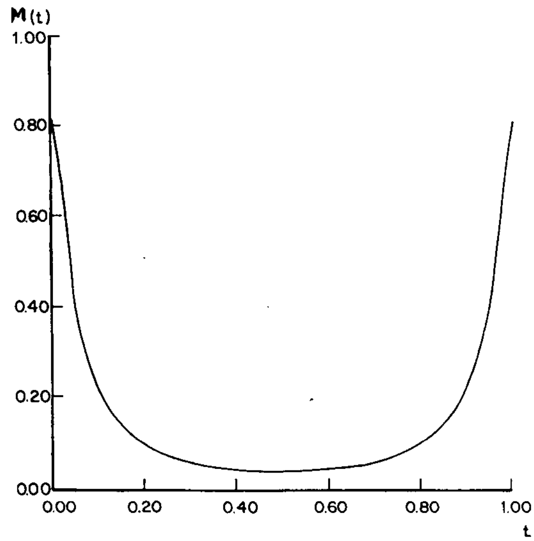

Fig. 3. The solution of the quadratic equation in the z domain for $\rho=0.9$ and $\theta=1$. The values that $m(z)$ takes on the unit circle U are plotted against t with domain $(0,1)$ according to $M(t)=m(e^{2\pi i t})=m(z)$.

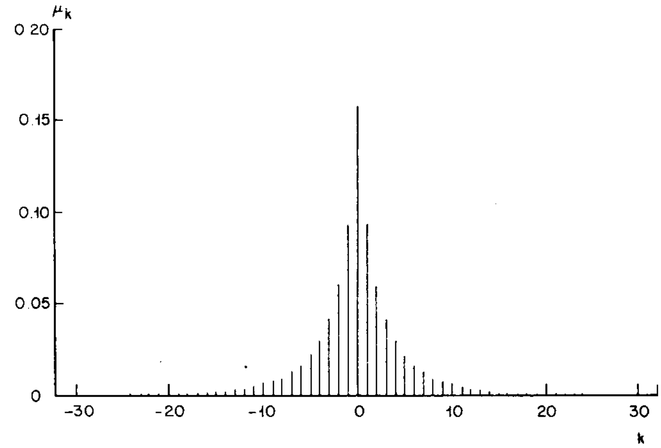

Fig. 4. The defining sequence $\{\mu_k\}$ of the gain matrix M (Toeplitz) of the Kalman–Bucy filter. This is calculated by applying the inverse z transform to $m(\cdot)$. (See Fig. 3.)

For example if $n=2$ and $N=512$ we have $40\ N^2 \cong$ 10 000 000 multiplications for the hierarchic recursive smoother and $109\ N^2 \cong 38$ 000 000 multiplications for the FFT smoother.

Furthermore, the hierarchic recursive smoother works columnwise so that the image can be held in the backing store and one column at a time can be brought to the central memory.

Remark: The above computational considerations are provisional and do not take into account all kinds of optimizations that may be made. However, they do give some idea of the on-line computational requirements.

B. Numerical Example

In this section, some results are presented from the computation of the parameters of the smoother and the impulse response of the smoother is given. This is done for the case $\rho=0.9$ and $\theta=1$, i.e., where the covariance of two points $\xi_{k,l}$ and $\xi_{k+p,l+q}$ is $E\{\xi_{k,l}\xi_{k+p,l+q}\}=0.9^{p+q}$ and the intensity of the noise is $E\{\omega_{k,l}^2\}=1$, which implies a signal-to-noise ratio of one.

The Toeplitz matrix M occurs in the filter (6.1), the reconstructor (6.2), and the weighter (6.3). It has defining sequence $\{\mu_k\}$ and the corresponding Toeplitz function $m(\cdot)$ is the z transform of this sequence. The (real) function values $m(z)$ were computed, according to (5.7) with $r(z)$ given by (5.5), for 256 equidistant points on the unit circle U in the complex plane. They are plotted on the interval $(0,1)$ in Fig. 3. The defining sequence $\{\mu_k\}$ can be obtained very efficiently from $m(z)$ by using an FFT algorithm. Although this only gives a defining sequence of finite length, it is accurate enough for our purposes since we always deal with Toeplitz matrices which have defining sequences that rapidly go to zero. The computed sequence $\{\mu_k\}$ is shown in Fig. 4. Once the defining sequence of the Toeplitz matrix M has been calculated, the dimension n of the subsystem that approximately realizes the product $M\tilde{y}$ has to be chosen. The relative approximation error ϵ_M, for the case $\rho=0.9$, $\theta=1$, according to the definition in Section V-C, is plotted in Fig. 5.

Finally, for the case $n=3$, the parameters F, G, H, and J of the subsystem were computed according to the proposition presented in Section V-C and are given below; the impulse response to be realized $\{\mu_k\}^+$, the actual impulse response $\{\tilde{\mu}_k\}^+$ and their difference are also shown.

$$F=\begin{pmatrix} 0.65345 & 1.00000 & 0.00000 \\ 0.02720 & 0.56095 & 1.00000 \\ -0.00126 & 0.02853 & 0.48770 \end{pmatrix}$$

$$G=\begin{pmatrix} 0.09193 \\ 0.00000 \\ 0.00000 \end{pmatrix}$$

$$H=\begin{pmatrix} 1.00000 & 0.00000 & 0.00000 \end{pmatrix}$$

$$J=\begin{pmatrix} 0.07754 \end{pmatrix},$$

k	μ_k	$\tilde{\mu}_k$	$\mu_k-\tilde{\mu}_k$
0	0.07754	0.07754	-
1	0.09193	0.09193	-
2	0.06007	0.06007	-
3	0.04175	0.04175	-
4	0.03020	0.03020	-
5	0.02245	0.02245	-
6	0.01701	0.01701	-
7	0.01308	0.01308	0.00000
8	0.01017	0.01017	0.00000
9	0.00798	0.00798	0.00000
10	0.00631	0.00630	0.00001
11	0.00502	0.00499	0.00003
12	0.00402	0.00398	0.00004
13	0.00323	0.00317	0.00006
14	0.00260	0.00254	0.00006
15	0.00211	0.00203	0.00008
16	0.00171	0.00163	0.00008
17	0.00139	0.00131	0.00008
18	0.00114	0.00105	0.00009
19	0.00093	0.00085	0.00008
20	0.00076	0.00068	0.00008

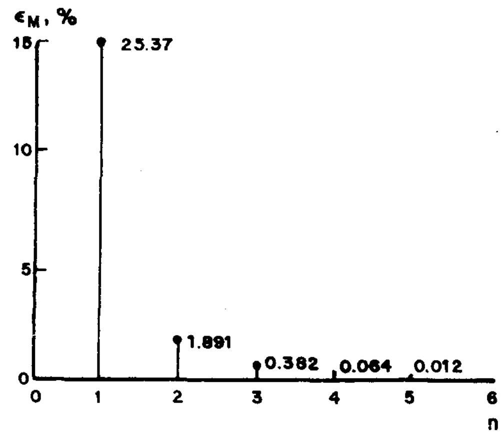

Fig. 5. The relative approximation error ϵ_M resulting from an n-dimensional realization of the subsystem instead of a ∞-dimensional subsystem (for $\rho=0.9$ and $\theta=1$).

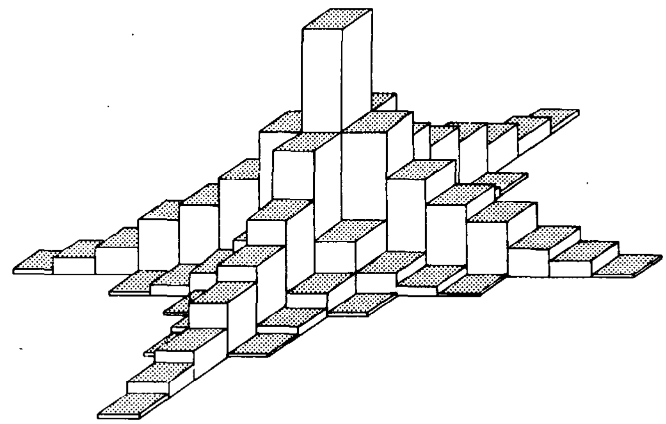

Fig. 6. The impulse response of the smoother. The height of the bar indicates the gray level at that point of the grid. The figure shows the result when an "image" with one black point in the center of a white field is smoothed with $\rho=0.9$ and $\theta=1$. The intensity at the center is reduced to 12 percent.

The impulse response of the smoother is the sequence $\{\gamma_{k,l}\}$ that is obtained by smoothing an "image" which has an element 1 in position (0,0) and all further elements zero, i.e., the observation $\{\eta_{k,l}\}=\delta_k\cdot\delta_l$. This is done numerically for the case $\rho=0.9$, $\theta=1$ and the result is shown in Fig. 6. The order of the subsystem was $n=3$ and the impulse response is symmetrical as expected.

The transfer function of the smoother is a function in z_1 and z_2 defined by

$$G(z_1,z_2)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\gamma_{k,l}z^{-l}z^{-k}.$$

In our case this becomes

$$G(z_1,z_2)=\frac{m(z_2)z_1}{2+m(z_2)(\theta r^{-1}(z_2)-1)}\left\{\frac{\rho(1-m(z_2))}{1-\rho(1-m(z_2))z_1}+\frac{1}{z_1-\rho(1-m(z_2))}\right\} \tag{6.10}$$

where $m(\cdot)$ is given by (5.7) and $r(\cdot)$ by (5.5).

Using methods analogous to Davenport and Root [21], we can prove that the transfer function of an infinite-lag smoother is given by

$$G(z_1,z_2)=\frac{\text{signal}}{\text{signal}+\text{noise}}.$$

In our case,

$$G(z_1,z_2)=\frac{r(z_1)\cdot r(z_2)}{r(z_1)\cdot r(z_2)+\theta} \tag{6.11}$$

with $r(\cdot)$ given by (5.5).

Expressing $r(z_2)$ in terms of $m(z_2)$ with (5.7) and $r(z_1)$ in terms of ρ and z_1 with (5.5), one may verify that (6.10) is indeed equal to (6.11).

VII. Applications

The images are processed with a multiplicative system, as recommended in [22]. The densities (the logarithm of the light intensities) are used directly as input for the smoother. The conversion to intensities is done after the processing. Using a CDC 6600 computer, the time required to process a 128×128 image (with $n=3$) was 40 s central processor time.

The variance of the noise, θ was estimated from a separate observation with a known signal. The factor ρ was estimated by taking an average of

$$\rho_p=\left[\frac{\overline{\eta_{k,l}\cdot\eta_{k+p,l}}}{\overline{\eta_{k,l}^2}-\theta}\right]^{1/p},\qquad \text{for } p=1,\cdots,6 \tag{7.1}$$

where $\eta_{k,l}$ is the observation scaled following (2.1) and the bar denotes the average. The signal-to-noise ratio SNR is defined as

$$\text{SNR}=\frac{\text{variance of signal}}{\text{variance of noise}}. \tag{7.2}$$

Fig. 7. Original photograph.

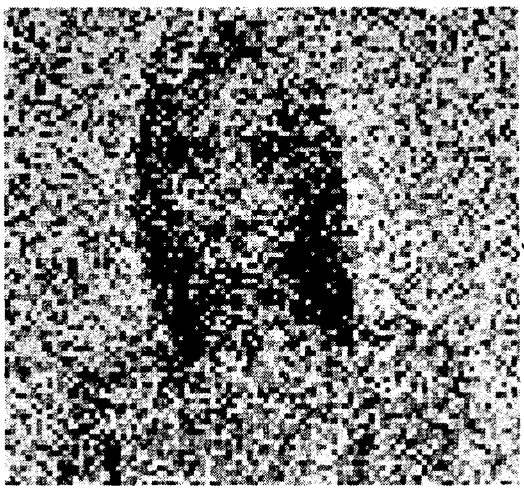

Fig. 8. Photograph with artificial noise (SNR is 1/2).

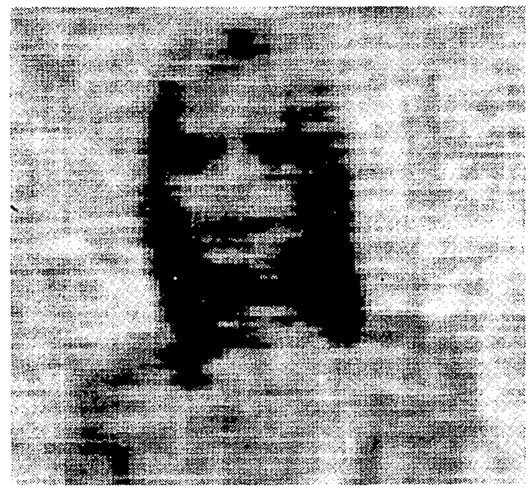

Fig. 9. Filter estimate of the hierarchic recursive smoother.

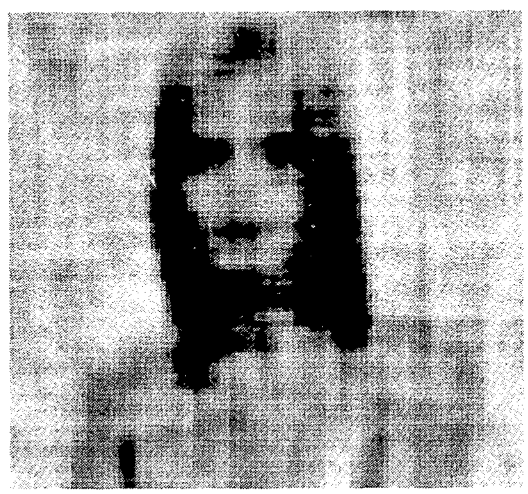

Fig. 10. Smoother estimate of the hierarchic recursive smoother.

In this case, $\mathrm{SNR}=\theta^{-1}$ since the variance of the signal is scaled to one.

To the original picture (Fig. 7 in which $\rho=0.90$), artificial noise was added (giving Fig. 8) making $\mathrm{SNR}=0.5$, which is a bad ratio in practice. It can be seen that the filtered image (Fig. 9) is already very close to the final estimate of the smoother (Fig. 10). This is because the filter uses the data from the line it is estimating.

VIII. Concluding Remarks

An efficient hierarchic recursive algorithm has been developed to enhance a large class of images which are degraded by additive noise.

Only the most simple case, where the covariance of the picture brightness is exponentially decreasing in the horizontal plus vertical distance and where the noise is white, is treated explicitly. However, the method can easily be extended to cover any covariance structure in the signal and the noise, which can be generated by a linear vector model with Toeplitz matrices. In particular, the extension to the case where the covariance of the picture brightness is of the form

$$\boldsymbol{E}\{\beta_{k,l}\beta_{k+p,l+1}\}=\sigma^2\cdot\phi_p\cdot\rho^{|q|}$$

with $0<\rho<1$ and

$$\lim_{p\to\infty}\phi_p=\lim_{p\to-\infty}\phi_p=0$$

is very simple and requires only the replacement of the function $r(\cdot)$ by the z transform of the sequence $\{\phi_p\}$.

The z domain technique of Section V can also be used to find the successive elements of the sequence $\{M_l\}$, the gain of the non-steady-state Kalman filter.

The algorithm may possibly be extended to multivariable filtering where every element $\beta_{k,l}$ is a vector (e.g., color pictures). The Toeplitz matrices will become block-Toeplitz matrices.

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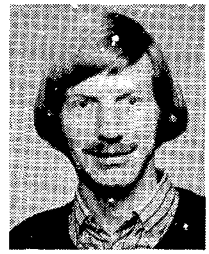

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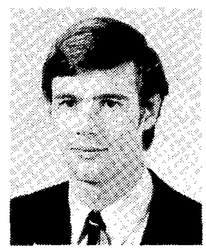

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