Realization of Systems with Internal Passivity and Symmetry Constraints

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ABSTRACT: *This expository paper describes the main results and applications of the theory of realization of linear stationary dynamical systems in the case that there are certain internal constraints on the parameters of the state space realization. The internal constraints considered here are those derived from passivity and symmetry requirements. Applications to stability theory, electrical network synthesis, and other areas are outlined.*

I. Introduction

The purpose of the present paper is to give an expository survey of the main concepts, results and applications related to the problem of obtaining certain special realizations of linear stationary dynamical systems. These realizations are constrained in the sense that the internal parameters of the state space representation are required to reflect some of the qualitative properties of the input/output system which they represent. The properties which will be considered here are those of passivity and symmetry. These questions thus lead to the problem of finding state space representations with internal constraints which adds an interesting theoretical twist to the usual realization problem. The results have some interesting applications. The main ones may be found in the areas of feedback stability and electrical network synthesis. However, there are also some less expected areas, particularly in physics, where these results are of relevance. Some indication of such applications will be given here. Since questions of representation have always been of very much importance in physics we believe that with the increased understanding of the representation of dynamical phenomena as exemplified by abstract realization theory there is a possibility that some progress could be made toward a more global conceptualization and solution of some such problems. However, this will almost certainly require a non-trivial extension of some of the results to nonlinear and/or stochastic systems. These applications constitute, in our opinion, one of the more promising and challenging areas where realization theory ideas could be applied and extended.

Since this paper is expository in nature it draws on a historical development to which many authors have contributed and there seems to be little point in trying here to do justice to all. The main fiber in this development started with the work in the area of feedback stability (**1–3**) which used in an essential way the so-called Kalman–Yacubovich–Popov lemma (sometimes called the

Positive Real lemma). This work was further developed by Anderson (**4**), Brockett and Lee (**5**) and many others. The applicability of these ideas to network synthesis was soon realized by Kalman (**6**) and further developed by Youla and Tissi (**7**), who also brought in the question of internally symmetric realization. However, the question of existence of realizations which are simultaneously internally passive and symmetric was left open until it was resolved first by Vongpanitlerd and Anderson (**8**) and later, independently, by the author (**9**).

We use the following notation: $R(R^+)$ denotes the (nonnegative) real numbers, R^n denotes n-dimensional Euclidean space, $R^{m\times p}$ denotes the real $(m\times p)$ matrices, $^{\mathrm{T}}$ denotes transposition, a dot or a superscript in parentheses denotes differentiation and $\geqslant 0\;(>0)$ means that a symmetric matrix is nonnegative (positive) definite. Other notation will be introduced as it is needed.

This paper is concerned with stationary linear dynamical systems with a finite number of inputs, outputs and internal degrees of freedom (i.e. states). From an input–output point of view the response of such systems may be described by

$$\Sigma_{I/O}\colon y(t) = W_0 u(t) + \int_{-\infty}^{t} W(t-\tau)\,u(\tau)\,\mathrm{d}\tau,$$

where $u(\cdot)\colon R\to R^m$ denotes the *input*, $y(\cdot)\colon R\to R^p$ denotes the *output*,

$$W_0\,\delta(\cdot)+W(\cdot)\colon R^+\to R^{p\times m}$$

denotes the *impulse response* ($\delta(\cdot)$ denotes the Dirac delta function). We will assume throughout that $u(\cdot)$ (and thus $y(\cdot)$) is a locally square integrable function with bounded support on the left. Corresponding to the assumption that $\Sigma_{I/O}$ has a finite number of internal degrees of freedom we will also assume that $W(t)$ is a *Bohl function*, i.e. that every entry is a finite sum of products of a polynomial, an exponential and a sine or a cosine. The system $\Sigma_{I/O}$ may equivalently be described by its *transfer function* $G(s)\triangleq W_0+(\mathscr{L}W(\cdot))$ ($\mathscr{L}$ denotes Laplace transform) or by its *Hankel matrix*, $H\triangleq[W^{(i+j-2)}(0)]$ $(i,j=1,2,\ldots)$, together with the *feedthrough* component W_0. It is well known that the system $\Sigma_{I/O}$ admits the internal description:

$$\Sigma\colon \dot{x} = Ax+Bu;\quad y = Cx+Du,$$

where $x\in R^n$ denotes the *state* of the system and the matrices $\{A,B,C,D\}$ define the state space parameter matrices of Σ. We call Σ a *state space realization* of $\Sigma_{I/O}$ provided Σ induces the same input–output map as $\Sigma_{I/O}$, i.e. provided $D=W_0$ and $C\,\mathrm{e}^{At}B = W(t)$ for $t\geqslant 0$, or, equivalently, provided $D+C(Is-A)^{-1}B = G(s)$. It is well known that there exist many realizations Σ of a given $\Sigma_{I/O}$. Those which have the additional property that the corresponding vector space dimension of the state space is as small as possible (this corresponds to $n=\operatorname{rank} H$) are called *minimal*. We make frequent use of the following basic result, called the state space isomorphism theorem (**10**): All minimal realizations Σ of $\Sigma_{I/O}$ may be recovered from one minimal realization with parameter matrices $\{A, B, C, D\}$ by the action of the transformation group $\{A, B, C, D\} \xrightarrow[|S| \neq 0]{S} \{SAS^{-1}, SB, CB^{-1}, D\}$ with S an arbitrary invertible element of $R^{n \times n}$ ($|\cdot|$ denotes determinant). Moreover, the matrix S which thus associates one minimal realization to another is unique.

II. Passivity

In this section we consider the concept of passivity. Intuitively speaking a system is called passive (from an external point of view) if at all times the net flow of energy is into the system. Consistent with the input–output point of view, we will assume that the instantaneous power is a function of the instantaneous input and output and, consistent with the linearity, we will assume this function to be a quadratic one. Although it would take us very little additional effort to consider a general quadratic form in u and y we will take the instantaneous power to be given by $u^T y$ (thus $m = p$) since this is the most frequent case encountered in applications. We thus arrive at the following definition:

Definition 1. $\Sigma_{I/O}$ is said to be *passive* if $\int_{-\infty}^{t} u^T(\tau)\, y(\tau)\, d\tau \geqslant 0$ for all inputs $u(\cdot)$ and $t \geqslant 0$.

The passivity of $\Sigma_{I/O}$ may be expressed in terms of its transfer function. This leads to the following class of complex valued functions:

Definition 2. Let $F(s)$ be a rational $(m \times m)$ matrix valued function of the complex variable s. Then $F(s)$ is said to be *positive real* if $F(s)$ is real for s real and if $F(\sigma + j\omega) + F^T(\sigma - j\omega) \geqslant 0$ (for all $\sigma \in R^+$, $\omega \in R$, $\sigma + j\omega \neq$ singularities of F).

Positive real functions are extensively studied in classical electrical network synthesis. Various other characterizations and a proof of the following proposition may be found in (**11**).

Proposition 1

$\Sigma_{I/O}$ is passive if and only if $G(s)$ is positive real.

It is also possible to give a time-domain condition for passivity. This leads to the following proposition which is, essentially, the content of the Kalman–Yacubovich–Popov lemma:

Proposition 2

Let Σ be a minimal realization of $\Sigma_{I/O}$. Then $\Sigma_{I/O}$ is passive if and only if there exists a solution $Q = Q^T > 0$ to the matrix inequality

$$\left[\begin{array}{c|c} A^T Q + QA & QB - C^T \\ \hline B^T Q - C & -D - D^T \end{array}\right] \leqslant 0. \tag{LMI}$$

Moreover, the set of solutions Q to this inequality is convex, compact and attains its maximum and its minimum (in the positive definite sense).

Proof: (i) Let $Q = Q^{\mathrm{T}} > 0$ be a solution of (LMI). Then $(\mathrm{d}/\mathrm{d}t)\, \frac{1}{2}x^{\mathrm{T}} Qx \leqslant u^{\mathrm{T}} y$ along solutions of Σ. Thus $\frac{1}{2}x^{\mathrm{T}}(t_0)Qx(t_0) + \int_{t_0}^{t_1} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \geqslant \frac{1}{2}x^{\mathrm{T}}(t_1)Qx(t_1)$ for all $t_1 \geqslant t_0$ and thus $\int_{-\infty}^{t} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \geqslant \frac{1}{2}x^{\mathrm{T}}(t)\, Qx(t) \geqslant 0$, which shows passivity as claimed.

(ii) Conversely, assume that $\Sigma_{I/O}$ is passive. Let

$$S_a(x_0) \triangleq \sup_{u(\cdot);\, T>0} - \int_0^T u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \quad \text{and} \quad S_r(x_0) \triangleq \inf_{u(\cdot);\, T \geqslant 0} \int_{-\mathrm{T}}^{0} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau,$$

both computed for solutions of Σ with $x(-T) = 0$ and $x(0) = x_0$. Now, S_a and S_r exist (by reachability and the fact that

$$-\int_0^T u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \geqslant \int_{-\infty}^{0} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau),$$

are quadratic functions of x_0 (since we are taking the supremum and the infimum of quadratic functionals subject to linear constraints) and positive whenever $x_0 \neq 0$ (since $S_r \geqslant S_a$ and since the feedback control law $u(\cdot) = -ky(\cdot)$, i.e. $u(\cdot) = -(I + kD)^{-1} kx(\cdot)$, with $k > 0$ such that $I + kD$ is invertible, shows, by observability, that $S_a(x_0) \neq 0$ for $x_0 \neq 0$). It remains to be shown that Q^+ and Q^-, defined by $S_a(x_0) = \frac{1}{2}x_0^{\mathrm{T}} Q^- x_0$ and $S_r(x_0) = \frac{1}{2}x_0^{\mathrm{T}} Q^+ x_0$, satisfy also (LMI). This matrix inequality is equivalent to $\mathrm{d}/\mathrm{d}t\, \frac{1}{2}x^{\mathrm{T}} Qx \leqslant u^{\mathrm{T}} y$ or $\frac{1}{2}x^{\mathrm{T}}(t_0)Qx(t_0) + \int_{t_0}^{t_1} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \geqslant \frac{1}{2}x^{\mathrm{T}}(t_1)Qx(t_1)$ for all $t_1 \geqslant t_0$. However, this last inequality is obvious when applied with $Q = Q^-$ or Q^+ since it simply states the sub-optimality of an arbitrary input on the internal $[t_0, t_1]$.

(iii) That the solution set Q is convex and closed is obvious and that it satisfies $Q^- \leqslant Q \leqslant Q^+$ follows immediately from using the definition of S_a and S_r on the inequality $\frac{1}{2}x^{\mathrm{T}}(t_0)Qx(t_0) + \int_{t_0}^{t_1} u^{\mathrm{T}}(\tau)\, y(\tau)\, \mathrm{d}\tau \geqslant \frac{1}{2}x^{\mathrm{T}}(t_1)Qx(t_1)$. □

It is now easy to turn Proposition 2 into a claim about a state space realization of $\Sigma_{I/O}$. We therefore introduce the following definition:

Definition 3. Σ is said to be *internally passive* if

$$\left[\begin{array}{c|c} -A & -B \\ \hline C & D \end{array}\right] + \left[\begin{array}{c|c} -A & -B \\ \hline C & D \end{array}\right]^{\mathrm{T}} \geqslant 0$$

(i.e. if $\mathrm{d}/\mathrm{dt}\, \frac{1}{2}x^{\mathrm{T}} x \leqslant u^{\mathrm{T}} y$).

We thus arrive at the following result:

Theorem I

The following conditions are equivalent:

(i) $\Sigma_{I/O}$ is passive;
(ii) $G(s)$ is positive real;
(iii) for any minimal realization Σ of $\Sigma_{I/O}$ there exists a $Q = Q^{\mathrm{T}} > 0$ such that (LMI) is satisfied; and
(iv) $\Sigma_{I/O}$ admits an internally passive (minimal) realization.

Proof: (i), (ii) and (iii) have been proven in Propositions 1 and 2. To prove (iv), assume that Q satisfies (LMI). Factoring Q into $S^{\mathrm{T}} S$ with $|S| \neq 0$ and transforming the minimal realization $\{A,B,C,D\}$ into $\{SAS^{-1},SB,CS^{-1},D\}$ leads to the desired internally passive realization. That (iv) in turn implies (iii) is obvious. □

Remarks. 1. As could be expected from the definition of Q^{+} and Q^{-}, the matrix inequality (LMI) has close connections with the Riccati equation (see (9) for details).

2. All passive realizations may be obtained by transforming a given minimal realization $\{A,B,C,D\}$ into $\{SAS^{-1},SB,CS^{-1},D\}$ with $S^{\mathrm{T}}S$ a solution of (LMI).

III. Symmetry

In this section we discuss another qualitative assumption of systems. This assumption is the one of symmetry (or, as it is often called, reciprocity). We believe that this condition is at this point in time only very vaguely understood but that it is one of the deeper and more important properties of large classes of physical systems. It seems to be tied up with variational principles, time reversal and the existence of potential functions. We will discuss here these concepts in the context of stationary linear dynamical systems but we would like to emphasize that it seems far from clear how one should generalize these ideas to nonlinear and, if it should prove applicable, to time-varying systems.

Definition 4. Let Σ_e be a *signature matrix* (i.e. a diagonal matrix with diagonal entries $+1$ or -1). Then $\Sigma_{I/O}$ is said to be (externally) *symmetric* with (external) signature Σ_e if the inputs with respective components $\tilde{u}_k(t) = \delta_{ki}\, v(t)$ and $\tilde{\tilde{u}}_k(t) = \delta_{kj}\, v(t)$ (δ denotes the Kronecker delta) yields the following relationship among the components of the corresponding components $\tilde{y}(t)$ and $\tilde{\tilde{y}}(t)$: $\sigma_j \tilde{y}_j(t) = \sigma_i \tilde{\tilde{y}}_i(t)$ (where σ_k denotes the kth element on the diagonal of Σ_e).

Various equivalent statements of Definition 4 may be found in (**11**). An interesting interpretation of reciprocity in terms of time reversal is discussed in (**12**).

Proposition 3

$\Sigma_{I/O}$ is symmetric with signature Σ_e if and only if $\Sigma_e G(s) = G^{\mathrm{T}}(s)\,\Sigma_e$.

The proof of Proposition 3, which is straightforward, is deleted. One can also give a time-domain condition for symmetry:

Proposition 4

Let Σ be a minimal realization of $\Sigma_{I/O}$. Then $\Sigma_{I/O}$ is symmetric with signature Σ_e if and only if there exists a nonsingular symmetric matrix T such that $A = T^{-1}A^{\mathrm{T}}T$, $B = T^{-1}C^{\mathrm{T}}\Sigma_e$, and $D = \Sigma_e D^{\mathrm{T}}\Sigma_e$. Moreover, T is unique.

Proof: The "if" part follows from a simple computation. To show the "only if" part assume thus the symmetry. Then by Proposition 3, $G(s) = \Sigma_e G^{\mathrm{T}}(s) \Sigma_e$ which shows that if $\{A, B, C, D\}$ is a minimal realization, then so is $\{A^{\mathrm{T}}, C^{\mathrm{T}} \Sigma_e, \Sigma_e B^{\mathrm{T}}, D^{\mathrm{T}}\}$. Hence there exists a *unique* nonsingular matrix S such that $A^{\mathrm{T}} = SAS^{-1}$, $C^{\mathrm{T}} \Sigma_e = SB$, $\Sigma_e B^{\mathrm{T}} = CS^{-1}$, $D = \Sigma_e D^{\mathrm{T}} \Sigma_e$. It is easily verified that S^{T} also satisfies these relations. Uniqueness thus yields $S = S^{\mathrm{T}}$ and the result follows with $T = S$. That T is unique follows from uniqueness of S. □

Next, as was done with passivity, we turn external symmetry into a representation question. This leads to the following definition:

Definition 5. Σ is said to be internally *symmetric* with *internal* signature Σ_i and *external* signature Σ_e if the matrix $\left[\begin{array}{c|c} \Sigma_i & 0 \\ \hline 0 & \Sigma_e \end{array}\right] \left[\begin{array}{c|c} -A & -B \\ \hline C & D \end{array}\right]$ is symmetric.

This leads to the following result:

Theorem II

The following conditions are equivalent:

(i) $\Sigma_{I/O}$ is symmetric with external signature Σ_e;

(ii) $\Sigma_e G(s) = G^{\mathrm{T}}(s) \Sigma_e$;

(iii) for any minimal realization Σ of $\Sigma_{I/O}$ there exists a (unique) nonsingular symmetric matrix T such that $A = T^{-1} A^{\mathrm{T}} T$, $B = T^{-1} C^{\mathrm{T}} \Sigma_e$ and $D = \Sigma_e D^{\mathrm{T}} \Sigma_e$; and

(iv) $\Sigma_{I/O}$ admits an internally symmetric (minimal) realization with external signature Σ_e.

Proof: (i), (ii) and (iii) are proven in Propositions 3 and 4. To prove (iv) it suffices to factor T into $-S^{\mathrm{T}} \Sigma_i S$ and, again, apply the transformation $\{A, B, C, D\} \to \{SAS^{-1}, SB, CB^{-1}, D\}$ to obtain an internally symmetric representation. That (iv) implies (iii) is obvious. □

Remarks. 3. All minimal internally symmetric realizations of $\Sigma_{I/O}$ may thus be obtained from one minimal realization $\{A, B, C, D\}$ by factoring the matrix T of Proposition 4 into $T = -S^{\mathrm{T}} \Sigma_i S$ and applying the basis transformation induced by S. Thus, in particular, the internal signature Σ_i is an invariant modulo permutation of its diagonal elements.

4. The quantity $\frac{1}{2} x^{\mathrm{T}} \Sigma_i x$ associated with an internally symmetric realization of $\Sigma_{I/O}$ is called the *Lagrangian*. It is an invariant of the representation in the sense that it depends on the input but not on the actual realization (provided, of course, that it is internally symmetric). In terms of a (not necessarily internally symmetric) realization the Lagrangian equals $\frac{1}{2} x^{\mathrm{T}} T x$, with T as in Proposition 4.

IV. Simultaneous Passivity and Symmetry

In this section we solve a problem which was until recently an open problem. As we have seen in Theorem I, a system which is externally passive always admits an internally passive realization; similarly, as we have seen in Theorem II, a system which is externally symmetric always admits an internally symmetric realization. An important question which, as demonstrated in Section VI, has many applications is whether a system which is simultaneously externally symmetric and passive always admits a realization which is internally also simultaneously symmetric and passive. The following lemma reduces this question to one about the solutions of (LMI).

Lemma 1

Let $Q = Q^T > 0$ and $T = T^T$ with $|T| \neq 0$. Then there exists a nonsingular matrix S such that $Q = S^T S$ and $T = -S^T \Sigma_i S$ with Σ_i a signature matrix if and only if $Q = TQ^{-1}T$.

Proof: The "only if" part is obvious. The "if" is proven in (**9**), p. 372. □

We now prove the following proposition which shows the existence of a matrix Q as required in Lemma 1:

Proposition 5

Assume that $\Sigma_{I/O}$ is passive and symmetric with signature Σ_e, and let Σ be a minimal realization of $\Sigma_{I/O}$. Let T be as in Proposition 4. Then there exists a solution $Q = Q^T > 0$ of (LMI) such that $Q = TQ^{-1}T$.

Proof: Assume that Q satisfies (LMI). Then since $A = T^{-1}A^T T$ and $B = T^{-1}C^T \Sigma_e$ it follows that

$$\left[\begin{array}{c|c} TAT^{-1}Q + QT^{-1}A^T T & QT^{-1}C^T\Sigma_e - TB\Sigma_e \\ \hline \Sigma_e CT^{-1}Q - \Sigma_e B^T T & -D - D^T \end{array}\right] \leqslant 0.$$

Post-multiplication of this matrix by $\left[\begin{array}{c|c} Q^{-1}T & 0 \\ \hline 0 & -\Sigma_e \end{array}\right]$ and pre-multiplication by $\left[\begin{array}{c|c} TQ^{-1} & 0 \\ \hline 0 & -\Sigma_e \end{array}\right]$ shows that $TQ^{-1}T$ also satisfies (LMI). Now the mapping $Q \to TQ^{-1}T$ is continuous and well-defined on the solution set of (LMI) since there $Q \geqslant Q^- > 0$. Since this set is, as shown in Proposition 2, convex and compact it follows from Brouwer's fixed point theorem that this mapping has a fixed point, i.e. there exists, as claimed, a solution $Q = Q^T > 0$ of (LMI) such that $Q = TQ^{-1}T$. □

We thus arrive at the following result:

Theorem III

The following conditions are equivalent:

(i) $\Sigma_{I/O}$ is passive and symmetric with external signature Σ_e;

(ii) $G(s)$ is positive real and $\Sigma_e G(s) = G^{\mathrm{T}}(s)\Sigma_e$;
(iii) for any minimal realization Σ of $\Sigma_{I/O}$ there exists a (unique) non-singular symmetric matrix T such that $A = T^{-1}A^{\mathrm{T}}T$, $B = T^{-1}C^{\mathrm{T}}\Sigma_e$ and $D = \Sigma_e D^{\mathrm{T}}\Sigma_e$, and a matrix $Q = Q^{\mathrm{T}} > 0$ which satisfies (LMI) and $Q = TQ^{-1}T$; and
(iv) $\Sigma_{I/O}$ admits a (minimal) realization which is internally passive and symmetric and which has external signature Σ_e.

Proof: (i), (ii) and (iii) have been proven in Theorems I, II and Proposition 5. To prove (iv) use the Q of (iii) in Lemma 1 to obtain a S which through the usual transformation $\{A,B,C,D\}\xrightarrow[|S|\neq 0]{S}\{SAS^{-1}, SB, CB^{-1}, D\}$ will yield the desired realization. That (iv) implies (iii) is obvious. □

Remark. 5. The proof of Proposition 5 has the disadvantage of not being constructive. However, it is also possible to give formulas for the desired matrix Q (**8**, **9**).

V. Relaxation Systems, Lossless Systems, etc.

In this section we will discuss briefly some other classes of systems. The first one we have called relaxation systems. These correspond to physical systems which have only one "type" of energy storage possibility, e.g. only potential energy or only kinetic energy, but not both, or only electric energy or only magnetic energy, but not both. Typical examples of relaxation systems are thus RC or RL electrical networks, viscoelastic materials (inertia is usually irrelevant for the description of the behaviour of such systems) and chemical reactions. The characteristic feature of such systems is that their response function indicates the complete absence of oscillatory tendencies. This is expressed in the following definition:

Definition 6. $\Sigma_{I/O}$ is said to be a *relaxation* system if its impulse response satisfies:

$$W_0 = W_0^{\mathrm{T}} \geqslant 0 \quad \text{and} \quad (-1)^n \frac{\mathrm{d}^n W(t)}{\mathrm{d}t^n} \geqslant 0 \quad \text{for } t \geqslant 0 \quad \text{and} \quad n = 0,1,2,\ldots$$

(i.e. if the impulse response is a *completely monotonic* function).

The corresponding internal characterization is defined as follows:

Definition 7. Σ is said to be *internally of the relaxation type* if

$$A = A^{\mathrm{T}} \leqslant 0, \quad B = C^{\mathrm{T}} \quad \text{and} \quad D = D^{\mathrm{T}}.$$

Various equivalent characterizations of relaxation systems are given in the following theorem. Its proof, which would take us too far, may be found in (**9**) or (**13**).

Theorem IV

The following conditions are equivalent:

(i) $\Sigma_{I/O}$ is a relaxation system;

(ii) $G(s)$ may be written as:

$$G(s) = G_\infty + \sum_{k=0}^{N} \frac{G_k}{s+\lambda_k}$$

with $G_\infty = G_\infty^T \geqslant 0$, $G_k = G_k^T \geqslant 0$ and $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_N$;

(iii) $W_0 = W_0^T \geqslant 0$ and its Hankel matrix satisfies $H = H^T \geqslant 0$ and $\sigma H = \sigma H^T \leqslant 0$ (σH denotes the shifted Hankel matrix, i.e. the Hankel matrix H with the first block row and column deleted or, equivalently, the Hankel matrix associated with $\dot{W}(t)$);

(iv) for any minimal realization Σ of $\Sigma_{I/0}$ there exists a matrix $T = T^T > 0$ such that $A = T^{-1} A^T T$, $B = TC^T$ and $D = D^T$;

(v) $\Sigma_{I/O}$ admits a (minimal) realization which is internally of the relaxation type;

(vi) $\Sigma_{I/O}$ admits an internally symmetric (minimal) realization with external signature I and internal signature $-I$ (or vice versa).

Note that (as follows immediately from condition (v) and Theorem I(iii)) relaxation systems are automatically passive. Another special class of passive systems which we will consider now are the lossless systems. These are systems which are passive but for which the energy which has been supplied to them is always recoverable:

Definition 8. $\Sigma_{I/O}$ is said to be *lossless* if it is passive and if to every input u_1 there corresponds and input u_2 such that

$$\int_{-\infty}^{+\infty} u^T(\tau)\, y(\tau)\, d\tau = 0,$$

where

$$u(t) = \begin{cases} u_1(t) & \text{for } t \leqslant 0, \\ u_2(t) & \text{for } t > 0 \end{cases}$$

and $y(\cdot)$ is the output corresponding to this input $u(\cdot)$.

The corresponding internal characterization is defined as follows:

Definition 9. Σ is said to be *internally lossless* if

$$A + A^T = 0, \quad B = C^T \quad \text{and} \quad D + D^T = 0.$$

The following theorem gives some equivalent conditions for losslessness. For a proof, we refer the reader again to (**9**) or (**13**).

Theorem V.

The following conditions are equivalent:

(i) $\Sigma_{I/O}$ is lossless;

(ii) $G(s)$ is positive real with $G(j\omega) + G^T(-j\omega) = 0$ for all ω real, $\omega \neq$ singularities of G;

(iii) $G(s)$ may be written as

$$G(s) = G_\infty + \sum_{k=0}^{N} \frac{G_k s}{s^2+\omega_k^2}$$

with $G_\infty + G_\infty^T = 0$, $G_k = G_k^T \geqslant 0$ and $0 = \omega_0 < \omega_1 < \ldots < \omega_N$;

(iv) for any minimal realization Σ of $\Sigma_{I/O}$ there exists a $Q = Q^T > 0$ such that $A^T Q + QA = 0$, $QB = C^T$ and $D + D^T = 0$;

(v) $\Sigma_{I/O}$ admits an internally lossless (minimal) realization.

There are various meaningful other such classes of systems. Most of them will have applications particularly in electrical network analysis and synthesis or in stability theory. We simply list here a few which are extensions, specializations and recombinations of the classes of systems considered earlier.

We present here only the most illustrative characterization:

(1) *Pseudo-passive systems* which are systems for which $G(j\omega) + G^T(-j\omega) \geqslant 0$ for all $\omega \in R$ or equivalently for which there exists any symmetric matrix Q such that (LMI) is satisfied.

(2) *Pseudo-lossless systems* which are systems for which

$$G(j\omega) + G^T(-j\omega) = 0 \quad \text{for all } \omega \in R.$$

(3) *Completely symmetric systems* which are systems which admit a realization for which $A = A^T$, $B = C^T$ and $D = D^T$.

(4) *Reversible systems* (**9, 13**) which are systems which are simultaneously lossless and symmetric.

(5) As a variant of (4) one could also consider systems with

$$\Sigma_e G(s) = G^T(-s) \Sigma_e$$

which are systems which exhibit an invariance under time reversal.

Remark 6. An interesting point which has implications in several applications is the classification of all solutions Q to the inequality (LMI) subject to the various constraints as they come up (for example in the symmetric case one also requires, as shown in Proposition 5, $Q = TQ^{-1}T$). The functions $\frac{1}{2}x^T Qx$ correspond indeed to the internal energy storage or to Lyapunov functions, and in any case they determine those matrices S which bring the state space parameters $\{A, B, C, D\}$ into "canonical" form. In general, the set of Q's is convex and compact (this holds also for case (1) above) and it is unique for relaxation systems, for lossless systems, and in cases (2) and (4) above.

VI. Applications

In this section we will outline some of the applications of the above theory.

(1) *Stability*

Consider the system described by the implicit equation

$$y(t) = \int_{-\infty}^{t} W(t-\tau)\,(u(\tau)-f(y(\tau),\tau))\,d\tau$$

which is the equation of a feedback system with $\Sigma_{I/O}$ in the forward loop and the time-varying nonlinearity with characteristic $f(\cdot,t)$ in the feedback loop. Its zero input behaviour leads to the autonomous system Σ_a described by the differential equation:

$$\Sigma_a\colon \dot{x} = Ax - Bf(Cx,t),$$

where $\{A,B,C\}$ is a minimal realization of the impulse response $W(t)$.

We will derive conditions under which Σ_a shows a *stable* behaviour. Rather than introducing formal definitions to this effect, assume that we are looking for conditions under which there exists a $M<\infty$ such that every solution $x(t)$ of Σ_a satisfies $\sup_{t\geqslant t_0}\|x(t)\|\leqslant M\|x(t_0)\|$ for all $t_0\in R$. The type of "non-explosiveness" implies stability in the sense of Lyapunov and boundedness, and, with a little bit more, also leads to asymptotic stability in the large. It essentially also implies L_p-input–output stability for $1\leqslant p\leqslant\infty$. (These various implications have been extensively treated in the literature; see (**14**) for details and other references.) It is easily seen that the above inequality will be satisfied if there exists a matrix $Q=Q^{\mathrm{T}}>0$ such that, with $V(x)=\tfrac{1}{2}x^{\mathrm{T}}Qx$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}V(x,t)\right]_{\Sigma_a} \triangleq \frac{\partial V}{\partial x}(x)\,(Ax-Bf(Cx,t))\leqslant 0.$$

It is clear that the stability property does not depend on which minimal realization of $W(t)$ we are using, since it is obviously invariant under the transformation $x \xrightarrow[|S|\neq 0]{S} Sx$. Thus we may choose this realization to suit our convenience.

Now, if the system $\Sigma_{I/O}$ admits a passive realization, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}x^{\mathrm{T}}x = x^{\mathrm{T}}Ax - x^{\mathrm{T}}Bf(B^{\mathrm{T}}x,t)$$

which shows, by Theorem I, that Σ_a will satisfy the stability condition introduced earlier provided:

(i) $G(s)$ is positive real, and

(ii) $\sigma^{\mathrm{T}}f(\sigma,t)\geqslant 0$ for all $\sigma\in R^m$ and $t\in R$.

If we now assume f to be linear and thus that we are considering the autonomous system:

$$\Sigma_a^L\colon \dot{x} = Ax - BK(t)Cx$$

then one may refine these results by exploiting the external symmetry of $\Sigma_{I/O}$. Thus one obtains a series of interesting average value criteria for the stability of Σ_a^L. For details, see (**13**). One such result is:

Theorem VI

Let $\Sigma_{I/O}$ be passive and symmetric with external signature Σ_e (i.e. let $G(s)$ be symmetric positive real) and let $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ be an internally symmetric and passive representation of $G(s)$. Then Σ_a^L is stable provided $K(t) = K^T(t)$ and $\limsup_{T\to\infty} \int_{t_0}^{t_0+T} \lambda(\tau)\,d\tau < \infty$ where

$$\lambda(t) \triangleq \max\{\lambda_{\max}[\tilde{A}_{--} - \tilde{B}_- K(t)\tilde{B}^T], \lambda_{\max}[\tilde{A}_{++}]\}.$$

In here

$$\tilde{A} = \left[\begin{array}{c|c} \tilde{A}_{++} & \tilde{A}_{+-} \\ \hline -\tilde{A}_{+-}^T & \tilde{A}_{--} \end{array}\right] \quad \text{and} \quad \tilde{B} = \left[\begin{array}{c} \tilde{B}_+ \\ \hline \tilde{B}_- \end{array}\right]$$

is a partitioning of A and B which is conformable with that of

$$\Sigma_i = \left[\begin{array}{c|c} I_+ & 0 \\ \hline 0 & -I_- \end{array}\right],$$

and $\lambda_{\max}[M]$ denotes the maximum eigenvalue of the symmetric matrix M.

The interesting feature of the above result is that by assuming symmetry of $G(s)$ and $K(t)$ one is thus able to obtain a criterion which only puts requirements on average values. Note that in the case $m = p = 1$ these symmetry requirements are of no consequence and thus Theorem 6 then becomes a complete generalization of the positivity conditions given before. In fact, as is shown in (**15**), this result may be extended in such a way that it yields a complete generalization of the well-known circle criterion.

Remark. 7. It is possible to extend the positivity result to an instability result. Thus if there exists a solution $Q = Q^T$ (but $Q \not\geq 0$) to (LMI) then Σ_a will be unstable. Such a solution exists if $\Sigma_{I/O}$ is pseudo-passive (but not passive).

(2) *Electrical network synthesis*

A formal definition of a synthesis question is as follows: Given certain (ideal) elements which may be interconnected according to certain interconnection laws, what systems may thus be realized and, if so, how? In (linear) passive electrical network synthesis these elements are taken to be resistors ($V = RI, R > 0$) (R's), capacitors ($I = C(\mathrm{d}V/\mathrm{d}t), C > 0$) ($C$'s), inductors ($V = L(\mathrm{d}I/\mathrm{d}t), L > 0$) ($L$'s), transformers ($V_2 = nV_1$; $I_2 = -(1/n)I_1$) (T's) and gyrators ($I_1 = gV_2$; $I_2 = -gV_1$) (G's). The interconnection laws are the usual electrical interconnections specified by Kirchhoff's laws. In the driving point synthesis question one asks to synthesize a network which induces a pre-specified voltage–current behaviour at the ports of the network.

Assume that this voltage–current behaviour is given in the form of system $\Sigma_{I/O}$, with

$$u = \left[\begin{array}{c} V_1 \\ \hline I_2 \end{array}\right] \quad \text{and} \quad y = \left[\begin{array}{c} I_1 \\ \hline T_2 \end{array}\right],$$

where $V_1, I_1 \in R^{n_1}$ and $V_2, I_2 \in R^{n_2}$. The ports $1, 2, \ldots, n_1$ are the voltage-controlled ports and the ports $n_1+1, n_1+2, \ldots, n_1+n_2 = m$ are the current controlled ports (V_i is the voltage across the ith port and I_i is the current flowing in and out of the ith port. These are oriented in such a way that $V_i I_i$ represents the instantaneous power *into* the network). Such a representation is called a *hybrid* representation. It specializes to an *impedance* representation if $n_2 = 0$ and to an *admittance* representation if $n_1 = 0$. It is possible to show that a linear passive electrical n-port always admits a hybrid representation and thus the form $\Sigma_{I/O}$ represents no loss of generality. Furthermore, if the network is at all synthesizable using a finite number of R, L, C, T and G's, then the assumption regarding the finite number of degrees of freedom is also satisfied and we may thus represent $\Sigma_{I/O}$ by a system of the type Σ.

We will show here how one may use the results of the previous sections in order to eliminate the dynamic component out of the problem, i.e. in reducing the dynamic question to a static one. The static synthesis question is easily solvable directly and it is possible to show that every port specification of the form:

(i) $V_1 = NV_2$; $I_2 = -N^{\mathrm{T}} I_1$, is synthesizable using transformers;

(ii) $\left[\begin{array}{c} V_1 \\ \hline I_2 \end{array}\right] = M \left[\begin{array}{c} I_1 \\ \hline V_2 \end{array}\right]$ is synthesizable using transformers and gyrators provided $M = -M^{\mathrm{T}}$;

(iii) using resistors and transformers provided $M + M^{\mathrm{T}} \geqslant 0$ and

$$\Sigma N + M^{\mathrm{T}} \Sigma = 0$$

where

$$\Sigma = \left[\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & -I_{n_2} \end{array}\right];$$

(iv) using resistors, transformers and gyrators provided $M + M^{\mathrm{T}} \geqslant 0$.

Assume now that we are asked to synthesise $\Sigma_{I/O}$. If $\Sigma_{I/O}$ is passive then there exists, by Theorem 1, an internally passive realization. It follows then from (iv) that the $(n + n_1 + n_2)$ port behaviour defined by

$$\left[\begin{array}{c} I \\ \hline I_1 \\ \hline V_2 \end{array}\right] = \left[\begin{array}{c|c} -A & -B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} V \\ \hline V_1 \\ \hline I_2 \end{array}\right]$$

may be realized using resistors, transformers and gyrators. Terminating now the first n ports by unit capacitors yields a synthesis of the terminal behaviour of $\Sigma_{I/O}$. Since passivity of $\Sigma_{I/O}$ is easily shown to be a necessary condition of realizability this shows that $\Sigma_{I/O}$ is synthesizable using R, L, C, T and G's if and only if $G(s)$ is positive real and, if so, it gives a procedure for going about it.

If $\Sigma_{I/O}$ is passive and symmetric with external signature

$$\Sigma_e = \left[\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & -I_{n_2} \end{array}\right]$$

then it admits, by Theorem III, an internally passive and symmetric representation. It follows then from (iii) that the $(n_+ + n_- + n_1 + n_2)$ port behaviour defined by

$$\begin{bmatrix} I_+ \\ \hdashline V_- \\ \hdashline I_1 \\ \hdashline V_2 \end{bmatrix} = \left[\begin{array}{c:c} -A & -B \\ \hdashline C & D \end{array}\right] \begin{bmatrix} V_+ \\ \hdashline I_- \\ \hdashline V_1 \\ \hdashline I_2 \end{bmatrix},$$

may be realized using R, L, C and T's. Terminating now the first n_+ ports by unit capacitors and the next n_- ports by unit inductors yields a synthesis of the terminal behaviour of $\Sigma_{I/O}$. Since external symmetry and passivity of $\Sigma_{I/O}$ is easily shown to be also a necessary condition for realizability this shows that $\Sigma_{I/O}$ is synthesizable using R, L, C and T's if and only if $\Sigma_e G(s) = G^T(s)\Sigma_e$ and, if so, it gives a procedure for going about it.

Analogous results may be obtained for *RLT*, *RCT*, *LCT* and *LCTG* synthesis. These involve the various classes of systems discussed in Section V.

(3) *Mechanics*

The well-known Onsager–Casimir relations from mechanics and thermodynamics may be formulated in such a way that they emerge out of the previous development. These relations may be stated as follows: Given the system $\Sigma_{I/O}$ then there exists a choice of the state such that the stored energy is $\frac{1}{2}x^T x$ and the force $j = \begin{bmatrix} x \\ \hdashline u \end{bmatrix}$ and the flux $f = \begin{bmatrix} -\dot{x} \\ \hdashline y \end{bmatrix}$ are related by the linear transformation $f = Lj$ such that

(i) $\langle f, j \rangle = f^T j =$ the dissipation rate $= u^T y - (\mathrm{d}/\mathrm{d}t)\,\frac{1}{2}x^T x$; and

(ii) L is signature symmetric, i.e. there exists a signature matrix Σ such that $L\Sigma = (L\Sigma)^T$.

[A somewhat more general but in the end equivalent formulation of these relations may be found in (**16**)].

Since

$$L = \left[\begin{array}{c:c} -A & -B \\ \hdashline C & D \end{array}\right]$$

it is easily seen that these relations claim the existence of a realization which is simultaneously internally passive and internally symmetric. Thus by Theorem III these relations are satisfied if and only if $\Sigma_{I/O}$ is externally passive and symmetric. The case that $\Sigma_{I/O}$ is of the relaxation type and thus all external and internal variables are of the same parity corresponds to the case in which Onsager's relations are satisfied.

(4) *Covariance generation*

Let $y_t(\omega)$ $(t \in R;\, y \in R^p)$ be a zero mean stationary Gaussian random vector process defined on a probability space $\{\Omega, \mathscr{A}, P\}$. The correlation function is defined by $R(\tau) \triangleq \mathscr{E}\{y_{t+\tau} y_t^{\mathrm{T}}\}$ $(\tau \in R)$ and obviously satisfies $R(\tau) = R^{\mathrm{T}}(-\tau)$. Furthermore, as is easily derived from its definition, $R(\cdot)$ is *positive definite* in the sense that $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v^{\mathrm{T}}(\sigma) R(\tau - \sigma) v(\tau) \, d\tau \, d\sigma \geqslant 0$ for all R^p-valued functions $v(\cdot)$ for which this integral exists.

The stochastic realization theory question is the problem of finding a Markov representation of the process $y_t(\omega)$. (Since we are concerned here with Gaussian processes, we will only consider linear realizations.) There are two possible interpretations of what one would mean by such a realization. (This distinction is by and large not recognized in the literature). The first interpretation is:

(1) Find a zero mean stationary Gaussian vector process $x_t(\omega)$ $(t \in R; x \in R^n)$ defined on $\{\Omega, \mathscr{A}, P\}$, and a $(p \times n)$ matrix C, such that:
  (ii) x_t is Markov; and
  (ii) $y_t = Cx_t$ almost surely $(\forall t \in R)$.

The second interpretation is:

(2) Find a zero mean stationary Gaussian vector process $x_t(\omega)$ $(t \in R, x \in R^n)$ defined on a probability space $\{\bar{\Omega}, \bar{\mathscr{A}}, \bar{P}\}$, and a $(p \times n)$ matrix C, such that:
  (i) x_t is Markov; and
  (ii) $\bar{y}_t \triangleq Cx_t$ has the same statistics as y_t, i.e. $\mathscr{E}\{\bar{y}_{t+\tau} \bar{y}_t^{\mathrm{T}}\} = R(\tau)$.

The first approach is probabilistic in nature, the second is statistical. Although the first approach would appear to be more natural and relevant, it is particularly the second approach which has received attention in the literature (**17–19**). A recent paper by Picci (**20**), however, considers the first approach. A derived question is the representation (this may again be understood in both senses) of a Markov process by a stochastic differential equation. In our context this requires finding a $(n \times n)$-matrix A, a $(n \times m)$ matrix B and an initial zero mean Gaussian random vector $x_{t_0}(\omega)$ such that $x_t(\omega)$ (for $t \geqslant t_0$) is represented by

$$dx_t(\omega) = Ax_t(\omega) \, dt + B \, dw_t(\omega)$$

with w_t a normalized R^m-valued Wiener process defined for $t \geqslant t_0$ and independent of x_{t_0}.

We will consider here briefly the second approach. We are thus looking for matrices $\{A, B, C\}$ and a covariance matrix P such that

$$dx_t(\omega) = Ax_t(\omega) \, dt + B \, dw_t(\omega), \quad \bar{y}_t(\omega) = Cx_t(\omega)$$

with $x_0(\omega)$ zero mean Gaussian with covariance P and independent of w_t (defined for $t \geqslant 0$), yields a stationary random variable $x_t(\omega)$ such that

$$\xi\{\bar{y}_{t+\tau} \bar{y}_t^{\mathrm{T}}\} = R(\tau) \quad (t, \tau \geqslant 0).$$

This requires, as is easily verified by direct calculation, that

$$PA^{\mathrm{T}} + AP = -BB^{\mathrm{T}} \quad \text{and} \quad R(\tau) = Ce^{A\tau} PC^{\mathrm{T}}$$

which is indeed possible provided $R(t)$, $t \geqslant 0$, is the impulse response of a system of the form $\Sigma_{I/O}$ which admits a realization $\{A, PC^{\mathrm{T}}, C\}$ with

with $P = P^{\mathrm{T}} \geqslant 0$ and $PA^{\mathrm{T}} + AP \leqslant 0$. Such a realization indeed exists (by Theorem I with $P = Q^{-1}$) since the positivity of $R(\cdot)$ implies the passivity of the system $\Sigma_{I/O}$ with impulse response $R(t)$, $t \geqslant 0$.

VII. Conclusions

We have tried to give in this paper a brief survey of the main concepts and results concerning the representation of stationary linear systems which satisfy certain qualitative conditions on their input–output behaviour. This leads to a theory which has some interesting applications, particularly in physics oriented problems. Much work remains to be done in this area. Particularly the treatment of nonlinear phenomena and the incorporation of at least the theory of Hamiltonian and Lagrangian systems seems to be very promising indeed.

References

(1) R. E. Kalman, "Lyapunov functions for the problem of Lur'e in automatic control", *Proc. natn. Acad. Sci., U.S.A.*, Vol. 49, pp. 201–205, 1963.

(2) V. A. Yacubovich, "Absolute stability of nonlinear control in critical cases, I and II", *Automation and Remote Control*, Vol. 24, pp. 273–283, 1963 and pp. 655–668, 1964.

(3) V. M. Popov, "Hyperstability and optimality of automatic systems with several control functions", *Rev. Roumaine Sci. Tech. Electrotechn. et Energ.* Vol. 9, pp. 629–630, 1964. See also: V. M. Popov, "Hyperstability of Control Systems", Springer, New York, 1973.

(4) B. D. O. Anderson, "A system theory criterion for positive real matrices", *SIAM J. Control*, Vol. 5, pp. 171–182, 1967.

(5) R. W. Brockett and H. B. Lee, "Frequency domain instability criteria for time-varying and nonlinear systems", *Proc. IEEE*, Vol. 55, pp. 604–619, 1967.

(6) R. E. Kalman, "On a new characterization of linear passive systems", *Proc. First Allerton Conf. on Circuit and System Theory*, Monticello, III., pp. 456–470, 1963.

(7) D. C. Youla and P. Tissi, "*N*-port synthesis via reactance extraction—Part I", 1966 *IEEE Int. Conv. Rec.*, Pt. 7, pp. 183–208, 1966.

(8) S. Vongpanitlerd and B. D. O. Anderson, "Scattering matrix synthesis via reactance extraction", *IEEE Trans. on Circuit Theory*, Vol. CT–17, pp. 511–517, 1970. See also: B. D. O. Anderson and S. Vongpanitlerd, "Network Analysis and Synthesis", Prentice-Hall, Englewood Cliffs, N.J., 1973.

(9) J. C. Willems, "Dissipative dynamical systems, Part I: General theory, Part II: Linear systems with quadratic supply rates", *Archs Rational Mech. and Anal.*, Vol. 45, pp. 321–351 and pp. 352–392, 1972.

(10) R. E. Kalman, P. L. Falb and M. A. Arbib, "Topics in Mathematical System Theory", McGraw-Hill, New York, 1969.

(11) R. W. Newcomb, "Linear Multiport Synthesis", McGraw-Hill, New York, 1966.

(12) W. A. Day, "Time-reversal and the symmetry of the relaxation function of the linear viscoelastic material", *Archs Rational Mech. Anal.*, Vol. 40, pp. 149–159, 1971. See also: W. A. Day, "The Thermodynamics of Simple Materials with Fading Memory", Springer, New York, 1972.

(13) S. I. Marcus and J. C. Willems, "Nonstationary network synthesis via state space techniques", *IEEE Trans. on Circuits & Syst.*, Vol. CAS–22, No. 9, pp. 713–720, 1975.

(**14**) J. C. Willems, "Mechanisms for the stability and instability in feedback systems", *IEEE Proc.*, Vol. 64, No. 1, 1976.

(**15**) J. C. Willems and R. W. Brockett, "Average value stability criteria for symmetric systems", *Ricerche di Automatica*, Vol. 4, Nos. 2–3, pp. 87–108, 1973.

(**16**) J. C. Willems, "Consequences of a dissipation inequality in the theory of dynamical systems", in "Physical Structure in Systems Theory", ed. by J. J. van Dixhoorn and F. J. Evans, pp. 193–218, Academic Press, New York, 1974.

(**17**) R. E. Kalman, "Linear stochastic filtering—reappraisal and outlook", *Symp. on System Theory*, Polytechnic Institute of Brooklyn, pp. 197–205, 1965.

(**18**) B. D. O. Anderson, "The inverse problem of stationary covariance generation", *J. Stat. Physics*, Vol. 1, No. 1, pp. 133–147, 1969.

(**19**) P. Faurre, "Réalisations Markoviennes de processus stationnaires", *IRIA Rapport de Recherche*, No. 13, March 1973.

(**20**) G. Picci, "Stochastic realization for stochastic processes", *IEEE Proc.*, Vol. 64, No. 1, pp. 112–122, 1976.