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Brief Paper

Feedback Stabilizability for Stochastic Systems with State and Control Dependent Noise*

JACQUES L. WILLEMS† and JAN C. WILLEMS‡

Summary—This paper deals with linear stochastic systems with state and control dependent noise. Conditions are derived for which there exists a state feedback control such that the closed loop system is stable in the mean square sense. Particular attention is paid to the case in which there is only state dependent noise or only control dependent noise and to cases in which the noise intensities are arbitrarily large.

Nomenclature

x	state
u	input
β_i, γ_j	noises
σ_i, ρ_j	noise intensities
A	system matrix
B	control input matrix
F_i, G_j	noise input matrices
K	feedback control gain
P	solution of Riccati equation
P_0	limiting solution of (9) for $\beta \to 0$
P^*	limiting solution of (19) for $\alpha \to 0$
V	Lyapunov function
L	differential generator (in Section 1)
L	linear operator (in Section 2)
E	expectation operator
N	null space
R	range space
R^n	Euclidean space
det	determinant
$g(s)$	transfer function
Ω	subspace defined in Section 2
$D = (d/dt) =$	derivative
s	complex variable
T	transposition
dim	dimension (of a subspace)

Introduction

THE CLASS of systems considered in this paper may be described by the Itô differential equation

$$dx = (Ax + Bu)\,dt + \sum_{i=1}^{k} \sigma_i F_i x\, d\beta_i + \sum_{j=1}^{l} \rho_j G_j u\, d\gamma_j. \quad (1)$$

In here $x \in R^n$ denotes the state, $u \in R^m$ denotes the control input, while β_i $(i = 1, 2, \ldots, k)$ and γ_j $(j = 1, 2, \ldots, l)$ denote the disturbances. These are assumed to be zero mean uncorrelated stationary normalized Wiener processes. Thus

$$E\{d\beta_i\} = 0, \quad E\{d\gamma_j\} = 0, \quad E\{d\beta_i^2\} = dt, \quad E\{d\gamma_j^2\} = dt,$$

$$E\{d\beta_{i_1}\, d\beta_{i_2}\} = 0(i_1 \neq i_2), \quad E\{d\gamma_{j_1}\, d\gamma_{j_2}\} = 0(j_1 \neq j_2),$$

and

$$E\{d\beta_i\, d\gamma_j\} = 0$$

for $i, i_1, i_2 = 1, 2, \ldots, k$ and $j, j_1, j_2 = 1, 2, \ldots, l$. The factors σ_i and ρ_j indicate the intensities of the disturbances. The constant system matrices A, B, F_i and G_j are of dimension $(n \times n)$, $(n \times m)$, $(n \times n)$, and $(n \times m)$, respectively. Thus the first term of the right hand side of (1) denotes the drift term, the second term denotes the control term, the third term denotes the diffusion term due to the state dependent noise, and the last term denotes the diffusion term due to the control dependent noise. It is easy to see that the independence and zero mean assumptions on the processes β_i and on the processes γ_j entail no loss of generality. Thus the model under consideration describes a general linear dynamical system with white noise coefficients and for which the state dependent noise and the control dependent noise terms are independent.

The analysis of systems with multiplicative stochastic disturbances has attracted a good deal of attraction in recent years, motivated, at least partly, by various areas of application, for example, to system with human operators, economic system which model some of the uncertainties as stochastically varying lags, mechanical systems subject to random vibrations (e.g. earthquakes), etc. Thus there has been research on control[1–3], least squares filtering[1–4], and the stability analysis of stochastic Itô systems[4–6]. The problem which we will consider in this paper is to derive conditions on the parameters (A, B, F_i, G_j, σ_i and ρ_j) for which there exists a feedback control law of the form

$$u = Kx \quad (2)$$

such that the closed loop system, i.e. the system described by the Itô equations

$$dx = (A + BK)x\, dt + \sum_{i=1}^{k} \sigma_i F_i x\, d\beta_i + \sum_{j=1}^{l} \rho_j G_j K x\, d\gamma_j \quad (3)$$

is stable in the mean square sense. In here K is a constant $(m + n)$ matrix which may be chosen arbitrarily. In fact, it is easily seen from the optimal control interpretation which will be given later that this stabilizability will not be enhanced if we were to allow more complex feedback control strategies for example by admitting control laws which are nonlinear and/or time-varying and/or control laws with memory. Stabilization of stochastic systems has frequently been studied, mainly in America[1–3, 14], in the USSR[15–17], and Japan[4, 18]. The main contribution of the present paper is the derivation of explicit criteria for the stabilizability of system (1).

The problem under consideration is formally described in the following definition:

Definition 1. The system (3) is said to be stable in the mean square sense if all initial states $x(0)$ yield

$$\lim_{t\to\infty} E(x(t)x(t)^T) = 0.$$

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†Engineering School, University of Gent, Gent, Belgium.

‡Mathematical Institute, University of Groningen, Groningen, The Netherlands.

System (1) is said to be stabilizable in the mean square sense if there exists a matrix K such that (3) is stable in the mean square sense. We use the concept of stability here as what would in other contexts usually be called asymptotic stability. Note that stability in the mean square sense implies stability of the mean (i.e. $\lim_{t\to\infty} E\{x(t)\} = 0$ for all $x(0)$) and almost sure stability (i.e. $\lim_{t\to\infty} x(t) = 0$ almost surely). In particular, this implies that stabilizability of the deterministic system:

$$\dot{x} = Ax + Bu \tag{4}$$

is a necessary condition for mean square stabilizability of (1). (This condition is satisfied if (4) is, say, controllable[8].) The stabilizability of (4) is not sufficient, however, for stabilizability of (1), and, as we shall see later, it is not always possible to stabilize (1) for large noise intensities, even if (4) is stabilizable.

The problem considered here is of interest in its own right, but it also enters in a natural way in stochastic control as a necessary condition for the existence of an optimal state feedback policy for the problem of minimizing the usual quadratic criterion

$$E\left\{\int_0^\infty (u^T Ru + x^T Qx)\,dt\right\}$$

for the linear stochastic system (1)[1-3]. Two important particular cases of the problem under discussion are the case in which there is only state dependent noise

$$dx = (Ax + Bu)\,dt + \sum_{i=1}^{k} \sigma_i F_i x \, d\beta_i \tag{5}$$

and the case in which there is only control dependent noise

$$dx = (Ax + Bu) + \sum_{i=1}^{l} \rho_i G_i x \, d\gamma_i. \tag{6}$$

Moreover, since in many applications one cannot *a priori* be sure of the intensity of the noises, we will pay particular attention to the problem of determining which systems are stabilizable for all noise intensities σ_i and ρ_i. Note also that in the case that (1) is to be considered in the sense of Stratonovich, then essentially the same theory goes through, (after introducing the necessary correction terms, in the case that there is only state dependent noise), but it requires essential modifications in the case that there is also control dependent noise, although, presumably, the methodology to treat this case would not need to be very much different.

The paper is structured as follows: in Section 1 a general, but not very explicit, criterion for stabilizability is given. In Section 2 this criterion is applied in the case of only state dependent noise and explicit conditions are derived for stabilizability for all noise intensities together with some criteria which give the maximum admissible noise intensity for mean square stabilizability. Section 3 deals with systems in which there is only control dependent noise. Finally, Section 4 contains some results for systems with both control and state dependent noise elements.

1. *Fundamental theorem*

In this Section we will derive a theorem which gives a necessary and sufficient condition for the mean square stabilizability of (1). It is stated in terms of the nonlinear matrix equation

$$SA + A^T S - SB\left(R + \sum_{i=1}^{l} \rho_i^2 G_i^T S G_i\right)^{-1} B^T S + \sum_{i=1}^{k} \sigma_i^2 F_i^T S F_i = -Q \tag{7}$$

in the symmetric matrix S for given symmetric Q and R, of dimension $n \times n$, $n \times n$, and $m \times m$ respectively.

Theorem 1. A sufficient condition for mean square stabilizability of (1) is that there exists positive definite matrices Q and R for which (7) has a positive definite solution S. A necessary condition for mean square stabilizability of (1) is that (7) has a positive definite solution S for any given positive definite matrices Q and R.

Proof. To prove the sufficiency part of the theorem, consider the feedback control law $u = Kx$ with

$$K = -\left(R + \sum_{i=1}^{l} \rho_i^2 G_i^T S G_i\right)^{-1} B^T S.$$

The claim is that this control law will stabilize (1). This will be proven by means of a Lyapunov argument, see, for example, [7]. Consider therefore the Lyapunov function $V(x) = x^T Sx$ and let L denote the differential generator[7] associated with (3) and with K chosen as above. It is easily calculated that $LV(x) = x^T Mx$ where $M = -Q - K^T RK$. Since $S > 0$ and $M < 0$ this shows the mean square stabilizability of (1).

To prove the necessity part of the theorem, we consider the problem of minimizing the performance criterion

$$= E\left\{\int_0^\infty (x^T Qx + u^T Ru)\,dt\right\}$$

over all feedback control policies $u = Kx$. Because the system is assumed to be mean square stabilizable, a minimizing K exists. By the results of [1-3] this implies that (7) then has a positive definite solution S which yields the optimum K given by

$$-\left(R + \sum_{i=1}^{l} \rho_i^2 G_i^T S G_i\right)^{-1} B^T S.$$

Since the condition derived in Theorem 1 involves the solution of a nonlinear matrix equation it is not particularly useful in applications. The theorem will be used in the remainder of the paper to derive explicit criteria for a number of interesting special cases.

2. *Systems with state dependent noise only*

In this Section we will consider systems described by the Itô equation (5). The matrix equation involved in the application of Theorem 1 correspondingly becomes

$$SA + A^T S - SBR^{-1}B^T S + \sum_{i=1}^{k} \sigma_i^2 F_i^T S F_i = -Q. \tag{8}$$

Consider also the algebraic matrix Riccati equation:

$$PA + A^T P - \frac{1}{\beta} PBB^T P = -Q \tag{9}$$

with $\beta > 0$ and $Q = Q^T \geq 0$. It is well known[8] that if the linear system (4) is stabilizable then there exists a unique symmetric solution P^+ to (9) which has the property that $A - (1/\beta)BB^T P^+$ is a Hurwitz matrix, i.e. its eigenvalues have negative real parts.

Moreover, P^+ is at least positive semi-definite. If $Q = CC^T$ and if (A, C) is observable, then P^+ is actually positive definite and is moreover the unique positive definite solution of (9). Finally P^+ is monotone nonincreasing with decreasing β [9, 10]. Thus

$$P_0 \triangleq \lim_{\beta\to 0} P^+$$

is well-defined for all fixed Q and is positive semi-definite. Straightforward numerical techniques exist to compute P^+, since it is the solution of a steady state Riccati equation[8]. P_0 can then be obtained by repeating the computation for decreasing β. The direct computation of P_0 is briefly discussed by Kwakernaak and Sivan[9]; more straightforward procedures for determining this limig have been developed by Nakamizo[19] and Friedland[20].

Let Ω denote the subspace of R^n spanned by the columns of the matrices F_i^T, $i = 1, 2, \ldots k$, i.e., $\Omega \triangleq \{x \in$

$R^n|x \perp N(F_i)$ for all $i\}$, where N denotes the null space. Application of Theorem 1 to the case under consideration leads to the following criterion for stabilizability:

Theorem 2. System (5) is mean square stabilizable if and only if

(i) System (4) is stabilizable (in the deterministic sense); and

(ii) there exists a matrix $Q^* = Q^{*T}$ with $Q^* \geq 0$ but $Q^* > 0$ on Ω such that

$$G \triangleq \sum_{i=1}^{k} \sigma_i^2 F_i^T P_0 F_i \leq Q^* \text{ but } G < Q^* \text{ on } \Omega.$$

Outline of the proof. The necessity of the condition is obvious. To prove the sufficiency, consider the solution P of (9) with $Q = Q^* + \epsilon Q_1$ where $\epsilon > 0$ and Q_1 a positive definite symmetric matrix. Then, for ϵ and β sufficiently small, $\sum_{i=1}^{k} \sigma_i^2 F_i^T P F_i \leq Q^*$. But P also satisfies

$$PA + A^T P - \frac{1}{\beta} PBB^T P + \sum_{i=1}^{k} \sigma_i^2 F_i^T P F_i = -\epsilon Q_1 - \left(Q^* - \sum_{i=1}^{k} \sigma_i^2 F_i^T P F_i\right)$$

which is an equation of the type (8) with $R, Q > 0$.

2.1 *Stabilizability for arbitrary noise intensities.* In this Section we will derive conditions on the parameter matrices A, B, and F_i ($i = 1, 2, \ldots, k$) of system (5) such that for all values of the noise intensities σ_i^2 there exists a stabilizing feedback gain matrix. This feedback gain matrix is however allowed to be a function of σ_i^2. In the next section we will consider the case in which this feedback gain matrix need not be a function of the noise intensities. Thus the idea is that σ_i is first measured and then the feedback gain matrix K which stabilizes is chosen.

The following result is an immediate consequence of Theorem 2:

Theorem 3. System (5) is mean square stabilizable for all noise intensities σ_i^2 if (4) is stabilizable and if there exists a symmetric matrix Q with $Q \geq 0$ but >0 on Ω such that $F_i^T P_0 F_i = 0$ ($i = 1, 2, \ldots, k$). Necessary conditions for (5) to be mean square stabilizable for all σ_i^2 are that (4) is stabilizable and that $F_i^T P_0 F_i = 0$ ($i = 1, 2, \ldots, k$) for some semi-definite matrix Q. Note that Theorem 3 gives a necessary and sufficient condition if Ω is one-dimensional.

Conditions for P_0 to be zero have been derived by Kwakernaak and Sivan[9]. Using a condition from their paper the following corollary is obtained:

Corollary 3.1. Assume that $m = \dim\{\sum_{\oplus} R(F_i)\} < \dim\{R(B)\}$ and let C be an $(m \times n)$ matrix such that $R(C) = \sum_{\oplus} R(F_i)$. Then system (4) is mean square stabilizable for all noise intensities σ_i^2 if there exists an $(n \times m)$ matrix B_1 such that $R(B_1) \subset R(B)$ and such that the polynomial $\det[C(Is - A)^{-1}B_1] \det[Is - A]$ has no zeroes with positive real part.

However, theorem 3 does not require P_0 to vanish; it only requires that $P_0 F_i$ is zero. Hence P_0 must be singular, which is always true, and the columns of the matrices F_i must belong to the null space of P_0. To check this, the following equivalence property is very useful: The condition $P_0 x_0 = 0$ is equivalent to the existence of vector $u(s)$ whose components are rational functions of s without poles in $Re(s) > 0$, such that

$$C(Is - A)^{-1}Bu(s) + C(Is - A)^{-1}x_0$$

vanishes identically, where $Q = C^T C$. From this statement, or from the consideration of (9) for $\beta \to 0$, one readily sees $\lim_{\beta \to 0} B^T P^+ B = 0$. This leads to the following interesting corollary which shows that systems in which the control enters the system "at more points" than the disturbance are always stabilizable.

Corollary 3.2. System (5) is mean square stabilizable for all noise intensities σ_i^2 if (4) is stabilizable, in the deterministic sense, and if $R(B) \supset R(F_i)$ for all i, where R denotes range space.

Consider now as a special case of (5) the following system with a single input, a single noise term, and a matrix F_1 of rank one:

$$dx = (Ax + bu)\,dt + b_1 c_1 x\, d\beta \tag{10}$$

where b and b_1 are column vectors, c_1 is a row vector, β is a (zero mean) Wiener process, and σ is a scalar which indicates the intensity of the disturbance. Then we have:

Corollary 3.3. Assume that the system $\dot{x} = Az$; $y = cx$ is detectable[8]. Then system (10) is stabilizable for all noise intensities σ^2 if and only if

(i) $\dot{x} = Ax + bu$ is stabilizable, in the deterministic sense; and

(ii) the rational function

$$\frac{c_1(Is - A)^{-1}b_1}{c_1(Is - A)^{-1}b}$$

has no poles whose real part is positive, after cancellation of common factors.

This corollary follows from the fact that, as shown in[9] there exists an input $u(t)$ ($t \geq 0$) which does not increase exponentially and for which $\int_0^\infty y^2(t)\,dt = 0$, with y and u related by $\dot{x} = Ax + bu$, $y = c_1 x$, and $x(0) = b_1$. Note that the detectability assumption in Corollary 3.3 entails no real loss of generality since by (i) there always exists a feedback gain which will induce detectability.

Note that the conditions of Corollaries 3.1 and 3.3 may be systematically verified by simple algebraic manipulations which involve the division algorithm to cancel common factors and the Routh-Hurwitz test[11] to check that no roots of the part of the denominator polynomial that is relatively prime with the numerator has positive real part. Note also that the conditions of Corollary 3.3 are obviously feedback invariant in the sense that they remain unchanged if A is replaced by $A + bk$, for any row vector k.

2.2 *Stabilizability for arbitrary noise intensity with the same feedback.* In this Section we will derive conditions on the parameter matrices A, B, and F_i ($i = 1, 2, \ldots, k$) of system (5) for which there exists a feedback gain matrix K such that the closed loop system described by the Itô equation

$$dx = (A + BK)x\,dt + \sum_{i=1}^{k} \sigma_i F_i x\, d\beta_i. \tag{11}$$

is mean square stable for all noise intensities σ; we start with a lemma which appears to be of interest in its own right. Consider therefore the system described by the Itô equation

$$dx = \tilde{A}x\,dt + \sum_{i=1}^{k} \sigma_i F_i x\, d\beta_i. \tag{12}$$

Let the matrix F defined by $F \triangleq [F_1|F_2|\ldots|F_k]$. Consider now the pair $(\tilde{A}, F)$. In general this pair will not be controllable. It is well-known that there then exists a change of basis on R^n such that with respect to this basis, the matrices A and F_i take the form

$$\tilde{A} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{12} & A_{22} \end{array}\right); \quad \left(\begin{array}{c|c} 0 & 0 \\ \hline F_{i,12} & F_{i,22} \end{array}\right).$$

Let

$$F_{12} \triangleq [F_{1,12} F_{2,12} \ldots F_{k,12}]$$

and

$$F_{22} \triangleq [F_{1,22}|F_{2,22}|\ldots|F_{k,22}].$$

Thus we may take $(A_{22}, [F_{12}|F_{22}])$ to be controllable. If $F_{22} \neq 0$ and if the pair (A_{22}, F_{22}) is not controllable, then we may repeat the above procedure. It thus follows that after a number of steps the matrices $\bar{A}$ and F_i will be brought into the form

$$\bar{A} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 \,.\,. & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & 0 \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}$$

$$F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ F_{i,21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \\ F_{i,p1} & \cdots & F_{i,p(p-1)} & F_{i,pp} \end{bmatrix}$$

where $F_{pp} \triangleq [F_{1,pp}|F_{2,pp}|\ldots,|F_{k,pp}]$ is either zero or is such that (A_{pp}, F_{pp}) is controllable. This reduction algorithm leads to the following explicit necessary and sufficient conditions for (12) to the mean square stable for all σ_i:

Lemma 1. System (12) is mean square stable for all noise intensities σ_i^2 if and only if A is a Hurwitz matrix and the matrix F_{pp} defined above is zero.

Outline of the proof. The fact these conditions are sufficient is relatively easy to prove and is therefore deleted. That $\bar{A}$ must be Hurwitz is also obvious. To show that also $F_{pp} = 0$ is a necessary condition it follows from the above reduction algorithm that it is enough to prove that (12) cannot be stable for all σ_i^2 if $(\bar{A}, F)$ is controllable. Mean square stability implies that there exists a positive definite symmetric solution P to the linear equation

$$P\bar{A} + \bar{A}^{\mathrm{T}}P + \sum_{i=1}^{k} \sigma_i^2 F_i^{\mathrm{T}} P F_i = -I.$$

The existence of a positive definite solution to this equation is equivalent to the existence of a positive definite solution to the related equation:

$$M - L(M) = I$$

where $L(M)$ denotes the linear operator defined by

$$L(M) \triangleq \int_0^\infty \left(\sum_{i=1}^{k} \sigma_i^2 F_i^{\mathrm{T}} \, \mathrm{e}^{\bar{A}^{\mathrm{T}}t} M \, \mathrm{e}^{\bar{A}t} \, F_i\right) \mathrm{d}t.$$

Since we need stabilizability for all σ_i^2, L needs to have all its eigenvalue zero. Moreover since L maps the cone of positive semidefinite symmetric matrices into itself, it has an eigenvector in this cone with associated eigenvalue, of course, zero. However, $M = M^{\mathrm{T}} \geq 0$ and $L(M) = 0$ imply, by controllability of $(\bar{A}, F)$, that M equals 0. Thus $(\bar{A}, F)$ controllable implies that (12) cannot be mean square stable for all σ_i^2, as claimed.

Returning now to the problem introduced in the beginning of this section, we see that the reduction algorithm associates with every K a matrix $F_{p(K)p(K)}$. Thus Lemma 1 leads to the following theorem:

Theorem 4. System (5) is mean square stabilizable with the same feedback gain matrix for all noise intensities σ_i^2 if and only if there exists a matrix K such that the conditions of Lemma 1 are satisfied with $\bar{A} = A + BK$.

Of course, Theorem 4 as it presently stands is a very unsatisfactory result, since one would like to reduce the problem of the existence of K to a condition about A, B and the F_i's. Except for some results given below we have not been able to reduce the conditions much further. The problem discussed in this section is closely related to the stable disturbance isolation problem discussed by Wonham in [12]. The problem considered there is to find conditions on A, B, C, D such that the control law $u = Kx$ for the system

$$\dot{x} = Ax + Bu + Dz\,; \quad y = Cx \tag{13}$$

has the property that

(i) $A + BK$ is a Hurwitz matrix; and

(ii) the transfer function from z to y,

$$C(Is - A - BK)^{-1}D$$

is zero. This problem is completely solved in [12].

Corollary 4.1. Consider system (5) and let F_i be factorizable as

$$F_i = D_i C_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Then this system is mean square stabilizable with the same feedback gain for all noise intensities σ_i^2 if the stable disturbance isolation problem for system (13) with

$$D = [D_1|D_2|\ldots|D_k]$$

and

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{bmatrix}$$

is solvable.

The above corollary is quite easy to prove directly. The condition is however not necessary, not even in the case that there is only one noise term (i.e. $k = 1$), unless, as is shown in the next corollary, the noise matrix F is of rank one. Consider therefore the system

$$\mathrm{d}x = (Ax + Bu)\,\mathrm{d}t + \sigma b_1 c_1 \,\mathrm{d}\beta \tag{14}$$

where b_1 is a column vector, c_1 is a row vector, β is a scalar (zero mean) Wiener process, and σ is a scalar which indicates the intensity of the disturbance. Associated with this system we have the deterministic system

$$\dot{x} = Ax + Bu + b_1 z\,; \quad y = c_1 x. \tag{15}$$

For this system we may then prove the following result:

Corollary 4.2. System (14) is stabilizable with the same feedback gain for all noise intensities σ^2 if and only if the stable disturbance isolation problem for (15) is solvable.

Proof. The easiest way for proving this corollary is to consider the operator L introduced in the proof of theorem 4. Here

$$L(M) = c_1^{\mathrm{T}} c_1 \int_0^\infty b_1^{\mathrm{T}} \, \mathrm{e}^{A^{\mathrm{T}}t} M \, \mathrm{e}^{At} \, b_1 \, \mathrm{d}t.$$

Thus L has an eigenvalue $\int_0^\infty w^2(t)\,\mathrm{d}t$, where $w(t) = c_1 \, \mathrm{e}^{At} b_1$. Thus all eigenvalues being zero implies that $w(t) = 0$ for all t which yields the desired result. The above corollary may in fact be generalized to the situation in which all the F_i have rank one.

2.3 *Computation of the maximum admissible noise intensity.* In this Section we will discuss the computation of the maximum admissible noise intensity which allows mean square stabilizability for system (10). For the notation see the introduction to Section 2. From Theorem 2 it follows that, if $b_1^{\mathrm{T}} P_0 b_1 \neq 0$, mean square stabilizability follows if and only if system (4) is stabilizable, in the deterministic sense, and

$$\sigma^2 < (b_1^{\mathrm{T}} P_0 b_1)^{-1}$$

where $P_0 = \lim_{\substack{\beta \to 0 \\ \beta > 0}} P$ and P is the unique symmetric solution of

$$A^TP + PA - \frac{1}{\beta} Pbb^TP = c_1^Tc_1$$

which is such that $A - (1/\beta)bb^TP$ is a Hurwitz matrix. Much information on the computation of $b_1^TP_0b_1$ may be found in [9]. Thus

$$b_1^TP_0b_1 = \inf_k \int_0^\infty y^2(t)\,dt$$

subject to the conditions $\dot{x} = Ax + bu$, $x(0) = b_1$, $x(\infty) = 0$, and $u = kx$. An alternative, more explicit expression is given in the case that (A, b) is in standard controllable form [13] by the path integral

$$b_1^TP_0b_1 = \int_{t_1}^{t_2} [q^*(D)z(t))^2 - (q(D)z(t))^2]\,dt \qquad (16)$$

where $q(s) = c_1(Is - A)^{-1}b.\det(Is - A)$ and $q^*(s)$ is derived from $q(s)$ replacing its zeros in the right half plane by their mirror images with respect to the imaginary axis. The above integral is a path integral in the sense that it only depends on the values of $z(t)$ and its derivatives at the end points $t = t_1$ and $t = t_2$, but that it is otherwise independent of the particular path which joins these points. In (16) these end points are given by

$$\left(z(t_1), \frac{dz(t_1)}{dt}, \ldots, \frac{d^{n-1}z(t_1)}{dt^{n-1}}\right) = 0$$

for $t = t_1$ and this vector equals b_1^T for $t = t_2$. Expression (16) offers an extremely convenient method for computing $b_1^TP_0b_1$, and thus the maximal admissible noise intensity, particularly if $q(s)$ has few zeroes in the right half plane. For a discussion of these path integrals, see [11] or [13].

2.4 *An example.* Consider the second order stochastic system:

$$dx_1 = x_2\,dt + \alpha\sigma(\gamma x_1 + \delta x_2)\,d\beta \qquad (17)$$
$$dx_2 = -ax_2\,dt + u\,dt + \beta\sigma(\gamma x_1 + \delta x_2)\,d\beta$$

This is a particular case of (10) with

$$c_1(Is - A)^{-1}b = \frac{\delta s + \gamma}{s(s + a)}$$
$$c_1(Is - A)^{-1}b_1 = \frac{(\alpha\gamma + \beta\delta)s + a\alpha\gamma + \beta\gamma}{s(s + a)}.$$

The results of Sections 2.1, 2.2 and 2.3 yield the following

(i) System (17) is mean square stabilizable for all σ^2 provided either α vanishes or γ and δ do not have opposite signs.

(ii) If (i) is not satisfied, then system (17) is mean square stabilizable provided

$$\sigma^2 < -\frac{1}{2\alpha^2\gamma\delta}.$$

(iii) System (17) is mean square stabilizable with the same feedback gain for all noise intensities σ^2 provided γ and δ have the same sign and $\alpha\gamma + \beta\delta = 0$. A suitable feedback control law is then $u = -k_1x_1 - k_2x_2$, where $k_1 > 0$ is arbitrary and

$$k_2 = -a + \frac{k_1\delta^2 + \gamma^2}{\gamma\delta}.$$

3. *Systems with control dependent noise only*

In this Section we will consider systems described by the Itô equation (6). The matrix equation involved in the application of Theorem 1 correspondingly becomes

$$SA + A^TS - SB\left(R + \sum_{j=1}^{l} \rho_j^2G_j^TSG_j\right)^{-1}B^TS = -Q. \qquad (18)$$

Consider also the algebraic matrix Riccati equation

$$PA + A^TP - PBM^{-1}B^TP = -\alpha T \qquad (19)$$

with $M = M^T > 0$, $T = T^T > 0$ and $\alpha > 0$. If (4) is stabilizable in the deterministic sense then there exists a unique positive definite symmetric solution P^+ to (19) which is monotone decreasing with α [10]. Thus

$$P^* \triangleq \lim_{\alpha\to 0} P^+$$

is well-defined for all fixed M, $T > 0$ and at least positive semi-definite. Application of Theorem 1 to the case under consideration leads to the following criterion for stabilizability:

Theorem 5. System (6) is mean square stabilizable if and only if

(i) System (4) is stabilizable (in the deterministic sense); and

(ii) there exists a matrix $M = M^T > 0$ such that

$$\sum_{j=1}^{l} \rho_j^2G_j^TP^*G_j < M.$$

The matrix P^* depends on M (but not on T). If Theorem 5 is used as a sufficient condition for mean square stabilizability, then the result obtained will depend on the choice of M. Hence there remains an unresolved possibility of optimizing the choice of M.

The matrix P^* admits a simple optimal control interpretation. Indeed

$$x_0^TP^*x_0 = \inf_u \int_0^\infty u^T(t)Mu(t)\,dt$$

subject to

$$\dot{x} = Ax + Bu;\ x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) = 0.$$

For the special case that there is only one control, i.e. for the system

$$dx = (Ax + bu)\,dt + \sum_{j=1}^{l} \rho_jg_ju\,d\gamma_j \qquad (20)$$

with b and g_j ($j = 1, 2, \ldots, l$) column vectors, then one can carry the computation further. Let $P_1 = \lim_{\alpha\to 0, \alpha>0} P$ where P is the unique positive definite solution of the algebraic Riccati equation:

$$PA + A^TP - Pbb^TP = -\alpha T$$

with $T = T^T > 0$.

Corollary 5.1. System (20) is mean square stabilizable if and only if

(i) System (4) is stabilizable (in the deterministic sense); and

(ii)
$$\sum_{j=1}^{l} \rho_j^2g_j^TP_1g_j < 1$$

The computation of P_1 may be carried out by the path integral method if (A, b) is in standard controllable form. Then

$$x_0^TP_1x_0 = \int_{t_1}^{t_2} [p^*(D)z(t))^2 - (p(D)z(t))^2]\,dt \qquad (21)$$

with $p(s) = \det(Is - A)$ and $p^*(s)$ is derived from $p(s)$ by replacing its zeroes in the right half plane by their mirror images with respect to the imaginary axis. The end points in (21) are

$$\left(z(t), \frac{dz(t)}{dt}, \ldots, \frac{d^{n-1}z(t)}{dt^{n-1}}\right) = 0 \quad \text{for } t = t_1$$

and this vector equals x_0^T for $t = t_2$.

3.1 *Stabilizability for arbitrary noise intensities.* A complete solution of the problem of finding conditions on the parameter matrices A, B, and G_j $(j = 1, 2, \ldots, l)$ of system (6) such that for all values of the noise intensities ρ_j^2 there exists a stabilizing control has been given in [14]. We will not repeat these results here but present some interesting special cases which may be derived from Theorem 5.

Corollary 5.2. System (20) is mean square stabilizable for all noise intensities ρ_j^2 if and only if

(i) System (4) is stabilizable, in the deterministic sense; and

(ii) the vectors g_j, $(j = 1, 2, \ldots, k)$ belong to the invariant subspace of A spanned by its (generalized) eigenvectors corresponding to eigenvalues with nonpositive real parts.

In the multivariable case however this condition is only sufficient, but not necessary:

Corollary 5.3. System (6) is mean square stabilizable for all noise intensities ρ_j^2 if

(i) System (4) is stabilizable, in the deterministic sense; and

(ii) The columns of G_j $(j = 1, 2, \ldots, l)$ belong to the invariant subspace of A spanned by its (generalized) eigenvectors corresponding to eigenvalues with nonpositive real parts.

3.2 *An example.* Consider the second order stochastic system

$$\begin{aligned} dx_1 &= x_2\, dt + \rho\alpha u\, d\gamma \\ dx_2 &= x_1\, dt + u\, dt + \rho\beta u\, d\gamma. \end{aligned} \tag{22}$$

The results of Section 3 yield the following:

(i) System (22) is mean square stabilizable for all ρ^2 provided $\alpha + \beta = 0$. The vector $[\alpha\, \beta]^T$ is then indeed an eigenvector of A corresponding to the eigenvalue -1.

(ii) If $\alpha + \beta \neq 0$, then the system is mean square stabilizable provided

$$\rho^2 < \frac{1}{2(\alpha + \beta)^2}.$$

4. *Systems with state and control dependent noise*

For the case in which one wants to obtain stabilizability criteria for system (1) with both the third and the fourth term present, it is necessary to study the full nonlinear matrix equation (7). It is again possible to state the results in terms of the algebraic Riccati equation

$$SA + A^TS - SBR^{-1}B^TS = -Q. \tag{23}$$

Recall that for the case in which there was only state dependent noise we were led to study the limiting solution to this equation for $R \to 0$, and that for the case in which there was only control dependent noise we were led to study the limiting solution to this equation for $Q \to 0$. For the case in which both state and control dependent noise are present we will have to consider the solution to equation (23) for non-zero R and Q.

Ths discussion of this Section will be limited to some particular cases for which we have been able to obtain some rather explicit criteria. Consider therefore the stochastic system

$$dx = (Ax + bu)\, dt + \sigma b_1 c_1 x\, d\beta + \sum_{j=1}^{l} \rho_j g_j u\, d\gamma_j \tag{24}$$

with b, b_1, and g_j $(j = 1, 2, \ldots, l)$ row vectors and c_1 a column vector. Together with this system we will consider the associated algebraic Riccati equation

$$SA + A^TS - \frac{1}{\alpha} Sbb^TS = -c_1^Tc_1 \tag{25}$$

with $\alpha > 0$ a parameter. If $\{A, b, c_1\}$ is minimal[13] (i.e. if (A, b) is controllable and (A, c_1) is observable) then, as is well-known, there exists for each $\alpha > 0$ a unique symmetric positive definite solution to (25), which we will denote by $S(\alpha)$. A sufficient condition for mean square stabilizability of (24) is that there exists an $\alpha > 0$ such that:

$$\sigma^2 b_1^T S(\alpha) b_1 < 1 \quad \text{and} \quad \sum_{j=1}^{l} \rho_j^2 g_j^T S(\alpha) g_j < \alpha.$$

These conditions are conflicting since $S(\alpha)$ will increase with increasing α, but less than linearly, i.e. $(S(\alpha)/\alpha)$ will decrease with increasing α. Thus we will consider that particular α for which the first inequality is satisfied with equality and verify whether the second inequality is satisfied for that α. This leads to the following theorem:

Theorem 6. let $\{A, b, c_1\}$ be minimal. Then system (24) is mean square stabilizable if and only if

(i) There exists a solution $\alpha^* > 0$ to the equation

$$\sigma^2 b_1^T S(\alpha^*) b_1 = 1; \tag{26}$$

(ii) the inequality:

$$\sum_{j=1}^{l} \rho_j^2 g_j^T S(\alpha^*) g_j < \alpha^* \tag{27}$$

hold for this α^*.

Note that if (26) does not have a solution, then the system is not stabilizable even if only the state dependent noise were present.

If b_1 is proportional to b, it is possible to verify (26) directly in terms of the system parameters since then

$$b^TS(\alpha)b = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log(1 + \alpha |g(j\omega)|^2)\, d\omega + \zeta$$

where $g(s) = c_1(Is - A)^{-1}b$ and ζ is a constant, independent of α, which equals twice the sum of the absolute values of the real parts of the eigenvalues of A in the right half plane. The criterion may still further be simplified if there is only one g which is moreover proportional to b. This situation occurs for example if (24) is derived from a higher order differential equation with white noise stochastic coefficients which has an input through a coefficient which is itself independent white noise.

Finally, note that it is clear from Theorem 6 that mean square stabilizability of (24) for all noise intensities σ^2 and ρ_j^2 will require at least that $\{A, b, c_1\}$ is not minimal or, equivalently, that the rational function $c_1(Is - A)^{-1}b$ has common factors in its numerator and its denominator.

4.1 *An example.* Consider the second order stochastic system

$$\begin{aligned} dx_1 &= x_2\, dt + \sigma x_1\, d\beta + a\rho u\, d\gamma \\ dx_2 &= u\, dt + b\rho u\, d\gamma \end{aligned} \tag{28}$$

The relevant Riccati equation has the solution:

$$S(\alpha) = \left[\begin{array}{c|c} \dfrac{\sqrt{(2)^4}\sqrt{(\alpha)}}{\sqrt{(\alpha)}} & \dfrac{\sqrt{(\alpha)}}{\sqrt{(2)^4}\sqrt{(\alpha^3)}} \end{array}\right]$$

Consequently, $\alpha^* = (1/4\sigma^8)$. Theorem 5 yields:

(i) System (28) is mean square stabilizable provided:

$$2\sigma^2\rho^2(2\sigma^4a^2 + 2\sigma^2ab + b^2) < 1$$

This shows that:

(ii) if $\rho = 0$ (only state dependent noise), then system (28) is mean square stabilizable for all intensities σ^2 of the state dependent noise; and

(iii) if $\sigma = 0$ (only control dependent noise), then system (28) is mean square stabilizable for all intensities ρ^2 of the control dependent noise. This result is of course obvious since there are no eigenvalues of A in the (open) right half plane.

(iv) if $\sigma \neq 0$ and $\rho \neq 0$, than there is a trade-off among the parameters of the system but stabilizability becomes more likely for σ and ρ small.

Conclusions

In this paper we have presented some results on mean square stabilizability of systems which contain white noise coefficients both in their dynamics and their input gains. A necessary and sufficient condition for stabilizability in terms of a nonlinear matrix equation was presented. For several special cases explicit condition for mean square stabilizability were given. Special attention was paid to the problem of stabilizability for all noise intensities. The conditions in the case of stabilizability for all noise intensities can probably be worked out in more detail using the presently available geometric theory of multivariable systems.

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