Nonstationary Network Synthesis via State-Space Techniques

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Abstract—A theory is developed for nonstationary network synthesis via state-space techniques. The method is based on direct realization algorithms from a Hankel matrix (input/output) description of the system. Passive, lossless, reciprocal, reversible, and relaxation systems are considered. For each of these classes of systems, necessary and sufficient conditions on the Hankel matrix for the existence of a representation which can be synthesized with certain types of network elements are derived. In addition, algorithms for computing these representations are presented.

I. Introduction

The concepts of linear systems theory provide a natural and valuable framework for the study of network analysis and synthesis. Consequently, this approach has been investigated extensively in the recent literature [1]–[10], [23], [26] ([10] and [23] provide especially good background for this paper). In these papers the approach to synthesis consists of first constructing an arbitrary minimal state-space realization, and then transforming this representation into one which is synthesizable by a particular type of network. This paper provides, for some particular classes of systems, algorithms for the construction of the desired minimal realization directly from the associated Hankel matrix. Thus we eliminate the necessity of solving a matrix equation and performing the associated coordinate basis change; in addition, we extend the theory to cover time-varying lossless systems and stationary reversible and relaxation systems.

II. Linear Systems and Realization Theory

The purpose of this section is to review some basic facts from linear system theory [11], [14]. We include these primarily for ease of reference and notation. We will consider the finite dimensional linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{2}$$

where x, the *state*, is an n-vector valued function; u, the *input*, is an m-vector valued function; and y, the *output*, is a p-vector valued function. We will always consider systems with $m = p$. We assume moreover that the system is defined for all $-\infty < t < \infty$, and that $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are matrices of dimension compatible with (1) and (2), and that they are at least continuous as functions of t. We will denote the dynamical system defined by (1) and (2) by $(A(t),B(t),C(t),D(t))$. The *state transition function* and the *read-out function* of this system are given by the well-known formulas [11]

$$x(t) = \phi(t,t_0,x_0,u) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \tag{3}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{4}$$

where $x_0 = x(t_0)$ and $\Phi(t,t_0)$ is the transition matrix defined by $\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$ and $\Phi(t_0,t_0) = I$. The system (1) and (2) is said to be *stationary* (or *time-invariant*) if $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are constant matrices.

If $A(t)$, $B(t)$, and $C(t)$ are sufficiently differentiable, we define the *j-controllability* and *j-observability matrices* [13], [14]

$$Q_j(t) = [P_0(t) \quad P_1(t) \quad \cdots \quad P_{j-1}(t)]$$

$$R_j'(t) = [S_0'(t) \quad S_1'(t) \quad \cdots \quad S_{j-1}'(t)]$$

where

$$P_{k+1}(t) = -A(t)P_k(t) + \dot{P}_k(t), \qquad P_0(t) = B(t) \tag{5}$$

$$S_{k+1}(t) = S_k(t)A(t) + \dot{S}_k(t), \qquad S_0(t) = C(t) \tag{6}$$

(here prime denotes the transpose). The system $(A(t),B(t),C(t),D(t))$ is said to be of *constant rank* and order n if there exist positive integers α, β, q_c, and q_0 such that $A(t)$, $B(t)$, and $C(t)$ are $\max(\alpha,\beta) - 1$, $\max(\alpha,\beta)$, and $\max(\alpha,\beta)$ times continuously differentiable, respectively, and such that

$$\operatorname{rank} Q_\alpha(t) = \operatorname{rank} Q_{\alpha+1}(t) = q_c \le n, \qquad \text{for all } t \tag{7}$$

$$\operatorname{rank} R_\beta(t) = \operatorname{rank} R_{\beta+1}(t) = q_0 \le n, \qquad \text{for all } t. \tag{8}$$

For stationary systems, we define the stationary j-controllability and j-observability matrices by

$$Q_j = [B \quad AB \quad \cdots \quad A^{j-1}B]$$

$$R_j' = [C' \quad A'C' \quad \cdots \quad (A')^{j-1}C'].$$

(Notice, however, that Q_j and R_j are *not* equal to $Q_j(t)$ and $R_j(t)$ which in this case will be independent of t, since there are some differences in sign. This somewhat confusing notation is in keeping with the standard nomenclature.)

The system $(A_1(t),B_1(t),C_1(t),Q_1(t))$ is said to be *algebraically equivalent* [14] to $(A_2(t),B_2(t),C_2(t),D_2(t))$ if

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there exists a continuously differentiable matrix $T(t)$ with $\det T(t) \neq 0$ for all t such that

$$\begin{aligned}(A_2(t),B_2(t),C_2(t),D_2(t)) &= (T(t)A_1(t)T^{-1}(t) + \dot{T}(t)T^{-1}(t),\\ &\qquad T(t)B_1(t),C_1(t)T^{-1}(t),D(t)).\end{aligned}$$

Two common representations of the response function from the input to the output of (1) and (2) are by its *weighting pattern* $W(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$ (defined for all $-\infty < t, \tau < \infty$), together with $D(t)$, or by its *impulse response* $K(t,\tau) = W(t,\tau) + D(t)\delta(t-\tau)$ for $t \geq \tau$ and $= 0$ for $t < \tau$, where $\delta(\cdot)$ denotes the Dirac delta function. The output is thus given in terms of the input (if $x(t_0) = 0$) by

$$y(t) = \int_{t_0}^{t} W(t,\tau)u(\tau)\,d\tau + D(t)u(t). \tag{9}$$

An alternate input/output description involves $D(t)$ and the *Hankel matrix* [14] defined by

$$\mathscr{H}_{ij}(t,\tau) = \begin{bmatrix} H_{00}(t,\tau) & \cdots & H_{0,j-1}(t,\tau) \\ \vdots & . & \vdots \\ H_{i-1,0}(t,\tau) & \cdots & H_{i-1,j-1}(t,\tau)\end{bmatrix}$$

where

$$H_{kl}(t,\tau) = \frac{\partial^k}{\partial t^k}\frac{\partial^l}{\partial \tau^l} W(t,\tau).$$

If we let $\Gamma_{ij}(t) = \mathscr{H}_{ij}(t,t)$, then it is easy to verify that

$$\Gamma_{ij}(t) = R_i(t)Q_j(t).$$

For stationary systems we will write $W(t)$ for $W(t-\tau, 0)$.

The Hankel matrix of a stationary system is defined by

$$\Gamma_{ij} = \begin{bmatrix} H_0 & H_1 & \cdots & H_{j-1} \\ H_1 & H_2 & \cdots & H_j \\ \vdots & \vdots & & \vdots \\ H_{i-1} & H_i & \cdots & H_{i+j-2}\end{bmatrix} \tag{10}$$

where the sequence $\{H_i\}$ is defined by $H_i = (d^i/dt^i)W(0)$ (we have used the standard definition of the stationary Hankel matrix; it is *not* equal to $\Gamma_{ij}(t)$ which in this case is independent of t, since there are some differences in sign). Stationary systems may also be described by their *transfer function* $G(s) = C(Is - A)^{-1}B + D$. The Hankel matrix coefficients $\{H_i\}$ are related to $G(s)$ by the Laurent expansion around $s = \infty$

$$G(s) = \sum_{i=1}^{\infty} H_{i-1}s^{-i}.$$

It is well known that the system (1) and (2) is not uniquely determined by its input/output descriptions. In fact, algebraically equivalent systems induce the same input/output descriptions [14]. The problem of representing an input/output description of a system by means of the form (1) and (2) is called the problem of *realization*. Thus $(A(t),B(t),C(t),D(t))$ is said to be a *realization* (or *representation*) of $(W(t,\tau),D(t))$ if $W(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$. It is said to be a *minimal* realization if every other realization of the form $(A_1(t),B_1(t),C_1(t),D_1(t))$ of $(W(t,\tau),D(t))$ has a state space of greater or equal dimension. Notice that $D(t)$ is an input/output quantity; in fact, it is the coefficient of $\delta(t-\tau)$ in the impulse response (or $D = G(\infty)$ in the stationary case). Thus any realization $(A_1(t), B_1(t),C_1(t),D_1(t))$ of $(W(t,\tau),D(t))$ will have $D_1(t) = D(t)$, so we need only consider the realization problem for $W(t,\tau)$.

For stationary systems (i.e., $W(t,\tau) = W(t-\tau)$) the fundamental condition for the existence of a representation of the form of (1) and (2) [14] is that there should exist nonnegative integers α, β, and n such that

$$\operatorname{rank} \Gamma_{\beta,\alpha} = \operatorname{rank} \Gamma_{\beta+1,\alpha+j} = n, \qquad j = 1,2,\cdots. \tag{11}$$

If (11) is satisfied, then n is the dimension of the state space of any minimal realization. Various algorithms for the construction of a stationary minimal realization from the Hankel matrix are presented in [2], [14], [15]; however, for our purposes, an appropriate algorithm for both stationary and nonstationary systems is based on Silverman and Meadows' [13] realization of nonstationary systems, as described in the following lemma.

Lemma 1: $W(t,\tau)$ has a constant rank realization if there exist positive integers α, β, and n such that $W(t,\tau)$ is α,β times continuously differentiable with respect to t,τ and

a) $\operatorname{rank} \mathscr{H}_{\beta,\alpha}(t,\tau) = \operatorname{rank} \mathscr{H}_{\beta+1,\alpha+1}(t,\tau) = n,$

$$\text{for all } t,\tau \tag{12}$$

b) $\Gamma_{\beta,\alpha}(t)$ can be factored in the form

$$\Gamma_{\beta,\alpha}(t) = N(t)M(t) \tag{13}$$

where N and M are continuously differentiable, $N(t)$ is $(m\beta \times n)$ with rank n for all t, and $M(t)$ is $(n \times m\alpha)$ with rank n for all t.

The proof (i.e., the realization algorithm) proceeds by performing the factorization (see [12, p. 191])

$$\Gamma_{\beta,\alpha+1}(t) = N(t)\tilde{M}(t) \tag{14}$$

where $N(t)$ is $(m\beta \times n)$ and $\tilde{M}(t)$ is $(n \times m(\alpha+1))$ (both have rank n). The matrices $N(t)$ and $\tilde{M}(t)$ are the β-observability and $(\alpha+1)$-controllability matrices for the minimal realization to be constructed. Let $M_1(t)$ denote the first m columns of $\tilde{M}(t)$, $N_1(t)$ the first m rows of $N(t)$, $M_\alpha(t)$ the first $m\alpha$ columns of $\tilde{M}(t)$, and $M_{\alpha+1}(t)$ the last $m\alpha$ columns of $\tilde{M}(t)$. Then

$$\begin{aligned}&(A(t),B(t),C(t))\\ &= \left(\left[-M_{\alpha+1}(t) + \frac{d}{dt}M_\alpha(t)\right]M_\alpha^{\dagger}(t),M_1(t),N_1(t)\right)\end{aligned} \tag{15}$$

(where $M_\alpha^{\dagger} = M_\alpha'(M_\alpha M_\alpha')^{-1}$ is a right inverse for M_α) defines a minimal constant-rank realization of $W(t,\tau)$ (see [13]).

For stationary weighting patterns $W(t)$ the algorithm proceeds as follows.

1) Perform the factorization

$$\Gamma_{\beta,\alpha+1} = NM$$

where N is $(m\beta \times n)$ and M is $(n \times m(\alpha + 1))$ (both have rank n).

2) Construct the minimal realization

$$(A,B,C) = (M_{\alpha+1}M_\alpha^\dagger, M_1, N_1)$$

where M_α, $M_{\alpha+1}$, M_1, and N_1 are defined as in Lemma 1.

III. Some Special Classes of Linear Systems

In certain problems in the analysis and synthesis of linear systems and networks one is led to consider models which, in input/output or in state-space form, exhibit special properties. The purpose of this section is to provide theorems regarding the compatibility of external and internal properties of a number of important systems of this kind.

We will consider linear systems via the state-space representation $(A(t),B(t),C(t),D(t))$ or the input/output representation $(W(t,\tau),D(t))$. *We will assume throughout that* $(A(t),B(t),C(t),D(t))$ *is a minimal constant rank representation and that* $(W(t,\tau),D(t))$ *satisfies the realizability condition* (11) *in the stationary case or* (12) *and* (13) *in the nonstationary case.*

A. Passive and Lossless Systems

Definition 1 (cf. [16], [23]): Let $w(t) = u'(t)y(t)$ where $y(t)$ is given by (9). Then $(W(t,\tau),D(t))$ is said to be *externally passive* if $\int_{t_0}^{t} w(\tau)\,d\tau \geq 0$ for all inputs and $t \geq t_0$. It is said to be *externally lossless* if it is externally passive and if $\int_{t_0}^{t_1} w(\tau)\,d\tau = 0$ whenever u is such that $y(t) = 0$ for all $t \geq t_1 \geq t_0$.[1] The representation $(A(t),B(t),C(t),D(t))$ is said to be *internally passive* if

$$\begin{bmatrix} A(t) + A'(t) & B(t) - C'(t) \\ B'(t) - C(t) & -D(t) - D'(t) \end{bmatrix} \leq 0, \qquad \text{for all } t \quad (16)$$

(i.e., $(d/dt)(\frac{1}{2}x'(t)x(t)) \leq u'(t)y(t)$ along solutions of (1) and (2)). It is said to be *internally lossless* if (16) holds with equality.

The above definitions become natural if one identifies $w(t) = u'(t)y(t)$ with the power supplied to the system and $\frac{1}{2}x'(t)x(t)$ with the stored energy in the system. Passive systems may be viewed as a special case of dissipative systems [16]. The following lemma relating external passivity to internal properties of the system is proved in [17].

Lemma 2: A sufficient condition for the external passivity (losslessness) of $(W(t,\tau),D(t))$ is that for every minimal representation $(A(t),B(t),C(t),D(t))$, there exists a solution $Q(t) = Q'(t) \geq 0$ to the matrix inequality (equality)

$$\begin{bmatrix} \dot{Q}(t) + A'(t)Q(t) + Q(t)A(t) & Q(t)B(t) - C'(t) \\ B'(t)Q(t) - C(t) & -D(t) - D'(t) \end{bmatrix} \leq 0. \quad (17)$$

If $D(t) + D'(t) > 0$ for all t, then $(W(t,\tau),D(t))$ is externally passive, and this condition is also necessary for external passivity. Every solution $Q(t)$ of (17) satisfies $Q(t) > 0$ and obeys the dissipation inequality $(d/dt)\cdot(\frac{1}{2}x'(t)Q(t)x(t)) \leq u'(t)y(t)$ along solutions of (1) and (2).

If $D(t) + D'(t)$ is singular, it is necessary that there exist a solution $Q(t) = Q'(t) \geq 0$ to the Stieltjes integral inequality (equality)

$$\int_{t_0}^{t_1} [x'(t)u(t)] \begin{bmatrix} dQ(t) + (A'(t)Q(t) + Q(t)A(t))\,dt & (Q(t)B(t) - C'(t))\,dt \\ (B'(t)Q(t) - C(t))\,dt & (-D(t) - D'(t))\,dt \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq 0$$

for all continuous $x(\cdot)$ and $u(\cdot)$ and all t_0 and t_1.

The Stieltjes integral arises in the necessary condition when $D(t) + D'(t)$ is singular, because it is then not possible to prove the differentiability of $Q(\cdot)$. Anderson and Moylan [17] present an algorithm for the computation of $Q(\cdot)$ in this case, but it involves *ad hoc* conditions throughout the algorithm to ensure differentiability; these conditions cannot be checked *a priori.*

Lemma 2 and the following lemma establish the compatibility of external and internal passivity and losslessness. The proof of Lemma 3 is analogous to the stationary case [16]–[20].

Lemma 3: A sufficient condition for the external passivity (losslessness) of $(W(t,\tau),D(t))$ is that there exist a minimal realization which is internally passive (lossless). If $D(t) + D'(t) > 0$, then $(W(t,\tau),D(t))$ is externally passive, and this condition is also necessary for external passivity.

In the stationary case, the conditions in Lemmas 2 and 3 are necessary and sufficient without the restriction $D + D' > 0$ (see [16]–[20]).

B. Reciprocal Systems

Since no adequate definition of time-varying reciprocal systems is presently known (see [6], [21], [27], [31]), we will restrict our discussion in this section to stationary systems.

Definition 2: The transfer function $G(s)$ is said to be *externally reciprocal* with external signature matrix Σ_e if $\Sigma_e G(s) = G'(s)\Sigma_e$, where Σ_e is a diagonal matrix with entries either $+1$ or -1. The stationary representation (A,B,C,D) is said to be *internally reciprocal* with internal signature matrix Σ_i and external signature matrix Σ_e if

$$\begin{bmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{bmatrix} \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}' \begin{bmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{bmatrix}. \quad (18)$$

The following well-known lemma ([2], [16], [23]) shows the compatibility of external and internal reciprocity.

Lemma 4: The transfer function $G(s)$ is externally reciprocal with external signature Σ_e if and only if there exists an internal signature Σ_i such that $G(s)$ admits a stationary minimal realization which is internally reciprocal (and thus satisfies (18)).

[1] We note that this definition is not the only one used in the literature. For example, one common definition [23] involves taking $t_1 = \infty$, with u and y both square integrable.

C. Reversible Systems

Definition 3: The transfer function $G(s)$ is said to be *externally reversible* with external signature matrix Σ_e [16] if $\Sigma_e G(s) = -G(-s)\Sigma_e$.

Lemma 5: $G(s)$ is externally passive and externally reversible with signature Σ_e if and only if there exists an internal signature Σ_i such that $G(s)$ admits a stationary minimal realization which is internally lossless and internally reciprocal (and thus satisfies (18)).

D. Relaxation Systems

Definition 4: The stationary system $(W(t),D)$ is a *relaxation system* ([16], [28]) if a) $D = D' \geq 0$; b) $W(t) = W'(t)$ for all $t \geq t_0$; c) $(-1)^n(d^n/dt^n)W(t) \geq 0$ for all $t \geq t_0$, $n = 0,1,2,\cdots$.

The relationship between the external and internal structure of relaxation systems is proved [16].

Lemma 6: The system $(W(t),D)$ is a relaxation system if and only if it admits a stationary minimal realization which is internally passive and reciprocal with $\Sigma_e = I_m$ and $\Sigma_i = -I_n$ (where I_n is the n-dimensional identity matrix).

IV. Realization Algorithms for Systems with Internal Constraints

In the previous section various classes of linear systems with internal or external constraints were introduced, and it was shown that the external and internal formulations of such systems were equivalent in a well-defined sense. The proofs of the lemmas stating that a system with a particular external property admits a realization with the analogous internal property are, in fact, constructive. However, the procedure followed is that of first constructing an arbitrary minimal representation (as in Lemma 1, or [2], [13]–[15]), and then transforming to an algebraically equivalent representation with the desired constraints. In this section it is shown how, in many cases, one may bypass the intermediate realization and proceed directly from the Hankel matrix to the desired realization. We will first translate the external constraints into conditions on the Hankel matrix, and then using these special properties of the Hankel matrix we will specialize the algorithm of Lemma 1 to construct a realization with the required constraints. We will implicitly use Lemmas 2 to 6 throughout this section.

Theorem 1

The system $(W(t,\tau),D(t))$ is internally lossless if and only if $\Gamma_{\gamma\gamma}(t) = \Gamma_{\gamma\gamma}'(t) \geq 0$ and $D(t) + D'(t) = 0$ for all t (where $\gamma = \max(\alpha+1, \beta+1)$ and α and β are defined in Lemma 1). In this case, the following algorithm yields an internally lossless minimal realization.

1) Perform the factorization (by means of a congruence reduction [21], [25, pp. 209–216])

$$\Gamma_{\gamma\gamma}(t) = M'(t)M(t) \tag{19}$$

where $M(t)$ is an $(n \times \gamma m)$ matrix with rank n for all t.

2) Construct the internally lossless minimal realization

$$(A(t),B(t),C(t),D(t)) = ([-M_\gamma(t) + \dot{M}_{\gamma-1}(t)]M_{\gamma-1}^{\dagger}(t), M_1(t), M_1'(t), D(t)) \tag{20}$$

where M_1 denotes the first m columns of $M(t)$, $M_{\gamma-1}(t)$ the first $m(\gamma-1)$ columns of $M(t)$, and M_γ the last $m(\gamma-1)$ columns of $M(t)$.

Proof:

Necessity: If $(A(t),B(t),C(t),D(t))$ is internally lossless, it may be shown by induction that $P_i(t) = S_i'(t)$ (cf. (5), (6)). Thus $Q_k(t) = R_k'(t)$ and

$$\Gamma_{\gamma\gamma}(t) = R_\gamma(t)Q_\gamma(t) = Q_\gamma'(t)Q_\gamma(t) \geq 0.$$

Sufficiency: Since $\Gamma_{\gamma\gamma}(t) = \Gamma_{\gamma\gamma}'(t) \geq 0$, and (12) and (13) is satisfied, it may be factored in the form (19) (see [25]). Considering (20), it is obvious that $B(t)$ and $C(t)$ satisfy (16) with equality; using the fact that $M_{\gamma-1}(t)$ and $M_{\gamma-1}'(t)$ satisfy (5) and (6), it may be shown that $A(t) + A'(t) = 0$. Q.E.D.[2]

In the case of stationary lossless systems, certain modifications of previous definitions' must be made. Define the *modified j-observability matrix* by

$$\tilde{R}_j' = [C' \quad (-1)^1 A'C' \quad (-1)^2(A')^2C' \quad \cdots \quad (-1)^{j-1}(A')^{j-1}C'].$$

In addition, let the *modified Hankel matrix* $\tilde{\Gamma}_{ij}$ be defined by

$$\tilde{\Gamma}_{ij} = \begin{bmatrix} \tilde{H}_{00} & \tilde{H}_{01} & \cdots & \tilde{H}_{0,j-1} \\ \tilde{H}_{10} & \tilde{H}_{11} & \cdots & \tilde{H}_{1,j-1} \\ \vdots & & & \vdots \\ \tilde{H}_{i-1,0} & \tilde{H}_{i-1,1} & \cdots & \tilde{H}_{i-1,j-1} \end{bmatrix}$$

where $H_{l,m} = (-1)^l \tilde{H}_{l+m}$ (see (10)). The modified Hankel matrix $\tilde{\Gamma}_{ij}$ differs from Γ_{ij} only in the respect that every odd-numbered "block row" is multiplied by -1. With these preliminaries, we state the following theorem, which is proved analogously to Theorem 1 [21]

Theorem 1(a)

The transfer function $G(s)$ is externally lossless if and only if $\tilde{\Gamma}_{\gamma\gamma} = \tilde{\Gamma}_{\gamma\gamma}' \geq 0$ and $D + D' = 0$ (where $\gamma = \max(\alpha+1, \beta+1)$ and α and β are defined in (11)). In this case, the following algorithm yields an internally lossless stationary minimal realization.

1) Perform the factorization (by means of a congruence reduction [21], [25, pp. 209–216])

$$\tilde{\Gamma}_{\gamma\gamma} = M'M$$

where M is an $(n \times m\gamma)$ matrix of rank n.

2) Construct the desired realization

$$(A,B,C,D) = (M_\gamma M_{\gamma-1}^{\dagger}, M_1, M_1', D)$$

[2] As suggested by a reviewer, an alternative realization procedure is provided by noting that the weighting pattern matrix for an internally lossless system can always be written in the form $W(t,\tau) = M'(t)M(\tau)$, from which the internally lossless minimal realization $(0,M(t),M'(t))$ is immediate.

where M_1, $M_{\gamma-1}$, and M_γ are defined as in Theorem 1.

The next theorem, which was proved for the case $\Sigma_e = I_m$ in [2], [22] deals with reciprocal systems.

Theorem 2

The transfer function $G(s)$ is externally reciprocal with external signature Σ_e if and only if $\Sigma_e D = D'\Sigma_e$ and the $(m\gamma \times m\gamma)$ matrix $\Gamma_{\gamma\gamma}^* \triangleq (\Sigma_e \dot{+} \cdots \dot{+} \Sigma_e)\Gamma_{\gamma\gamma}$ is symmetric (where $\gamma = \max(\alpha + 1, \beta + 1)$, α and β are defined in (11), and $\dot{+}$ denotes a block diagonal matrix). In this case, the following algorithm yields an internally reciprocal stationary minimal realization.

1) Perform the factorization (by means of a congruence reduction ([21], [25, pp. 209–216])

$$\Gamma_{\gamma\gamma}^* = M'(-\Sigma_i)M \tag{21}$$

where M is an $(n \times m\gamma)$ matrix of rank n.

2) Construct the required realization

$$(A,B,C,D) = (\Sigma_i M_\gamma M_{\gamma-1}^\dagger \Sigma_i, -\Sigma_i M_1, \Sigma_e M_1', D) \tag{22}$$

where M_1, $M_{\gamma-1}$, and M_γ are defined as in Theorem 1.

Proof:

Necessity: This condition is obvious [2], [21], [22].

Sufficiency: The factorization (21) is valid because (11) is satisfied and $\Gamma_{\gamma\gamma}^*$ is symmetric [25]. It is easy to see that B and C of (22) satisfy (18). Notice that

$$M_\gamma'\Sigma_i M_{\gamma-1} = M_{\gamma-1}'\Sigma_i M_\gamma. \tag{23}$$

Premultiplying (23) by $\Sigma_i(M_{\gamma-1}M_{\gamma-1}')^{-1}M_{\gamma-1}$ and postmultiplying by the transpose of this expression yields $A'\Sigma_i = \Sigma_i A$. Q.E.D.

For systems which are simultaneously externally passive and reciprocal, the algorithm of Theorem 2 significantly improves the procedures of Yarlagadda [10]. Instead of first constructing an arbitrary realization and then performing two equivalence transformations, one may construct an internally reciprocal realization *directly* from the Hankel matrix and then apply the transformation of [10] (Method 2) to construct an internally passive and reciprocal realization.

For the study of reversible systems, it is assumed without loss of generality that $\Sigma_i = [I_{n_1} \dot{+} (-I_{n_2})]$ and $\Sigma_e = [I_{m_1} \dot{+} (-I_{m_2})]$, where $n_1 + n_2 = n$ and $m_1 + m_2 = m$. Then internal losslessness and reciprocity imply [16] that

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|cc} 0_{n_1,n_1} & -A_1 & 0_{n_1,m_1} & -B_2 \\ A_1' & 0_{n_2,n_2} & -B_1 & 0_{n_2,m_2} \\ \hline 0_{m_1,n_1} & B_1' & 0_{m_1,n_1} & D_1 \\ B_2' & 0_{m_2,n_2} & -D_1' & 0_{m_2,n_2} \end{array}\right] \tag{24}$$

where $0_{r,s}$ denotes the $(r \times s)$ zero matrix.

Definition 5: The matrix $\Gamma_{\gamma\gamma}^*$ is said to be in *reversible form* if

$$\Gamma_{\gamma\gamma}^* = \begin{bmatrix} V_1 & V_2 & \cdots & V_{\gamma/2} \\ V_2 & & & \\ \vdots & & & \vdots \\ V_{\gamma/2} & \cdots\cdots & & V_{\gamma-1} \end{bmatrix} \tag{25}$$

where, for $i = 1,\cdots,\gamma - 1$, V_i is a $(2m \times 2m)$ matrix of the form

$$V_i = \begin{bmatrix} V_{i1} & 0_{m_1,m_2} & 0_{m_1,m_1} & V_{i4} \\ 0_{m_2,m_1} & V_{i2} & V_{i3} & 0_{m_2,m_2} \\ 0_{m_1,m_1} & V_{i4} & V_{i5} & 0_{m_1,m_2} \\ V_{i3} & 0_{m_2,m_2} & 0_{m_2,m_1} & V_{i6} \end{bmatrix}. \tag{26}$$

Also, we define the $(m \times m)$ matrices

$$\hat{V}_i = \begin{bmatrix} V_{i1} & V_{i4} \\ V_{i3} & V_{i6} \end{bmatrix} \qquad \bar{V}_i = \begin{bmatrix} V_{i2} & V_{i3} \\ V_{i4} & V_{i5} \end{bmatrix}$$

and the $(\frac{1}{2}m\gamma \times \frac{1}{2}m\gamma)$ matrices

$$\hat{\Gamma}_{\gamma\gamma}^* = \begin{bmatrix} \hat{V}_1 & \hat{V}_2 & \cdots & \hat{V}_{\gamma/2} \\ \hat{V}_2 & & & \\ \vdots & & & \vdots \\ \hat{V}_{\gamma/2} & \cdots\cdots & & \hat{V}_{\gamma-1} \end{bmatrix}$$

$$\bar{\Gamma}_{\gamma\gamma}^* = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \cdots & \bar{V}_{\gamma/2} \\ \bar{V}_2 & & & \\ \vdots & & & \vdots \\ \bar{V}_{\gamma/2} & \cdots\cdots & & \bar{V}_{\gamma-1} \end{bmatrix}.$$

With these preliminaries, the major result on the realization of passive and reversible systems will now be proved. In order to simplify the proof, it will be assumed in the proof that $\gamma = \max(\alpha + 1, \beta + 1)$ is an even integer; the proof may easily be modified to cover the case that γ is odd.

Theorem 3

The transfer function $G(s)$ is externally passive and reversible with external signature Σ_e if and only if

a) $\Gamma_{\gamma\gamma}^*$ is in reversible form,
b) $\hat{\Gamma}_{\gamma\gamma}^* = (\hat{\Gamma}_{\gamma\gamma}^*)' \geq 0$,
c) $\bar{\Gamma}_{\gamma\gamma}^* = (\bar{\Gamma}_{\gamma\gamma}^*)' \leq 0$,
d) $D = -\Sigma_e D \Sigma_e = -D'$.

In this case, the following algorithm yields a stationary minimal realization which is internally lossless and reciprocal.

1) Perform the factorizations

$$\hat{\Gamma}_{\gamma\gamma}^* = \hat{M}' I_{n_2} \hat{M} \tag{27}$$

$$\bar{\Gamma}_{\gamma\gamma}^* = \bar{M}'(-I_{n_1})\bar{M}. \tag{28}$$

2) Partition $\hat{M}$ and $\bar{M}$ according to

$$\hat{M} = [\hat{M}_1 \quad \hat{M}_2 \quad \cdots \quad \hat{M}_\gamma]$$

$$\bar{M} = [\bar{M}_1 \quad \bar{M}_2 \quad \cdots \quad \bar{M}_\gamma] \tag{29}$$

where $\hat{M}_{2i}$ is $(n_2 \times m_2)$, $\hat{M}_{2i+1}$ is $(n_2 \times m_1)$, $\bar{M}_{2i}$ is $(n_1 \times m_1)$, and $\bar{M}_{2i+1}$ is $(n_1 \times m_2)$.

3) Form the matrix

$$M = \left[\begin{array}{cc|cc|c|cc} 0_{n_1,m_1} & \bar{M}_1 & \bar{M}_2 & 0_{n_1,m_2} & \cdots & \bar{M}_\gamma & 0_{n_1,m_2} \\ -\hat{M}_1 & 0_{n_2,m_2} & 0_{n_2,m_1} & -\hat{M}_2 & & 0_{n_2,m_1} & -\hat{M}_\gamma \end{array}\right].$$

4) Construct the required realization

$$(A,B,C,D) = (\Sigma_i M_\gamma M_{\gamma-1}^\dagger \Sigma_i, -\Sigma_i M_1, \Sigma_e M_1', D). \tag{30}$$

Proof:

Necessity: If (A,B,C,D) satisfies (24), then

$$\Sigma_e CA^{2i}B = \begin{bmatrix} (-1)^i B_1'(A_1'A_1)^i B_1 & 0_{m_1,m_2} \\ 0_{m_2,m_1} & (-1)^{i+1} B_2'(A_1A_1')B_2 \end{bmatrix}$$

$$\Sigma_e CA^{2i+1}B = \begin{bmatrix} 0_{m_1,m_1} & (-1)^{i+1} B_1'A_1'(A_1'A_1)^i B_2 \\ (-1)^{i+1} B_2'A_1(A_1'A_1)^i B_1 & 0_{m_2,n_2} \end{bmatrix}.$$

Hence, $\Gamma_{\gamma\gamma}^*$ is in reversible form. Also $\bar{\Gamma}_{\gamma\gamma}^*$ and $\hat{\Gamma}_{\gamma\gamma}^*$ may be expressed as in (27)–(29), where

$$\hat{M}_{2i} = (-1)^i A_1'(A_1A_1')^{i-1}B_2$$

$$\hat{M}_{2i+1} = (-1)^i (A_1'A_1)^i B_1$$

$$\bar{M}_{2i} = (-1)^{i+1} A_1(A_1'A_1)^{i-1}B_1$$

$$\bar{M}_{2i+1} = (-1)^i (A_1A_1')^i B_2.$$

Thus $\hat{\Gamma}_{\gamma\gamma}^* \geq 0$ and $\bar{\Gamma}_{\gamma\gamma}^* \leq 0$.

Sufficiency: If a)–c) are assumed, then the factorizations (27), (28) are valid [25]. It can be easily verified that

$$\Gamma_{\gamma\gamma}^* = M'[(-I_{n_1}) \dotplus I_{n_2}]M = M'(-\Sigma_i)M.$$

Thus Theorem 2 implies that (30) defines a minimal internally reciprocal realization. Some simple calculations will show that the realization defined in (30) also satisfies (24).

The following result on relaxation systems is also discussed in [28].

Theorem 4

The system $(W(t),D)$ (or, equivalently, $G(s)$) is a relaxation system if and only if

a) $D = D' \geq 0$,
b) $\Gamma_{\gamma\gamma} = \Gamma_{\gamma\gamma}' \geq 0$,
c) $\sigma\Gamma_{\gamma\gamma} = (\sigma\Gamma_{\gamma\gamma})' \leq 0$,

where

$$\sigma\Gamma_{\gamma\gamma} = \begin{bmatrix} H_1 & H_2 & \cdots & H_\gamma \\ H_2 & H_3 & \cdots & H_{\gamma+1} \\ \vdots & \vdots & \cdots & \vdots \\ H_\gamma & H_{\gamma+1} & \cdots & H_{2\gamma-1} \end{bmatrix}$$

is the "shifted" Hankel matrix [29, p. 289].

In this case, the algorithm of Theorem 2 yields a minimal stationary realization which is internally passive and internally reciprocal (with $\Sigma_e = I_m$, $\Sigma_i = -I_n$).

Proof:

Necessity: If (A,B,C,D) is internally passive and reciprocal with $\Sigma_e = I_m$, $\Sigma_i = -I_n$, then $A = A' \leq 0$, $B = C'$, and $D = D'$. It is clear from the results on reciprocal systems that this implies $\Gamma_{\gamma\gamma} = \Gamma_{\gamma\gamma}' \geq 0$ and $\sigma\Gamma_{\gamma\gamma} = (\sigma\Gamma_{\gamma\gamma})'$. Also,

$$\sigma\Gamma_{\gamma\gamma} = Q_\gamma' A Q_\gamma \tag{31}$$

so $A \leq 0$ implies $\sigma\Gamma_{\gamma\gamma} \leq 0$.

Sufficiency: The algorithm of Theorem 2 ((21), (22)) yields the internally reciprocal realization $(A,B,C,D) = (M_\gamma M_{\gamma-1}^\dagger, M_1, M_1', D)$. Now consider the expression (31) for $\sigma\Gamma_{\gamma\gamma}$. Postmultiplying (31) by $Q_\gamma'(Q_\gamma Q_\gamma')^{-1}$ and premultiplying by the transpose of this expression shows that c) implies $A \leq 0$. Hence, the representation is also internally passive. The application of Lemma 6 completes the proof.

Remark: The apparent similarity between the conditions on the Hankel matrices in Theorems 1 and 4 is due to the sign difference in the definitions of stationary and nonstationary Hankel matrices.

V. Applications to Electrical Network Synthesis

In this section we will outline how the results of the previous sections may be used in the synthesis of electrical networks. The method employed is based on the concept of *reactance extraction* [2], [6], [7], [23]. In this approach, a linear time-varying network is analyzed by first extracting all the inductors, capacitors, and independent sources thereby leaving a purely memoryless network. The state equations are then determined in terms of the port description of this remaining memoryless network, in which the state variables are the inductor currents and capacitor voltages. For synthesis purposes, we will assume without loss of generality [30] that all inductors and capacitors are time invariant; hence, all time-varying elements are memoryless (resistors, transformers, and gyrators).

We thus define i_L and v_L, the vectors of inductor current and voltage; i_c and v_c, the capacitor currents and voltages; i_1 and v_1, the currents and voltages of the voltage sources, and i_2 and v_2, the currents and voltages of the current sources. We assume that the memoryless network which interconnects the inductors, capacitors, and sources has a hybrid description of the form (explicit time dependent notation will be omitted)

$$\begin{bmatrix} i_c \\ v_L \\ \hline i_1 \\ v_2 \end{bmatrix} = \left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array}\right] \begin{bmatrix} v_c \\ i_L \\ \hline v_1 \\ i_2 \end{bmatrix}. \tag{32}$$

If C and L are diagonal nonsingular matrices with entries equal to the various reactances, the reactive elements may be described by

$$C \frac{d}{dt} v_C = -i_C$$
$$L \frac{d}{dt} i_L = -v_L. \tag{33}$$

Substituting (33) into (32) and defining

$$x = \begin{bmatrix} \sqrt{C} & v_C \\ \sqrt{L} & i_L \end{bmatrix} \qquad u = \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \qquad y = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{34}$$

yields state equations of the form of (1) and (2), where

$$(A(t),B(t),C(t),D(t)) = (-QP_1(t)Q, -QP_2(t), P_3(t)Q, P_4(t)) \tag{35}$$

with

$$Q = \begin{bmatrix} \sqrt{C} & 0 \\ 0 & \sqrt{L} \end{bmatrix}^{-1}.$$

For the purpose of synthesis, we will also assume without loss of generality that the reactive portion of the network contains only $1 - H$ inductors and $1 - F$ capacitors. This is possible because, for example, an $l - H$ inductor may be replaced by a transformer of turns ratio $\sqrt{l}:1$ terminated in a $1 - H$ inductor; the normalizing transformer may then be incorporated into the memoryless portion of the network.

Thus we have reduced the problem to: given the input/output description $(W(t,\tau),D(t))$, construct the hybrid matrix

$$\begin{bmatrix} -A(t) & -B(t) \\ C(t) & D(t) \end{bmatrix}$$

where $W(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$, and terminate this memoryless network in $1 - H$ inductors and $1 - F$ capacitors. In addition, constraints on $(W(t,\tau),D(t))$ may be translated, via the lemmas of Section 3, into constraints on $(A(t),B(t),C(t),D(t))$, and consequently on the types of elements used in the reactance extraction synthesis. For instance, $(W(t,\tau),D(t))$ may be synthesized as a passive (time-varying) *RLCTG* network if and only if it is externally passive;[3] furthermore, the realization $(A(t),B(t),C(t),D(t))$ must be internally passive. A similar relationship holds between external (and internal) losslessness and *LCTG* networks, reciprocity and (possibly active) *RLCT* networks, reversibility and *LCT* networks, and relaxation systems and passive *RCT* or *RLT* networks.

In Section 4 we have given algorithms for the computation of realizations satisfying the appropriate internal constraints for the various types of external constraints; the procedure is as follows. Given $(W(t,\tau),D(t))$, construct (via the algorithms of Section 4), the appropriate realization $(A(t),B(t),C(t),D(t))$. Then synthesize the memoryless hybrid matrix

$$\begin{bmatrix} -A(t) & -B(t) \\ C(t) & D(t) \end{bmatrix}$$

(see [30]), and terminate this network in $1 - H$ inductors and $1 - F$ capacitors. The relationships between the various types of networks and their realization algorithms are summarized in Table I.

[3] For a discussion of time-varying network elements, including gyrators and transformers, see Newcomb [30].

TABLE I

Network Elements	External Condition	Internal Condition	Realization Algorithm
RLCTG	Passive (Def.1)	Passive (Def.1)	
LCTG	Lossless (Def.1)	Lossless (Def.1)	Theorem 1
RLCT	Reciprocal (Def.2)	Reciprocal (Def.2)	Theorem 2
LCT	Passive and Reversible (Def.3)	Lossless and Reciprocal	Theorem 3
RCT (RLT)	Relaxation System (Def. 4)	Passive and Reciprocal ($\Sigma_e = I_n$, $\Sigma_i = -I_n$)	Theorem 4

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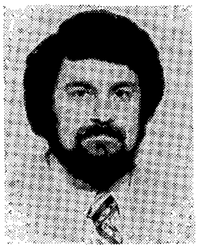

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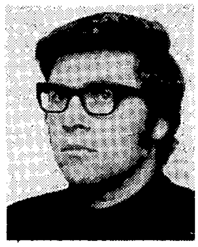

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