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THE GENERATION OF LYAPUNOV FUNCTIONS FOR INPUT-OUTPUT STABLE SYSTEMS*

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Abstract. This paper discusses the relationship between properties of input-output descriptions and state space models for dynamical systems. It is shown that a state space realization of an input-output stable dynamical system is globally asymptotically stable in the sense of Lyapunov if it is uniformly observable and if every state is reachable. This result is proved in the context of abstract dynamical systems and leads to the equivalence of input-output stability and asymptotic stability for uniformly controllable and uniformly observable linear finite-dimensional systems. The generation of Lyapunov functions is subsequently considered, and variational techniques for the construction of Lyapunov functions are presented. Passivity and related energy concepts are particularly exploited in this context. These results yield the Lyapunov functions used in the proofs of the circle criterion and the Popov criterion as particular cases. The generality of the approach, however, makes these ideas applicable to much more general situations. Examples illustrating the results and the unifying point of view are included.

1. Introduction. "Dynamical systems" as they are studied and defined[1] in modern system theory distinguish themselves from arbitrary operators in mathematics by one basic property: they are causal, i.e., nonanticipatory: future values of the input do not influence past values of the output. This basic realizability property of physical systems may be incorporated in the mathematical model in two ways: either by appropriately restricting the operator defining the input-output relationship, or by working with a state space description which will then automatically ensure this causality. This last approach has proved particularly useful in optimal control theory due to the fact that any deterministic optimal controller can always be implemented with a memoryless function of the state in the feedback. It is therefore very advantageous to work with a state space model from the very start.

This duality in the possible description of systems has reflected itself in other areas of system theory and is particularly prevalent in stability theory. The input-output approach leads to the concept of input-output stability and has been developed mainly in the last decade, especially following the work of Sandberg [4] and Zames [5]. The state space description leads to concepts such as global asymptotic stability in the sense of Lyapunov and poses the stability problem in a setting which does not involve inputs, thus making use of the theory of classical dynamical systems. Which of the two approaches is to be preferred depends on

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[1] There appears to be no agreement as to the use of the term "dynamical system." For the purpose of this paper any causal input-output relation will be termed a dynamical system. It will be shown that this is equivalent to the existence of a state. Zadeh [1] and Balakrishnan [2] appear to reserve the term for systems in which the state evolution is governed by a differential equation. The dynamical systems studied in classical mechanics [3] correspond to the state evolution equations in the absence of inputs.

the particular application. (For a discussion of these issues, see the survey paper by the author [6].) It has become clear, however, that the input-output approach leads to more powerful and general results. There are, in fact, a number of interesting stability criteria available which have been obtained in an input-output stability setting, but for which no proofs using Lyapunov methods exist.

Notwithstanding this success of input-output stability, there are certain aspects of Lyapunov stability theory which make the study and development of this "internal" approach to stability theory both useful and important. Not the least of these advantages is the possibility of obtaining estimates on the domain of attraction of an equilibrium in the case of nonglobal stability, a concept which has not even been satisfactorily formulated, let alone developed in the context of input-output stability. The main stumbling block in applying Lyapunov methods to the stability analysis particularly of nonlinear systems remains the absence of general methods for the construction of Lyapunov functions. This paper in part addresses itself to this problem.

The paper is concerned with the implications of input-output stability to global stability of dynamical systems, and with the construction of Lyapunov functions for input-output stable systems. The converse question, i.e., the implications of global stability to input-output stability, will not be considered, in view of space limitations and in view of the fact that such implications are much easier to obtain.

The first part of the paper introduces the concept of a dynamical system, which is defined as a causal operator between signal spaces, and the concept of a realization in which state space concepts become relevant. Some important properties of dynamical systems and realizations are then introduced: they are those of stability, controllability, observability, reachability, connectedness, and irreducibility. These notions play an important role in the sequel.

The second part of the paper discusses the generation of Lyapunov functions for input-output stable systems. Particular emphasis is placed on passive systems and on concepts such as available energy, required energy, and cyclic energy, the latter of which is very reminiscent of certain notions in thermodynamics.

The third part of the paper is concerned with feedback systems. Feedback systems are very important in control, and their stability is, of course, the main qualification on the performance of a feedback structure as a controller. Moreover, for design purposes, it is extremely desirable that properties of feedback systems be concluded from considerations of the open-loop elements. This aspect makes the results of the previous sections not easily applicable to the analysis of feedback systems, and a somewhat different approach is thus required. The ensuing Lyapunov functions are defined in terms of variational problems.

The paper ends with a list of examples. They illustrate the viewpoint adopted here and lead to the Lyapunov functions used to prove the circle criterion and the Popov criterion.

The work reported here has been directly inspired by a very interesting paper by Baker and Bergen [7] which appeared recently. They indeed posed the problem of constructing Lyapunov functions as a variational problem, an approach which has been fully exploited in the context presented here. Some of these ideas already appeared in the work of Popov [8], Kalman [9], and Anderson [10], [11]. The author obtained a great deal of insight from the work of Brockett on passivity and stability [12]. It is interesting to note that the ingenious independence of path argument as exploited by the latter author in his construction of Lyapunov functions follows here as a rather logical consequence of the variational problems which lead to the desired Lyapunov functions.

The paper also indicates what may be the basic reason why stability conditions appear to be easier to obtain using input-output methods than through the construction of suitable Lyapunov functions: both input-output stability and Lyapunov stability can be posed as variational minimization problems, and whereas Lyapunov methods need the explicit solution of these variational problems (thus the boundedness *and* the value of an infimum), input-output stability only requires the boundedness of this infimum. This observation is due to Zames (private communication).

2. Dynamical systems. A dynamical system is usually defined on a subset of the real line as a mapping between function spaces satisfying an appropriate set of axioms. This paper will be concerned with continuous time systems only. Moreover, it will be assumed that the inputs and the outputs take their values in appropriate inner product spaces and that their norm is a locally square integrable function of time. This restriction precludes a certain amount of generality and is made mainly for expository purposes since the results of the paper generalize to much more general situations. In particular, the assumption that the input and output spaces are inner product spaces is of no consequence to many of the results in the paper. One of the reasons for treating systems in this setting is the possibility of introducing and exploiting concepts related to energy and passivity of systems. Indeed, these have far-reaching implications in stability theory.

There are two main avenues for obtaining mathematical models of systems: the first one starts with an internal model in which physical laws and interconnections are used to describe the dynamics and which then yield the relation between the influence variables (the inputs) and the variables of interest (the outputs). The second approach starts with an input-output relation as the basic mathematical model to be used. Such a model is usually the logical consequence from identification experiments at the input-output terminals.

Besides inputs and outputs there is an additional set of variables which is of fundamental importance in the description of dynamical systems. These are the so-called states which summarize the effect of past inputs. The internal modeling approach, in fact, usually displays a state explicitly. More often than not the state has no immediate physical significance and there is never any uniqueness as to its choice. Although the basic mechanism of interest in system theory is the generation of outputs from inputs, it is very often advantageous, however, to view this process as taking place through this intermediate variable, the state. This point of view has been particularly useful in such fields as dynamic optimization theory and the study of Markov processes.

These concepts are formally introduced in the present section, and it is shown that input-output descriptions and state space descriptions of dynamical systems are essentially equivalent.

The first notion is that of signal spaces which will be the input and output function spaces.

Let V be an inner product space and let R denote the real line. Let f be a V-valued function defined on R. Then the *causal truncation* of f at T is defined to be the result of the projection operator P_T defined by

$$(P_T f)(t) \triangleq \begin{cases} f(t) & \text{for } t \leqq T, \\ 0 & \text{otherwise}. \end{cases}$$

The *anticausal truncation* of f at T is defined as $Q_T f = f - P_T f$. Consider now the vector space of V-valued functions on R with $\int_{-\infty}^{+\infty} \|f(t)\|_v^2 \, dt < \infty$. This vector space is itself an inner product space with

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{+\infty} \langle f_1(t), f_2(t) \rangle_V \, dt$$

as the inner product. It will be denoted by $L_2(V)$ and is complete if V is. As usual, no attention is paid to the fact that a function in $L_2(V)$ actually represents the equivalence class of functions which are equal to it almost everywhere with respect to Lebesgue measure.

As Wiener remarked when extending Fourier transforms, $L_2(V)$ is not a very interesting class of functions since it consists of functions which were small in the remote past and are destined to become small in the remote future. This last aspect, in particular, makes this function space of very limited use in stability studies which precisely refer to this remote future, and any a priori limitations on the future would therefore be very inappropriate.

A useful extension of $L_2(V)$ is its so-called *causal extension* denoted by $L_{2e}(V)$, which consists of all V-valued functions on R whose causal truncations belong to $L_2(V)$, i.e.,

$$L_{2e}(V) \triangleq \{f : R \to V | P_T f \in L_2(V), \textit{all } T \in R\}.$$

The *anticausal extension* of $L_{2e}(V)$ is similarly defined as

$$\{f : R \to V | Q_T f \in L_2(V), \text{all } T \in R\}.$$

Since all time functions considered in this paper will be assumed to start at some finite time, very little use of this anticausal extension will be made.

DEFINITION 1. Let $L_{2e}(V)$ denote the causal extension of $L_2(V)$. Then the subspace of $L_{2e}(V)$ defined by

$$S(V) \triangleq \{f \in L_{2e}(V) | Q_T f = 0 \text{ for } \textit{some } T \in R\}$$

will be called a *signal space*. Elements of $S(V)$ will be called *signals*, and elements of $L_2(V) \cap S(V)$ will be called *small signals*.

Thus, signal spaces consist of functions which vanish in the remote past and which have, in a sense, no finite escape, but are otherwise quite arbitrary. As is customary in the related literature, it will be assumed that inputs are applied to systems starting at some finite time in the past. This time need not be a priori fixed and will, in general, be different for each experiment. Note that signal spaces are closed under concatenation and that any "reasonable" physical signal belongs to a signal space. For the purposes of this paper, signal spaces represent a very convenient abstraction of reality. The fact that signals are required to have their support on a half-line is very important and results in mild conditions for the well-posedness of mathematical models. In other words, given a mathematical model for a system (e.g., an integral equation or a differential equation), it will, in general, be relatively easy to establish that inputs in a given class generate well-defined outputs, provided, however, that these inputs have their support on a half-line. For inputs defined on the whole real line $(-\infty, +\infty)$, establishing this existence and uniqueness of outputs usually leads to stringent conditions and requires typically input-output continuity of the system. This case is moreover of dubious physical significance. For details see [13], [14, § 4.6], [15].

Let $U = S(V_u)$ and $Y = S(V_y)$ be signal spaces. U will be called the *input space*, and Y will be called the *output space*. Elements of U and Y will be called respectively *input signals* and *output signals*. A mapping F from U into Y is said to be *causal* (or *nonanticipatory*) if for all $T \in R$ and all $u_1, u_2 \in U$ with $P_T u_1 = P_T u_2$, the equality $P_T F u_1 = P_T F u_2$ holds.[2] This condition is equivalent to requiring that $P_T F P_T = P_T F$ on U.

Note that the signal spaces as introduced above could have been called somewhat more consistently causal signal spaces. The analogous concepts of anticausal signal spaces and anticausal operators thus become straightforward. No use will be made of these concepts, however. An additional notion which is of some importance is that of a *memoryless* operator. This would most logically be defined as an operator which is *both* causal *and* anticausal but is easiest (although equivalently) defined as an operator F, defined by an element $r \in Y$ and an instantaneous map, f, from $V_u \times R$ into V_y with $f(0, t) = 0$ for all $t \in T$ and $Fu \triangleq r + Nu$, where $(Nu)(t) \triangleq f(u(t), t)$ is such that any function $u \in L_2(V_u)$ with compact support yields $Nu \in L_2(V_y)$ (consequently also with compact support).

DEFINITION 2. A *dynamical system* is defined as a causal mapping from the input signal space U into the output signal space Y. If this mapping is memoryless, then the dynamical system will similarly be called memoryless.

The above setting for the study of input-output relations is similar to the one employed by Balakrishnan in [16]. The definition eliminates the possibility of studying differentiators, for instance, but for the purposes of this paper (stability) such a restriction is not very disturbing. In the study of networks, however, one clearly wants a more general definition which admits singularity functions in the impulse response. Zemanian [17] and Balakrishnan [18] have studied systems in which the inputs are assumed to be infinitely smooth functions and the outputs are distributions. Extended spaces appeared first in the context of stability theory as a result of the work of Sandberg [4] and Zames [5].

For many purposes, it is convenient to impose some smoothness conditions on the operators in question. Note that U and Y have, as signal spaces, no topology since they are, although derived from normed spaces, not normed themselves. However, causality enables one nevertheless to make a suitable definition of local continuity. Although simple continuity is the most logical smoothness condition to impose, it is very often advantageous to require somewhat stronger conditions, more specifically Lipschitz continuity. Recall that a (in general nonlinear) map F between normed spaces is said to be *Lipschitz continuous* if there

[2] Note the abuse of notation in the fact that the symbol P_T is used to denote an operator on U *and* an operator on Y. This ambiguity, however, causes no difficulty.

exists a real constant $K < \infty$ such that for all x_1, x_2 in the domain of F, $\|Fx_1 - Fx_2\| \leqq K\|x_1 - x_2\|$. Let F be a causal map from the input space U into the output space Y. Then F is said to be *locally Lipschitz continuous* if for all $t_0, t_1 \in R$, $P_{t_1}FQ_{t_0}$ is Lipschitz continuous as a map from $L_2(V_u)$ into $L_2(V_y)$.

DEFINITION 3. A dynamical system is said to be *smooth* if the defining map G is locally Lipschitz continuous. It is said to be *uniformly smooth* if for any given $T > 0$, $P_{t+T}GQ_t$ is Lipschitz continuous uniformly in t.

Convention. For convenience it will be assumed that all dynamical systems under consideration are *unbiased*, i.e., that they map the zero input into the zero output; hence $G0 = 0$. This absence of a bias term can always be obtained by a trivial redefinition of G and assumes, for instance, that for memoryless operators the element $r \in Y$ appearing in the definition is the zero element.

Now that the definition of input-output models of dynamical systems has been introduced, attention is focused on the formalism for the state space description of dynamical systems. Let R_2^+ denote the *causal sector* of R_2 defined as $R_2^+ \triangleq \{(t_2, t_1) | t_2, t_1 \in R, t_2 \geqq t_1\}$.

DEFINITION 4. A (mathematical model of a) dynamical system is said to be in *state space form* if it is determined by an abstract set X (the *state space*) and two maps, ϕ, the *state transition map*, and y, the *output reading map*, satisfying the following axioms:

(i) ϕ maps $R_2^+ \times X \times U$ into X;

(ii) (*Causality*): $\phi(t, t_0, x_0, u) = \phi(t, t_0, x_0, P_tQ_{t_0}u)$ for all $(t, t_0) \in R_2^+$, $x_0 \in X$, and $u \in U$;

(iii) (*Consistency*): $\phi(t_0, t_0, x_0, u) = x_0$ for all $t_0 \in R, x_0 \in X$, and $u \in U$;

(iv) (*Composition law or semi-group property*): $\phi(t_2, t_0, x_0, u) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u)$ for all $(t_1, t_0), (t_2, t_1) \in R_2^+, x_0 \in X$, and $u \in U$;

(v) y maps $R \times X \times V_u$ into V_y and the value of the output at time t is given by $y(t, x(t), u(t))$;

(vi) X is a subset of an inner product space V_x;

(vii) (*Unbiasedness*): $\phi(t, t_0, 0, 0) = 0$ for all $(t, t_0) \in R_2^+$, and $y(t, 0, 0) = 0$ for all $t \in R$;

(viii) Let $\tilde{X}$ denote the signal space induced by V_x (i.e., $\tilde{X} = S(V_x)$); it is then assumed that the functions

$$x(t) = Q_{t_0}\phi(t, t_0, x, u) = \begin{cases} \phi(t, t_0, x, u) & \text{for } t \geqq t_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$Q_{t_0}y(t, x(t), u(t)) = \begin{cases} y(t, x(t), u(t)) & \text{for } t \geqq t_0, \\ 0 & \text{otherwise,} \end{cases}$$

belong to $\tilde{X}$ and Y respectively for all $t_0 \in R$, $x_0 \in X$, and $u \in U$.

Axioms (i)–(v) are the usual axioms involved in describing dynamical systems from a state space point of view. Axiom (vi) induces a topology on the state space and will be needed in the definitions of Lyapunov stability, for instance.[3] Axiom

[3] The assumption that X is an inner product space is restrictive and inconvenient for many applications, more so than would appear at first sight. For a study of dynamical systems whose state space is a group manifold see [26].

(vii) is in keeping with the unbiasedness convention introduced above, and axiom (viii) guarantees the absence of a finite escape and results in the fact that locally square integrable inputs produce locally square integrable outputs.

A dynamical system in state space form thus views the generation of outputs from inputs as occurring through the composition of two maps, G_x and G_y, with $G_x\colon U \to X$ and $G_y\colon X \times U \to Y$. The map G_x is a dynamical system in its own right (with X viewed as the output space) but satisfies a richer set of axioms than merely those implied by the dynamical system axioms: in addition, it is required that this map have the Markov property which results in a decoupling of the past from the future in the sense that the present value of the state has sufficient information in it so as to summarize the effect of past inputs. The state space thus represents an adequate memory-bank. The map G_y is memoryless (with input space $X \times U$) and the dependence of $y(t)$ on past values of u is obtained through the dependence on $x(t)$. It is a simple matter to verify that the composite system $y = G_y(G_x u, u)$ is indeed a dynamical system in the input-output sense. Note also that $G_x 0$ satisfies the axioms for dynamical systems without inputs as studied in classical mechanics and its extensions.

The next definitions refer to the smoothness of the state transition map and the output reading map. These smoothness conditions are generally quite important aspects of a particular dynamical system in state space form. For instance, it can be shown that otherwise any finite-dimensional dynamical system can be realized by a one-dimensional dynamical system if this latter is not required to have any smoothness. It suffices therefore to consider a one-to-one map from R^n into R and appropriately modify the state space and the maps defining the dynamical system.

DEFINITION 5. A dynamical system in state space form is said to be *smooth* if for any $(t_1, t_0) \in R_2^+$ there exist $K_1, K_2, K_3, K_4 < \infty$ such that

$$\|P_{t_1}Q_{t_0}(\phi(t, t_0, x_1, u_1) - \phi(t, t_0, x_2, u_2))\| \leqq K_1\|x_1 - x_2\| + K_2\|P_{t_1}Q_{t_0}(u_1 - u_2)\|$$

for all $x_1, x_2 \in X$ and $u_1, u_2 \in U$, and

$$\|P_{t_1}Q_{t_0}(y(t, x_1(t), u_1(t)) - y(t, x_2(t), u_2(t)))\| \leqq K_3\|P_{t_1}Q_{t_0}(x_1 - x_2)\| + K_4\|P_{t_1}Q_{t_0}(y_1 - u_2)\|$$

for all $x_1, x_2 \in \widetilde{X}$ and $u_1, u_2 \in U$.

It is said to be *uniformly smooth* if for any $T \geqq 0$ and $t_1 = t_0 + T$ in the above inequalities, the constants K_1, K_2, K_3, K_4 may be taken independent of t_0.

These definitions are entirely analogous to those imposed for input-output systems. It is a simple matter to verify that (uniformly) smooth dynamical systems in state space form define (uniformly) smooth input-output dynamical systems. It is also clear that uniform smoothness and smoothness are equivalent for time-invariant systems.

The final discussion of this section involves the relationship between the above definitions of dynamical systems. As might be expected they are indeed equivalent.

DEFINITION 6. Consider the dynamical system G and a dynamical system in state space form with defining maps ϕ and y. Then the dynamical system in state space form is said to be a *realization* of G if any $u \in U$ with $t_0 \in R$ such that $P_{t_0}u = 0$, yields $(Gu)(t) = y(t, \phi(t, t_0, 0, u), u(t))$ for all $t \in R$. The dynamical system in state space form thus defines the same input-output relation as G.

THEOREM 1. *Every (smooth, uniformly smooth) dynamical system has a (smooth, uniformly smooth) realization in state space form.*

Proof. The proof proceeds by construction. The state space $X = V_x$ will be taken to be the collection of all functions in $L_2(V_u)$ with compact support, and the state at time t will be taken to be $P_t u$, where $u \in U$ is the input to the system. Thus, for instance,

$$\phi(t, x_0, t_0, u) = S_t P_t u \overset{\Delta}{=} x_0 + S_t P_t Q_{t_0} u,$$

where S_T denotes the shift operator $(S_T z)(t) = z(t - T)$, and

$$y(t, x, u) = (Gu)(t) = (GP_t u)(t) = (GS_{-t} S_t P_t u)(t) = \overset{\Delta}{=} (GS_{-t} x(t))(t).$$

It is left to the reader to verify that these maps indeed satisfy the axioms of dynamical systems in state space form. The smoothness claims are also easily verified directly in view of the simplicity of the state transition map and the fact that the output map and the original dynamical system are essentially identical.

The above theorem, although too trivial and general to be of significance in specific instances, yields a rather interesting canonical decomposition of nonlinear dynamical systems into a linear, time-invariant, reachable dynamical part followed by a memoryless nonlinear part.[4] Note also that the dynamical part in this decomposition may be described by the partial differential equation

$$\partial x(z, t)/\partial t = \partial x(z, t)/\partial z, \qquad z \leqq 0,$$

with the boundary control $x(0, t) = u(t)$ and with solutions defined in an appropriate sense. The function $x(z, t)$ for $z \geqq 0$ then plays the role of the state at time t, and the partial differential equation describes the evolution of the initial state $x(z, t_0)$ resulting from the input $u(t)$. Notice also that in the above realization the map y inherits linearity and time-invariance of G.

It should be noted that the equivalence of a dynamical system and a state space realization of a dynamical system might nevertheless lead the latter model to produce an output which could not be the result of any input to the former model. Such outputs result from initial states which are somewhat artificial in the sense that they cannot be produced by past inputs. The equivalence of a dynamical system and one of its realizations is thus really a zero initial state equivalence.

Notation. Let G denote a dynamical system in state space form, $x_0 \in X$, $t_0 \in R$, and $u \in U$. Then the function defined by

$$\begin{cases} y(t, \phi(t, t_0, x_0, u), u(t)) & \text{for } t \geqq T \geqq t_0, \\ 0 & \text{otherwise,} \end{cases}$$

[4] It was pointed out to the author that similar decomposition due to Wiener [19] and Balakrishnan [16] have appeared in the literature.

will be denoted by $Q_T G(t_0, x_0, u)$ (or $Q_T y(t_0, x_0, u)$ when there is no danger of confusion).

3. Fundamental properties of dynamical systems. A number of fundamental concepts related to dynamical systems and their state space realizations are introduced and discussed in this section: they relate to the influence of the control on the state (reachability, controllability, and connectedness), of the state on the output (observability and irreducibility), and of the input on the output (stability and continuity).

DEFINITION 7. The state space of a dynamical system in state space form is said to be *reachable* if given any $x \in X$ and $t \in R$, there exists a $t_0 \in R$, $t_0 \leqq t$, and a $u \in U$ such that $\phi(t, t_0, 0, u) = x$. A dynamical system in state space form is said to be *controllable* if given any $x_0 \in X$ and $t_0 \in R$, there exists a $t \in T$, $t \geqq t_0$, and a $u \in U$ such that $\phi(t, t_0, x_0, u) = 0$. The state space of a dynamical system in state space form is said to be *connected* if given any $x_0, x_1 \in X$ there exists an element $(t_1, t_0) \in R_2^+$ and a $u \in U$ such that $\phi(t_1, t_0, x_0, u) = x_1$.

Reachability thus requires the map $\phi(t, \cdot, 0, \cdot)$ to be onto X, whereas controllability requires that 0 be in the range space of $\phi(\cdot, t_0, x_0, \cdot)$. Note that reachability, controllability, and time-invariance imply connectedness.

DEFINITION 8. A dynamical system in state space form is said to be *observable* if for any $t_0 \in R$, knowledge of $Q_{t_0} y(t_0, x_0, 0)$ (uniquely) determines $x_0 \in X$. The state space of a dynamical system in state space form is said to be *irreducible* if for any given $t_0 \in R$ and $x_0 \in X$ there exists a $Q_{t_0} u \in U$, such that knowledge of $Q_{t_0} y(t_0, x_0, u)$ (uniquely) determines $x_0 \in X$.

Observability thus requires the map $y(t_0, \cdot, 0)$ to be one-to-one on X, whereas irreducibility requires the map $y(t_0, \cdot, u)$ to be one-to-one on X by choosing $(Q_{t_0} u)(x)$. It is clear that observability implies irreducibility and that the nomenclature "irreducible" is quite appropriate since if the state space is not irreducible, then there exist at least two initial states which will be completely indistinguishable under experimentation: these two states are thus entirely equivalent, and nothing will be lost by eliminating one of them from the state space.

The above nomenclature is common (although far from standard) in the related literature with the possible exception of irreducibility which is often taken to indicate the set-theoretic minimality of the state space. Observability and irreducibility are equivalent for linear systems with a finite-dimensional smooth state space realization. The simplest example of systems in which these concepts are different are systems with multiplicative control described, e.g., by $\dot{x} = uAx$. It should also be remarked that the above definitions, although natural, are not the most convenient ones for certain applications. Although it can be shown that every dynamical system has a realization[5] with a reachable state space, it is sometimes very difficult to discover exactly what states are reachable (and to define X then appropriately). For instance, in systems described by partial differential equations these reachable states have certain smoothness properties which are not a priori known; therefore, in certain applications it is much more convenient to adopt an "almost" reachability requirement. The same remark holds for the following

[5] See, for example, the proof of Theorem 1.

definitions which, in addition, require an appropriate choice of the topology on the state space.

DEFINITION 9. The state space of a dynamical system in state space form is said to be *uniformly reachable* if there exist a continuous function $\alpha: R^+ \to R^+$ (R^+ denotes the nonnegative real numbers) with $\alpha(0) = 0$ and a constant $T \geqq 0$, such that for any $x \in X$ and $t \in R$, there exists a $u \in U$ with $\|P_t u\|^2 \leqq \alpha(\|x\|)$ such that $\phi(t, t - T, 0, u) = x$. *Uniform controllability* and *uniform connectedness* are similarly defined. A dynamical system is said to be *uniformly observable* if there exist a strictly monotone increasing continuous function $\beta: R^+ \to R$ with $\beta(0) = 0$ and $\lim_{\sigma \to +\infty} \beta(\sigma) = +\infty$ and a constant $T \in R$, $T \geqq 0$, such that for any $x \in X$ and $t_0 \in R$,

$$\|P_{t_0+T} Q_{t_0} y(t_0, x, 0)\|^2 \geqq \beta(\|x\|).$$

The state space of a dynamical system in state space form is said to be *uniformly irreducible* if with β and T as before, the inequality

$$\|P_{t_0+T} Q_{t_0}(y(t_0, x_1, u) - y(t_0, x_2, u))\|^2 \geqq \beta(\|x_1 - x_2\|)$$

holds for all $x_1, x_2 \in X$, $t_0 \in R$, and some $u \in U$.

The above definitions differ somewhat from those in the literature. Most of the papers concerned with uniform controllability for linear systems follow Kalman's [20] original definition, which imposes many more restrictions than the definitions used here. In particular, it requires any control which makes the transfer from state 0 at time $t - T$ to state x at time t to be such that $\|P_t Q_{t-T} u\|^2 \geqq \alpha(\|x\|) > 0$.

The most efficient realization of a dynamical system is one in which the state space is reachable *and* irreducible. This indeed guarantees that every output which can be observed as a result of initial conditions and inputs could have been observed by properly choosing the past input and that two different initial conditions will lead to different outputs by properly choosing the input. Two realizations which are both reachable *and* irreducible are thus isomorphic. They differ in the sense that their state spaces are labeled differently. The one-to-one onto map between these state spaces may, in general, be a function of time, however. A realization of a dynamical system in which the state space is reachable and irreducible can thus properly be called *minimal*, a notion which has many more substantive implications for linear systems. In looking for reachable *and* irreducible realizations it is natural to consider as the candidate for the state space the equivalence classes of those inputs up to time t which yield the same output after time t, regardless of the input after time t; more precisely, by considering the equivalence class $\{P_t u | \, Q_t y$ is fixed for all $Q_t u\}$ as a typical element of the state space. The difficulty with this representation is that, in general, the state space itself then becomes a function of time. There are two methods of getting around this difficulty: one is to modify the original axioms and definitions so as to allow for a state space which is itself a function of time; the other is to define a dynamical system as a causal *and* a noncausal map depending on whether one considers time moving forward or backward from the initial time. The state is thus alternatively required to summarize past and future, and the state space thus has many more invariant properties with respect to time. This device has been used successfully by Kalman

[21] and others in their study of systems described by the Volterra integral equation

$$y(t) = \int_{t_0}^{t} w(t, \tau)u(\tau)\, d\tau$$

with separable kernel w. This principle rests on dubious physical grounds, however, and leads to technical difficulties for infinite-dimensional systems. The above problems do not occur in stationary systems.

Recall that a mapping between normed spaces is said to be *bounded* if it maps bounded sets into bounded sets. It is said to have a *finite gain* if there exists a $K < \infty$ such that for any $\rho \geqq 0$ the ball with radius ρ gets mapped into the ball with radius $K\rho$. The infimum of all real numbers K achieving the above inequality is usually called the *gain* of the operator.

DEFINITION 10. A dynamical system, G, is said to be *input–output stable* if it maps bounded sets of small signals in U into bounded sets of small signals in Y. It is said to be *finite-gain input-output stable* if it is stable and if there exists a $K < \infty$ such that for any small signal $u \in U$, $\|Gu\| \leqq K\|u\|$. The infimum of all such real numbers K will be denoted by $\|G\|$. A dynamical system G is said to be *input-output continuous* if it is stable and if the map G is continuous (in the topology induced by $L_2(V_u)$ and $L_2(V_y)$) as a map from $U \cap L_2(V_u)$ into $Y \cap L_2(V_y)$. It is said to be *input-output Lipschitz continuous* if G is actually Lipschitz continuous.

It can be shown [14, § 2.4] that a dynamical system is finite-gain stable if and only if the gain of $P_{t_1}GQ_{t_0}$ is bounded for all $t_0, t_1 \in R$, uniformly in t_0 and t_1. In fact,

$$\|G\| = \lim_{-t_0, t_1 \to \infty} \|P_{t_1}GQ_{t_0}\|,$$

and this limit is approached monotonically. A similar relationship holds for Lipschitz continuity.

Related, but not identical, are the following more familiar Lyapunov stability concepts for dynamical systems in state space form.

DEFINITION 11. The equilibrium state of a dynamical system in state space form is said to be *globally attractive* if for any $x_0 \in X$ and $t_0 \in R$,

$$\lim_{T \to \infty} \phi(t_0 + T, t_0, x_0, 0) = 0.$$

It is said to be *uniformly globally attractive* if this limit is uniform in t_0. It is said to be *stable* if for any $\varepsilon > 0$ and $t_0 \in R$ there exists a $\delta(\varepsilon, t_0)$ such that $\|\phi(t_0 + T, t_0, x_0, 0)\| \leqq \varepsilon$ for all $T \geqq 0$ whenever $\|x_0\| \leqq \delta$. It is said to be *uniformly stable* if $\delta(\varepsilon, t_0)$ may be chosen independent of t_0. A dynamical system in state space form is said to be *bounded* if for any $x_0 \in X$ and $t_0 \in R$, $\phi(t_0 + T, t_0, x_0, 0)$ is bounded on the half-line $T \geqq 0$. It is said to be *uniformly bounded* if this bound may be chosen independent of t_0. A dynamical system in state space form is said to be *globally asymptotically stable* if the equilibrium state is globally attractive and stable. It is said to be *uniformly globally asymptotically stable* if the equilibrium state is uniformly globally attractive, uniformly stable, and uniformly bounded.

The usual method of proving stability of systems in state space form is to consider an appropriate Lyapunov function. The following definition of a Lyapunov function is a convenient one for the present discussion.

DEFINITION 12. Let V be a mapping from $X \times R$ into R^+, with $V(x, t) = 0$ if and only if $x = 0$. Then V is said to be a *Lyapunov function* for a dynamical system in state space form if for any $x_0 \in X$ and $t_0 \in R$,

(i) $V(\phi(t, t_0, x_0, 0), t)$ is a monotone nonincreasing function of t for $t \geqq t_0$;

(ii) $\lim_{T \to \infty} V(\phi(t_0 + T, t_0, x_0, t), t_0 + T) = 0$.

The function V will be called a *uniform Lyapunov function* if, in addition, the limit in (ii) is uniform in t_0 and if $V(x, t)$ is bounded in t for all $x \in X$. The function V is said to be *decrescent* if there exists a continuous function $\alpha: R^+ \to R^+$ with $\alpha(0) = 0$ such that $V(x, t) \leqq \alpha(\|x\|)$ for all $x \in X$ and $t \in R$. It is said to be *positive definite* if there exists a monotone increasing continuous function $\beta: R^+ \to R^+$ with $\beta(0) = 0$ such that $V(x, t) \geqq \beta(\|x\|)$ for all $x \in X$ and $t \in R$. It is said to be *radially unbounded* if there exists a continuous function $\gamma: R^+ \to R^+$ with $\lim_{\sigma \to +\infty} \gamma(\sigma) = +\infty$ such that $V(x, t) \geqq \gamma(\|x\|)$ for all $x \in X$ and $t \in R$. If V is a Lyapunov function for a dynamical system in state space form, then the equilibrium state is globally asymptotically stable if V is positive definite, and the dynamical system is bounded if V is radially unbounded. If V is a uniform Lyapunov function, then the equilibrium state is uniformly globally attractive if V is positive definite, and uniformly stable if V is positive definite and decrescent; the system is uniformly bounded if V is radially unbounded, and uniformly globally asymptotically stable if V is radially unbounded, positive definite, and decrescent. Notice also that decrescency implies the last condition in the definition of a uniform Lyapunov function, and thus a decrescent Lyapunov function for a uniformly globally asymptotically stable system is a uniform Lyapunov function.

The main purpose of this paper is to study the relations between input-output stability and global stability. It seems reasonable to expect that an input-output stable system will be globally stable if inputs sufficiently influence states and if states sufficiently influence outputs. Then internal instability should reflect into external instability. That this can be made precise is shown in the next section.

4. Input-output stability and global stability. This section establishes the fundamental relationship between input-output stability and global stability. In trying to obtain these internal stability implications from external data, one defines certain functions which depend on the external variables only. In order for these functions to be well-defined and to qualify as suitable Lyapunov functions, a number of additional assumptions have to be made, and it is at this point that reachability, controllability, observability, and input-output stability become relevant. There are two natural functions to consider for this purpose:

(i) $V_r(x, t) \triangleq \inf \|P_t Q_{t_0} u\|^2$, where the infimum is to be taken over all $t_0 \leqq t$ and $u \in U$ with $\phi(t, t_0, 0, u) = x$ (the infimum (supremum) over the void set is by assumption $+\infty(-\infty)$), and

(ii) $V_0(x, t) \triangleq \|Q_t y(t, x, 0)\|^2$.

The symbolism is clear: the first function is inspired by reachability, and the second by observability. V_r is well-defined if the state space is reachable, and V_0 is well-defined if the state space is reachable and if the dynamical system is input-output stable. Indeed, let $u \in U$ be such that $\phi(t, t_0, 0, u) = x$. Then

$$\|Q_t y(t, x, 0)\|^2 \leqq \|G P_t u\|^2 \leqq \|G\|^2 \|P_t u\|^2.$$

If $V_0(x, t)$ is well-defined, then it is clearly monotone nonincreasing along undriven solutions and approaches zero when $t \to \infty$ since

$$V_0(\phi(t, t_0, x_0, 0), t) = \int_t^\infty \|y(\tau, \phi(\tau, t_0, x_0, 0), 0)\|_{V_y}^2 \, d\tau.$$

The function V_r is also monotone nonincreasing along undriven solutions, although this is not as immediate. The argument used in the demonstration of this fact will repeatedly be used in the sequel and will therefore only here be done explicitly. Thus, consider $V_r(\phi(t_1, x, t, 0), t_1)$ with $t_1 \geqq t$, and denote $\phi(t_1, x, t, 0)$ by x_1. Then $V_r(x_1, t_1) = \inf \|P_{t_1} Q_{t_0} u\|^2$. The state of the dynamical system can be driven to x_1 at time t_1 by first driving it from 0 at time t_0 to x at time t and then applying zero control until time t_1. This is, in general, a suboptimal control for reaching x_1 at time t_1, even when it is driven to x at time t in an optimal fashion. This suboptimal strategy thus shows that $V_r(x_1, t_1) \leqq V_r(x, t)$.

The basic relationships between input-output stability and global stability are stated in the following theorems.

THEOREM 2. *A uniformly observable realization of an input-output stable dynamical system with a reachable state space is globally asymptotically stable and bounded, and V_0 is a positive definite radially unbounded Lyapunov function for it.*

Proof. It suffices to show that V_0 is a positive definite radially unbounded Lyapunov function. By reachability and input-output stability, V_0 is well-defined. By observability, $V_0 = 0$ if and only if $x = 0$; it is monotone nonincreasing and approaches zero along undriven solutions since

$$\int_t^\infty \|y(\tau, \phi(\tau, t, x, 0), 0)\|_{V_y}^2 \, d\tau < \infty.$$

It remains to be shown that V is positive definite and radially unbounded. This, however, is an immediate consequence of uniform observability.

Note that in the above theorem the uniform observability condition cannot simply be relaxed to only observability, even if at the same time one assumes uniform reachability rather than merely reachability.

THEOREM 3. *A uniformly observable realization of a finite-gain input-output stable dynamical system with a uniformly reachable state space is uniformly globally asymptotically stable, and V_r and V_0 are positive definite radially unbounded decrescent uniform Lyapunov functions for it.*

Proof. From finite-gain input-output stability it follows that

$$\|P_t y(t, 0, u)\|^2 \leqq \|G\|^2 \|P_t u\|^2$$

and thus that

$$V_0(x, t) \leqq \|G\|^2 V_r(x, t).$$

V_r is well-defined and decrescent by uniform reachability, and V_0 is well-defined, positive definite, and radially unbounded by uniform observability. Thus both V_0 and V_r are positive definite, radially unbounded, and decrescent, and the theorem follows if it can be shown that $V_0(\phi(t_0 + T, t_0, x_0, 0), t_0 + T)$ approaches zero as $T \to \infty$, uniformly in t_0. Notice that with T as in the definition of uniform observability it follows that

$$V_0(\phi(t_0 + T, t_0, x_0, 0), t + T)|V_0(x_0, t_0) \leqq 1 - \frac{\beta(\|x_0\|)}{\|G\|^2\alpha(\|x_0\|)},$$

where α and β are as in the definitions of uniform reachability and uniform observability. This implies uniform convergence of the Lyapunov function V_0 since for any $\varepsilon > 0$ and $M < \infty$ there exists a $\delta > 0$ such that $\beta(\sigma)|\alpha(\sigma) \geqq \delta > 0$ for all $0 < \varepsilon \leqq \sigma \leqq M < \infty$. Hence the system is uniformly globally asymptotically stable, which in turn implies that V_r is a uniform Lyapunov function.

Note that finite-gain stability (or uniform smoothness) and uniform observability yield that every control u transferring state 0 at time t_0 to state x at time t requires

$$\|P_t Q_{t_0} u\|^2 \geqq \|G\|^{-2}\beta(\|x\|),$$

a condition which is usually part of the definition of uniform controllability [20]. As a final remark in this section, note that the fact that the inputs and outputs take their values in inner product spaces is inessential and that the results hold, mutatis mutandis, if these spaces are merely normed spaces. The inner product structure becomes very important in the next section, which is concerned with passivity.

5. Lyapunov functions for passive systems. The notions which will be introduced in this section are those of passivity and certain concepts related to energy. It will be assumed in this section that the inner product spaces under consideration are real.[6]

DEFINITION 13. Let $U = Y$ and let G be a dynamical system from U into Y. Then G is said to be *passive* if for all $u \in U$ and $t \in R$, $\langle P_t u, P_t G u\rangle \geqq 0$. It is said to be *strictly passive* if $G - \varepsilon I$ is passive for some $\varepsilon > 0$.

This terminology is to be interpreted as follows: $\langle u(t), y(t)\rangle_{V_u}$ represents the instantaneous power delivered to the system from the outside. Thus $\langle P_t u, P_t y\rangle$ represents the total energy at time t delivered to the system from the outside. If regardless of the termination and in the absence of initial excitations this energy is nonnegative, then the system is passive viewed from its input-output terminals.

It can be shown (see [14, § 2.17]) that if G is input-output stable, then it is passive if and only if $\langle u, Gu\rangle \geqq 0$ for all small input signals u.

DEFINITION 14. Let G be a dynamical system in state space form. Then the *required energy*, E_r, is defined on $X \times R$ as

$$E_r(x, t) \triangleq \inf \langle P_t u, P_t G u\rangle,$$

where the infimum is to be taken over all $t_0 \leqq t$ and $u \in U$ with $P_{t_0} u = 0$ which yield $\phi(t, t_0, 0, u) = x$. The *available energy*, E_a, is defined on $X \times R$ as

$$E_a(x, t) \triangleq \sup_{\substack{u \in U \\ t_1 \geqq t}} - \langle P_{t_1} Q_t u, P_{t_1} Q_t y(t, x, u)\rangle.$$

The *cycle energy*, E_c, is defined on $X \times R$ as $E_r - E_a$. Thus

$$E_c(x, t) = \inf \langle P_{t_1} u, P_{t_1} G u\rangle,$$

[6] Complex inner product spaces can be treated equally well by considering the real part of the inner product in the definitions of passivity and energy.

where the infimum is to be taken over all t_0 and t_1 with $t_0 \leqq t \leqq t_1$ and $u \in U$ with $P_{t_0}u = 0$, which yields $\phi(t, t_0, 0, u) = x$.

The available energy is thus the maximum energy which can be extracted from a system, whereas the required energy is the energy needed to excite a system to a given set of initial conditions. The cycle energy is the minimum energy it takes to cycle a system between the equilibrium and a given state. Note that all of the above energies are defined in terms of input-output relations.

LEMMA 1. *Consider a realization of a passive dynamical system and assume that the state space is reachable. Then E_a, E_r and E_c exist (i.e., E_a, E_r, $E_c < \infty$) and are nonnegative. Moreover, $0 \leqq E_a, E_c \leqq E_r$.*

Proof. That E_r and E_c are finite and nonnegative follows immediately from passivity and reachability. Hence, since $E_a + E_c = E_r$, $E_a \leqq E_r$.

It remains to be shown that E_a is nonnegative. This follows by considering $Q_t u = 0$, which shows that the supremum in the definition of E_c is taken over a set which contains zero. This completes the proof of the lemma.

The inequality $E_a \leqq E_r$ formalizes the intuitive notion that passive systems cannot supply more energy to the outside than has previously been supplied to them from the outside. Note that none of the above notions satisfactorily defines the *stored energy*, $E_s(x, t)$, which is an internal property of a dynamical system and thus usually a function of the realization. The passivity definition employed here is purely input–output. Similar definitions of internal passivity can be made, and the theory for linear time-invariant dissipative systems [22] is available. One can then pose the question of whether or not every input-output passive system has a passive realization. These ramifications fall beyond the scope of the present paper. It would be interesting to verify that the stored energy in a passive realization of a passive system satisfies the inequality $E_a \leqq E_s \leqq E_r$, as it should.

The cycle energy E_c is a measure of the degree of irreversibility of a system. This is the intuitive basis for the following definitions.

DEFINITION 15. A passive dynamical system in state space form is said to be *irreversible* if $E_c(x, t) = 0$ only if $x = 0$. It is said to be *uniformly irreversible* if there exists a monotone increasing function $\gamma: R^+ \to R^+$ with $\gamma(0) = 0$ and $\lim_{\sigma \to +\infty} \gamma(\sigma) = +\infty$ such that for all $x \in X$ and $t \in R$, $E_c(x, t) \geqq \gamma(\|x\|)$. It is said to be *reversible* if $E_c = 0$, i.e., if $E_r = E_a$.

THEOREM 4. *The available energy, E_a, and the required energy, E_r, are decrescent uniform Lyapunov functions for a uniformly observable realization of a passive finite-gain input-output stable dynamical system with a uniformly reachable state space.*

Proof. It will first be shown that E_r is decrescent. By the Schwarz inequality,

$$|\langle P_t u, P_t G u \rangle| \leqq \|P_t u\| \, \|P_t G u\|.$$

It thus follows from finite-gain stability that there exists a constant $K < \infty$ such that $E_r(x, t) \leqq K \inf \|P_t u\|^2$, where the infimum is to be taken over all $t_0 \leqq t$ and $u \in U$ with $P_{t_0}u = 0$, $\phi(t, t_0, 0, u) = x$. By uniform reachability, $E_r(x, t)$ is thus decrescent. It will now be shown that E_r is a Lyapunov function. That $E_r(x, t)$ is nonincreasing along undriven solutions follows from its definition and by letting $u = 0$ from t_0 until t_1, using an analogous argument to the one used in § 4 in showing that V_r is monotone nonincreasing. By Theorem 3,

$$\phi(t_0 + T, t_0, x_0, 0) \to 0 \quad \text{as } T \to \infty,$$

uniformly in t_0, which by decrescence indeed implies that E_r is a uniform Lyapunov function. Since $E_a \leqq E_r$, it remains to be shown that E_a is monotone nonincreasing along undriven solutions. This follows from an analogous argument to the one used to show that E_r is nonincreasing. This completes the proof.

Theorem 4 is not as convincing as one might like it to be since it does not make any claims about the positive definiteness of the Lyapunov functions. This positive definiteness can be obtained using somewhat stronger hypotheses. The available energy E_a will be positive definite if the feedback system with the dynamical system G in the forward and some constant gain $k > 0$ in the feedback loop remains a well-defined dynamical system. This is the case under weak additional assumptions on G. The resulting control to be used to show definiteness of E_a is the solution e of the feedback equation $Q_t(e + kG(t, x, e)) = 0$. This corresponds to the input which results from a termination of the system with a positive resistor. The required energy E_r will be positive definite if E_a is or if the system is strictly passive (rather than merely passive). A third possibility is to require uniform irreversibility, since $E_c \leqq E_r$.

6. Feedback systems. One of the main reasons for being interested in stability stems from its importance in feedback control. The canonical form of the feedback system considered in this paper is shown in Fig. 1, and the closed loop system is thus described by the implicit equations

(FE) $$(I + G)e = u, \qquad y = Ge.$$

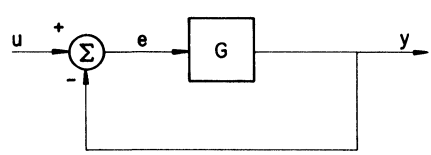

FIG. 1. *The feedback system under consideration*

It will be assumed that the input signal space U and the output signal space Y are the same and that G is a dynamical system from U into Y. Two questions related to the well-posedness of this feedback system arise: first, whether or not the closed loop feedback system still represents a well-defined dynamical system in its own right and, second, whether or not the state space induced by a realization of G will also qualify as the state space for the closed loop system. These issues fall beyond the scope of this paper, and it will be explicitly assumed rather that these well-posedness conditions are satisfied. It is thus assumed that:

(i) $(I + G)^{-1}$ exists (as a map from U into itself) and is causal. This implies that the closed loop system $G(I + G)^{-1} = I - (I + G)^{-1}$ is itself a dynamical system from the input space U into the output space Y.

(ii) If G is described in state space form with state space X, then X also qualifies as the state space for the closed loop dynamical system $G(I + G)^{-1}$, and a unique solution to the feedback equations exists for any initial condition $x_0 \in X$, initial time $t_0 \in R$, and input $u \in U$.

These well-posedness questions have been investigated in the literature [14, § 4.2], [15] and the simplest sufficient conditions essentially impose a restriction on the feedthrough in G, in addition to some smoothness conditions on the open loop system. They are satisfied in most models and, in particular, whenever G contains a pure or generalized delay.

In the study of feedback systems it is important to establish conditions on the open loop operator in order to draw conclusions about the closed loop system. The first questions thus answered are those related to controllability and irreducibility.

THEOREM 5. *Consider the feedback system described by equations* (FE). *Then reachability* (*controllability, connectedness, irreducibility*) *of the state space realization of the open loop system implies reachability* (*controllability, connectedness, irreducibility*) *of the associated state space realization of the closed loop system. Uniform reachability* (*controllability, connectedness, irreducibility*) *of the state space realization of the open loop system implies uniform reachability* (*controllability, connectedness, irreducibility*) *of the associated state space realization of the closed loop system provided the open loop system is in addition uniformly smooth.*

Proof. Preservation of reachability, controllability, or connectedness is essentially obvious. Indeed, let u_1 be a control which results in the desired transfer for the open loop system. Then the control $u = u_1 + Gu_1$ will clearly result in the same transfer for the closed loop system. Irreducibility of the closed loop system will be established by contradiction. Assume therefore that there exist $x_1, x_2 \in X, x_1 \neq x_2$, and $t_0 \in R$ such that

$$Q_{t_0}G(I + G)^{-1}(t_0, x_1, u) = Q_{t_0}G(I + G)^{-1}(t_0, x_2, u) \qquad \text{for all } u \in U.$$

This implies that

$$Q_{t_0}G(t_0, x_1, u_1) = Q_{t_0}G(t_0, x_2, u_1)$$

for all u_1 which can be written as

$$u_1 = u - Q_{t_0}G(I + G)^{-1}(t_0, x_1, u), \qquad u \in U.$$

Since u_1 can thus be taken to be any element of U by choosing $u = u_1 + Q_{t_0}G(t_0, x_1, u_1)$, this shows that

$$Q_{t_0}G(t_0, x_1, u_1) = Q_{t_0}G(t_0, x_2, u_1) \qquad \text{for all } u_1 \in U.$$

Hence, the open loop system is not irreducible whenever the closed loop system is not irreducible. To show uniform reachability, let u be a control such that $\phi(t, t_0, 0, u) = x$ with $t_0 = t - T$ and T as in the definition of uniform reachability. The control $u + Gu$ then transfers the closed loop system from state 0 at $t_0 = t - T$ to state x at t. Since

$$\|P_tQ_{t-T}u + P_tGQ_{t-T}u\| \leqq (1 + \|P_tGQ_{t-T}\|)\|P_tQ_{t-T}u\|,$$

uniform reachability of the closed loop system thus becomes a consequence of uniform reachability of the open loop system if $\|P_tGQ_{t-T}\|$ is uniformly bounded, which in turn is a consequence of the uniform smoothness assumption. Preservation of uniform irreducibility is shown in a similar way. This completes the proof.

The above theorem contains no surprises with the possible exception that it does not state the preservation of observability under feedback. This is in fact untrue, and it is necessary to consider nonlinear systems to obtain satisfactory counterexamples. The system $\dot{x} = uAx$, $y = Cx$ will lead to a counterexample for the contention that closed loop observability follows from open loop observability. This phenomenon is by and large a consequence of the definition of observability which is really observability under zero input. If one were to modify this definition and require that for all given inputs $u \in U$ the response to different initial states should be different, then this observability under arbitrary inputs would indeed by preserved under feedback. These two types of observability are equivalent for linear systems. Note that the above theorem states the equivalence of reachability (controllability, connectedness, irreducibility) of the open loop and the closed loop system since putting positive unit feedback around the closed loop system gives back the open loop system. The fact that only feedback systems with unit feedback are being considered is also inessential to the basic result.

7. Lyapunov functions for feedback systems. In trying to define Lyapunov functions for input-output stable feedback systems, one can of course apply the techniques developed by § 4 and § 5. Such an approach is not very promising since the computation of some of the Lyapunov functions defined there requires detailed knowledge of the closed loop system, whereas it is desirable to pose the calculations and variational questions entirely in terms of the open loop dynamical system. This holds, in particular, for the function V_0 defined in § 4.

It can be shown [13] that a feedback system is finite-gain input-output stable if and only if there exists a constant $\varepsilon > 0$ such that the inequality $\|P_t(I + G)u\| \geqq \varepsilon\|P_tu\|$ holds for all $u \in U$ and $t \in R$. In fact, ε^{-1} may be taken as any real number larger than the gain of $(I + G)^{-1}$. It should also be remarked that for linear feedback systems stability, continuity and finite-gain stability are equivalent.

Now consider the following two functions which are defined as variational problems and will lead to Lyapunov functions for feedback systems:

(i) $V_r(x, t) = \inf \|P_tQ_{t_0}(u + y(t_0, 0, u))\|^2$ when the infimum is to be taken over all $u \in U$ and $t_0 \in R$ such that $\phi(t, t_0, 0, u) = x$ (ϕ and y denote the state transition and output reading map of the open loop dynamical system).

(ii) $V_\varepsilon(x, t) = -\inf(\|P_{t_1}Q_t(u + y(t, x, u))\|^2 - \varepsilon^2\|P_{t_1}Q_tu\|^2)$, where the infimum is to be taken over all $u \in U$ and $t_1 \geqq t$.

THEOREM 6. *Assume that the feedback system described by equation* (FE) *is uniformly observable, finite-gain input-output stable, and that the state space is uniformly reachable. Let K denote the gain of $(I + G)^{-1}$ and let $0 < \varepsilon \leqq K^{-1}$. Then the feedback system is uniformly globally asymptotically stable and V_r and V_ε are positive definite radially unbounded decrescent uniform Lyapunov functions for it.*

Proof. Note that uniform global asymptotic stability and the claims about V_r follow from Theorem 3. It will now be shown that V_ε is finite.

Let $u_1 \in U$ and $t_0 \leqq t$ be such that $P_{t_0}u_1 = 0$ and $\phi(t, t_0, 0, u_1) = x$. Then

$$P_{t_1}Q_t(u + y(t, x, u)) = P_{t_1}(Q_t u + P_t u_1 + y(t_0, 0, Q_t u + P_t u_1)) - P_t(u_1 + y(t_0, 0, P_t u_1)).$$

Thus,

$$\begin{aligned}\|P_{t_1}Q_t(u + y(t, x, u))\|^2 &\geqq \varepsilon^2\|P_{t_1}(Q_t u + P_t u_1)\|^2 - \|P_t(u_1 + y(t_0, 0, P_t U_1))\|^2 \\ &= \varepsilon^2\|P_{t_1}Q_t u\|^2 + \varepsilon^2\|P_t u_1\|^2 - \|P_t(u_1 + y(t_0, 0, P_t u_1))\|^2\end{aligned}$$

and

$$-V_\varepsilon(x, t) \geqq \varepsilon^2\|P_t u_1\|^2 - \|P_t(u_1 + y(t_0, 0, P_t u_1))\|^2.$$

Since the right-hand side of this inequality depends on $P_t u_1$ only, the result follows. The fact that V_ε is nonnegative follows from taking $u = G(I + G)^{-1}(t, x, 0)$. That V_ε is monotone nonincreasing along solutions follows from the usual argument explained earlier. To show that

$$V_\varepsilon(\phi(t_0 + T, t_0, x_0, 0), t_0 + T) \to 0 \quad \text{as } T \to \infty$$

uniformly in t_0, it suffices to show that V_ε is decrescent since $\phi(t_0 + T, t_0, x_0, 0) \to 0$ as $T \to \infty$ uniformly in t_0. It follows from the above inequality that

$$V_\varepsilon(x, t) \leqq \inf(\|P_t(u_1 + y(t_0, 0, P_t u_1))\|^2 - \varepsilon^2\|P_t u_1\|^2),$$

where the infimum is to be taken over all $t_0 \leqq t_1$ and $u_1 \in U$ with $P_{t_0}u_1 = 0$ and $\phi(t, t_0, 0, u_1) = x$. Decrescence thus follows from uniform reachability and finite-gain stability. Positive definiteness and radial unboundedness follows by considering $u = e = -G(I + G)^{-1}(t, x, 0)$ which yields $V_\varepsilon(x, t) \geqq \beta(\|x\|)$ by uniform observability. This completes the proof.

The standard methods for proving stability of feedback systems is to show that the open loop gain is less than unity (small loop gain theorem) or to show that the open loop dynamical system may be viewed as the cascade of two passive systems (positive operator conditions). These cases admit special consideration and are treated in the remainder of this section.

Consider therefore the following two functions:

(i) $V_1(x, t) = -\inf(\|Q_t u\|^2 - \|Q_t y(t, x, u)\|^2)$, where the infimum is to be taken over all $u \in U$ with $\|u\| < \infty$, and

(ii) $V_2(x, t) = \inf(\|P_t u\|^2 - \|P_t G u\|^2)$, where the infimum is to be taken over all $t_0 \in R$ and $u \in U$ with $P_{t_0}u = 0$ and $\phi(t, t_0, 0, u) = x$.

THEOREM 7. *Assume that the feedback system described by equations* (FE) *is uniformly observable, that the open loop dynamical system has gain less than unity and that the state space is uniformly reachable. Then the feedback system is finite-gain input-output stable and uniformly globally asymptotically stable and V_1 and V_2 are positive definite radially unbounded decrescent uniform Lyapunov functions for it.*

The case in which the open loop consists of the composition of a dynamical system G_1 followed by a memoryless dynamical system G_2 whose gain product is less than unity has received a great deal of attention and leads to a Lyapunov function which only depends on G_1. Consider therefore the following functions:

(i) $V_1^*(x, t) = -\inf(\|Q_t u\|^2 - \|G_1\|^{-2}\|Q_t G_1(t, x, u)\|^2)$, where the infimum is to be taken over all $u \in U$ with $\|u\| < \infty$, and

(ii) $V_2^*(x, t) = \inf(\|P_t u\|^2 - \|G_1\|^{-2}\|P_t G u\|^2)$, where the infimum is to be taken over all $t_0 \in R$ and $u \in U$ with $P_{t_0} u = 0$ and $\phi(t, t_0, 0, u) = x$.

THEOREM 8. *Assume that the feedback system described by equations* (FE) *is uniformly observable and that the state space is uniformly reachable. Let* $G = G_2 G_1$, *where* G_1 *is a uniformly controllable, uniformly observable dynamical system and* G_2 *is a memoryless dynamical system. Assume that the product of the gains of* G_1 *and* G_2, $\|G_1\|\,\|G_2\|$, *is less than unity. Then the feedback system is finite-gain input-output stable and uniformly globally asymptotically stable and* V_1^* *and* V_2^* *are positive definite radially unbounded decrescent uniform Lyapunov functions for it.*

Proof. The proofs of Theorems 7 and 8 offer no surprises considering the previous theorems, and the details will be omitted. The stability claims follow from the so-called small-gain theorem [5] for input-output stability and Theorem 2. Existence of V_1, V_2, V_1^* and V_2^* follows from the small gain condition, decrescence from uniform reachability, and positive definiteness by taking $u = 0$. Monotonicity along undriven solutions requires a minor modification of the usual argument. Consider, for instance, the function V_1^* of Theorem 8. Let $t_1 \geqq t_0$ and $x_1 = \psi(t_1, t_0, x_0, 0)$, with ψ the state transition map of the closed loop feedback system. Choose u on (t_0, t_1) to equal $e = P_{t_1} Q_{t_0}(I + G)^{-1}(t_0, x_0, 0)$. Thus,

$$V_1^*(x_0, t_0) \geqq -\|e\|^2 + \|G_1\|^{-2}\|G_1(t_0, x_0, e)\|^2 + V_1^*(x_1, t_1).$$

Since, however,

$$\begin{aligned} -\|e\|^2 + \|G_1\|^{-2}\|P_{t_1}Q_{t_0}G_1(t_0, x_0, e)\|^2 &= -\|G_2 P_{t_1}Q_{t_0}G_1(t_0, x_0, e)\|^2 \\ &\quad + \|G_1\|^{-2}\|P_{t_1}Q_{t_0}G_1(t_0, x_0, e)\|^2 \\ &\geqq (-\|G_2\|^2 + \|G_1\|^{-2})\|P_{t_1}Q_{t_0}G_1(t_0, x_0, e)\|^2 \end{aligned}$$

and $\|G_1\|^{-2} - \|G_2\|^2 > 0$, $V_1^*(x_0, t_0) \geqq V_1^*(x_1, t_1)$ as desired. This completes the proof.

Theorem 8 is particularly useful, for instance, when G_1 is linear and G_2 is nonlinear or when G_1 is linear and time invariant and G_2 is time-varying. The variational problems which then result are indeed much simpler if one applies Theorem 8 than those needed in Theorem 7.

The two theorems which follow are the counterparts of the preceding ones, but with passivity conditions replacing the small gain condition. The stability theorem which lies at the basis of these results states that a feedback system is finite-gain input-output stable if the open loop dynamical system G is the composition of a passive system, G_1, and a strictly passive finite-gain input-output stable system, G_2. This decomposition is usually not the result of physical considerations, but rather a mathematical device which allows one to prove stability of the closed loop system.

Assume thus that G, G_1 and G_2 with $G = G_2 G_1$ are dynamical systems in state space form with state spaces X, X_1 and X_2 respectively. The space $X_1 \times X_2$ certainly qualifies as another state space for G but will, in general, be much larger than X, particularly if the latter is minimal (i.e., reachable and irreducible). In general, this is true in stability applications since the factors G_1 and G_2 are usually not natural decompositions of G but are constructed with the aid of so-called "multipliers," which usually results in this inflation of the state space. Assume now that the dynamical system G in state space form has a reachable and irreducible state space X. Consider the dynamical system $G = G_2 G_1$ with state space $X_1 \times X_2$ and assume that the state $(x_1, x_2) \in X_1 \times X_2$ is reachable at $t \in R$, i.e., that there exist a $t_0 \leqq t$ and $u \in U$ with $P_{t_0} u = 0$ such that $\phi_1(t, t_0, 0, u) = x_1$ and $\phi_2(t, t_0, 0, G_1 u) = x_2$ (ϕ_1 and ϕ_2 denote the state transition maps of G_1 and G_2 respectively). Consider now on this subset of reachable states at time t the equivalence classes of those which yield the same output after time t for all $u \in U$, i.e., the reachable states (x_1', x_2') and (x_1'', x_2'') will be considered equivalent if

$$Q_t G_2(t, x_2', G_1(t, x_1', u)) = Q_t G_2(t, x_2'', G_1(t, x_1'', u)) \quad \text{for all } u \in U.$$

There is (by minimality) a one-to-one and onto correspondence between these equivalence classes and the space X. Denote by $X_t(x_1, x_2)$ the element of X corresponding in this sense to the equivalence class derived from the reachable state (x_1, x_2). The map X_t may in general depend explicitly on t. Assume furthermore that there exist constants k and K such that

$$k(\|x_1\|^2 + \|x_2\|^2) \leqq \|X_t(x_1, x_2)\|^2 \leqq K(\|x_1\|^2 + \|x_2\|^2) \quad \text{for all } t \in R$$

and reachable states $(x_1, x_2) \in X_1 \times X_2$. The decomposition of G into $G = G_2 G_1$ will then be called a *compatible factorization* of the dynamical system G.

The statement of the theorem which follows involves Lyapunov functions defined on $X_1 \times X_2$, but these can, by the above remarks, also be considered as Lyapunov functions on the state space X provided one only considers pairs (x_1, x_2) which are reachable. The following theorem statement then becomes clear.

THEOREM 9. *Assume that the feedback system described by equations* (FE) *is uniformly observable and that the state space is uniformly reachable. Assume also that the open loop dynamical system G has a uniformly reachable and uniformly irreducible state space and that it admits a compatible factorization $G = G_2 G_1$ into the uniformly observable dynamical systems G_1 and G_2 with uniformly reachable state spaces X_1 and X_2 respectively. Assume that one of these factors is passive and uniformly smooth and that the other is strictly passive and finite-gain input-output stable. Then the closed loop feedback system is finite-gain input-output stable and uniformly globally asymptotically stable, and the total available energy, $E_a = E_a^{(1)} + E_a^{(2)}$, and the total required energy, $E_r = E_r^{(1)} + E_r^{(2)}$, are decrescent uniform Lyapunov functions for it.* (The superscripts refer to the dynamical systems composing G.)

Proof. Decrescence of E_r on $X_1 \times X_2$ follows from Theorem 4 with an appropriate modification in the proof in order to replace finite-gain stability by the uniform smoothness condition. Decrescence on $X_1 \times X_2$ then implies decrescence on X by the inequality in the definition of a compatible factorization. Since $E_a \leqq E_r$, E_a is also decrescent. The stability claims about the feedback system are well known [5], and it remains to be shown that the energy functions are monotone nonincreasing along undriven solutions. This will only be shown for the required energy. The proof for the available energy is similar.

Let $(x_1', x_2') \in X_1 \times X_2$ and $t_0 \in R$ be given, and let $(x_1'', x_2'') \in X_1 \times X_2$ denote the state of the dynamical systems G_2G_1 at time $t_1 \geqq t_0$ resulting from the transfer along solutions of the undriven feedback system. Then

$$\inf_{\rightarrow x_1'', t_1} \langle P_{t_1}u_1, P_{t_1}G_1u_1\rangle + \inf_{\rightarrow x_2'', t_2} \langle P_{t_1}u_2, P_{t_1}G_2u_2\rangle$$
$$\leqq \inf_{\rightarrow x_1', t_0} \langle P_{t_0}u_1, P_{t_0}G_1u_1\rangle + \inf_{\rightarrow x_2', t_0} \langle P_{t_0}u_2, P_{t_0}G_2u_2\rangle$$
$$+ \langle P_{t_1}Q_{t_0}e_1, P_{t_1}Q_{t_0}G_1(t_0, x_1', e_1)\rangle + \langle P_{t_1}Q_{t_0}e_2, P_{t_1}Q_{t_0}G_2(t_0, x_2', e_2)\rangle,$$

where the notation $\inf_{\rightarrow x_1'', t_1}$, for instance, denotes the infimum over all $t \leqq t_1$ and $u_1 \in U$ with $P_t u_1 = 0$ and $\phi_1(t_1, t, 0, u_1) = x_1''$. The other symbolism is to be interpreted in an analogous way. The inputs e_1 and e_2 denote respectively $(I + G)^{-1}(t_0, (x_1', x_2'), 0)$ and $G_1(I + G)^{-1}(t_0, (x_1', x_2'), 0)$. The desired result then follows if one notices that $Q_{t_0}e_2 = Q_{t_0}G_1(t_0, x_1', e_1)$ and that $Q_{t_0}e_1 = -Q_{t_0}G_2(t_0, x_2', G_1(t_0, x_1', e_1))$ since this shows that the contributions of the last two terms in the above inequality cancel. This completes the proof.

The reader is referred to the remark following Theorem 4 for conditions to ensure positive definiteness and radial unboundedness. Notice again that positive definiteness on $X_1 \times X_2$ suffices for positive definiteness on X by the definition of a compatible factorization. The case in which the operator G_2 is memoryless leads, as in the small gain case, to a simplification. This is stated in the following final theorem.

THEOREM 10. *Assume that the feedback system described by equations* (FE) *is uniformly observable and that the state space is uniformly reachable. Assume that the open loop dynamical system, $G = G_2G_1$, consists of the composition of a uniformly observable, finite-gain input-output stable, strictly passive dynamical system, G_1, with a uniformly reachable state space, followed by a memoryless passive dynamical system, G_2. Then the closed loop feedback system is finite-gain input–output stable and uniformly globally asymptotically stable, and the available energy, $E_a^{(1)}$, and the required energy, $E_r^{(1)}$, are decrescent uniform Lyapunov functions for it.*

Proof. The proof combines the ideas in the proofs of Theorems 8 and 9 and is left to the reader.

The theorems developed here treat the small gain stability conditions and the passive operator stability conditions. The methods can, however, easily be extended to treat conic operators as well.

8. Examples.[7]

Example 1. Let $G(s)$ be a $p \times m$ matrix of rational functions of s with $\lim_{s \rightarrow \infty} G(s) = 0$, and assume that $\{A, B, C\}$ is a minimal[8] realization of $G(s)$. Assume that

[7] The norms and inner products involved in these examples are the usual norms and inner products of Euclidean spaces. Prime denotes transposition. For the calculations involved in the solution of the variational problems in this section, see [23, §§ 21, 22, 23 and 25]. Although some of the problems are not treated explicitly there, the modifications merely require algebraic manipulation and no new methodology.

[8] Algebraically this means that A is an $n \times n$ matrix, that the $n \times nm$ and $n \times np$ matrices $(B, AB, \cdots, A^{n-1}B)$ and $(C', A'C', \cdots, (A')^{n-1}C')$ are of full rank n, and that $G(s) = C(Is - A)^{-1}B$. The full

the poles of $G(s)$, which by minimality equal the eigenvalues of A, are in $\operatorname{Re} s < 0$. The system $\dot{x} = Ax + Bu$; $y = Cx$ is thus finite-gain input–output stable and globally asymptotically stable, and Theorem 3 yields as positive definite decrescent radially unbounded Lyapunov functions the quadratic forms $x'K_1x$ and $x'K_2x$, where K_1 is the (unique, positive definite) solution of the linear matrix equation $A'X + XA = -C'C$, and K_2^{-1} is the (unique, positive definite) solution of the linear matrix equation $AX + XA' = -BB'$. Clearly, these are only two of many possible Lyapunov functions for this asymptotically stable dynamical system.

Example 2. Let $G(s)$ be an $m \times m$ matrix of rational functions of s with $\lim_{s\to\infty} G(s) < \infty$, and assume that $\{A, B, C, D\}$ is a minimal realization of $G(s)$. Assume that the poles of $G(s)$ are in $\operatorname{Re} s < 0$, that $G(j\omega) + G'(-j\omega)$ is Hermitian positive definite for all $\omega \in R$, and that $D + D'$ is positive definite. The n-dimensional system $\dot{x} = Ax + Bu$; $y = Cx + Du$ and thus strictly passive, finite-gain input-output stable, and globally asymptotically stable. The available energy, $E_a(x_0, t_0)$, is given by

$$- \inf_{u \in L_2(0,\infty)} \eta, \qquad \text{where } \eta = \int_0^\infty u'(t)y(t)\,dt,$$

subject to the constraint $\dot{x} = Ax + Bu$; $y = Cx + Du$, $x(0) = x_0$, and is independent of t_0. This variational problem is a least squares problem and, by Lemma 1, an infimum exists. This infimum is, in fact, attained by the feedback control

$$u = -(D + D')^{-1}(C + B'K)x$$

and

$$\min_{u \in L_2(0,\infty)} \eta = \eta^* = -x_0'Kx_0/2,$$

where $K = K'$ is the (unique) negative definite solution of the algebraic Riccati equation

$$0 = -A'X - XA + (C + B'X)'(D + D')^{-1}(C + B'X).$$

Note [8], [9], [10] that this implies the existence of an $n \times n$ positive definite matrix $P = P'$ $(P = -K)$, and $n \times m$ matrix L, and an $m \times m$ matrix W_0 such that (Kalman–Yakubovich–Popov):

$$A'P + PA = -LL',$$
$$PB = C' - LW_0,$$
$$W_0'W_0 = D + D'.$$

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rank condition on $(B, AB, \cdots, A^{n-1}B)$ is equivalent to controllability, reachability, and connectedness, and the full rank condition on $(C', A'C', \cdots, (A')^{n-1}C')$ is equivalent to observability and irreducibility, where these notions refer to the linear time-invariant finite-dimensional system $\dot{x} = Ax + Bu$; $y = Cx$. For these systems, the observability considered here is equivalent to observability under arbitrary inputs, and all of these properties hold uniformly whenever they hold. Global asymptotic stability requires all the eigenvalues of A to be in $\operatorname{Re} \lambda < 0$ and is equivalent to input-output continuity if the system is minimal.

Since, moreover, in this (linear) case the passivity is necessary and sufficient for the existence of the required infimum, and since passivity is equivalent to positive realness of $G(s)$, the above conditions (existence of K or P) are also necessary for positive realness. Thus the available energy $E_a(x, t) = x'Px/2$ is a positive definite radially unbounded decrescent Lyapunov function. The required energy, $E_r(x_0, t_0)$, is somewhat more involved to calculate and is defined by

$$\inf_{T \geqq 0} \inf_{u \in L_2(-T,0)} \eta,$$

where

$$\eta = \int_{-T}^{0} u'(t)y(t)\, dt,$$

subject to the constraint[9] $\dot{x} = Ax + Bu$; $y = Cx + Du$, $x(-T) = 0$, $x(0) = x_0$, and is independent of t_0. The above variational problem is again a least squares problem, and by Lemma 1, an infimum exists. This infimum can be characterized as follows:

$$\eta^* = \inf_{T \geqq 0} \min_{u \in L_2(-T,0)} \eta = -x_0'\Sigma x_0/2,$$

where $\Sigma = P + W^{-1}$ and $P = P'$ is the (unique) positive definite solution of the algebraic Riccati equation

$$0 = A'X + XA + (C - B'X)'(D + D')^{-1}(C - B'X).$$

In fact, this matrix is the same as the one appearing in the calculation of the available energy and is such that $A_1 = A - B(D + D')^{-1}(C - B'P)$ is an asymptotically stable matrix. W is the (unique) solution of the linear matrix equation $A_1X + XA_1' = -B(D + D')^{-1}B'$, and is symmetric positive definite. The required energy $E_r(x, t) = \frac{1}{2}x'Px + \frac{1}{2}x'W^{-1}x$ is also a positive definite radially unbounded decrescent Lyapunov function. The cycle energy $E_c = E_r - E_a$ is given by $E_c(x, t) = \frac{1}{2}x'W^{-1}x$. The system is thus lossy.

Example 3. Let $g(s)$ be a rational function of s with $\lim_{s\to\infty} g(s) = 0$ and assume that $\{A, b, c'\}$ is a minimal realization of $g(s)$. Let k be a scalar. Assume that the Nyquist locus of $g(s)$ does not intersect but encircles the $-1/k$ point in the complex plane $-\rho$ times in the clockwise direction, where ρ is the number of poles of g in $\operatorname{Re} s \geqq 0$. Then the closed loop system $\dot{x} = (A - kbc')x$ is globally asymptotically stable, and Theorem 6 yields as a positive definite radially unbounded decrescent Lyapunov function $-x'Rx$, where $R = R'$ is the (unique) negative definite solution of the algebraic matrix Riccati equation

$$0 = -A'X - XA + \frac{(kc + Xb)(kc + Xb)'}{1 - \varepsilon^2} - cc'$$

with $\varepsilon > 0$ such that $|1 + kg(j\omega)| \geqq \varepsilon_1 > \varepsilon$ for all $\omega \in R$.

[9] It is important to realize that this variational problem is *not* equivalent to the simpler one which asks to evaluate $\inf_{u \in L_2(-\infty,0)} \int_{-\infty}^{0} u'(t)y(t)\, dt$ subject to $\dot{x} = Ax + Bu$; $y = Cx + Du$, $x(0) = x_0$. (This latter variational problem leads again to the available energy.)

Example 4. Let $G(s)$ be a $p \times m$ matrix of rational functions of s with $\lim_{s \to \infty} G(s) = 0$ and assume that $\{A, B, C\}$ is a minimal realization of $G(s)$. Assume that the poles of $G(s)$ are in $\operatorname{Re} s < 0$, and that for all $\omega \in R$ the eigenvalues of the matrix $G'(-j\omega)G(j\omega)$ are inside the open ball with radius ρ^{-2} in the complex plane. Let $f(\sigma, t)$ be a R^m-valued function defined on $R^p \times R$, Lipschitz continuous on R^p, uniformly in t, and satisfying, for some $\alpha < \rho$, the inequality $\|f(\sigma, t)\| \leqq \alpha\|\sigma\|$ for all $(\sigma, t) \in R^p \times R$. Consider now the nonlinear differential equation

$$\dot{x}(t) = Ax(t) - Bf(Cx(t), t).$$

This differential equation may be viewed as the mathematical model of the undriven feedback system studied in § 6 with the open loop dynamical system described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = f(Cx(t), t),$$

and the closed loop dynamical system determined by the equations

$$\dot{x}(t) = Ax(t) - Bf(Cx(t), t) + Bu(t), \qquad y(t) = f(Cx(t), t).$$

This system satisfies all the assumptions for Theorem 8 to be applicable, and the nonlinear differential equation is thus uniformly globally asymptotically stable by the small gain theorem. Consider now

$$\inf_{u \in L_2(0,\infty)} \int_0^\infty [u'(t)u(t) - \rho^2 y'(t)y(t)]\, dt,$$

subject to the constraint $\dot{x} = Ax + Bu$; $y = Cx$, $x(0) = x_0$. This infimum exists and is given by $x_0'Kx_0$ when K is the (unique) negative definite solution of the matrix Riccati equation

$$0 = -A'X - XA + XBB'X + \rho^2 C'C.$$

Theorem 8 thus states that $-x'Kx$ is a positive definite radially unbounded decrescent uniform Lyapunov function for this nonlinear differential equation. Theorem 8 yields as another positive definite radially unbounded decrescent uniform Lyapunov function $x'(K + W^{-1})x$, where K is as defined above and W is the (unique, positive definite) solution of the linear matrix equation

$$(A - BB'K)X + X(A - BB'K)' = -BB'.$$

Example 5. Let $g(s)$ be a rational function of s with $\lim_{s \to \infty} g(s) = 0$, and assume that $\{A, b, c'\}$ is a minimal realization of $g(s)$. Assume that the poles of $g(s)$ are in $\operatorname{Re} s < 0$ and that there exists a real number $\alpha \geqq 0$ such that for some constant $\varepsilon > 0$, $\operatorname{Re}(\alpha + j\omega)g(j\omega) \geqq \varepsilon > 0$ for all $\omega \geqq 0$. Let $f(\sigma)$ be a real-valued function defined on R, Lipschitz continuous on R, and satisfying for some $\delta > 0$ the inequality $f(\sigma)/\sigma > \delta > 0$ for all $\sigma \in R$, $\sigma \neq 0$. Consider now the nonlinear differential equation $\dot{x}(t) = Ax(t) - bf(c'x(t))$. This differential equation may be viewed as the mathematical model of the undriven feedback system studied in § 6 with the open loop dynamical system described by the equation

$$\dot{x} = Ax + bu; \qquad y = f(c'x),$$

and the closed loop dynamical system described by the equations

$$\dot{x} = Ax - bf(c'x) + bu, \qquad y = f(c'x).$$

The open loop dynamical system can be viewed as the cascade of the two systems: a linear time-invariant system with transfer function $(s + \alpha)g(s)$ followed by a nonlinear system which is the cascade of a linear time-invariant system with transfer function $1/(s + \alpha)$ followed by the memoryless nonlinearity $f(\cdot)$. The state equations of the first system are

$$\dot{x} = Ax + bu, \qquad y = \alpha c'x + c'Ax + c'bu.$$

This system satisfies the assumptions of the system of Example 3 which thus yields expressions for the available energy, the required energy, and the cycle energy. The state equations for the second system are

$$\dot{z} = -\alpha z + u, \qquad y = f(z).$$

The available energy $E_a(z, t)$ for this system is independent of t_0 and is defined by

$$E_a(z_0) = -\inf_{T \geqq 0} \inf_{u \in L_2(0,T)} \eta,$$

where

$$\eta = \int_0^T u(t)y(t)\,dt,$$

subject to $\dot{z} = -\alpha z + u$, $y = f(z)$, $z(0) = z_0$. Thus

$$\eta = F(z(T)) + \alpha \int_0^T z(t)f(z(t))\,dt - F(z_0)$$

with

$$F(z) = \int_0^z \sigma f(\sigma)\,d\sigma.$$

Since $F(\sigma) \geqq 0$ and $\sigma f(\sigma) \geqq 0$ for all $\sigma \in R$, and since the value of

$$\alpha \int_0^T z(t)f(z(t))\,dt + F(z(T))$$

can be made arbitrarily small by proper choice of u, it follows that

$$E_a(z_0) = F(z_0) = \int_0^{z_0} \sigma f(\sigma)\,d\sigma.$$

Similarly, the required energy

$$E_r(z_0) = F(z_0) = \int_0^{z_0} \sigma f(\sigma)\,d\sigma.$$

The cycle energy E_c for this first order nonlinear system is thus zero, and the system is reversible. It is a simple matter to show that the above factorization of the system $\dot{x} = Ax + bu$, $y = f(c'x)$ is a compatible factorization as defined in § 7. Notice that reachable states satisfy the condition $z_0 = c'x_0$, which defines a hyperplane in the space $R^n \times R$. Theorem 9 thus yields as positive definite radially unbounded decrescent Lyapunov functions for the nonlinear differential equation $\dot{x} = Ax - bf(c'x)$ satisfying the conditions enumerated earlier (i.e., the conditions of the Popov criterion):

(i) $V(x) = \frac{1}{2}x'Px + \int_0^{c'x} f(\sigma)\,d\sigma$, where $P = P'$ is the (unique) positive definite solution of the algebraic Riccati equation[10]

$$0 = A'X + XA + \frac{1}{2c'b}(\alpha c' + c'A - b'X)'(\alpha c' + c'A - b'X).$$

(ii) $V(x) + \frac{1}{2}x'W^{-1}x$, where W is the (unique) solution of the linear matrix equation

$$A_1X + XA_1' = -\frac{bb'}{2c'b} \quad \text{with} \quad A_1 = A - \frac{b}{2c'b}(\alpha c' + c'A - b'P),$$

and is symmetric positive definite.

Example 6. *Path integrals* [23, § 26], [25].

LEMMA 2 [23, p. 170]. *Assume that $x(t)$ is an n times differentiable function of t and that α_{ij}, $i, j = 0, 1, \cdots, n$, are constants. Then the*

$$\eta = \int_{t_0}^{t_1} \sum_{i,j=0}^{n} \alpha_{ij} \frac{d^ix(t)}{dt^i}\frac{d^jx(t)}{dt^j}\,dt$$

is independent of path (i.e., it depends only on the values of $x(t)$ and its derivatives at $t = t_0$ and $t = t_1$) if and only if the polynomial

$$h(s) = \sum_{i,j=0}^{n} \alpha_{ij}(s^i(-s)^j + (-s)^is^j)$$

vanishes identically.

This lemma leads to rather specific formulas for the Lyapunov functions described in this paper. For instance, Theorem 3 thus yields as a Lyapunov function for the differential equation $p(D)x(t) = 0$ with $D = d/dt$ and $p(s)$ a polynomial with all its roots in $\mathrm{Re}\, s < 0$,

$$V(x, x^{(1)}, \cdots, x^{(n-1)}) = \inf_{T \geqq 0} \inf_{\substack{x(t)|x(-T) = \cdots = x^{(n-1)}(-T) = 0 \\ x(0) = x, \cdots, x^{(n-1)}(0) = x^{(n-1)}}} \eta,$$

where $\eta = \int_{-T}^{0} (p(D)x(t))^2\,dt$. Let $r(s)$ be a solution of the polynomial equation $p(s)p(-s) = r(s)r(-s)$, and let $(p\bar{p})^+(s) = p(-s)$ and $(p\bar{p})^-(s) = p(s)$ denote the solutions with poles respectively in $\mathrm{Re}\, s > 0$ and $\mathrm{Re}\, s < 0$. Now rewriting η as

$$\eta = \int_{-T}^{0} [(p(D)x(t))^2 - (r(D)x(t))^2]\,dt + \int_{-T}^{0} (r(D)x(t))^2\,dt,$$

one observes that by Lemma 2 the first integral is independent of path and thus depends on the values of $x, x^{(1)}, \cdots, x^{(n-1)}$ only. The integrand in the second integral is nonnegative and should hence be made as small as possible. By choosing

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[10] Compare with the results of [24].

$r(s) = (p\bar{p})^+(s) = p(-s)$ and letting $T \to +\infty$, the contribution of this second integral can indeed be made arbitrarily small and yields a positive definite radially unbounded decrescent Lyapunov function for $p(D)x(t) = 0$, the quadratic form

$$V(x, \dot{x}, \cdots, x^{(n-1)}) = \int_{-\infty}^{0} [(p(D)x(t))^2 - (p(-D)x(t))^2]\, dt,$$

with $x(t)$ any n times differentiable function such that

$$x(0) = x, \cdots, x^{(n-1)}(0) = x^{(n-1)}$$

and

$$\lim_{t \to -\infty} x(t) = \cdots = x^{(n-1)}(t) = 0.$$

If $g(s)$ is chosen such that

$$\lim_{s\to\infty} \frac{q(s)}{p(s)} < \infty \quad \text{and} \quad \operatorname{Re} \frac{q(j\omega)}{p(j\omega)} \geqq 0 \quad \text{for all } \omega,$$

then Theorem 4 yields as Lyapunov functions

$$E_r(x, \dot{x}, \cdots, x^{(n-1)}) = \int_{-\infty}^{0} [p(D)x(t)q(D)x(t) - \frac{1}{2}((p\bar{q} + \bar{p}q)^+(D)x(t))^2]\, dt$$

and

$$E_a(x, \dot{x}, \cdots, x^{(n-1)}) = \int_{0}^{\infty} [p(D)x(t)q(D)x(t) - \frac{1}{2}((p\bar{q} + \bar{p}q)^-(D)x(t))^2]\, dt$$

with $x(t)$ any n times differentiable function such that

$$x(0) = x, \cdots, x^{(n-1)}(0) = x^{(n-1)}$$

and

$$\lim_{t \to \pm\infty} x(t) = \cdots = x^{(n-1)}(t) = 0.$$

9. Conclusions. The development of the results and the techniques described in this paper evolves in three stages: the first one introduces and compares the input-output description with the state space description of dynamical systems and shows their equivalence. The second part in the development leads to the equivalence of input-output stability and global stability under appropriate controllability and observability conditions; the third issue is the construction of Lyapunov functions.

The methods for constructing Lyapunov functions involve, for the most part, variational problems and are posed in the framework of systems with inputs and outputs; this notwithstanding the fact that the system for which global asymptotic stability (in the sense of Lyapunov) is to be shown is an autonomous (undriven) system. The results thus obtained serve as a further relationship between the areas of dynamic optimization and stability theory and focus interest on a class of optimization problems, some of which will, in fact, lead to singular controls.

It is felt that the importance of this paper lies in its theoretical contribution in demonstrating the equivalence between global asymptotic stability and input-output stability, which, as expected, merely requires appropriate controllability and observability (more precisely: reachability and uniform observability). It also serves to unify the two main approaches to stability theory: input-output stability and Lyapunov stability. In this latter class it unifies and generalizes the various available results by posing the construction of these Lyapunov functions as variational problems.

The results of the paper could also serve as a starting point to develop techniques which will lead to suitable Lyapunov functions for estimating the domain of attraction for nonglobal stable systems. This is a problem of great practical importance, and the methods of the paper lead to tractable variational problems which could be used in such an analysis.

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