The Application of Lyapunov Methods to the Computation of Transient Stability Regions for Multimachine Power Systems

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Abstract—**This paper outlines a method for computing a stability region for a multimachine power system. It is shown that the dynamic equations describing this system are of the type for which some rather elegant stability criteria are known, of which the Popov criterion is the most useful. It can be proved by means of a Lyapunov function which will, together with a method for computing the domain of attraction of an equilibrium point of a dynamic system, lead to a systematic procedure for obtaining an estimate of the region of stability for a multimachine power system. The procedure is very well suited for automatic computation and can take into consideration the effects of damping and of fast governors.**

I. Introduction

With the advent of large power systems came a renewed interest in the stability properties of such systems. Indeed, the tendency of a system to lose synchronism and the possibility of the existence of oscillations in the power transfer between interconnected systems appears to be much more prevalent for large systems than for relatively isolated groups.

Most stability studies are based on direct simulation: the postfault system behavior is simulated and the stability properties of the solutions are considered for various values of the switching time; that is, when normal operating conditions are restored. For low values of this switching time the system regains synchronism. The largest acceptable value of the switching time, i.e., the largest value for which stability prevails, is generally called the critical switching time. The critical switching time is, in fact, considered the most important stability limit.

For complex systems this simulation becomes cumbersome and very costly, since an almost prohibitive amount of computation is required in its execution. Thus the need increases for the development of more direct methods for the computation of the critical switching time.

For a system consisting of a single machine connected to an infinite bus, a direct method called the equal-area or the energy-integral criterion [1] has been known for a long time. This method, however, has no obvious analog for larger systems. A direct method for estimating the domain of attraction of a given equilibrium point (i.e., the set of initial conditions for which the resulting motion approaches this equilibrium) is given by the direct method of Lyapunov. This method, which can be applied to any dynamic system, has in fact been used in studies concerning the single-machine and multimachine power systems in several recent papers [2]–[4].

The difficulty in the application of Lyapunov's direct method is that in general there is no obvious choice for a function suitable for use as a Lyapunov function. In most systems describing a physical (mechanical or electrical) system, the energy stored in the system appears to be a natural candidate. The above mentioned works are in fact led by energy considerations for the selection of the Lyapunov function. This is not necessary, however, and many examples of stable systems are known for which the energy is *not* a suitable Lyapunov function. Whether or not the multimachine power system represents such a case is an open question. In any event, there is no apparent reason for the energy to be always decreasing when the system is in a stable transient condition.

This paper outlines a method for estimating the domain of attraction of an equilibrium point based on another choice for the Lyapunov function for the multimachine stability problem. This choice is inspired by the Lyapunov function used in proving the so-called Popov criterion and its generalizations[1] which has recently received a great deal of attention in the automatic control literature. The Popov criterion gives a sufficient condition for the stability of a feedback system with a linear time-invariant system in the forward loop and a memoryless, time-invariant but possibly nonlinear element in the feedback loop.

The paper starts by showing that a multimachine power system can indeed be modeled mathematically in a form for which the Popov criterion is applicable. This then leads to a Lyapunov function that establishes at least the local stability of the equilibrium point; i.e., it establishes that there is a domain of attraction of the equilibrium point. A method is then outlined to use the Lyapunov function to estimate this domain of attraction. This systematic method, which appears to be of interest in its own right, does not give the domain of attraction in "closed" form but is very well suited for automatic computation. It will also be indicated that the effects of salient poles, damping, and fast governor action can actually be taken into account without theoretical difficulties, of course at the expense of increased complexity.

II. Dynamic Equations

Consider a multimachine power system. It will be assumed that the synchronous machines can be represented by a constant voltage behind their transient reactance. In other words, it is

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[1] For a general treatment of Lyapunov stability, see [5] and [6]. For a detailed treatment of the Popov criterion, see [8].

assumed that the flux linkages of the various synchronous machines are constant. This assumption is valid in almost all practical situations since the flux decay is generally much slower than the transient phenomena that of interest in transient stability analysis.

The differential equation describing the ith machine is then given by

$$M_i \frac{d^2\delta_i(t)}{dt^2} + a_i \frac{d\delta_i(t)}{dt} + P_{ei}(t) - P_{mi}(t) = 0, \quad i = 1,2,\cdots,n \tag{1}$$

where

- t time
- δ_i angle between rotor shaft of ith machine and shaft running at synchronous speed (electrical degrees)
- M_i inertia of ith machine
- a_i damping coefficient of ith machine, mainly due to asynchronous torque
- P_{ei} electrical power delivered by ith machine
- P_{mi} mechanical power delivered to ith machine.

In most transient stability investigations, the mechanical input power is assumed to be constant since the governor time constants are usually at least one order of magnitude larger than the transient periods. It will be indicated at the end of the paper how the effects of fast governors can be incorporated in the analysis. To keep matters simple, however, it will be assumed for the moment that this condition is satisfied and that the mechanical input can be assumed constant. The electrical power output for round-rotor machines is given by

$$P_{ei} = E_i^2G_i + \sum_{\substack{j=1\\ j\neq i}}^{n} E_iE_jY_{ij}\cos\left[\theta_{ij} - (\delta_i - \delta_j)\right], \quad i = 1,2,\cdots,n \tag{2}$$

where E_i is the internal voltage of the ith machine, G_i is the short-circuit conductance of the ith machine, and Y_{ij} and θ_{ij} are, respectively, the modulus and the phase angle of the short-circuit transfer admittance between the ith and the jth machines.

In almost all practical situations, the transfer conductances $G_{ij} = Y_{ij}\cos\theta_{ij}$, $i \neq j$, are negligible, and only the transfer susceptances $B_{ij} = Y_{ij}\sin\theta_{ij}$, $i \neq j$, have to be taken into consideration. Then the electrical power output is thus given by

$$P_{ei} = E_i^2G_i + \sum_{\substack{j=1\\ j\neq i}}^{n} E_iE_jB_{ij}\sin(\delta_i - \delta_j), \quad i = 1,2,\cdots,n. \tag{3}$$

The above expression for the electrical power will be used throughout the paper, although the method is equally well suited to deal with the general case. This expression is not quite correct for machines with salient poles since the dependence of the electrical power on the angle differences $\delta_i - \delta_j$ is not sinusoidal. This however can be taken into consideration in a straightforward way since the methods outlined below do not depend on this dependence in an essential way. This is shown in some detail for the one-machine system in [7].

The equilibrium states are the solutions of the set of equations $d\delta_i(t)/dt = 0$ and $P_{ei} = P_{mi}$, $i = 1,2,\cdots,n$. These equations yield the equilibrium values of the load angles up to an arbitrary constant, since the above equations feature the differences $\delta_i - \delta_j$ only. The above n equations are not overdetermined however, since

$$\sum_{i=1}^{n} P_{mi} = \sum_{i=1}^{n} E_i^2G_i$$

is a necessary condition for the existence of an equilibrium, and hence there is a redundancy in the above equations for the load angles.

Let $(\delta_1^\circ, \delta_2^\circ, \cdots, \delta_n^\circ)$ be the equilibrium load angles for which stability has to be determined. The dynamic equations describing the motion of the multimachine power system about these load angles can then be written in state form as

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{f}[\boldsymbol{C}\boldsymbol{x}(t)] \tag{4}$$

where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}$ is a $2n$-dimensional column vector which is the state of the system. The components of the n-dimensional column vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ are

$$y_i = \omega_i = \frac{d\delta_i}{dt}, \quad i = 1,2,\cdots,n$$

which is the difference between the actual angular rotor speed (in electrical degrees) of the ith machine and the synchronous speed, and

$$z_i = \delta_i - \delta_i^\circ, \quad i = 1,2,\cdots,n$$

is the difference between the actual and the equilibrium load angle of the ith machine.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{M}^{-1}\boldsymbol{R} & \boldsymbol{O}_n \\ \boldsymbol{I}_n & \boldsymbol{O}_n \end{bmatrix}$$

is a $(2n \times 2n)$ matrix where $\boldsymbol{O}_n$ and $\boldsymbol{I}_n$ are, respectively, the $(n \times n)$ identity and zero matrices, and $\boldsymbol{R}$ and $\boldsymbol{M}$ are the diagonal $(n \times n)$ matrices

$$\boldsymbol{M} = \operatorname{diag}(M_i),\ \boldsymbol{R} = \operatorname{diag}(-a_i), \quad i = 1,2,\cdots,n.$$

$$\boldsymbol{C} = [\boldsymbol{O}_{mn} \quad \boldsymbol{D}]$$

is an $(m \times 2n)$, $m = [n(n-1)]/2$ matrix where $\boldsymbol{O}_{mn}$ is a rectangular $(m \times n)$ matrix with all zero elements, and $\boldsymbol{D}$ is an $(m \times n)$ matrix such that $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{z}$ has as its components $\sigma_1 = z_1 - z_2$, $\sigma_2 = z_1 - z_3, \cdots, \sigma_{n-1} = z_1 - z_n$, $\sigma_n = z_2 - z_3$, $\sigma_{n+1} = z_2 - z_4, \cdots, \sigma_m = z_{n-1} - z_n$.

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{M}^{-1}\boldsymbol{D}^T\boldsymbol{E} \\ \boldsymbol{O}_{nm} \end{bmatrix}$$

is a $(2n \times m)$ matrix where $\boldsymbol{O}_{nm} = \boldsymbol{O}_{mn}^T$ (T denotes transpose) and $\boldsymbol{E}$ is a diagonal $(m \times m)$ matrix, $\boldsymbol{E} = \operatorname{diag}(e_k)$ with $e_k = E_iE_jB_{ij}$, where i and j are the indices of the components of z on which σ_k is dependent, i.e., $\sigma_k = z_i - z_j$.

The vector-valued function $\boldsymbol{f}(\boldsymbol{\sigma})$ has m elements and is of the diagonal type, which means that the ith component σ_i of $\boldsymbol{\sigma}$ only depends on the ith component of $\boldsymbol{f}(\boldsymbol{\sigma})$. Indeed, $f_i(\sigma_i) = \sin(\sigma_i + \sigma_i^\circ) - \sin\sigma_i^\circ$, $i = 1, 2, \cdots, m$ where σ_i° is the ith component of

$$\sigma^\circ \triangleq \boldsymbol{D}\begin{bmatrix} \delta_1^\circ \\ \delta_2^\circ \\ \cdot \\ \cdot \\ \cdot \\ \delta_n^\circ \end{bmatrix}.$$

It is a simple matter to verify the equivalence of the state vector differential equation (4) and the system of differential equations (1). For a three-machine system the above vectors and matrices become

$$A = \begin{bmatrix} -a_1/M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_2/M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3/M_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} z_1 - z_2 \\ z_1 - z_3 \\ z_2 - z_3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} E_1E_2B_{12}/M_1 & E_1E_3B_{13}/M_1 & 0 \\ -E_1E_2B_{12}/M_2 & 0 & E_2E_3B_{23}/M_2 \\ 0 & -E_1E_3B_{13}/M_3 & -E_2E_3B_{23}/M_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The stability properties of the equilibrium solution $\boldsymbol{x} = \boldsymbol{0}$ of systems described by this class of differential equations have been studied quite intensively in the last decade. The general approach of establishing conditions for stability for such systems is the direct method of Lyapunov. This procedure is outlined in the next section.

The model obtained above also holds for the system consisting of a single machine connected to an infinite bus. This system is actually a two-machine system, but one of the machines (the infinite bus) has an infinite inertia ($M_2 \to \infty$). Its load angle is hence constant and can be taken as the reference angle.

III. Stability of Equilibrium State

The local stability properties of equilibrium solutions can be derived from linearization of the equations of motion around the equilibrium solution. Since this linearized system is linear and time invariant, it suffices to consider the eigenvalues of the $(2n \times 2n)$ matrix $\boldsymbol{D} = \boldsymbol{A}\text{–}\boldsymbol{BSC}$ where $\boldsymbol{S} = \text{diag}(\cos \sigma_i^\circ)$ is an $(m \times m)$ diagonal matrix. For the reason mentioned in the previous section (i.e., the equilibrium angles are only determined up to an arbitrary constant), $\boldsymbol{D}$ has at least one zero eigenvalue. The equilibrium state is locally asymptotically stable if (and only if) the $2n - 1$ remaining eigenvalues of $\boldsymbol{D}$ have their real parts negative.

It is simple to show that, for a single machine tied to an infinite bus, the damping coefficient and the synchronizing torque must be positive. For a three-machine system, the eigenvalues of the matrix

$$D = \begin{bmatrix} M^{-1}R & M^{-1}T \\ I_n & O_n \end{bmatrix}$$

with

$$T = \begin{bmatrix} -(\alpha_{12} + \alpha_{13}) & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & -(\alpha_{12} + \alpha_{23}) & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & -(\alpha_{13} + \alpha_{23}) \end{bmatrix}$$

and $\alpha_{ij} = E_iE_jB_{ij} \cos(\delta_i^\circ - \delta_j^\circ)$ are to be computed. One eigenvalue will always be zero. If the others have negative real parts, then local asymptotic stability follows.

Local stability only allows the conclusion that for initial conditions sufficiently close to the equilibrium state of the post-fault system, the solutions tend to this equilibrium. It is a basic requirement for acceptable steady-state operating conditions. Multimachine power systems are in general not asymptotically stable in the large (i.e., not all solutions approach the equilibrium solution). There are very often multiple equilibrium solutions, periodic solutions, and initial conditions that lead to asynchronism.

It is thus of great importance not only to establish the local stability properties of the equilibrium solution, but to estimate the actual domain of attraction; i.e., the set of initial conditions for which the solutions approach the equilibrium should somehow be estimated. This then allows for a more meaningful judgment of the desirability of the steady-state operating conditions and for the design of some relevant parameters, e.g., the critical switching time. A method of obtaining such an estimate is outlined in the remainder of the paper.

IV. Lyapunov Function

Consider the dynamic system described by the differential equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{f}[\boldsymbol{C}\boldsymbol{x}(t)] \tag{5}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are, respectively, $(2n \times 2n)$, $(2n \times m)$, and $(m \times 2n)$ matrices, and $\boldsymbol{f}(\sigma)$ maps the m-dimensional vector $\boldsymbol{\sigma} = \text{col}(\sigma_1, \sigma_2, \cdots, \sigma_m)$ into the m-dimenensional vector $\boldsymbol{f}(\boldsymbol{\sigma}) = \text{col}[f_1(\sigma_1), f_2(\sigma_2), \cdots, f_m(\sigma_m)]$. Note that the nonlinearity $\boldsymbol{f}(\boldsymbol{\sigma})$ is time invariant, memoryless, and of the diagonal type (as explained in Section II). It is furthermore assumed that $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$ and that all the eigenvalues of the matrix $\boldsymbol{A}$ have nonpositive real parts.

As was shown above, the motion of the state of a multimachine power system around an equilibrium state is described by an equation such as (5).

The stability of the null-solution of (5) has been studied by several authors in the recent control theory literature. The following theorem is a generalization of the celebrated Popov stability theorem to systems with multiple nonlinearities. It concerns the asymptotic stability in the large of the solution $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ of (5) and thus ensures that for any initial condition $\boldsymbol{x}(t = 0) = \boldsymbol{x}_0$, the ensuing solution $\boldsymbol{x}(t)$ satisfies $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$.

Theorem 1

The null solution of the differential equation (5) is asymptotically stable in the large if

1) $0 < \sigma_i f_i(\sigma_i)$ for all $i = 1,2, \cdots, m$ and $\sigma_i \neq 0$, and
2) there exists a diagonal $(m \times m)$ matrix $\boldsymbol{Q} = \text{diag}(q_i)$, $q_i \geq 0$,

such that $\boldsymbol{Z}(s) = (\boldsymbol{I}_m + \boldsymbol{Q}s)\, \boldsymbol{C}(s\boldsymbol{I}_{2n} - \boldsymbol{A})^{-1}\boldsymbol{B}$ is a positive real matrix; i.e., $\boldsymbol{Z}(j\omega) + \boldsymbol{Z}^T(-j\omega)$ is a nonnegative definite Hermitian matrix for all real $\omega \geq 0$.

The question of whether or not, for particular matrices, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, $\boldsymbol{Q}$ exist, can in general be resolved quite readily using graphical techniques or directly using analytical means. The properties of positive real functions and of positive real functions of rational functions have been extensively studied in the electrical network synthesis literature. At this point, let it suffice to remark that this problem can be resolved without much difficulty. An appropriate reference where problems of this type are discussed is [8], where further references can be found.

The above stability theorem can be proven in a number of ways. One of them is by constructing a suitable Lyapunov function that involves solving certain algebraic matrix equations involving the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{Q}$. The above reference gives methods of (numerically) obtaining this Lyapunov function. The computational difficulties encountered are rather mild, and the reader is referred to the specialized literature for details. The solution can be obtained in a very systematic way and the procedure is very well suited for automatic computation. The form of the Lyapunov function and methods to obtain it are given in Appendix I.

The nonlinearities appearing in the mathematical description of the multimachine power system do not satisfy the conditions 1) of the above theorem since they are of the type $f_i(\sigma_i) = \sin(\sigma_i + \sigma_i^\circ) - \sin \sigma_i^\circ$, and hence the inequality $\sigma_i f_i(\sigma_i) \geq 0$ is only satisfied for a range of values σ. Hence asymptotic stability in the large cannot be concluded. However with the aid of the function $V(\boldsymbol{x})$ which is to be found as explained above, one can obtain an estimate (through consideration of $\dot{V}(\boldsymbol{x})$) of the domain of attraction. This estimate will be large if the Lyapunov function $V(\boldsymbol{x})$ "fits" well to the system. The reason for advertising the particular choice given by the Popov criterion is that this Lyapunov function has given such excellent results for asymptotic stability in the large, and can thus be considered as the optimal choice. However, the following analysis can be modified rather easily if other Lyapunov functions are used.

For the three-machine system it is hence required to find a diagonal matrix $\boldsymbol{Q} = \operatorname{diag}(q_i)$, $q_i \geq 0$, such that $\boldsymbol{Z}(s) = (\boldsymbol{I} + \boldsymbol{Q}s)\boldsymbol{G}(s)$ is a positive real matrix where

$$\boldsymbol{G}(s) = \frac{1}{s}\cdot\begin{bmatrix} \frac{1}{M_1 s + a_1} + \frac{1}{M_2 s + a_2} & \frac{1}{M_1 s + a_1} & -\frac{1}{M_2 s + a_2} \\ \frac{1}{M_1 s + a_1} & \frac{1}{M_1 s + a_1} + \frac{1}{M_3 s + a_3} & \frac{1}{M_3 s + a_3} \\ -\frac{1}{M_2 s + a_2} & \frac{1}{M_3 s + a_3} & \frac{1}{M_2 s + a_2} + \frac{1}{M_3 s + a_3} \end{bmatrix} \begin{bmatrix} E_1E_2B_{12} & 0 & 0 \\ 0 & E_1E_3B_{13} & 0 \\ 0 & 0 & E_2E_3B_{23} \end{bmatrix}.$$

If all the damping coefficients a_i are zero, then $q_i \to \infty$ is the only possible choice. If $a_1, a_2, a_3 > 0$, then $\boldsymbol{Z}(s)$ can be made positive real not only with $q_i \to \infty$, but also for finite values of the constants q_i, and many possible functions $V(\boldsymbol{x})$ can thus be constructed.

V. Determination of Region of Asymptotic Stability

If the nonlinear functions in (5) do not satisfy the conditions $\sigma_i f_i(\sigma_i) > 0$ for all $\sigma_i \neq 0$ but only for a finite (or semi-infinite) interval containing zero, then the Lyapunov functions constructed in the preceding section can still be used to compute the region of asymptotic stability of the null solution.

Assume thus that $\sigma_i f_i(\sigma_i) > 0$ for $\sigma_i^m < \sigma_i < 0$ and $0 < \sigma_i < \sigma_i^M$, $i = 1,2, \cdots, m$, and that the second condition of Theorem 1 is satisfied. Let the ith row of the matrix $\boldsymbol{C}$ be denoted by the $2n$-dimensional vector $\boldsymbol{c}_i$. Let Γ denote the region determined by all $2n$-dimensional vectors $\boldsymbol{x}$ such that $\sigma_1^m \leq \boldsymbol{c}_1\boldsymbol{x} \leq \sigma_1^M$, $\sigma_2^m \leq \boldsymbol{c}_2^T\boldsymbol{x} \leq \sigma_2^M, \cdots, \sigma_m^m \leq \boldsymbol{c}_m^T\boldsymbol{x} \leq \sigma_m^M$. Let $V(\boldsymbol{x})$ denote the Lyapunov functions constructed in Section IV. $\dot{V}(\boldsymbol{x})$ (the derivative of V along solutions) will be nonpositive for $\boldsymbol{x}$ in Γ, but not necessarily for other values of $\boldsymbol{x}$.

Theorem 2

Let $\delta\Gamma$ denote the boundary of Γ and let V_1 denote the minimum of $V(\boldsymbol{x})$ over all $\boldsymbol{x}$ in $\delta\Gamma$. The equation $V(\boldsymbol{x}) = V_1$ defines a bounded surface inside Γ and contains $\boldsymbol{0}$. The region R_1 enclosed by this surface belongs to the domain of attraction of $\boldsymbol{0}$.

The above theorem follows rather easily from the fact that $\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$ in Γ and the usual estimates of the domain of attraction based on Lyapunov functions.

It is possible to obtain a larger domain of attraction by taking into consideration the particular structure of the nonlinear differential equation under consideration and using the methods outlined in [9] for differential equations containing a single nonlinear element.

Let S_i be the part of $\delta\Gamma$ where either $\boldsymbol{c}_i\boldsymbol{x} = \sigma_i^m$ or $\boldsymbol{c}_i\boldsymbol{x} = \sigma_i^M$, and let L_i be the intersection of S_i and the set of all $\boldsymbol{x}$ for which $\boldsymbol{c}_i^T[\boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x})] = 0$.

Theorem 3

Let V_2 denote the minimum of $V(\boldsymbol{x})$ over all $\boldsymbol{x}$ in L_i, $i = 1,2, \cdots, m$. (Clearly $V_2 \geq V_1$ and equality holds exceptionally.) The equation $V(\boldsymbol{x}) = V_2$ defines a bounded surface inside Γ and contains $\boldsymbol{0}$. The region R_2 enclosed by this surface belongs to the domain of attraction of $\boldsymbol{0}$.

Clearly the above theorem gives a larger domain of attraction than what is predicted by Theorem 2. The proof is a relatively straightforward extension of the methods outlined in [9]. The application of Theorem 3 generally leads to a considerable enlargement of the domain of attraction compared to what can be obtained from Theorem 2. This will be illustrated later on by means of an example.

The procedures for estimating the domain of attraction as given by the above theorems is very well suited for automatic computation. The main computational difficulties are the minimizations that appear in the determination of the regions R_1 and R_2. (A related but not as general method for estimating the domain of attraction is given in Appendix II.)

VI. Transient Stability Region and Critical Switching Time

The following procedure can be used for determining the critical switching time based on stability considerations.

1) Determine the prefault steady-state operation of the system.

2) Integrate numerically the dynamic equations of the faulted system with the prefault steady-state operating point as initial conditions.

3) Determine the postfault steady-state operation of the system and its local stability.

4) Compute the Lyapunov function using Theorem 1 and estimate the domain of attraction using Theorem 2 or Theorem 3.

5) If the system state is within the domain of attraction at the moment of fault clearance, then the postfault system does not lose synchronism.

6) Otherwise the switching time should be reduced at least to the time when the fault swing trajectory leaves the domain of attraction.

VII. Effect of Fast Governors

If the relative value of the time constants of the governors and the transients does not warrant the assumption that the mechanical power input is constant, then it is usually assumed that the mechanical power input to the ith machine is related to $d\delta_i/dt = \omega_i$ by $P_{mi}=P_{mi}^0 + p_{mi}$, with $p_{mi} = -H_i(s)\omega_i(s)$, and where

$$H_i(s) = \frac{q_{i,k-1}s^{k-1} + \cdots + q_{i,0}}{s^k + \cdots + p_{i,0}}$$

is the transfer function of the governor of the ith machine and P_{mi}^0 is a constant.

The equations of the ith machine are then given by

$$\frac{d\boldsymbol{y}_i(t)}{dt} = \boldsymbol{A}_i\boldsymbol{y}_i + \boldsymbol{b}_i\omega_i$$

$$\frac{d\delta_i}{dt} = \omega_i$$

$$M_i\frac{d\omega_i}{dt} = -a_i\omega_i - P_{ei} + P_{mi}^0 + \boldsymbol{c}_i^T\boldsymbol{y}_i$$

where $\boldsymbol{A}_i$ is a $(k \times k)$ matrix and $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ are k-dimensional vectors such that $H_i(s) = \boldsymbol{c}_i^T(\boldsymbol{I}s - \boldsymbol{A}_i)^{-1}\boldsymbol{b}_i$, and where $\boldsymbol{y}_i$ is a k-dimensional vector, representing the state of the governor of the ith machine. There are several ways of choosing the $\boldsymbol{A}_i$, $\boldsymbol{b}_i$, and $\boldsymbol{c}_i$. Procedures for obtaining such representations can be found in standard texts on the state-space description of linear systems.

The equilibrium states are then to be determined through the equations $\omega_i = 0$, $\boldsymbol{y}_i = 0$, and $P_{ei} = P_{mi}^0$ for $i = 1,2,\cdots,n$. The stability properties and the domain of attraction can then again be determined by bringing the system in the form of (5) and using the methods outlined above.

VIII. Examples

A. One-Machine System with Constant Power Input [7]

Consider a round-rotor machine connected to an infinite bus described by the differential equations

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + B\{\sin[y(t) + \delta^\circ] - \sin\delta^\circ\} = 0$$

where $y = \delta - \delta^\circ$. Let $x_2 = dy/dt$ and $x_1 = y$. The method outlined in Section IV yields the Lyapunov function

$$V(x_1,x_2) = Ax_1^2 + \alpha x_2^2 + 2x_1x_2 + 2\alpha B(\cos\delta^\circ - \cos(x_1 + \delta^\circ) - x_1\sin\delta^\circ)$$

where $\alpha \geq 1/A$ is a constant. The stability region obtained by applying theorem 2 is given by the set R_1, which consists of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that satisfy $V(x_1, x_2) < (A - 1/\alpha)(\pi - 2\delta^\circ)^2 + 2\alpha B[2\cos\delta^\circ - (\pi - 2\delta^\circ)\sin\delta^\circ]$. Theorem 3 yields as an estimate of the domain of attraction the set R_2, consisting of all

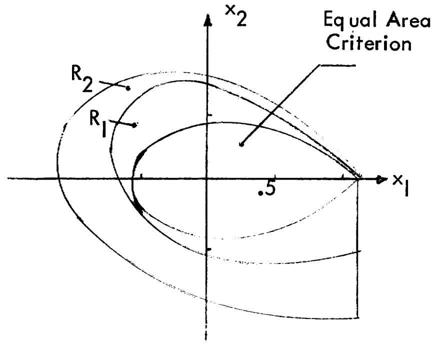

Fig. 1. Regions of stability for example in Section VIII-A.

vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that satisfy $x_1 < \pi - 2\delta^\circ$ and $V(x_1, x_2) < A(\pi - 2\delta^\circ)^2 + 2\alpha B[2\cos\delta^\circ - (\pi - 2\delta^\circ)\sin\delta^\circ]$. For the numerical example $\delta^\circ = 60^\circ$, $A = 0.5$ pu, and $B = 1$ pu, this yields the regions R_1 and R_2 as shown in Fig. 1. The damping coefficient chosen in this numerical example is much larger than those encountered in practice but was chosen for the sake of argument and to illustrate the possibilities obtained through application of Theorem 3.

B. One-Machine System with Fast Governor

Consider the same system and numerical values considered in Section VIII-B, but assume that the power input is regulated by a fast governor with transfer function

$$H(s) = \frac{K}{1 + Ts}$$

with $K = 1$ pu, $T = 2$ pu. The Lyapunov function is given by $V(x_1, x_2, x_3) = x_1^2 + \alpha x_2^2 + \beta x_3^2 + 2x_1x_2 - 2x_1x_3 + \alpha[1 - 2\cos(x_1 + \pi/3) - \sqrt{3}x_1]$ where $x_1 = \delta-\delta^\circ$, $x_2 = d\delta/dt$, $x_3 = p_{mi}/P_{mi}^0 = p_{mi}/B\sin\delta^\circ$, $\beta = 3\alpha - 2 + 2\sqrt{2\alpha(\alpha - 2)}$, and $\alpha \geq 2$ is a constant.

Theorem 2 yields the stability region R_1: all $\boldsymbol{x}$ such that $V(x_1, x_2, x_3) < \pi^2/9\,(1 - 1/\alpha - 1/\beta) + \alpha(2 - \pi/\sqrt{3})$, whereas Theorem 3 yields R_2: all $\boldsymbol{x}$ such that $x_1 < \pi/3$ and $V(x_1, x_2, x_3) < \pi^2/9\,(1 - 1/\beta) + \alpha(2 - \pi/\sqrt{3})$. The intersections of these stability regions and the planes $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ are shown in Fig. 2. Also shown in Fig. 2 are the curves obtained by the method suggested in [3].

C. n-Machine System Without Governors

It is apparent that $(\boldsymbol{I} + \boldsymbol{Q}s)\boldsymbol{G}(s)$ with $\boldsymbol{G}(s) = \boldsymbol{C}(\boldsymbol{I}s - \boldsymbol{A})^{-1}\boldsymbol{B}$ is positive real if $q_i \to \infty$ for all i. The Lyapunov function thus obtained is given by

$$V(\boldsymbol{x}) = \sum_{i=1}^{n}\frac{M_i\omega_i^2}{2} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E_iE_jB_{ij}[\cos(\delta_i^\circ - \delta_j^\circ) - \cos(\delta_i - \delta_j) - (\delta_i - \delta_j - \delta_i^\circ + \delta_j^\circ)\sin(\delta_i^\circ - \delta_j^\circ)].$$

This Lyapunov function is equivalent to the total system energy and was considered in [3].

The above considerations thus show that our methods represent a generalization of existing methods. The previous example indicated that these generalizations might in fact lead to considerable improvements for the estimation of the domain of attraction.

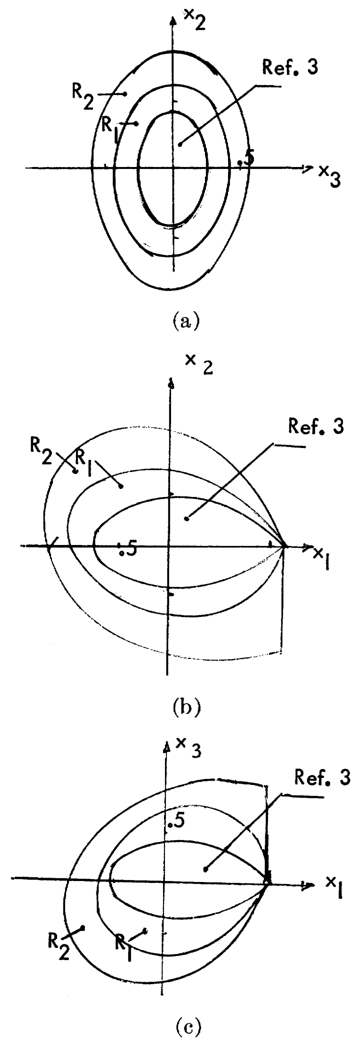

Fig. 2. Regions of stability for example in Section VIII-B. (a) $x_1 = 0$. (b) $x_3 = 0$. (c) $x_2 = 0$.

IX. Concluding Remarks

This paper has outlined a method of obtaining an estimate of the domain of attraction of a multimachine power system. The system equations are expressed such that the nonlinear terms appear as diagonal nonlinearities, depend linearly on the state, and influence the derivative of the state system linearly. This then allows, through the application of some well-known stability criteria, the construction of a Lyapunov function that is claimed to "fit" the system optimally. This Lyapunov function is then used to obtain a region of asymptotic stability of the equilibrium solution, i.e., a set of initial conditions for which synchronism is restored. Two methods of obtaining such a region are presented. The first one is straightforward; the second is more complex, but it yields better results, particularly when damping is present.

The method is very well suited for automatic computation and yields an estimate of the critical switching time without relying on simulation, thus in general requiring a much smaller computational effort.

The application of the direct method of Lyapunov to the power system stability problem is not new and some fine papers [2], [3] on the subject have appeared. This paper uses the same philosophy, and is an attempt to indicate how better results (at the expense of a somewhat more complex procedure) can be obtained and how other effects such as fast governors, salient poles, damping, etc., can be taken into consideration without requiring essential modifications.

Appendix I

The Lyapunov Function

The Lyapunov function is given by

$$V(x) = x^T P x + \sum_{i=1}^{m} 2q_i \int_0^{c_i x} f_i(\sigma_i)\, d\sigma_i$$

where c_i denotes a $2n$-dimensional vector that is the ith row of C, i.e.,

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix}$$

and q_i denotes the ith entry of the $(m \times m)$ diagonal matrix Q which appears in the statement of Theorem 1.

P is a real $(2n \times 2n)$ symmetric positive definite matrix which is the solution of the (algebraic) matrix equations

$$PA + A^T P = -LL^T$$

$$PB = C^T - LW + A^T C^T Q$$

$$W^T W = QCB + B^T C^T Q$$

where W and L are (auxiliary) matrices. It can be shown that a sufficient (and actually also necessary) condition for the above nonlinear matrix algebraic equations to have a positive definite solution P is the $(I + sQ)C(Is - A)^{-1}B$ is a positive real matrix. This positive definite solution can be obtained by spectral factorization methods, by direct solution of the above nonlinear algebraic equations, or by methods involving finding the steady state of a nonlinear matrix differential equation. The particular method to be used depends primarily on the dimension of the system. For lower order systems, the spectral factorization methods are to be recommended and the computations can essentially be performed by hand. For higher order systems, however, one generally has to resort to a recursive scheme to solve these nonlinear algebraic equations.

Appendix II

Estimate of Domain of Attraction When Energy-Like Lyapunov Function Is Used

The Lyapunov function used in Theorem 1 is, as is apparent from Appendix I, indeed positive definite if $\sigma_i f_i(\sigma_i) \geq 0$ for all σ_i. It can also be shown that $\dot{V}(x)$ is then nonpositive along solutions. These conclusions however are in general false if the above inequality holds, as is the case for the multimachine power system, for only a range of values of σ_i.

It can nevertheless be true that even though $V(x)$ is then not positive definite—due to the terms $q_i \int_0^{c_i x} f_i(\sigma_i) d\sigma_i$, which need then not be positive—the derivative along solutions $\dot{V}(x)$ would nevertheless be nonpositive for all x. This is in particular the situation when all the damping coefficients in the dynamic equation are positive, which results in the positive realness of $sEC(Is - A)^{-1}B$, if one computes the Lyapunov function by letting $q_i \rightarrow \infty$ in Appendix I. It can be shown that $\dot{V}(x)$ then becomes a sum of squares, and $\dot{V}(x)$ is nonpositive definite in the whole state space. Under the circumstances it is then possible to make a more accurate estimate of the domain of attraction. This is the subject of Theorem 4.

Theorem 4

Let $V(x)$ be as given, and assume that $\dot{V}(x)$ along solutions is nonpositive for all x. Consider the surfaces $V(x) = k \geq 0$. For small values of k the surfaces are bounded since by assumption $\sigma_i f(\sigma_i) \geq 0$ for σ_i sufficiently small. Let $k = k_{\max}$ be the smallest nonzero value of k for which $\partial V(x)/\partial x_1 = \partial V(x)/\partial x_2 = \cdots = \partial V(x)/\partial x_{2n} = 0$ has a solution for some x such that $V(x) = k$. Then the region R_3 containing the origin and enclosed by the surface $V(x) = k_{\max}$ belongs to the domain of attraction of $\mathbf{0}$.

The proof of Theorem 4 argues that the function $V(x)$ vanishes at the origin and is cupshaped near the origin, i.e., for $||x||$ sufficiently small. The surafces $V(x) = k$ are thus bounded for small positive k. When k increases, then the surfaces remain bounded until either the surface $V(x) = k$ passes through a point where $V(x)$ has a relative maximum (i.e., $\partial V(x)/\partial x_1 = \cdots = \partial V(x)/\partial x_{2n} = 0$ for some x on this surface), or k assumes some value that is a limiting value of $V(x)$ as $x \to \infty$ along some line. The latter case, however, cannot happen for the function $V(x)$ considered here.

Theorem 4 thus states that the surfaces $V(x) = k \geq 0$ remain bounded until they hit a relative maximum of $V(x)$, which could then in this sense qualify for the equilibrium point of the system that is "closest" to the one considered, i.e., the origin 0. The application of Theorem 4 requires the solution of the equations $\partial V(x)/\partial x_1 = \cdots = \partial V(x)/\partial x_{2n} = 0$, but in some cases provides excellent estimates on the domain of attraction.

Theorem 4 is essentially a generalization of the argument used in [3] and [4]. In fact, the example in Section VIII-C gives the expression obtained when the energy (or equivalently $q_i \to \infty$) is considered in the choice of the Lyapunov function. The derivative along solutions $\dot{V}(x)$ is then nonpositive, and thus Theorem 4 is applicable. This yields the domain of attraction $V(x) < V(x^u)$ where x^u satisfies $\partial V(x^u)/\partial x_1 = \partial V(x^u)/\partial x_2 = \cdots = \partial V(x^u)/\partial x_{2n} = 0$, and is the closest equilibrium point in the sense precisely defined above (and is locally unstable).

Example

As an example for the application of Theorem 4, consider the three-machine system with per-unit values $M_1 = M_2 = M_3 = 1$, $E_1 = E_2 = E_3 = 1$, $B_{12} = 4$, $B_{13} = B_{23} = 2$, $P_1 = -P_3$, $P_2 = 0$, and $G_{11} = G_{22} = G_{33} = 0$. It thus consists of a machine (or a group of machines) delivering power to a bus of finite capacity, where reactive power is supplied by a synchronous condensor. Then $\delta_1^\circ - \delta_2^\circ = 18.8°$, $\delta_1^\circ - \delta_3^\circ = 59.9°$, $\delta_1^u - \delta_2^u = 26.7°$, and $\delta_1^u - \delta_3^u = 142.9°$. The stability region R_3 about the stable equilibrium obtained by the procedure suggested by Theorem 4 is hence given by $\omega_1^2 + \omega_2^2 + \omega_3^2 - 6(\delta_1 - \delta_3) - 8 \cos(\delta_2 - \delta_1) - 4 \cos(\delta_1 - \delta_3) - 4 \cos(\delta_2 - \delta_3) + 17.16 < 0$.

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Discussion

Gerald G. Richardson (Bonneville Power Administration, Portland, Ore.): The authors are to be commended for a very interesting paper.

The Bonneville Power Administration presently solves transient stability studies of the Pacific Northwest power system by the direct simulation method using a 500-bus digital computer program. With the advent of regional interconnections, the 500-bus program is no longer large enough to represent the high-voltage grid accurately. Our engineers now are developing a 2000-bus stability program to investigate larger systems. Because of the inertias involved and the long ties between systems, swings must be carried out for several seconds to obtain one full oscillation. It takes even longer for cases that are on the borderline of being unstable. This represents a lot of computer time. I am sure that the industry eagerly awaits a simpler and faster approach to the solution of its stability problems.

The examples in the paper, used to illustrate the application of Lyapunov methods, are excellent. However, there is a big difference between a one-machine problem and actual power system problems. I would be very interested in seeing a solution for a more complicated system using the Lyapunov method compared with the solution by direct simulation. Two examples of faults at different locations on the same system would be very desirable.

There are many happenings in a time sequence of events following a disturbance, such as circuit reclosing, generator dropping, or series capacitor switching. Other circuits may open by relay action due to the initial disturbance. The fault may be single line to ground instead of three phase. Please explain how the Lyapunov method can take these factors into account. Even if the method cannot evaluate such a successive occurrence of happenings, I think the stability evaluation offered by the Lyapunov method looks very promising.

Manuscript received June 18, 1969.

Jacques L. Willems and **Jan C. Willems:** The authors thank the discusser for his interest in the paper. As Mr. Richardson points out, the need for direct methods for stability investigations is urgent, since direct simulation of large systems requires an almost prohibitive amount of computer programming and computer time. The authors only included textbook-type examples of single-machine systems mainly because of their illustrative character. The direct introduction of the successive occurrence of events such as circuit reclosing, generator dropping, and series capacitor switching would lead to the consideration of nonautonomous differential equations, and for such systems stability theory is much less developed. The authors therefore proposed to integrate the system equations in order to obtain the postfault initial state, and to apply Lyapunov techniques in checking whether or not the postfault initial state is within the transient stability region of the equilibrium state of the postfault system. The sequence of events the discusser mentions would thus have to be considered when integrating the system equations during the fault conditions.

Manuscript received August 18, 1969.