Some Results on the L_p Stability of Linear Time-Varying Systems

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Abstract—**This paper considers the L_p stability of systems with a convolution operator in the forward loop and a time-varying gain in the feedback loop. A frequency-domain condition involving the frequency response of the forward loop and the Fourier coefficients of the time-varying gain of the feedback loop is presented. The paper also contains a somewhat more involved time-domain condition in terms of the open-loop weighting pattern.**

I. Introduction

THE STABILITY of systems with a convolution operator in the forward loop and a time-varying gain in the feedback loop has been considered by many authors, and the result that is best known is the so-called circle criterion which has evolved out of the work of Sandberg [1], Zames [2], and others. The problem considered in this paper is the same, but the approach is different in the sense that the estimates which are used here are not the usual L_2 estimates of the operators in the forward and the feedback loop separately but rather a direct estimate on the gain of their composition. This approach leads to an interesting frequency-domain condition and a more elaborate time-domain criterion.

The setting of the problem is the one used by Sandberg [1] and Zames [2] and uses the idea of extended spaces. The reader is assumed to have some familiarity with l_p and L_p spaces. The proofs require the notions of Banach spaces and Banach algebras. The reader is referred to [3] for details on these.

Let $x(t)$ be a complex-valued function of t, and let $T \in R$. Then $P_T x(t)$ is the complex-valued function of t defined by $P_T x(t) = x(t)$ for $t < T$ and $P_T x(t) = 0$ otherwise. L_{pe} is the space of all functions $x(t)$ for which $P_T x(t) \in L_p$ for all $T \in R$.

Let I, I^+, R, R^+ denote respectively the integers, the nonnegative integers, the real numbers, and the nonnegative real numbers. Consider now the linear feedback system shown in Fig. 1 with the convolution operator G_1 in the forward loop defined by

$$G_1 x(t) = \sum_{n \epsilon I^+} g_n x(t - t_n) + \int_{R^+} g(\tau) x(t - \tau)\, d\tau$$

and the time-varying gain G_2 in the feedback loop defined

$$G_2 x(t) = k(t) x(t)$$

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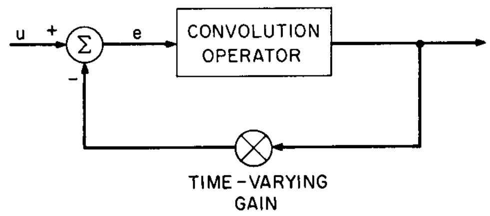

Fig. 1. Feedback system.

with $\{t_n\}$, $n \in I^+$, a sequence of nonnegative real numbers, $\{g_n\}$, $n \in I^+$, an l_1 sequence, $g \in L_1$ with $g(t) = 0$ for $t < 0$, and $k(t) \in L_\infty$.

A standard argument shows that G_1 and G_2 define bounded linear transformations from L_p into itself, $1 < p < \infty$, and that G_1 and G_2 are causal on L_p, i.e., $P_T G_2$ and $P_T G_1$ commute with P_T on L_p. Thus G_1 and G_2 map L_{pe} into itself.

Definition: The feedback system under consideration is said to be L_p *stable* if, for any $u \in L_p$, any $e \in L_{pe}$ which satisfies (in the L_p sense) $P_T u = P_T(I + G_2 G_1)e$ for all $T \in R$ actually belongs to L_p itself, and if $\| e \|_{L_p} \leq K \| u \|_{L_p}$ for some $K \in R$, independent of u. It is said to be L_p unstable if it is not L_p stable.

II. Main Results

Let

$$G(s) = \sum_{n \epsilon I} g_n \exp(-s t_n) + \int_{R^+} g(t) \exp(-st)\, dt$$

denote the Laplace transform of G_1. $G(s)$ is analytic in $\operatorname{Re} s > 0$. Note that whenever $g(t)$ is real-valued, $G(j\omega) = \overline{G(-j\omega)}$, for all $\omega \in R$, where the bar denotes the complex conjugate.

It is assumed that $k(t)$ can be expressed as

$$k(t) = \sum_{n \epsilon I} k_n \exp(-j\omega_n t) + \int_R \hat{k}(j\omega) \exp(-j\omega t)\, d\omega$$

with $\{k_n\}$, $n \in I$, an l_1 sequence, $\{\omega_n\}$, $n \in I$, a sequence of real numbers, and $\hat{k} \in L_1$. Note that whenever $k(t)$ is real-valued it follows that if $\omega_n \in \{\omega_n\}$ then so is $-\omega_n$. Furthermore, the values k_n associated with them are complex conjugate of each other, and $\hat{k}(-j\omega) = \overline{\hat{k}(j\omega)}$. The above restriction on $k(t)$ is a mild one and allows, e.g., every periodic and almost-periodic function with a l_1 Fourier series as the feedback gain.

Theorem 1

The feedback system under consideration is L_2 stable if

$$F_\infty(\omega) = \sum_{n\epsilon I+} | k_n G(j(\omega - \omega_n)) | + \int_R | \hat{k}(j\omega')G(j(\omega - \omega')) | d\omega'$$

and

$$F_1(\omega) = \sum_{n\epsilon I} | k_n G(j(\omega + \omega_n)) | + \int_R | \hat{k}(j\omega')G(j(\omega + \omega')) | d\omega'$$

satisfy $\| F_\infty \|_{L\infty}, \| F_1 \|_{L\infty} < 1$.

Remark: Note that if $g(t)$ and/or $k(t)$ are real valued, then $\| F_\infty \|_{L\infty} = \| F_1 \|_{L\infty}$ and there is thus only one L_∞ norm to be computed in the verification of Theorem 1.

It is, of course, possible to make the usual transformation on the feedback loop. Let k_0 denote the mean value of $k(t)$, i.e., the sum of the coefficients with $\omega_n = 0$ in the expansion of $k(t)$, and let $G'(j\omega) = (1 + k_0 G(j\omega))^{-1} G(j\omega)$.

Theorem 2

Assume that

$$\inf_{\mathrm{Re}\, s \geq 0} | 1 + k_0 G(s) | > 0.$$

Then the feedback system under consideration is L_2 stable if

$$F_\infty'(\omega) = \sum_{n\epsilon I;\omega_n \neq 0} | k_n G'(j(\omega - \omega_n)) | + \int_R | \hat{k}(j\omega')G'(j(\omega - \omega')) | d\omega'$$

and

$$F_1'(\omega) = \sum_{n\epsilon I;\omega_n \neq 0} | k_n G'(j(\omega + \omega_n)) | + \int_R | \hat{k}(j\omega')G'(j(\omega + \omega')) | d\omega'$$

satisfy $\| F_\infty' \|_{L\infty}, \| F_1' \|_{L\infty} < 1$.

Remark: Note again that if $g(t)$ and/or $k(t)$ are real-valued, then $\| F_\infty' \|_{L\infty} = \| F_1' \|_{L\infty}$.

As usual, the above theorems are all derived from some estimate on the norms of the operators $G_2 G_1$ and $G_1 G_2$. The above theorems consider the operators $G_2 G_1$ and $G_1 G_2$ as nonstationary operators in the frequency domain and make some L_1- and L_∞-type estimates on the norms. From these conditions L_2 stability is then concluded. It is apparent that the roles of the operators G_2 and G_1 are dual in the sense that G_2 operates in the frequency domain precisely as G_1 operates in the time domain, and vice versa. Consequently, the question naturally arises what the corresponding criterion is when these estimates are made in the time domain. This leads to the following theorems.

Theorem 3

The feedback system under consideration is L_∞ stable if

$$f_\infty(t) = \sum_{n\epsilon I+} | g_n k(t - t_n) | + \int_{R+} | g(\tau)k(t - \tau) | d\tau$$

satisfies $\| f_\infty \|_{L\infty} < 1$. It is L_1 stable if

$$f_1(t) = \sum_{n\epsilon I+} | g_n k(t + t_n) | + \int_{R+} | g(\tau)k(t + \tau) | d\tau$$

satisfies $\| f_1 \|_{L\infty} < 1$. It is L_p stable $(1 \leq p \leq \infty)$ if $\| f_\infty \|_{L\infty} < 1$ and $\| f_1 \|_{L\infty} < 1$.

Let k_0 again denote the mean value of $k(t)$, let $k'(t) = k(t) - k_0$, and let

$$G'(j\omega) = (1 + k_0 G(j\omega))^{-1} G(j\omega) = \sum_{n\epsilon I+} g_n' \exp(-j\omega t_n') + \int_{R+} g'(t) \exp(-j\omega t)\, dt.$$

Theorem 4

Assume that

$$\inf_{\mathrm{Re}\, s \geq 0} | 1 + k_0 G(s) | > 0.$$

Then the feedback system under consideration is L_∞ stable if $\| f_\infty' \|_{L\infty} < 1$. It is L_1 stable if $\| f_1' \|_{L\infty} < 1$. It is L_p stable if $\| f_\infty' \|_{L\infty} < 1$ and $\| f_1' \|_{L\infty} < 1$. The functions $f_\infty'(t)$ and $f_1'(t)$ are defined exactly as $f_\infty(t)$ and $f_1(t)$ but with $k(t)$, $\{g_n\}$, $\{t_n\}$, and $g(t)$ replaced by $k'(t)$, $\{g_n'\}$, $\{t_n'\}$, and $g'(t)$.

As will be shown through an example, the above stability criteria are not implied by the circle criterion. (They by no means generalize the circle criterion, however.) This might be surprising since it is to be expected that L_2 estimates will give the best results for linear systems. The circle criterion is in fact based on L_2 estimates, and the above criteria are not. The reason why in some circumstances it is thus possible to improve on the circle criterion appears to be that in the circle criterion the gains of G_2 and G_1 are estimated separately and they yield an estimate of $G_2 G_1$ through the inequality $\| G_2 G_1 \| \leq \| G_2 \| \| G_1 \|$. This is not the case in the above criterion which is based on direct estimates of the norms of $G_2 G_1$ and $G_1 G_2$.

It should also be noted that particularly the L_∞ stability obtained above follows rather easily from the usual estimates which are made in proving bounded-input, bounded-output stability for linear time-varying systems. See [4] for a correct exposition of this relationship.

III. Examples

Example 1

Let $k(t) = k_0 + k_1 \cos(\omega_0 t + \phi)$, $\omega_0 > 0$. Theorem 2 then predicts L_2 stability whenever

$$\tfrac{1}{2} | k_1 | (| G'(j(\omega - \omega_0)) | + | G'(j(\omega + \omega_0)) |) < 1$$

for all $\omega \in R$.

Example 2

Let $k(t) = \sum_{n\epsilon I} k_n \exp(-jn\omega_0 t)$, $\omega_0 > 0$, and assume that $G(j\omega)$ satisfies a filtering condition of the type $|G'(j(\omega + n\omega_0))| \ll |G'(j\omega)|$ for all $n \in I$, $n \neq 0$, and $|\omega| \leq \omega_0/2$. Then L_2 stability results if

$$\max_{\omega\epsilon R} |G'(j\omega)| \max_{n\epsilon I; n\neq 0} |k_n| < 1.$$

Example 3

Let $g(t) \geq 0$ and $k(t) = A \sin \omega_0 t$. Theorem 3 yields L_p stability if

$$\sup_{0\leq\phi\leq\pi} |A| \int_0^\infty g(t) |\cos \omega_0(t + \phi)| \, dt < 1.$$

The circle criterion predicts L_2 stability if

$$|A| \int_0^\infty g(t) \, dt < 1.$$

More generally, whenever $g(t) \geq 0$ (or $g(t) \leq 0$), and $k(t)$ has symmetrical limits [i.e., $\operatorname{ess\,sup}_{t\epsilon R} k(t) = \operatorname{ess\,inf}_{t\epsilon R} k(t)$] then Theorem 3 will give at least as good an estimate for stability as the circle criterion.

IV. Proofs

Since the results of Theorems 1 through 4 are essentially concerned with the existence of a bounded inverse of a bounded linear transformation from a Banach space into itself, it appears useful to introduce the algebra of bounded linear transformations from a Banach space into itself.

Let B denote a complex Banach space and let $\mathcal{L}(B,B)$ denote the algebra of all bounded linear operators from B into itself, with addition and multiplication defined in the obvious way and with multiplication of elements defined as composition of maps. Let the norm on $\mathcal{L}(B,B)$ be the induced norm, i.e., for $L \in \mathcal{L}(B,B)$,

$$\|L\|_{\mathcal{L}(B,B)} \triangleq \sup_{x\epsilon B; x\neq 0} \frac{\|Lx\|_B}{\|x\|_B}.$$

Lemma 1

$\mathcal{L}(B,B)$ is a Banach algebra with a unit.

Proof: The proof of this standard result can be found, e.g., in Hille and Phillips [3, p. 51].

The open-loop operator characterizing the feedback system under consideration is a time-varying operator belonging to $\mathcal{L}(L_p,L_p)$, $1 \leq p \leq \infty$, and is for the purposes of the paper most easily characterized by its weighting pattern. This operator will be imbedded in a general class of time-varying operators with weighting patterns consisting of a function and a string of (time-varying) impulses.

Consider the space Y_∞ consisting of complex-valued (generalized) functions on $R \times R$ defined by

$$y(t,\tau) = w(t,\tau) + \sum_{n\epsilon I} g_n(t)\delta(\tau - t + \tau_n)$$

and the space Y_1 consisting of complex-valued (generalized) functions on $R \times R$ defined by

$$y(t,\tau) = w(t,\tau) + \sum_{n\epsilon I} h_n(\tau)\delta(t - \tau - \tau_n).$$

By defining addition and scalar multiplication in the obvious way, Y_∞ and Y_1 become vector spaces. Y_∞ and Y_1 can be equipped with a norm if some assumptions about the integrability in the t or τ direction is made. It is thus assumed that[1] if $y \in Y_\infty$, then $w(t,\cdot) \in L_1$ and $\{g_n(t)\} \in l_1$, for almost all $t \in R$, and if $y \in Y_1$, then $w(\cdot,\tau) \in L_1$ and $\{h_n(\tau)\} \in l_1$ for almost all $\tau \in R$. Let

$$f_\infty(t) = \|w(t,\cdot)\|_{L_1} + \|\{g_n(t)\}\|_{l_1}$$

and

$$f_1(\tau) = \|w(\cdot,\tau)\|_{L_1} + \|\{h_n(\tau)\}\|_{l_1}.$$

Let $y \in Y_\infty$ if $f_\infty \in L_\infty$ and $\|y\|_{Y_\infty} \triangleq \|f_\infty\|_{L_\infty}$, and $y \in Y_1$ if $f_1 \in L_\infty$ and $\|y\|_{Y_1} \triangleq \|f_1\|_{L_\infty}$. It can be verified that the spaces Y_∞ and Y_1 thus defined are in fact Banach spaces. This generalization of a well-known fact for time-invariant operators (see, e.g., [3, p. 153]) is left to the reader since it follows rather easily if one keeps the validity of this result for the time-invariant case in mind.

The intersection of Y_∞ and Y_1 will be an important space which will be considered in the sequel, and consists of all $y(t,\tau)$ which can be written in both the forms imposed by Y_∞ and Y_1. A few words of explanation of this characterization of $Y_\infty \cap Y_1$ appears necessary. It imposes a uniform integrability constraint on $w(t,\tau)$ in both the t and the τ direction. It can also easily be verified that if $y \in Y_\infty \cap Y_1$, then $g_n(t) = h_n(t - \tau_n)$.

The spaces Y_∞ and Y_1 are introduced for the reason that every element of Y_∞ and Y_1 defines a bounded linear transformation from L_∞ and L_1, respectively, into itself. Moreover, every element of $Y_\infty \cap Y_1$ defines a bounded linear transformation from L_p, $1 \leq p \leq \infty$, into itself. This is the subject of the following lemma. Consider thus the mapping formally defined by

$$Wx(t) = \sum_{n\epsilon I} g_n(t)x(t - \tau_n) + \int_R w(t,\tau)x(\tau) \, d\tau$$

$$= \sum_{n\epsilon I} h_n(t - \tau_n)x(t - \tau_n) + \int_R w(t,\tau)x(\tau) \, d\tau.$$

Let w denote the (generalized) function

$$w(t,\tau) + \sum_{n\epsilon I} g_n(t)\delta(\tau - t + \tau_n) = w(t,\tau) + \sum_{n\epsilon I} h_n(\tau)\delta(t - \tau - \tau_n)$$

where $g_n(t) = h_n(t - \tau_n)$.

Lemma 2

1) If $w \in Y_\infty$, then $W \in \mathcal{L}(L_\infty,L_\infty)$ and $\|W\|_{\mathcal{L}(L_\infty,L_\infty)} = \|w\|_{Y_\infty}$.

[1] The integrations involved are with respect to $(\cdot)$.

2) If $w \in Y_1$, then $W \in \mathcal{L}(L_1,L_1)$ and $\| W \|_{\mathcal{L}(L_1,L_1)} = \| w \|_{Y_1}$.

3) If $w \in Y_\infty \cap Y_1$, then $W \in \mathcal{L}(L_p,L_p)$, $1 \leq p \leq \infty$, and $\| W \|_{\mathcal{L}(L_p,L_p)} \leq \| w \|_{Y_\infty}{}^{1/p} \cdot \| w \|_{Y_1}{}^{1/q}$ with $1/p + 1/q = 1$.

Proof: For simplicity in notation, assume that $g_n(t) = 0$ for all $n \in I$ (the extension to cover the general case is straightforward), and assume that $1 < p < \infty$ (the case $p = 1$ or $p = \infty$ can easily be treated directly). Then

$$Wx(t) = \int_R w(t,\tau)x(\tau)\, d\tau$$

and thus

$$\| Wx(t) \|_{L_p}^p \leq \int_R \left[\int_R | w(t,\tau) | \, | x(\tau) | \, d\tau \right]^p dt$$

$$= \int_R \left[\int_R | w(t,\tau) |^{1/q} | w(t,\tau) |^{1/p} | x(\tau) | \, d\tau \right]^p dt.$$

Thus from Hölder's inequality it follows that

$$\| Wx(t) \|_{L_p}^p \leq \int_R \left[\int_R | w(t,\tau) | \, d\tau \right]^{pq^{-1}} \cdot \left[\int_R | w(t,\tau) | \, | x(\tau) |^p \, d\tau \right] dt$$

$$\leq \| w \|_{Y_\infty}{}^{pq^{-1}} \int_R \int_R | w(t,\tau) | \, | x(\tau) |^p \, d\tau \, dt$$

$$\leq \| w \|_{Y_\infty}{}^{pq^{-1}} \| w \|_{Y_1} \| x \|_{L_p}{}^p$$

and thus $\| Wx(t) \|_{L_p} \leq \| w \|_{Y_\infty}{}^{q^{-1}} \| w \|_{Y_1}{}^{p^{-1}} \| x \|_{L_p}$, as claimed.

The above shows that W is well defined and that inequality holds in 1), 2), and 3). To show that actually equality holds in 1) and 2), certain particular choices of $x(t)$ need to be made which yield an L_∞ or L_1 gain which is arbitrarily close to $\| w \|_{Y_\infty}$ or $\| w \|_{Y_1}$, respectively. The details of the resulting tedious inequality manipulations are left to the reader. Let it just be mentioned that for the L_∞ case, signum functions for $x(t)$, and for the L_1 case, delta-like functions, ought to be considered to obtain this tight estimate.

One more important fact which is needed about the relationship of Y_∞ and Y_1 with $\mathcal{L}(L_\infty,L_\infty)$ and $\mathcal{L}(L_1,L_1)$, respectively, is the algebraic structure of Y_∞ and Y_1 themselves. Let Y_∞ and Y_1 be made into algebras by defining multiplication of elements as composition of maps. This composition makes sense by the previous lemma and satisfies the norm inequalities by Lemmas 1 and 2. Closedness of Y_∞ and Y_1 under multiplication is immediate from consideration of the weighting patterns of the resulting map.

Lemma 3

Y_∞ is a closed subalgebra of $\mathcal{L}(L_\infty,L_\infty)$ and Y_1 is a closed subalgebra of $\mathcal{L}(L_1,L_1)$.

Proof: Since Y_∞ and Y_1 are Banach algebras themselves and equipped, by Lemma 2, with the norms of $\mathcal{L}(L_\infty,L_\infty)$ and $\mathcal{L}(L_1,L_1)$ respectively, the lemma follows.

A few more facts on existence of inverses are needed to complete the introductory material which goes into the proofs of Theorems 1 and 3. These are stated in Lemmas 4 and 5. The proofs are both immediate.

Lemma 4

Let W be an element of a Banach algebra B, with a unit I. Then $I + W$ is invertible if $\| W \|_B < 1$. In fact,

$$(I + W)^{-1} = \sum_{n \in I^+} (-1)^n W^n.$$

Lemma 5

Let W_1 and W_2 be elements of a Banach algebra B with unit I. Then $I + W_1W_2$ is invertible if and only if $I + W_2W_1$ is, and in fact,

$$(I + W_2W_1)^{-1} = I - W_2(I + W_1W_2)^{-1}W_1.$$

The stage is now set to attempt the proofs of Theorems 1 through 4.

Proof of Theorem 1: Consider the mappings in the frequency domain defined by $H_1x(j\omega) = G(j\omega)x(j\omega)$ and

$$H_2x(j\omega) = \sum_{n \in I} k_n x(j(\omega - \omega_n)) + \int_R \hat{k}(j(\omega - \omega'))x(j\omega')\, d\omega'$$

where G, $\{k_n\}$ and $\hat{k}$ are as defined in Section II. Clearly H_1 and H_2 agree with G_1 and G_2 on L_2. Let $M_{12} = H_1H_2$ and $M_{21} = H_2H_1$. Then M_{12} and M_{21} correspond to the time-varying weighting patterns (in the frequency domain) given by

$$\begin{aligned} m_{12}(\omega,\omega') &= G(j\omega)\hat{k}(j(\omega - \omega')) + \sum_{n \in I} k_n G(j\omega)\delta(\omega' - \omega + \omega_n) \\ &= G(j\omega)\hat{k}(j(\omega - \omega')) + \sum_{n \in I} k_n G(j(\omega' + \omega_n))\delta(\omega - \omega' - \omega_n) \\ m_{21}(\omega,\omega') &= \hat{k}(j(\omega - \omega'))G(j\omega') + \sum_{n \in I} k_n G(j(\omega - \omega_n))\delta(\omega' - \omega + \omega_n) \\ &= \hat{k}(j(\omega - \omega'))G(j\omega') + \sum_{n \in I} k_n G(j\omega')\delta(\omega - \omega' - \omega_n) \end{aligned}$$

and hence $m_{12}, m_{21} \in Y_\infty \cap Y_1$. Moreover

$$\| m_{12} \|_{Y_1} = \left\| \int_{-\infty}^{+\infty} | G(j\omega)\hat{k}(j(\omega - \omega')) | \, d\omega + \sum_{n \in I} | \hat{k}_n G(j(\omega' + \omega_n)) | \right\|_{L_\infty}$$

and

$$\| m_{21} \|_{Y\infty} = \left\| \int_{-\infty}^{+\infty} | k(j(\omega - \omega'))G(j\omega')) | \, d\omega' + \sum_{n\epsilon I} | k_n G(j(\omega - \omega_n)) | \right\|_{L\infty}.$$

Notice that thus $\| m_{12} \|_{Y_1} = \| F_1 \|_{L\infty}$ and $\| m_{21} \|_{Y\infty} = \| F_\infty \|_{L\infty}$, and thus that by assumption

$$\| m_{12} \|_{Y_1}, \| m_{21} \|_{Y\infty} < 1.$$

Hence m_{12} and m_{21} are by Lemmas 4 and 3 invertible on Y_1 and Y_∞, respectively. This then implies by Lemma 5 that m_{21} is thus invertible on $Y_\infty \cap Y_1$. This inverse thus has a weighting pattern which belongs to $Y_\infty \cap Y_1$. It induces by Lemma 2 an element of $\mathcal{L}(L_2,L_2)$ which obviously qualifies for the inverse of $I + G_2G_1$ on L_2. It remains to be shown that $(I + G_2G_1)^{-1}$ is causal. This, however, follows since at no point in the previous proof was the fact used that the $L_2(-\infty,+\infty)$ was being considered rather than $L_2(T,\infty)$, $T \in R$. Hence $I + G_2G_1$ has a bounded causal inverse on L_2. Let $e \in L_{2e}$ satisfy $P_T u = P_T(I + G_2G_1)e$, for all $T \in R$. Thus $P_T e = P_T(I + G_2G_1)^{-1}P_T u$. Hence $\| P_T e \|_{L_2} \leq \| (I + G_2G_1)^{-1} \| \| u \|_{L_2}$ which proves L_2 stability as claimed.

Proof of Theorem 3: The proof of Theorem 3 is completely analogous to the proof of Theorem 1 with the time domain replaced by the frequency domain and the roles of $k(t)$ and $G(j\omega)$ reversed. However, somewhat stronger conclusions can be made since in this case L_1 and L_∞ are of some intrinsic importance. The proof thus proceeds by demonstrating that the assumptions of the theorem assure that $I + G_2G_1$ has a bounded inverse on L_∞ or L_1 if $\| f_\infty \|_{L\infty}$ or $\| f_1 \|_{L\infty} < 1$, respectively. This merely involves application of Lemmas 2, 3, and 4. Lemma 4 also immediately shows that this inverse is in addition causal. If $\| f_\infty \|_{L\infty}, \| f_1 \|_{L\infty} < 1$ then the weighting pattern of this inverse will actually belong to $Y_\infty \cap Y_1$ and thus by Lemma 2 induces an element of $\mathcal{L}(L_p,L_p)$, $1 \leq p \leq \infty$. Causality is again immediate from Lemma 4 and stability then follows in all the previous cases by an identical argument to the one used in the proof of Theorem 1.

Proof of Theorems 2 *and* 4: Some well-known results (see [3], p. 150) ensure that $I + k_0G_1$ has a bounded causal inverse on L_p, $1 \leq p \leq \infty$. Since

$$[I + (G_2 - k_0I)G_1(I + k_0G_1)^{-1}](I + k_0G_1) = I + G_2G_1$$

it follows that this case can thus be reduced to Theorems 1 and 3.

Remark: The proofs of Theorems 1 and 3 for the case $1 < p < \infty$ are not based on the principle that the conditions of the theorems assure that the open-loop gain is less than unity. This might nevertheless be the case, but the invertibility results are not based on the contraction principle. The estimates are somewhat more delicate and are based on consideration of the weighting pattern of the inverse and proving some properties of this inverse weighting pattern. This is done through consideration of $I + G_1G_2$ and $I + G_2G_1$ simultaneously. Moreover, in Theorem 1 artificial spaces (i.e., L_1 and L_∞ functions of ω) are introduced which only in the L_2 case have a meaning in the time domain where stability is defined. It should also be noted that the introduction of Banach algebras in the proofs appears to be the natural setting for the analysis which goes into proving the results.

V. Concluding Remarks

1) It is a simple matter to generalize the preceding theorems to the case where G_1 is a matrix convolution operator and G_2 is a time-varying matrix multiplication. In particular, Theorem 2 then becomes (for the real-valued case)

a) $\| F' \|_{L\infty} < 1$, where

$$F'(\omega) = \sum_{n\epsilon I; \omega_n \neq 0} \| k_n \| \| G'(j(\omega - \omega_n)) \| + \int_R \| \hat{k}(j\omega) \| \| G'(j(\omega - \omega')) \| \, d\omega'$$

b) $$\inf_{\text{Re } s \geq 0} | \det (I + k_0G(s)) | > 0$$

or

$$\inf_{\text{Re } s \geq 0} | \det (I + G(s)k_0) | > 0.$$

2) The condition $\inf_{\text{Re } s \geq 0} | 1 + k_0G(s) | > 0$ can, at least when $g_n = 0$, for all $n \in I^+$, be reduced to

a) $$\inf_{\text{Re } s = 0} | 1 + k_0G(s) | > 0$$

b) $$k_0G(j\omega), \quad -\infty < \omega < \infty$$

does not encircle the $-1 + 0j$ point in the complex plane.

3) Theorems 2 and 4 can be stated as instability theorems if the condition

$$\inf_{\text{Re } s \geq 0} | 1 + k_0G(s) | > 0$$

is replaced by

$$\inf_{\text{Re } s \geq 0} | 1 + k_0G(s) | = 0 \quad \text{and} \quad \inf_{\text{Re } s = 0} | 1 + k_0G(s) | > 0$$

(see [5] for details).

4) It is of course not necessary to assume that k_0 in Theorems 2 and 4 is the mean value of $k(t)$. This was merely done because it apparently gives the best results.

5) If the system is described by an ordinary differential equation then L_2 stability implies asymptotic stability. For more results in that direction see Sandberg [1].

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References

[1] I. W. Sandberg, "Some results on the theory of physical systems governed by nonlinear functional equations," *Bell Sys. Tech. J.*, vol. 44, p. 871, 1965.
[2] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems, part I: conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Trans. Automatic Control*, vol. AC-11, pp. 228–238, April 1966.
——, "On the input-output stability of time-varying nonlinear feedback systems, part II: conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Trans. Automatic Control*, vol. AC-11, pp. 465–476, July 1966.
[3] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, 2nd ed. Providence, R. I.: Am. Math. Soc. Publ., 1957.
[4] C. A. Desoer and A. J. Tomasian, "A note on zero-state stability of linear systems," *Proc. 1st Ann. Allerton Conf. on Circuit and Systems Theory*, 1963, p. 50.
[5] J. C. Willems, "Stability, instability, invertibility and casuality," *SIAM J. Control*, vol. 7, no. 4, 1969.

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Minimum Sensitivity Design of Linear Multivariable Feedback Control Systems by Matrix Spectral Factorization

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Abstract—**A scalar measure of system sensitivity to plant parameter variations is employed in the design of linear lumped stationary multivariable feedback control systems. The plant parameters are treated as random variables, and design formulas are derived which lead to systems with the smallest expected value for the chosen scalar sensitivity measure. The design formulas give physically realizable feedback and tandem compensation network transfer function matrices provided the overall system transfer function matrix is properly specified. The solution of the minimum sensitivity design problem is obtained by first solving the multivariable semi-free-configuration Wiener problem.**

Introduction

THE RESULTS of an earlier effort [1] are extended to linear lumped stationary multivariable control systems in this paper. The system considered is shown in Fig. 1. The plant is represented by the rational transfer function matrix $G_p(s,\boldsymbol{\alpha})$. It is assumed that the plant is asymptotically stable. (When the plant is not asymptotically stable, but is completely controllable, it can always be made asymptotically stable with state variable feedback [2] or with output feedback through a compatible observer [3].) The N-dimensional column vector $\boldsymbol{\alpha}$ represents the mean or expected value of the plant parameters, and any deviation from the mean is denoted by $\delta\boldsymbol{\alpha}$. Thus,

$$E\{\delta\alpha_i\} = 0, \quad i = 1,2,\cdots,N \tag{1}$$

where $E\{\cdot\}$ denotes the expected value, and $\delta\alpha_i$ is the element in the ith row of $\delta\boldsymbol{\alpha}$. It is assumed that the covariance matrix (the prime denotes the transpose)

$$\Sigma = E\{\delta\boldsymbol{\alpha}\delta\boldsymbol{\alpha}'\} = [\sigma_{ij}], \sigma_{ji} = \sigma_{ij} = E\{\delta\alpha_i\delta\alpha_j\} \tag{2}$$

is known, and that the variations $\delta\alpha_i$ are small and independent of the signals in the system. The input $\boldsymbol{R}$ is generated by a stationary stochastic process with known power spectral density matrix.

The rational transfer function matrices $G_c(s)$ and $H(s)$ represent, respectively, the tandem compensation network

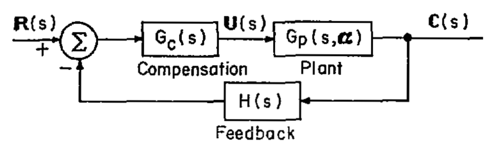

Fig. 1. System.

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