Some New Rearrangement Inequalities Having Application in Stability Analysis

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***Abstract*—The Sylvester test for establishing the positivity of quadratic forms is a basic tool. For nonquadratic forms, however, necessary and sufficient conditions for positivity are generally not known. Given here are some simple necessary and sufficient conditions for forms of the type**

$$\sum_{k,l=1}^{n} x_k m_{kl} f(x_l)$$

to be positive. These results are derived by combining a classic result of Hardy, Littlewood, and Polya with the Birkhoff characterization of doubly stochastic matrices. The results are applied to the difference equations governing a nonlinear feedback loop. In this setting they yield new and quite general conditions for stability.

I. Introduction

In the classic book on inequalities by Hardy, Littlewood, and Polya [1], Chapter 10 is devoted to questions relating the inner products of similarly ordered sequences to the inner products of rearranged sequences. The simplest result given there states that if $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ and if $y_{\pi(1)}, y_{\pi(2)}, \cdots y_{\pi(n)}$ is any rearrangement of the y sequence, then

$$\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\pi(k)}.$$

The informal explanation of this fact in [1] is that given a lever arm with hooks at distances $x_1, x_2, \cdots, x_n$ from a pivot, and weights $y_1, y_2, \cdots, y_n$ to hang on the hooks, the largest moment is obtained by hanging the largest weight on the farthest hook, the next largest weight on the next most distant hook, etc.

This result has an interpretation in terms of positive transformations. Recall that a finite dimensional linear transformation is called positive if the inner product between any vector and its image under the transformation is non-negative. Thus, $y = Qx$ defines a positive transformation if and only if the matrix Q plus its transpose is non-negative definite. Since simple necessary and sufficient conditions for a matrix to be non-negative definite are known, any linear finite dimensional transformation can be checked for postivity. For nonlinear transformations, the situation is quite different and this is where the rearrangement inequality is useful. Suppose that f is a scalar-valued function of a scalar argument, and that y_k is related to x_k by the (in general) nonlinear transformation

$$y_k = \sum_{l=1}^{n} m_{kl} f(x_l). \tag{1}$$

If f is a monotone nondecreasing function, then a suitable renumbering of the x's gives $x_1 \leq x_2 \leq \cdots \leq x_n$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$. Hence the transformation

$$y_k = f(x_k) - f(x_{\pi(k)})$$

satisfies

$$\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} x_k f(x_k) - x_k f(x_{\pi(k)}) \geq 0$$

and is positive. If $f(x)$ denotes the n-vector whose components are $f(x_i)$, then in language of positive transformations, the Hardy, Littlewood, and Polya rearrangement theorem says that the transformation

$$y = (I - P)f(x)$$

is positive where I is the identity matrix, P is any permutation matrix, and f is monotone nondecreasing. It will be shown that this result together with a result of Birkhoff on the decomposition of doubly stochastic matrices permits the derivation of a number of interesting positivity conditions for a class of transformations of the type defined by (1).

Why are positive transformations important? Many techniques involve establishing at a certain point that a certain function is non-negative definite, e.g., second variations in optimization and Liapunov functions and their derivatives, etc. This verification can, of course, often be reduced to establishing the positivity of a certain transformation. The particular transformations defined by (1) are of special interest in the study of systems whose nonlinear terms each depend on a single argument. Systems of this type have been extensively studied in connection with the so-called frequency power formulas of Manley and Rowe [2] and the Lur'e feedback loop stability problem [3]. The present paper is the result of trying to bring certain methods and results in these areas into harmony. In particular, Prosser [4] and Black [5] used the rearrangement theorem to get frequency power formulas. Quite independently O'Shea

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[6] announced a result which strongly resembles certain results of Page [7] on frequency power formulas. In a related paper, Zames and Falb [8] provided, using a factorization of certain operators, an important refinement of the usual positive operator argument to obtain stability criteria. This paper also provides the tools necessary to bring into nearly complete agreement the frequency power formula and the positive operator points of view.

The present paper starts with establishing positivity conditions for certain transformations which are both necessary and sufficient. In order to use the full power of these results in the stability problem, it is necessary to consider positive operators which are formed by a time-varying linear system and a monotone or an odd monotone nonlinearity. The results that are obtained by specializing to the time-invariant case are exactly those in the literature. The proof of the stability theorem follows the argument used by Zames and Falb [8], but a more difficult factorization theorem is needed. The factorization theorem given here is felt to be of interest in its own right. Its proof is inspired by a paper by Baxter [9] in probability theory.

For various technical reasons, the discussion is largely concerned with difference equations. With some modifications, similar results can be obtained for differential equations.

II. Generalizations of a Classical Rearrangement Inequality

Definitions

Two sequences of real numbers $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ are said to be *similarly ordered* if the inequality $x_k < x_l$ implies that $y_k \leq y_l$. Thus two sequences are similarly ordered if and only if they can be rearranged such that the resulting sequences are both monotone nondecreasing, i.e., there exists a permutation $\pi(k)$ of the n first integers [$\pi(k)$ takes on each of the values $1, 2, \cdots, n$ just once as k varies through the values $1, 2, \cdots, n$] such that *both* the sequences $\{x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)}\}$ and $\{y_{\pi(1)}, y_{\pi(2)}, \cdots, y_{\pi(n)}\}$ are monotone nondecreasing. Two sequences are said to be *unbiased* if $x_k y_k \geq 0$. Clearly two sequences are similarly ordered and unbiased if and only if the augmented sequences $\{x_1, x_2, \cdots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \cdots, y_n, y_{n+1}\}$ with $x_{n+1} = y_{n+1} = 0$ are similarly ordered. Two sequences are said to be *similarly ordered and symmetric* if they are unbiased and if the sequences $\{|x_1|, |x_2|, \cdots, |x_n|\}$ and $\{|y_1|, |y_2|, \cdots, |y_n|\}$ are similarly ordered.

Example: Let $f(\sigma)$ be a mapping from the real line into itself, and consider the sequences $\{x_1, x_2, \cdots, x_n\}$ and $\{f(x_1), f(x_2), \cdots, f(x_n)\}$. These two sequences will be similarly ordered for all sequences $\{x_1, x_2, \cdots, x_n\}$ if and only if $f(\sigma)$ is a monotone nondecreasing function of σ, i.e., if for all σ_1 and σ_2, $(\sigma_1 - \sigma_2)(f(\sigma_1) - f(\sigma_2)) \geq 0$. They will be unbiased if and only if $f(\sigma)$ is a first and third quadrant function, i.e., if for all σ, $\sigma f(\sigma) \geq 0$. They will be similarly ordered and symmetric if and only if $f(\sigma)$ is an odd monotone nondecreasing function of σ, i.e., if $f(\sigma)$ is monotone nondecreasing and $f(\sigma) = -f(-\sigma)$ for all σ.

Definitions[1]

A real $(n \times n)$ matrix $M = (m_{kl})$ is said to be *doubly hyperdominant with zero excess* if $m_{kl} \leq 0$ for $k \neq l$, and if

$$\sum_{k=1}^{n} m_{kl} = \sum_{l=1}^{n} m_{kl} = 0 \quad \text{for all } k, l.$$

It is said to be *doubly hyperdominant* if $m_{kl} \leq 0$ for $k \neq l$, and if

$$\sum_{k=1}^{n} m_{kl} \geq 0 \quad \text{and} \sum_{l=1}^{n} m_{kl} \geq 0 \quad \text{for all } k, l.$$

An $(n \times n)$ matrix M is said to be *doubly dominant* if

$$m_{ll} \geq \sum_{\substack{k=1 \\ k \neq l}}^{n} |m_{kl}| \quad \text{and } m_{kk} \geq \sum_{\substack{l=1 \\ l \neq k}}^{n} |m_{kl}|.$$

It is clear that all of the classes of matrices introduced above are subclasses of the class of all matrices whose symmetric part is non-negative definite, and that every doubly hyperdominant matrix is doubly dominant.

Two other classes of matrices which will be used in the sequel and which have received ample attention in the past are defined below.

Definitions

An $(n \times n)$ matrix M is said to be *doubly stochastic* if it is a non-negative matrix (i.e., $m_{kl} \geq 0$ for all k, l) and if its rows and columns sum to 1. An $(n \times n)$ matrix is said to be a *permutation* matrix if every row and column contains $n-1$ zero elements and an element which equals 1. The relation between the class of doubly stochastic matrices and permutation matrices is given in the following lemma due to Birkhoff.

Lemma 1 (Birkhoff): The set of all doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices, i.e., if M is a doubly stochastic matrix then

$$M = \sum_{i=1}^{N} \alpha_i P_i$$

with $\alpha_i \geq 0$,

$$\sum_{i=1}^{N} \alpha_i = 1$$

[1] The term dominant is standard. Hyperdominant is prevalent, at least in the electrical network literature. The term doubly is used by analogy with doubly stochastic where a property of a matrix also holds for its transpose. Beyond this the nomenclature originates with the authors.

and P_i a permutation matrix. This decomposition is, in general, not unique.

Proof: A short proof can be found in Marcus and Minc [10].

Theorem 1 states the main result of this paper and is a considerable generalization of a classical rearrangement inequality due to Hardy, Littlewood, and Polya [1]. This inequality is stated in Lemma 2.

Lemma 2 (Hardy, Littlewood, and Polya): Let $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ be two similarly ordered sequences, and let $\pi(k)$ be a permutation of the first n integers. Then

$$\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\pi(k)}.$$

Proof: A proof can be found in [1].

Theorem 1

A necessary and sufficient condition for the bilinear form

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l$$

to be non-negative for all similarly ordered sequences $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ is that the matrix $M=(m_{kl})$ be doubly hyperdominant with zero excess.

Proof:

1) *Sufficiency:* Let M be a doubly hyperdominant matrix with zero excess and let r be a positive number such that $r \geq m_{kl}$ for all k, l. Clearly $M=r(I-r^{-1}(rI-M))$. Since, however, $r^{-1}(rI-M)$ is a doubly stochastic matrix, it can be decomposed as

$$\sum_{i=1}^{N} \alpha_i P_i$$

with

$$\alpha_i \geq 0, \quad \sum_{i=0}^{N} \alpha_i = 1$$

and P_i a permutation matrix. Thus M can be written as

$$M = \sum_{i=1}^{N} \beta_i (I - P_i) \quad \text{with } \beta_i \geq 0.$$

This decomposition of doubly hyperdominant matrices with zero excess shows that it is enough to prove the sufficiency part of Theorem 1 for the matrices $I-P_i$. This, however, is precisely what is stated in Lemma 2.

2) *Necessity:* The matrix M may fail to be doubly hyperdominant with zero excess because $m_{kl}>0$ for some $k \neq l$, in which case the sequences with $n-1$ zero elements except $+1$ and -1 in, respectively, the kth and lth spots lead to

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l = - m_{kl} < 0.$$

Assume next that the matrix M fails to be doubly hyperdominant with zero excess because

$$\sum_{k=1}^{n} m_{kl} \neq 0$$

for some l (a similar argument holds if

$$\sum_{l=1}^{n} m_{kl} \neq 0$$

for some k), and consider the sequences $\{1, \cdots, 1, 1+\epsilon, 1, \cdots, 1\}$ and $\{0, \cdots, 0, \epsilon^{-1}, 0, \cdots, 0\}$ with $\epsilon \neq 0$, and the elements $1+\epsilon$ and ϵ^{-1} in the lth spot. This leads to

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l = \epsilon^{-1} \sum_{k=1}^{n} m_{kl} + m_{ll}.$$

By taking ϵ sufficiently small and of an appropriate sign,

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l$$

can thus be made negative.

The following two theorems are generalizations of Theorem 1 to similarly ordered unbiased and to similarly ordered symmetric sequences.

Theorem 2

A necessary and sufficient condition for the bilinear form

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l$$

to be non-negative for all similarly ordered unbiased sequences $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ is that the matrix $M=(m_{kl})$ be doubly hyperdominant.

Proof:

1) *Sufficiency:* Let M be a doubly hyperdominant matrix and define

$$m_{k,n+1} = - \sum_{l=1}^{n} m_{kl}, \; m_{n+1,l} = - \sum_{k=1}^{n} m_{kl} \quad \text{for } k, l \leq n,$$

and

$$m_{n+1,n+1} = \sum_{k,l=1}^{n} m_{kl}.$$

Then taking $x_{n+1}=y_{n+1}=0$, it follows from Theorem 1 that

$$\sum_{k,l=1}^{n} m_{kl} x_k y_l = \sum_{k,l=1}^{n+1} m_{kl} x_k y_l \geq 0$$

since the augmented matrix $M^*=(m_{kl})$, $k, l=1,2\cdots, n+1$ is doubly hyperdominant with zero excess and since the sequences $\{x_1, x_2, \cdots, x_n, x_{n+1}\}$ and $\{y_1, y_2, \cdots, y_n, y_{n+1}\}$ with $x_{n+1}=y_{n+1}=0$ are similarly ordered.

2) *Necessity:* The same sequences as in Theorem 1 can be used if the matrix M fails to be doubly hyperdominant because $m_{kl}>0$ for some $k\neq l$. Assume next that the matrix M fails to be doubly hyperdominant because

$$\sum_{k=1}^{n} m_{kl} < 0$$

for some l (a similar argument holds if

$$\sum_{l=1}^{n} m_{kl} < 0$$

for some k), and consider the sequences used in Theorem 1 with the additional restriction that $\epsilon>0$. Then by taking $\epsilon>0$ sufficiently small,

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l = \epsilon^{-1}\sum_{k=1}^{n} m_{kl} + m_{ll}$$

can be made negative.

Theorem 3

A necessary and sufficient condition for the bilinear form

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l$$

to be non-negative for all similarly ordered symmetric sequences $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ is that the matrix $M=(m_{kl})$ be doubly dominant.

Proof:

1) *Sufficiency:* Let M be a doubly dominant matrix. Clearly

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l \geq \sum_{\substack{k,l=1\\k=l}}^{n} m_{kl}|x_k||y_l| - \sum_{\substack{k,l=1\\k\neq l}}^{n} |m_{kl}||x_k||y_l|.$$

The right-hand side of the above inequality is non-negative by Theorem 2 since the matrix $M^*=(m^*_{kl})$ with $m^*_{kl}=m_{kl}$ when $k=l$ and $m^*_{kl}=-|m_{kl}|$ when $k\neq l$ is doubly hyperdominant and since the sequences $\{|x_1|, |x_2|, \cdots, |x_n|\}$ and $\{|y_1|, |y_2|, \cdots, |y_n|\}$ are similarly ordered and unbiased. This implies that

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l \geq 0.$$

2) *Necessity:* Assume that the matrix M fails to be doubly dominant because

$$m_{ll} - \sum_{\substack{k,l=1\\k\neq 1}}^{n} |m_{kl}| < 0$$

for some l (an analogous argument holds if

$$m_{kk} - \sum_{\substack{l=1\\l\neq k}}^{n} |m_{kl}| < 0$$

for some k), and consider the sequences $\{-\operatorname{sgn} m_{1l}, \cdots, -\operatorname{sgn} m_{l-1,l}, 1+\epsilon, -\operatorname{sgn} m_{l+1,l}, \cdots, -\operatorname{sgn} m_{nl}\}$ and $\{0, \cdots, 0, \epsilon^{-1}, 0, \cdots, 0\}$ with $\operatorname{sgn}\alpha=\alpha/|\alpha|$ if $\alpha\neq 0$, $\operatorname{sgn} 0=0$, $\epsilon>0$ and $1+\epsilon$ and ϵ^{-1} elements in the lth spots. This leads to

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l = \epsilon^{-1}(m_{ll} - \sum_{\substack{k=1\\k\neq l}}^{n} |m_{kl}|) + m_{ll}$$

which, by taking ϵ sufficiently small, leads to

$$\sum_{k,l=1}^{n} m_{kl}x_ky_l < 0.$$

III. Extension to l_2-Summable Sequences

Definitions

A sequence of real numbers $\{a_k\}$, $k=0, \pm 1, \pm 2, \cdots$, is said to be l_p *summable* ($p\geq 1$) if

$$\left(\sum_{k=-\infty}^{+\infty} |a_k|^p\right)^{1/p} < \infty.$$

The collection of all l_2-summable sequences forms a Hilbert space with the *inner product* of two elements $x=\{x_k\}$ and $y=\{y_k\}$, $k=0, \pm 1, \pm 2, \cdots$, defined as

$$\langle x, y\rangle = \sum_{k=-\infty}^{+\infty} x_ky_k.$$

An array of real numbers $R=\{r_{kl}\}$, $k, l=0, \pm 1, \pm 2, \cdots$, is said to *belong to* $\mathcal{L}(l_2, l_2)$ if for all l_2-summable sequences x the sequence $\{y_k\}$ defined by

$$y_k = \sum_{l=-\infty}^{+\infty} r_{kl}x_l$$

exists for all k, is l_2 summable, and if there exists a constant M such that

$$\left(\sum_{k=-\infty}^{+\infty} y_k^2\right)^{1/2} \leq M\left(\sum_{k=-\infty}^{+\infty} x_k^2\right)^{1/2}.$$

The greatest lower bound of all numbers which satisfy this inequality is called the *norm* of R, denoted by $\|R\|$. The *transpose* of R, denoted R', is the array $R'=\{r_{kl}'\}$ with $r_{kl}'=r_{lk}$. A standard result in the theory of bounded linear operators in Hilbert space [11, p. 52] states that R' belongs to $\mathcal{L}(l_2, l_2)$ if and only if R does, that $\|R\|=\|R'\|$, and that $\langle x, Ry\rangle=\langle R'x, y\rangle$ for all l_2 summable sequences x and y. An element R of $\mathcal{L}(l_2, l_2)$ is said to have a *bounded inverse* if there exists an element R^{-1} of $\mathcal{L}(l_2, l_2)$ such that $RR^{-1}=R^{-1}R=I$, with $I=(\delta_{kl})$ and $\delta_{kl}=0$ for $k\neq l$, $\delta_{kk}=1$. It is well known that $(R^{-1})'=(R')^{-1}$.

Lemma 3: Let $R=\{r_{kl}\}$ be such that the sequences $\{r_{kl}\}$ are l_1 summable for fixed k and l, uniformly in k and l, i.e., there exists an M such that

$$\sum_{l=-\infty}^{+\infty} |r_{kl}| \leq M \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} |r_{kl}| \leq M.$$

Then R belongs to $\mathcal{L}(l_2, l_2)$ and $\|R\|\leq M$.

Proof: The Schwartz inequality and Fubini's theorem for sequences [12, p. 245] yield the following inequalities:

$$\left(\sum_{k=-\infty}^{+\infty}\left|\sum_{l=-\infty}^{+\infty} r_{kl}x_l\right|^2\right)^{1/2}$$
$$\leq\left(\sum_{k=-\infty}^{+\infty}\left(\sum_{l=-\infty}^{+\infty}|r_{kl}||x_l|\right)^2\right)^{1/2}$$
$$\leq\left(\sum_{k=-\infty}^{+\infty}\left(\sum_{l=-\infty}^{+\infty}|r_{kl}|\right)\left(\sum_{l=-\infty}^{+\infty}|r_{kl}||x_k|^2\right)\right)^{1/2}$$
$$\leq M\left(\sum_{l=-\infty}^{+\infty}|x_l|^2\right)^{1/2}.$$

The previous definitions in this section and Lemma 3 are standard. In what follows an important role will be played by some particular elements of $\mathcal{L}(l_2, l_2)$ and some particular sequences which will now be introduced.

Definitions

The definitions of *similarly ordered*, *similarly ordered unbiased*, and *similarly ordered symmetric* l_2-summable sequences are completely analogous to the case of finite sequences and will not be repeated here. It is possible to show that two l_2-summable sequences are similarly ordered if and only if they are similarly ordered and unbiased. An element M of $\mathcal{L}(l_2, l_2)$ is said to be *doubly hyperdominant* if $m_{kl}\leq 0$ for $k\neq l$ and if

$$\sum_{k=-\infty}^{+\infty} m_{kl} \quad \text{and} \quad \sum_{l=-\infty}^{+\infty} m_{kl}$$

exist and are non-negative for all l and k. An element M of $\mathcal{L}(l_2, l_2)$ is said to be *doubly dominant* if

$$m_{ll} \geq \sum_{\substack{k=-\infty\\k\neq l}}^{+\infty} |m_{kl}| \quad \text{and} \quad m_{kk} \geq \sum_{\substack{l=-\infty\\l\neq k}}^{+\infty} |m_{kl}|.$$

It is clear from Lemma 3 that if M belongs to $\mathcal{L}(l_2, l_2)$ and is doubly hyperdominant or doubly dominant, then $\|M\|\leq 2\ \sup_k m_{kk}$. This supremum is finite since M belongs to $\mathcal{L}(l_2, l_2)$.

The following extension of Theorems 2 and 3 holds.

Theorem 4

Let $M=\{m_{kl}\}$, $k, l=0, \pm 1, \pm 2, \cdots$, be an element of $\mathcal{L}(l_2, l_2)$. Then a necessary and sufficient condition for the inner product $\langle x, My\rangle$ to be non-negative for all

a) similarly ordered unbiased l_2-summable sequences x and y, and
b) similarly ordered symmetric l_2-summable sequences x and y

is that M be

1) doubly hyperdominant, and
2) doubly dominant.

Proof: It is clear that all finite subsequences of x and y are similarly ordered and unbiased. Hence, by Theorems 2 and 3 all finite truncations of the infinite sum in the inner product $\langle x, My\rangle$ yield a non-negative number. Thus the limit, since it exists, is also non-negative.

Of particular interest are the arrays $R=\{r_{kl}\}$, $k, l=0, \pm 1, \pm 2, \cdots$, for which the entries depend only on the difference of the indices k and l. These arrays are said to be of the *Toeplitz type* and have been intensively studied in classical analysis (see, e.g., [13]). It follows from Lemma 3 that if $R=\{r_{k-l}\}$ is of the Toeplitz type, then it belongs to $\mathcal{L}(l_2, l_2)$ if $\{r_k\}$, $k=0, \pm 1, \pm 2, \cdots$, is l_1 summable. The previous theorem can be phrased somewhat simpler in this case. However, another definition is needed first.

Definition

An l_1-summable sequence $\{a_k\}$ is said to be *hyperdominant* if $a_k\leq 0$ for $k\neq 0$ and if

$$\sum_{k=-\infty}^{+\infty} a_k \geq 0.$$

It is said to be *dominant* if

$$a_0 \geq \sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty} |a_k|.$$

Theorem 5

Let $M=(m_{k-l})$ $k, l+0, \pm 1, \pm 2, \cdots$, be an element of $\mathcal{L}(l_2, l_2)$ which is of the Toeplitz type. Then a necessary and sufficient condition for the inner product $\langle x, My\rangle$ to be non-negative for all

a) similarly ordered unbiased l_2-summable sequences x and y
b) similarly ordered symmetric l_2-summable sequences x and y

is that $\{m_k\}$ be

1) hyperdominant
2) dominant.

Proof: This theorem is a special case of Theorem 4.

IV. Stability of Difference Equations

Consider the system defined by the relation between the *input* sequence $\{u_k\}$ and the *output* sequence $\{y_k\}$, $k=0, \pm 1, \pm 2, \cdots$, governed by the equation

$$y_k = \sum_{l=-\infty}^{+\infty} g_{kl}u_l + r_k \qquad k = 0, \pm 1, \pm 2, \cdots$$

where the array $\{g_{kl}\}$, $k, l=0, \pm 1, \pm 2, \cdots$, is the *weighting pattern*. It will be assumed that the system under consideration is *causal*, i.e., that $u_k=r_k=0$ for $k \leq N$ implies that $y_k=0$ for $k \leq N$. Thus g_{kl} is assumed to be zero for all $k<l$. The system defined above is slightly more general than the input–output relation governed by the n-dimensional difference equation

$$\begin{aligned} x_{k+1} &= A_k x_k + b_k u_k \\ y_k &= c_k' x_k + d_k u_k \qquad k = 0, 1, 2, \cdots \\ &x_0 \text{ given} \end{aligned}$$

where b_k and c_k are n vectors, d_k is a scalar, A_k is an $(n \times n)$ matrix and x_k is an n vector called the *state* of the system. This input–output relation is a particular case of the input–output relation defined above with

$$\begin{aligned} g_{kl} &= c_k' A_{k-1} \cdots A_{l+1} b_l && \text{for } k \geq l+2 \\ g_{kl} &= c_k' b_k && \text{for } k = l+1 \\ g_{kl} &= d_k && \text{for } k = l \\ g_{kl} &= 0 && \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} r_k &= c_k' A_{k-1} \cdots A_0 x_0 && \text{for } k \geq 1 \\ r_0 &= c_0' x_0 && \\ r_k &= u_k = 0 && \text{for } k < 0. \end{aligned}$$

The case in which the system is time invariant is of particular interest. The system is then defined by the equation

$$y_k = \sum_{l=-\infty}^{+\infty} g_{k-l}u_l + r_k \qquad k = 0, \pm 1, \pm 2, \cdots$$

where g_k is assumed to be zero for $k<0$. This system is slightly more general than the input–output relation governed by the n-dimensional difference equation

$$\begin{aligned} x_{k+1} &= A x_k + b u_k \\ y_k &= c' x_k + d \qquad k = 0, 1, 2, \cdots \end{aligned}$$

where b and c are constant n vectors, d is a scalar constant, A is a constant $(n \times n)$ matrix, and x_k is an n vector called the *state* of the system. This input–output relation is a particular case of the input–output relation defined above with

$$\begin{aligned} g_k &= c' A^{k-1} b && \text{for } k > 0 \\ g_0 &= d && \\ g_k &= 0 && \text{for } k < 0 \\ r_k &= c' A^k x_0 && \text{for } k \geq 0 \\ r_k &= u_k = 0 && \text{for } k < 0. \end{aligned}$$

Let the input be given as a function of the output by the feedback law

$$u_k = -f(y_k) + v_k$$

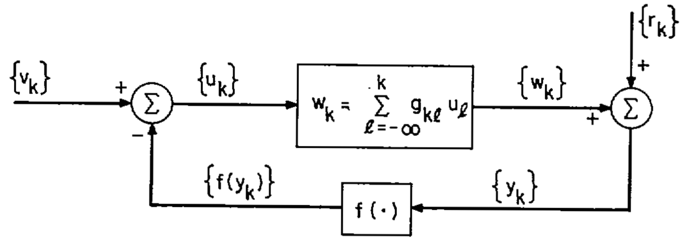

Fig. 1. The feedback loop.

where v_k, $k=0, \pm 1, \pm 2, \cdots$, is a sequence of real numbers and $f(\sigma)$ is a mapping from the real line into itself. The resulting feedback system is shown in Fig. 1, and the equation of motion becomes

$$y_k + \sum_{l=-\infty}^{+\infty} g_{kl} f(y_l) = \sum_{l=-\infty}^{+\infty} g_{kl} v_l + r_k \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{2}$$

Definition

The feedback system under consideration is said to be l_2-*stable* if for all l_2-summable sequences $r=\{r_k\}$ and $v=\{v_k\}$ $k=0, \pm 1, \pm 2, \cdots$, all solutions $\{y_k\}$ which are such that

$$\left(\sum_{k=-\infty}^{n} y_k^2 \right)^{1/2}$$

exists for all integers n are l_2-summable and satisfy the inequality

$$\left(\sum_{k=-\infty}^{+\infty} y_k^2 \right)^{1/2} \leq \rho_1 \left(\sum_{k=-\infty}^{+\infty} v_k^2 \right)^{1/2} + \rho_2 \left(\sum_{k=-\infty}^{+\infty} r_k^2 \right)^{1/2}$$

for some *constants* ρ_1 and ρ_2.

Remark: Notice that l_2-stability implies that $\lim_{k\to\infty} y_k=0$, and that for the n-dimensional difference equation described above it implies that if $v_k=0$ for all k, then

$$\lim_{x_0 \to 0} \sup_{k=0,1,2,\cdots} |y_k| = 0,$$

which in turn implies asymptotic stability in the sense of Liapunov provided the system is uniformly completely observable.

The remainder of this section is concerned with finding sufficient conditions for the feedback system (2) to be l_2-stable. First some additional definitions and notation.

Notation and Definitions

Let $x=\{x_k\}$, $k=0, \pm 1, \pm 2, \cdots$, be any sequence of real numbers. Then Fx *denotes* the sequence $\{f(x_k)\}$, $k=0, \pm 1, \pm 2, \cdots$. F is said to be *bounded* if there exists a constant K such that $|f(\sigma)| \leq K|\sigma|$ for all σ. If $f(\sigma)$ is invertible, then $F^{-1}x$ denotes the sequence $\{f^{-1}(x_k)\}$. F is said to be *monotone* (or *odd monotone*) if $f(\sigma)$ is a monotone (or an odd monotone) function of σ. F is said to be *strictly monotone* (or *strictly odd monotone*) if $f(\sigma)-\epsilon\sigma$ is a monotone (or an odd monotone) function of σ for some $\epsilon>0$. Clearly if F is strictly monotone, then F^{-1} is well defined, monotone, and bounded. If F is bounded, then it maps l_2-summable sequences into l_2-summable sequences. For any integer n, $P_n x$ *denotes* the sequence $\{y_k\}$ with $y_k=x_k$ for $k\leq n$ and $y_k=0$ otherwise. P_n belongs to $\mathcal{L}(l_2, l_2)$, $P_n^2=P_n$ and $\|P_n\|=1$. P_nF and FP_n commute whenever $f(0)=0$, and thus in particular when F is bounded.

Definitions

An element R of $\mathcal{L}(l_2, l_2)$ is said to *belong to* $\mathcal{L}^+(l_2, l_2)$ if $r_{kl}=0$ whenever $k<l$. It is said to *belong to* $\mathcal{L}^-(l_2, l_2)$ if R' belongs to $\mathcal{L}^+(l_2, l_2)$. The causality assumption on the system introduced above implies that if $G=\{g_{kl}\}$ belongs to $\mathcal{L}(l_2, l_2)$, then it belongs to $\mathcal{L}^+(l_2, l_2)$. Note the analogy between this notation and the notation used for spectral factorization in filtering theory (see, e.g., [14]).

Lemma 4: Let R be an element of $\mathcal{L}^+(l_2, l_2)$. Then P_nR and P_n commute for all l_2-summable sequences.

Proof: Let $y=P_nRx$. Then

$$y_k = \sum_{l=-\infty}^{+\infty} r_{kl}x_l \qquad \text{for } k\leq n$$

and $y_k=0$ otherwise. Since, however, $r_{kl}=0$ for $k<l$, the first summation reduces to

$$y_k = \sum_{l=-\infty}^{k} r_{kl}x_l$$

which shows that indeed $y=P_nRP_nx$.

An important step in the proof of the stability theorem that follows relies on the fact that certain elements of $\mathcal{L}(l_2, l_2)$ can be factored in a suitable fashion. This is stated in the next theorem, the proof of which will be given in Section V which is devoted to this factorization problem.

Theorem 6

Let Z be an element of $\mathcal{L}(l_2, l_2)$ which is such that $Z-\epsilon I$ is doubly dominant for some $\epsilon>0$. Then there exist elements M and N of $\mathcal{L}(l_2, l_2)$ such that

1) $Z=MN$
2) M has a bounded inverse M^{-1}
3) N belongs to $\mathcal{L}^+(l_2, l_2)$ and M^{-1} belongs to $\mathcal{L}^-(l_2, l_2)$.

Moreover, if Z is of the Toeplitz type, then M and N may be taken to be of the Toeplitz type.

Proof: The proof will be given in Section V.

Definition

An element R of $\mathcal{L}(l_2, l_2)$ is said to be *non-negative* if $\langle x, Rx\rangle\geq 0$ for all l_2-summable sequences x.

The road is now open for the following stability theorem which is the main result of this section and is an extension of similar results obtained by O'Shea and Younis [6], [15] and Zames and Falb [8].

Theorem 7

A sufficient condition for the feedback system under consideration to be l_2-stable is that

1) G belongs to $\mathcal{L}(l_2, l_2)$ and F is strictly monotone (strictly odd monotone), and bounded
2) there exists an element Z of $\mathcal{L}(l_2, l_2)$ such that $Z=\epsilon I$ is doubly hyperdominant (doubly dominant) for some $\epsilon>0$ and such that ZG is non-negative.

Proof: Let $y=\{y_k\}$ be any solution of the equation of motion which is such that

$$\left(\sum_{k=-\infty}^{n} y_k^2\right)^{1/2}$$

exists for all integers n. Let $x=Fy$. Notice that since F is strictly monotone and P_ny is l_2-summable, it follows that P_nx and $P_nF^{-1}x$ are l_2-summable. From the equation of motion it follows that

$$P_nF^{-1}x + P_nGP_nx = P_n(Gv+r) \quad n=0, \pm 1, \pm 2, \cdots.$$

Let Z be factored as in Theorem 6. From this theorem, Lemma 4, and the above equality it follows that

$$\langle P_nM'P_nx, P_nNF^{-1}P_nx\rangle + \langle P_nM'P_nx, P_nNGP_nx\rangle = \langle P_nM'P_nx, P_nN(Gv+r)\rangle.$$

However,

$$\begin{aligned}\langle P_nM'P_nx, P_nNF^{-1}P_nx\rangle &= \langle P_nM'P_nx, P_nNF^{-1}(M')^{-1}P_nM'P_nx\rangle\\ &= \langle M'(M')^{-1}P_nM'P_nx, NF^{-1}(M')^{-1}P_nM'P_nx\rangle\\ &= \langle (M')^{-1}P_nM'P_nx, ZF^{-1}(M')^{-1}P_nM'P_nx\rangle\\ &\geq \epsilon\langle (M')^{-1}P_nM'P_nx, F^{-1}(M')^{-1}P_nM'P_nx\rangle\\ &\geq \epsilon\langle P_nx, F^{-1}P_nx\rangle \geq \epsilon'\langle P_nx, P_nx\rangle.\end{aligned}$$

These equalities follow from the factorization and repeated use of Lemma 4. The inequalities hold for some $\epsilon'>0$ by Theorem 4 since $Z-\epsilon I$ is doubly hyperdominant (dominant) since F^{-1} is monotone (odd monotone) and since F is bounded. Similar manipulations and the fact that ZG is non-negative yield

$$\langle P_nM'Px, P_nNGP_nx\rangle \geq 0.$$

The Schwartz inequality and the triangle inequality for l_2-summable sequences yield

$$|\langle P_nM'P_nx, P_nNx(Gv+r)\rangle| \leq \|M\|\,\|N\|\langle P_nx, P_nx\rangle^{1/2}(\|G\|\langle P_nv, P_nv\rangle^{1/2} + \langle P_nr, P_nr\rangle^{1/2}).$$

From the above inequalities and the equality preceding them, it follows thus that

$$\langle P_nx, P_nx\rangle^{1/2} \leq \epsilon'^{-1}\|M\|\,\|N\|\,\|G\|\langle v, v\rangle^{1/2} + \epsilon'^{-1}\|M\|\,\|N\|\langle r, r\rangle^{1/2}$$

which, since this inequality holds for all n, since the right side is independent of n and since F is strictly monotone, yields l_2 stability. This ends the proof of the theorem.

The case in which the system is time invariant and the multiplier is of the Toeplitz type is, of course, of particular interest and yields the stability theorem obtained by O'Shea and Younis [15]. The positivity condition and the doubly hyperdominance (doubly dominance) condition can then be stated in terms of z transforms. This is done in the next theorem. But first the definition of a z transform will be introduced. For a discussion of limit in the mean transform ideas, see, e.g., Feller [16, p. 601].

Definition

The *z-transform* of an l_2-summable sequence $\{a_k\}$, $k=0, \pm 1, \pm 2, \cdots$, is defined by

$$A(z) = \text{l.i.m.} \sum_{k=-\infty}^{+\infty} a_k z^{-k}$$

and exists for $|z|=1$. The *inverse z-transform* of a function which is square integrable along $|z|=1$ is the l_2-summable sequence $\{a_k\}$ defined by

$$a_k = \frac{1}{2\pi} \oint_{|z|=1} A(z) z^{k-1} dz \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Lemma 5: Let $R=\{r_{k-l}\}$, k, $l=0, \pm 1, \pm 2, \cdots$, be an element of $\mathfrak{L}(l_2, l_2)$ which is of the Toeplitz type. Then a necessary and sufficient condition for the inner product $\langle x, Rx\rangle$ to be non-negative for all l_2-summable sequences x is that the z-transform of $\{r_k\}$, $R(z)$ satisfies Re $R(z) \geq 0$ for almost all z with $|z|=1$.

Proof: It is well known that

$$\begin{aligned}\langle x, Rx\rangle &= \frac{1}{2\pi} \oint_{|z|=1} R(z) \mid X(z) \mid^2 z^{-1} dz \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) \mid X(e^{j\omega}) \mid^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re } R(e^{j\omega}) \mid X(e^{j\omega}) \mid^2 d\omega\end{aligned}$$

and the conclusion follows.

Theorem 8

A sufficient condition for the feedback system under consideration to be l_2-stable is that

1) G is an element of $\mathfrak{L}(l_2, l_2)$ which is of the Toeplitz type and F is strictly monotone (strictly odd monotone) and bounded

2) there exists a $Z(z)$ such that $Z(z)-\epsilon$ is the z-transform of a hyperdominant (dominant) sequence for some $\epsilon>0$ and such that Re $G(z)Z(z) \geq 0$ for almost all z with $|z|=1$.

Proof: The theorem follows from Theorem 7 and Lemma 5.

Remarks and Comments

1) For the n-dimensional time-invariant difference equation above, it is quite simple to show that G will belong to $\mathfrak{L}(l_2, l_2)$ if all eigenvalues of A have magnitude less than unity.

2) Since in Theorems 7 and 8, F is assumed to be bounded, it follows that all solutions under consideration satisfy

$$\left(\sum_{k=-\infty}^{+\infty} f(y_k)^2 \right)^{1/2} \leq \rho_3 \left(\sum_{k=-\infty}^{+\infty} v_k^2 \right)^{1/2} + \rho_4 \left(\sum_{k=-\infty}^{+\infty} r_k^2 \right)^{1/2}$$

for some constants ρ_3 and ρ_4.

3) The fact that the $\epsilon > 0$ appears in the hyperdominance (dominance) condition is not essential and could be replaced by requiring $ZG-\epsilon I$ to be non-negative for some $\epsilon>0$.

V. Factorization of Operators

From the previous section the importance of obtaining a suitable factorization of certain elements of $\mathfrak{L}(l_2, l_2)$ is apparent. Similar factorizations have been studied in relation to probability theory and optimal control theory. The problem is one of considerable interest and difficulty, and the natural setting for the study of such factorizations appears to be a Banach algebra [8], [9]. The general factorization theorem thus obtained is then specialized to a class of elements of $\mathfrak{L}(l_2, l_2)$ and is shown to yield Theorem 6. It will be indicated that in case these elements of $\mathfrak{L}(l_2, l_2)$ are in addition of the Toeplitz type, the results are rather conservative and that a less restrictive factorization theorem due to Krein [17, p. 198][2] exists. The setting of the factorization problem is the same as used by Zames and Falb [8], but the results appear to be more general. The method of proof is inspired by a paper by Baxter [9].

Definitions

A *Banach algebra* is a normed linear space σ over the real or complex field which is complete in the topology induced by its norm and which has a mapping (*multiplication*) from $\sigma \times \sigma$ into σ defined. This multiplication is associative, is distributive with respect to addition, is related to scalar multiplication by $a(AB)=A(aB)=(aA)B$, and to the norm on σ by $\|AB\| \leq \|A\| \, \|B\|$ for all A, $B \in \sigma$ and all scalars a. A Banach algebra is said to have a *unit element* if there exists an element $I \in \sigma$ such that $AI=IA=A$ for all $A \in \sigma$. An element A of a Banach algebra with a unit element is said to be *invertible* if there exists an element, A^{-1}, of σ such that $AA^{-1}=A^{-1}A=I$. A bounded linear transformation π from σ into itself is said to be a *projection* on σ if $\pi^2=\pi$ and if the range of π is a sub-

[2] The authors are indebted to their colleague M. Gruber for pointing out this reference.

algebra of σ. Note that the range of a projection is thus closed under addition and multiplication. The *norm* of π, $\|\pi\|$ is defined in the usual way as the greatest lower bound of all numbers M which satisfy $\|\pi A\| \leq M\|A\|$ for all $A \in \sigma$. θ *denotes* the identity transformation on σ.

The following factorization theorem states the main result of this section.

Theorem 9

Let σ be a Banach algebra with a unit element and let π^+ and $\pi^- = \theta - \pi^+$ be projections on σ. Let σ^+ and σ^- be the ranges of π^+ and π^-, and assume that $\|\pi^+\| \leq 1$ and that $\|\pi^-\| \leq 1$. Let Z be an element of σ. If $\|Z\| < |\rho|$, then there exist elements $Z^+ \in \sigma$ and $Z^- \in \sigma$ such that

1) $M = \rho I - Z = Z^- Z^+$
2) Z^+ and Z^- are invertible
3) Z^+ and $(Z^+)^{-1}$ belong to[3] $\sigma^+ \oplus I$, and Z^- and $(Z^-)^{-1}$ belong to $\sigma^- \oplus I$.

Proof: Since the proof of the theorem is rather lengthy, it is subdivided into several lemmas.

Lemma 6: Let $\{A_k\}$, $\{P_k\}$ and $\{N_k\}$, $k = 1, 2, \cdots$, be sequences of elements of σ, σ^+, and σ^-, respectively, and assume that for some $r_0 > 0$ and all $|r| \leq r_0$,

1) the series

$$A = I + \sum_{k=1}^{\infty} A_k r^k, \qquad P = I + \sum_{k=1}^{\infty} P_k r^k$$

and

$$N = I + \sum_{k=1}^{\infty} N_k r^k$$

converge, and

2) $A = PN$.

Then A uniquely determines the sequences $\{P_k\}$ and $\{N_k\}$.

Proof: Equating coefficients of equal powers in r in the equality $A = PN$ leads to

$$P_n + N_n = A_n - \sum_{k=1}^{n-1} P_k N_{n-k} \qquad \text{for } n = 2, 3, \cdots.$$

Thus

$$P_n = \pi^+\left(A_n - \sum_{k=1}^{n-1} P_k N_{n-k}\right)$$

and

$$N_n = \pi^-\left(A_n - \sum_{k=1}^{n-1} P_k N_{n-k}\right)$$

which shows that A uniquely determines P_n and N_n provided it uniquely determines $P_1, \cdots, P_{n-1}$ and $N_1, \cdots, N_{n-1}$. Since A uniquely determines P_1 and N_1 by $P_1 = \pi^+ A_1$ and $N_1 = \pi^- A_1$, the result follows by induction.

Lemma 7: The equations

$$P = I + r\pi^+(ZP)$$

and

$$N = I + r\pi^-(NZ)$$

have a unique solution $P \in \sigma$ and $N \in \sigma$ for all $|r| \leq |\rho|^{-1}$. Moreover, these solutions are given by the convergent series

$$P = \sum_{k=0}^{\infty} P_k r^k \quad \text{and} \quad N = \sum_{k=0}^{\infty} N_k r^k$$

with $P_0 = N_0 = I$, $P_{k+1} = \pi^+(ZP_k)$ and $N_{k+1} = \pi^-(N_k Z)$. Notice that $P \in \sigma^+ \oplus I$ and that $N \in \sigma^- \oplus I$.

Proof: The result follows from the inequalities

$$\|r\pi^+(Z(A - B))\| \leq |\rho|^{-1}\|Z\| \, \|A - B\|$$
$$\|r\pi^-((A - B)Z)\| \leq |\rho|^{-1}\|Z\| \, \|A - B\|$$

and the contraction mapping principle [18, p. 43]. Moreover, it is easily verified that the successive approximations obtained by this contraction mapping with $P_0 = N_0 = I$ yield the power series expressions of P and N as claimed in the lemma.

Lemma 8: The solutions P and N to the equations of Lemma 8 are invertible for all $|r| \leq |\rho|^{-1}$ and

$$P^{-1} = I - r\pi^+(NZ)$$
$$N^{-1} = I - r\pi^-(ZP).$$

Moreover, $N^{-1}P^{-1} = I - rZ$ for all $|r| \leq |\rho|^{-1}$. Notice that $P^{-1} \in \sigma^- \oplus I$ and that $N^{-1} \in \sigma^+ \oplus I$.

Proof: From the equations defining P and N, it follows that for

$$|r| \leq |\rho|^{-1}, \qquad \|r\pi^+(NZ)\| \leq \frac{|r| \, \|Z\|}{1 - |r| \, \|Z\|}$$

and

$$\|r\pi^-(ZP)\| \leq \frac{|r| \, \|Z\|}{1 - |r| \, \|Z\|}.$$

Since all elements of σ which are of the form $I - B$ with $\|B\| < 1$ are invertible, it follows thus that $I - r\pi^+(NZ)$, $I = r\pi^-(ZP)$ and $I - rZ$ are invertible for $|r| \leq |\rho|^{-1}/2$ and their inverses are given by the convergent series

$$(I - r\pi^+(NZ))^{-1} = I + \sum_{k=1}^{\infty} (\pi^+(NZ))^k r^k$$

$$(I - r\pi^-(ZP))^{-1} = I + \sum_{k=1}^{\infty} (\pi^-(ZP))^k r^k$$

$$(I - rZ)^{-1} = I + \sum_{k=1}^{\infty} Z^k r^k.$$

[3] $\sigma^+ \oplus I$ denotes all elements of σ which are of the form $R + aI$ with $R \in \sigma^+$ and a a scaler. $\sigma^- \oplus I$ is defined analogously.

From the equations of P and N, it follows that for $|r| \leq |\rho|^{-1}$, $(I-rZ)P = I - r\pi^-(ZP)$ and $N(I-rZ) = I - r\pi^+(NZ)$, and thus that for $|r| < \rho^{-1}/2$

$$(I - rZ)^{-1} = P(I - r\pi^-(ZP))^{-1} = (I - r\pi^+(NZ))^{-1}N.$$

Since all elements in the above equalities are given by the series expansions given earlier and in Lemma 7, and since σ^+ and σ^- are closed under addition and multiplication, Lemma 6 is applicable. This yields for $|r| \leq |\rho|^{-1}/2$

$$P = (I - r\pi^+(NZ))^{-1}, \qquad N = (I - r\pi^-(ZP))^{-1}$$

and

$$PN = (I - rZ)^{-1}.$$

Thus for $|r| \leq |\rho|^{-1}/2$ the following equalities hold:

$$P(I - r\pi^+(NZ)) = (I - r\pi^+(NZ))P = I$$
$$N(I - r\pi^-(ZP)) = (I - r\pi^-(ZP))N = I$$
$$(I - r\pi^-(ZP))(I - r\pi^+(NZ)) = I - rZ.$$

Since for $|r| \leq |\rho|^{-1}$ all terms in the above equalities are given by geometrically convergent power series in r, they are analytic for $|r| \leq |\rho|^{-1}$. Since, however, equality holds for $|r| \leq |\rho|^{-1}/2$, it is concluded from the analyticity that equality holds for all $|r| \leq |\rho|$. This ends the proof of Lemma 8.

Proof of Theorem 9: Let $r = \rho^{-1}$ in Lemma 8. The theorem follows with $Z^- = \rho(I - \rho^{-1}\pi^-(ZP))$, $(Z^-)^{-1} = \rho^{-1}N$, $Z^+ = I - \rho^{-1}\pi^+(NZ)$, and $(Z^+)^{-1} = P$.

It will now be shown that under a suitable choice of the Banach algebra and the projection operators, the following corollaries to Theorem 9 hold. These corollaries then yield Theorem 6.

Corollary 1: Let Z be an element of $\mathcal{L}(l_2, l_2)$ which is such that $Z - \epsilon I$ is doubly dominant for some $\epsilon > 0$. Then there exist elements M and N of $\mathcal{L}(l_2, l_2)$ such that

1) $Z = MN$,
2) M and N have bounded inverses M^{-1} and N^{-1}
3) N and N^{-1} belong to $\mathcal{L}^+(l_2, l_2)$ and M and M^{-1} belong to $\mathcal{L}^-(l_2, l_2)$.

Corollary 2: Let $A(z) - \epsilon$ be the z-transform of a sequence which is dominant for some $\epsilon > 0$. Then there exist functions $A^+(z)$ and $A^-(z)$ such that

1) $A(z) = A^-(z)A^+(z)$
2) $A^+(z)$ and $(A^+(z))^{-1}$ are the z-transforms of l_1-summable sequences $\{a_k^+\}$ and $\{b_k^+\}$ with $a_k^+ = b_k^+ = 0$ for $k < 0$, and $A^-(z)$ and $(A^-(z))^{-1}$ are the z-transforms of l_1-summable sequences $\{a_k^-\}$ and $\{b_k^-\}$ with $a_k^- = b_k^- = 0$ for $k > 0$.

Proof: It will be shown that these corollaries follow from Theorem 9 under a suitable choice of the Banach algebra σ and the projections π^+ and π^-.

Corollary 1 follows from Theorem 9 with σ all members of $\mathcal{L}(l_2, l_2)$ such that if $A = \{a_{kl}\} \in \sigma$, then the sequences $\{a_{kl}\}$ are l_1-summable for fixed k and l, uniformly in k and l, i.e., there exists an M such that

$$\sum_{k=-\infty}^{+\infty} |a_{kl}| \leq M \quad \text{and} \quad \sum_{l=-\infty}^{+\infty} |a_{kl}| \leq M.$$

Multiplication is defined in the usual way. The norm is defined as the greatest lower bound of all numbers M satisfying the above inequalities. The nonobvious elements in the verification of the fact that σ forms a Banach algebra are that σ is closed under multiplication, that $\|AB\| \leq \|A\| \; \|B\|$ for all $A, B \in \sigma$, and that σ is complete. Closedness under multiplication, follows from Fubini's theorem for sequences [12, p. 245] and the inequalities

$$\sum_{k=-\infty}^{+\infty} \left| \sum_{i=-\infty}^{+\infty} a_{ki}b_{il} \right| \leq \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} |a_{ki}| \, |b_{il}|$$
$$= \sum_{i=-\infty}^{+\infty} |b_{il}| \sum_{k=-\infty}^{+\infty} |a_{ki}|$$
$$\leq \|A\| \; \|B\|$$

which also shows that $\|AB\| \leq \|A\| \; \|B\|$. Completeness follows from the fact that the set of all l_1-summable sequences is complete. The projection operator π^+ is defined by $\pi^+A = B$ with $A = \{a_{kl}\}$, $B = \{b_{kl}\}$ and $a_{kl} = b_{kl}$ for $k \geq l$, $b_{kl} = 0$ otherwise, and $\pi^- = \theta - \pi^+$. It is clear that $\|\pi^+\| = 1$ and that $\|\pi^-\| = 1$. The only fact that is left to be shown is that if for some $\epsilon > 0$ $Z - \epsilon I$ is doubly dominant, then Z can be written as $Z = \rho I - A$ with $\|A\| < \rho$. It is easily verified that any ρ with $|\rho| \geq \sup_{k=0,\pm1,\pm2} z_{kk}$ yields such a decomposition.

The proof of Corollary 2 is completely along the lines of the proof of Corollary 1, but with σ, all l_1-summable sequences, multiplication of $A = \{a_k\}$ and $B = \{b_k\}$, defined by $AB = C = \{c_k\}$ with

$$c_k = \sum_{l=-\infty}^{+\infty} a_{k-l}b_l \quad \text{and} \quad \|A\| = \sum_{k=-\infty}^{+\infty} |a_k|.$$

The projection operator π^+ is defined by $\pi^+A = B$ with $A = \{a_k\}$, $B = \{b_k\}$, and $b_k = a_k$ for $k \geq 0$, $b_k = 0$ for $k < 0$, and $\pi^- = \theta - \pi^+$.

Remark: The factorization in Corollary 2 is valid under much weaker conditions than stated. Indeed, although dominance of the involved sequence is certainly sufficient for the factorization to be possible, it is by no means necessary as is shown by the following theorem due to Krein [17, p. 198].

Theorem 10 (Krein)

Let $A(z)$ be the z-transform of an l_1-summable sequence. Then there exist functions $A^+(z)$ and $A^-(z)$ such that

1) $A(z) = A^-(z) \, A^+(z)$
2) $A^+(z)$ and $(A^+(z))^{-1}$ are the z-transforms of l_1-summable sequences $\{a_k^+\}$ and (b_k^+) with $a_k^+ = b_k^+ = 0$ for $k < 0$, and $A^-(z)$ and $(A^-(z))^{-1}$

are the z-transforms of l_1-summable sequences $\{a_k^-\}$ and $\{b_k^-\}$ with $a_k^- = b_k^- = 0$ for $k > 0$ if and only if

a) $A(z) \neq 0$ for $|z| = 1$

b) the increase in the argument of the function $A(z)$ as z moves around the circle $|z| = 1$ in counterclockwise direction is zero.

Moreover, all factorizations which satisfy conditions a) and b) differ only by a multiplicative constant.

Proof: A proof can be found in Krein [17].

It is clear that if $A(z) - \epsilon$ is the z-transform of a dominant sequence for some $\epsilon > 0$, then $A(z)$ satisfies the conditions of the above theorem since Re $A(z) \geq \epsilon > 0$ for $|z| = 1$.

VI. Conclusion

The conditions for stability derived here involve the multiplier idea in matrix form rather than in the form of scalar functions of a complex variable, as has been the case in earlier work since Popov. They apply to time-varying and time-invariant systems. However, the tests are difficult. Further research is required to identify some relatively simple special cases. The basic inequalities involving sums of terms of the type $\sum_{k,l} x_k m_{kl} f(x_l)$ appear to be of great interest in themselves. It is true, however, that for time-invariant systems and multipliers, the stability results of this paper do not go beyond those of O'Shea, and hence it appears that quite different arguments will be needed to extend his results if, indeed, they are shown to be only sufficient conditions. Basically the arguments are elementary if the discussion is limited to time-invariant systems and multipliers. In that case, all matrices are of the Toeplitz type and the required factorization is not hard, at least in the case of rational functions. It is felt that the approach used here results in a more elegant proof than those previously published. The general case, however, requires the factorization given in Theorem 9 and the argument is consequently more difficult. Both the factorization theorem and the basic inequalities are potentially useful in other areas of system theory. This requires further investigation.

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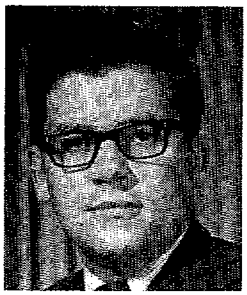

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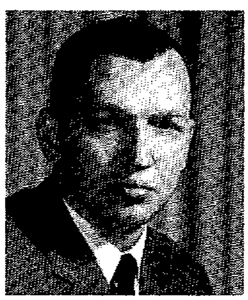

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