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Switched behaviors with impulses - a unifying framework

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***Abstract*— We present a new framework to describe and study switched behaviors. We allow for jumps and impulses in the trajectories induced either implicitly by the dynamics after the switch or explicitly by "impacts". With some examples from electrical circuit we motivate that the dynamical equations before and after the switch already uniquely define the "dynamics" at the switch, i.e. jumps and impulses. On the other hand, we also allow for external impacts resulting in jumps and impulses not induced by the internal dynamics. As a first theoretical result in this new framework we present a characterization for autonomy of a switched behavior.**

I. Introduction

In this note we aim to provide a unifying approach to handle switched and impulsive systems in the behavioral framework [16]. Different to the previous approaches [3], [4] and [10] our goal is a formalism which still allows us to write

$$w \in \mathfrak{B}_\sigma \Leftrightarrow \; \mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = 0,$$

where $\mathfrak{B}_\sigma$ denotes the switched behavior with switching signal σ and the nature of the linear (time-varying) operator $\mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})$ will be explained later. As a *universe* for the switched behavior we consider the space of piecewise-smooth distributions [13] which was also used (without using this name) in [3] as a universe. Our proposed framework will include the following special cases and combinations thereof:

Classical switched linear systems $\dot{x} = A_\sigma x + B_\sigma u$ with solutions in the sense of Carathéodory, see e.g. [8],

Switched DAEs $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with solutions in the space of piecewise-smooth distributions, see [14],

Linear systems with impulsive inputs $\dot{x} = Ax + Bu$ where u may contain Dirac impulses or derivatives thereof, see e.g. [5],

Impulsive systems $\dot{x} = Ax + Bu$, $x(t_i+) = J_i x(t_i-)$, for some ordered set $\{ t_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ and jump-matrices J_i, $i \in \mathbb{Z}$, see e.g. [7].

In this general framework we are able to show, see Corollary 10, that autonomy of the switched behavior for any switching signal is characterized by autonomy of the constituent behaviors, irrespective of possible impacts. To arrive at this main result we have to first review classical distribution theory and recall the definition of piecewise-smooth distributions in Section II. Our framework allows for Dirac-impulses in the coefficient matrices; in Section III we motivate this by showing that linear impulsive systems can equivalently be represented as a linear differential equation containing Dirac-impulses in its coefficient matrix (Lemma 2). After this motivation we present the formal definition of switched behaviors with impacts in Section IV. Within the space of piecewise-smooth distributions it can readily verified whether jumps or impulses at the switching times are compatible with the system description. Under a certain regularity assumption this implies that jumps and impulses induced by switches are already uniquely determined without explicitly stating a jump rule. We illustrate and motivate this uniqueness with examples based on electrical circuits. Section V presents the aforementioned main result of this note.

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II. Piecewise-smooth distributions

Only some minor knowledge about the formalities of distribution theory [11] are needed to understand the concept of piecewise-smooth distribution. We denote the space of distributions by $\mathbb{D}$ and formally the definition is as follows:

$$\mathbb{D} := \{ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continous} \},$$

where $\mathcal{C}_0^\infty$ denotes the set of smooth (i.e. arbitrarily often differentiable) functions with compact support (a.k.a. the space of test-functions, equipped with a certain topology which is not of further interest here). Distributions are also called generalized functions because of the following injective homomorphism

$$\mathcal{L}_{1,\mathrm{loc}} \ni f \;\mapsto f_\mathbb{D} \in \mathbb{D},$$

where $\mathcal{L}_{1,\mathrm{loc}}$ is the space of locally integrable functions and

$$f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R} : \varphi \mapsto \int_\mathbb{R} \varphi(t) f(t) \mathrm{d}t.$$

The importance of distributions is due to the capability to differentiate distributions arbitrarily often via the following rule, $D \in \mathbb{D}$,

$$D'(\varphi) := -D(\varphi')$$

which is consistent with classical differentiation of differentiable functions, i.e. for any differentiable f

$$(f_\mathbb{D})' = (f')_\mathbb{D}.$$

Note that we will use the dot above a function or distribution as an equivalent way to denote the derivative.

The most famous distribution is the Dirac impulse δ (a.k.a. Dirac-Delta or Delta-function) formally defined via

$$\delta(\varphi) = \varphi(0),\ \varphi \in \mathcal{C}_0^\infty.$$

It is easily seen that the Dirac impulse is the distributional derivative of the Heaviside function

$$\mathbb{1}_{[0,\infty)}(t) := \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Distributions can be multiplied with smooth functions via

$$(\alpha D)(\varphi) := D(\alpha\varphi), \quad D \in \mathbb{D}, \alpha \in \mathcal{C}^\infty, \tag{1}$$

where $\mathcal{C}^\infty$ denotes the space of smooth functions. In the following the space of *piecewise-smooth functions* will play an important role and is given by

$$\mathcal{C}^\infty_{\mathrm{pw}} := \left\{ \alpha = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \;\middle|\; \begin{array}{l} \alpha_i \in \mathcal{C}^\infty, i \in \mathbb{Z}, \\ \{\, t_i \mid i \in \mathbb{Z} \,\} \text{ ordered} \\ \text{and locally finite} \end{array} \right\},$$

where $\mathbb{1}_I$ denotes the characteristic function of the interval $I \subseteq \mathbb{R}$. Switched systems can be interpreted as time-varying linear systems with piecewise-constant coefficient matrices. The latter are of course a special case of piecewise-smooth coefficient matrices. Hence one might aim to generalize the multiplication (1) also to piecewise-smooth functions. However, for general distributions this is not possible in a consistent way [12, Thm. 2.2]. That is the reason we consider the smaller space of *piecewise-smooth distributions* instead of the whole space $\mathbb{D}$. The former is defined as

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_\mathbb{D} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}},\ T \text{ locally finite}, \\ D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\},$$

where δ_t denotes the Dirac-impulse at $t \in \mathbb{R}$, i.e. $\delta_t = \mathbb{1}'_{[t,\infty)}$. In other words, a piecewise-smooth distributions is the sum of a piecewise-smooth function and isolated impulses (composed of Dirac-impulses and their derivatives). It is easily seen, that the space of piecewise-smooth distributions is closed under differentiation and therefore recovers the essential property of the space of distributions. Furthermore, on each finite interval, every piecewise-smooth distribution is the finite derivative of a piecewise-smooth function. The space of piecewise-smooth distributions allows for an evaluation at a certain time $t \in \mathbb{R}$ in the following ways, where $D = f_\mathbb{D} + \sum_{t\in T} D_t$:

Left-sided evaluation $D(t-) := f(t-) = \lim_{\varepsilon\searrow 0} f(t-\varepsilon)$,
Right-sided evaluation $D(t+) := f(t+) = f(t)$,
Impulsive part $D[t] := D_t$ if $t \in T$ and $D[t] = 0$ otherwise; $D[\cdot] := \sum_{t\in T} D_t$.

Although the authors of [3] also used the space of piecewise-smooth distributions as the universe they were not aware that this space has the nice property that a unique multiplication with the following properties can be defined [12]:

Associative algebra $F(GH) = (FG)H$ and distributive laws hold, e.g. $(F+G)H = FH + GH$, for all $F, G, H \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$
Product rule $(FG)' = F'G + FG'$ for all $F, G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$,
Functions $(fg)_\mathbb{D} = f_\mathbb{D} g_\mathbb{D}$ for all $f, g \in \mathcal{C}^\infty_{\mathrm{pw}}$,
Causality $\mathbb{1}_{[t,\infty)}\delta_t = \delta_t$ for all $t \in \mathbb{R}$.

Note that we do not assume commutativity. With the above properties the multiplication of general piecewise-smooth distributions can be defined via

$$FG = (fg)_\mathbb{D} + F[\cdot]g_\mathbb{D} + f_\mathbb{D}G[\cdot],$$

where $F = f_\mathbb{D} + F[\cdot]$ and $G = g_\mathbb{D} + G[\cdot]$, for details see [12]. In particular, for $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and $t \in \mathbb{R}$,

$$\delta_t D = D(t-)\delta_t, \quad D\delta_t = D(t+)\delta_t, \tag{2}$$

and

$$\delta_t\delta_t = 0.$$

Furthermore, when causality is not required then one can show [1] that all possible multiplication have the property that either $\mathbb{1}_{[t,\infty)}\delta_t = \delta_t$ or $\mathbb{1}_{[t,\infty)}\delta_t = 0$. We will call the above multiplication (causal) Fuchssteiner multiplication (after [1], c.f. [2]) and the following example motivates the causality property.

Example 1: Consider the switched system given by

$$\mathbb{1}_{[0,\infty)}\dot{x} = \mathbb{1}_{(-\infty,0)}x,$$

i.e. $x = 0$ on $(-\infty, 0)$ and $\dot{x} = 0$ on $[0, \infty)$. The latter implies (also in the distributional framework) that $x = c$ on $(0, \infty)$ for some constant $c \in \mathbb{R}$. Hence $\dot{x} = c\delta$. With the causal Fuchssteiner multiplication $\mathbb{1}_{[0,\infty)}\dot{x} = c\delta$ and since $\mathbb{1}_{(-\infty,0)}x = 0$ it follows that $c = 0$ and the only solution of the switched system is $x = 0$, i.e. the past, $x = 0$ on $(-\infty, 0)$, uniquely determines the future, $x = 0$ on $[0, \infty)$. On the other hand, when using the anticausal Fuchssteiner multiplication then $\mathbb{1}_{[0,\infty)}\dot{x} = 0 = \mathbb{1}_{(-\infty,0)}x$ and $x = c\mathbb{1}_{[0,\infty)}$ would be a solution for the switched system for all $c \in \mathbb{R}$. In particular, the past does not uniquely determine the future, however the converse is true now. $\triangleleft$

For the forthcoming definition of switched behaviors, the ring of polynomials $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}[s]$ with coefficients in the ring of piecewise-smooth distributions will play a role. Different to classical polynomial rings the multiplication in $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is not commutative due to the time varying nature of the coefficients and the interpretation of the variable s as the differential operator. As with smoothly time-varying linear systems the commutation rule is given by

$$sD = Ds + D'.$$

III. Impulsive systems

We will now consider the impulsive system

$$\dot{x} = Ax + Bu, \quad x(t_i+) = J_i x(t_i-), \tag{3}$$

where $\{\, t_i \in \mathbb{R} \mid i \in \mathbb{Z} \,\}$ is ordered and locally finite, x is an n-dimensional vector and A, J_i, B are matrices of appropriate size. The ODE is assumed to hold on the open intervals (t_i, t_{i+1}), $i \in \mathbb{Z}$, only. Now let

$$\mathcal{A} := A + \sum_{i\in\mathbb{Z}} (J_i - I)\delta_{t_i} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$$

and consider the following ODE with distributional coefficients

$$\dot{x} = \mathcal{A}x + Bu, \tag{4}$$

which is assumed to be valid on the whole time axis $\mathbb{R}$ without exceptions. Now the following result states that the corresponding solution sets are isomorphic.

Lemma 2: Consider the impulsive system (3) and the associated ODE with distributional coefficients (4) and any input $u \in \mathcal{C}^\infty_{\mathrm{pw}}$. Then $x \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ is a solution of (3) if, and only if, $x_\mathbb{D}$ is a solution (within $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$) of (4) with input $u_\mathbb{D}$. Furthermore, all solution x of (4) with input $u_\mathbb{D}$ are impulse free, i.e. $x[\cdot] = 0$; in particular, there are no additional (distributional) solutions of (4) compared to (3).

Proof: Let $x = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} x_i \in \mathcal{C}^\infty_{\mathrm{pw}}$ with $x_i \in \mathcal{C}^\infty$ then

$$(x_\mathbb{D})' = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}(x_i')_\mathbb{D} + \sum_{i\in\mathbb{Z}} (x(t_i+) - x(t_i-))\delta_{t_i}.$$

If x is a solution of (3) then $\dot{x}_i = Ax_i + Bu$ on (t_i, t_{i+1}) and $x(t_i+) = J_i x(t_i-)$, hence, invoking (2),

$$\begin{aligned}(x_\mathbb{D})' &= \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}(Ax_{i\mathbb{D}} + Bu_\mathbb{D}) + \sum_{i\in\mathbb{Z}} (J_i - I)x(t_i-)\delta_{t_i} \\ &= Ax_\mathbb{D} + Bu_\mathbb{D} + \sum_{i\in\mathbb{Z}} (J_i - I)\delta_{t_i} x_\mathbb{D} = \mathcal{A}x_\mathbb{D} + Bu_\mathbb{D}.\end{aligned}$$

On the other hand if $x_\mathbb{D}$ is a solution of (4) then

$$\begin{aligned}(x_\mathbb{D})' &= \left(A + \sum_{i\in\mathbb{Z}} (J_i - I)\delta_{t_i}\right) x_\mathbb{D} + Bu_\mathbb{D} \\ &= \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}(Ax_i + Bu)_\mathbb{D} + \sum_{i\in\mathbb{Z}} (J_i - I)x(t_i-)\delta_{t_i},\end{aligned}$$

which shows that x solves (3). Finally, let $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ be a solution of (4) and seeking a contradiction assume $x[\cdot] \neq 0$. Let $t \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $\delta_t^{(p)}$ is the highest derivative of a Dirac-impulse contained in $x[\cdot]$. Then $\dot{x}[\cdot]$ contains $\delta_t^{(p+1)}$ while the right hand side $\mathcal{A}x + Bu_\mathbb{D}$ only contains $\delta_t^{(p)}$, as $u_\mathbb{D}$ doesn't contain any Dirac impulses. Hence we have arrived at a contradiction and $x[\cdot] = 0$ is shown. ∎

The previous result motivates to also allow for Dirac impulses in the coefficient matrices of the systems description. Another motivation is given by the following observation: When aiming for a normal or simplified form it is common to consider coordinate transformations. However, in the context of switched systems these coordinate transformation might also be switched (i.e. piecewise-constantly time-varying). So carrying out the transformation $x \mapsto T_\sigma z$ and plugging this into a differential equations leads to the need to differentiate T_σ which introduces Dirac impulses in the coefficient matrices. In the following we will also allow for derivatives of the Dirac-impulses within the coefficient matrices which corresponds to jump rules invoking derivatives of the state. In particular, the "glueing conditions" from [10] might also be formulated via distributional entries in the coefficient matrices.

IV. Switched behaviors with impulses

We are now ready to present our proposed new framework to describe switched behaviors with possible impulsive effects. Therefore let $R_p(s), J_p(s) \in \mathbb{R}^{m\times n}[s]$, $p \in \mathcal{P}$, be two families of polynomial matrices for a parameter set $\mathcal{P}$, where $\mathbb{R}[s]$ denotes the ring of real polynomials. For a switching signal $\sigma : \mathbb{R} \to \mathcal{P}$ with the locally finite ordered set of switching times $\{\, t_i \mid i \in \mathbb{Z} \,\}$ define the corresponding switched behavioral kernel operator $\mathcal{R}_\sigma(s) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times n}[s]$ as

$$\mathcal{R}_\sigma(s) := \sum_{i\in\mathbb{Z}} \left(\mathbb{1}_{[t_i,t_{i+1})} R_{\sigma(t_i+)}(s) + J_{\sigma(t_i+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(\delta_{t_i})\right). \tag{5}$$

The corresponding differential operator $\mathcal{R}_\sigma(\frac{\mathrm{d}}{\mathrm{d}t})$ is then given for $w \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ by

$$\mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = \sum_{i\in\mathbb{Z}} \left(\mathbb{1}_{[t_i,t_{i+1})} \sum_{j=0}^{d_{p_i}} R_{p_i}^j w^{(j)} + \sum_{k=0}^{\nu_{p_i}} J_{p_i}^k \delta_{t_i}^{(k)} w \right),$$

where $p_i := \sigma(t_i+)$, $R_p(s) = \sum_{j=0}^{d_p} R_p^i s^i$, $d_p \in \mathbb{N}$, $J_p(s) = \sum_{k=0}^{\nu_p} J_p^k s^k$, $\nu_p \in \mathbb{N}$, $p \in \mathcal{P}$. In particular, $R_\sigma(\frac{\mathrm{d}}{\mathrm{d}t})$ is applied to w and $J_\sigma(\frac{\mathrm{d}}{\mathrm{d}t})$ is applied to the corresponding Dirac-impulse and then multiplied with w. The overall switched behavior is

$$\mathfrak{B}_\sigma := \left\{\, w \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \;\middle|\; \mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = 0 \,\right\}.$$

The matrices polynomial $R_p(s)$ defines the *dynamics* between the switches (and in the regular case also at the switches, see our discussion later) and the corresponding behavior $\mathfrak{B}_p := \ker R_p(\frac{\mathrm{d}}{\mathrm{d}t})$ can be seen as the constituent non-switched behaviors. The polynomial matrices $J_p(s)$ define the *impacts* at certain switching times resulting in jumps and impulses not induced by the dynamics given by $R_p(s)$.

For an illustration we again consider the above mentioned systems descriptions and show how they are written as a switched behavior. In all cases $w = (x^\top, u^\top)^\top$.

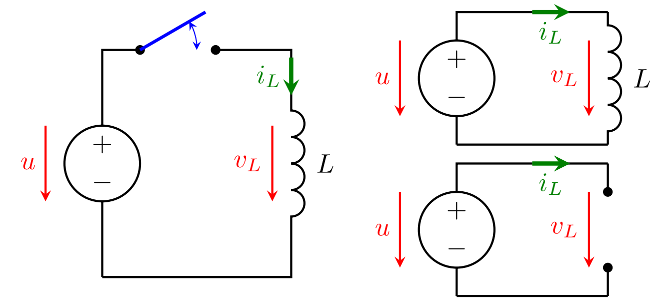
Switched linear systems $R_p(s) = s[I\ 0] - [A_p\ B_p]$ and $J_p(s) = 0$ for all $p \in \mathcal{P}$,

Switched DAEs $R_p(s) = s[E_p\ 0] - [A_p\ B_p]$ and $J_p(s) = 0$ for all $p \in \mathcal{P}$, note that here jumps and impulses can occur at the switching instance without introducing those explicitly via $J_p(s)$,

Systems with impulsive inputs $R_p(s) = s[I\ 0] - [A\ B]$ and $J_p(s) = 0$ independently of $p \in \mathcal{P}$ (note that now impulses in u are allowed due to the chosen universe),

Impulsive systems $R_p(s) = s[I\ 0] - [A\ B]$, $J_p(s) = [(J_p - I)\ 0]$, see Lemma 2.

In the previous approaches [3], [10], it was necessary to define the property of a trajectory w belonging to the switched behavior $\mathfrak{B}_\sigma$ piecewise via the individual intervals between the switching times and, additionally, how the pieces are connected. With our new approach this is not the case



(a) Electrical circuit with a physical switch

(b) Electrical circuit where coil simply "disappears"

Fig. 1: Disconnecting a coil from a voltage source: two different interpretations as electrical circuits

as the (global) condition

$$\mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) = 0$$

is already well defined in the space of piecewise-smooth distributions because the multiplication of a (distributional) trajectory w with piecewise-constant coefficients and with Dirac-impulses (and their derivatives) is already established. We do not deny that there are useful physical models of switched systems where the "dynamics" at the switch or impact (i.e. jumps and impulses) are modeled (mathematically) independently from the dynamics before and after the switch. However, the following examples (inspired by [15]) shows that jumps and impulses might already be induced by the dynamics before and after the switch.

Example 3 (Disconnecting a coil from a voltage source): Consider the simple electrical circuit as shown in Figure 1a composed of a coil, a voltage source and a switch. For simplicity we assume that the voltage source supplies a constant (but possibly unknown) voltage u which can be modeled via the differential equation $\dot{u} = 0$. Independently of the switch the inductivity rule holds:

$$L\tfrac{\mathrm{d}}{\mathrm{d}t} i_L = v_L.$$

If the switch is closed the algebraic relation $v_L = u$ holds, otherwise $i_L = 0$. Altogether this can be modeled via a switched behavior with $w = (u, i_L, v_L)$ and

$$R_{\mathrm{cl}}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & Ls & -1 \\ 1 & 0 & -1 \end{bmatrix}, \; R_{\mathrm{op}}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & Ls & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and, as we do not have modeled any external resetting of the variables, $J_{\mathrm{cl}}(s) = J_{\mathrm{op}}(s) = 0$. What would we expect as solutions? Consider therefore the situation where the switch is closed for $t < 0$ and at $t = 0$ the switch is opened and remains open. Then all solutions before the switch, i.e. $t < 0$, are given by

$$u(t) = c_1, \quad v_L(t) = u(t) = c_1,$$
$$i_L(t) = \int^t \frac{1}{L} v_L(\tau)\mathrm{d}\tau = \frac{c_1 t}{L} + c_2,$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary. What is happening now at a switch for some given past trajectory w? First of all, the switch does not effect the voltage source, i.e. the defining equation $\dot{u} = 0$ remains valid and $u(t) = c_1$ also for $t \geq 0$. Furthermore, the new algebraic constraint $i_L = 0$ gets active, which forces the current i_L to jump from its value $i_L(0-) = c_2$ just before the switch to zero. Note that the defining equation leaves no other choice, any additional jump rule would not be consistent with the equations. Furthermore, independently of the switch the inductivity rule holds, in particular

$$v_L = L\tfrac{\mathrm{d}}{\mathrm{d}t} i_L = \mathbb{1}_{(-\infty,0)} c_1 - c_2 \delta_0,$$

when the distributional derivative is used. Additionally to the jump in v_L also a Dirac-impulse occurs (if $c_2 \neq 0$), which can actually be observed in reality as a spark when opening the switch. Again these "dynamics" at the switch are already uniquely defined by the physical model. With the piecewise-smooth distributional solution framework as defined above we will get exactly this solution which we obtained by analyzing the circuit by hand. $\triangleleft$

The above example showed that, once one has decided how to model a physical phenomena, the behavior is uniquely specified at the switching instance including possible Dirac-impulses. However, a different model might produce different results as the following example shows.

Example 4 (Example 3, alternative model): Consider the two circuits from Figure 1b where the switched systems results from removing the coil from the upper circuit leading to the lower circuit with an open connection. Different to the analysis in the previous example the inductivity rule is now not invariant under switching, but the voltage loop rule $u = v_L$ remains invariant. This results in the switched behavior with

$$R_{\mathrm{cl}}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & Ls & -1 \\ 1 & 0 & -1 \end{bmatrix}, \; R_{\mathrm{op}}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

and, as before, $J_{\mathrm{cl}}(s) = J_{\mathrm{op}}(s) = 0$. Assume the coil is removed at $t = 0$. Carrying out a similar analysis as above we conclude that all solution are given by

$$u(t) = c_1, \quad v_L(t) = c_1, \quad i_L(t) = \begin{cases} \frac{c_1 t}{L} + c_2, & t < 0, \\ 0, & t \geq 0. \end{cases}$$

Now no Dirac impulse occurs in the solution. Nevertheless, the mathematics again dictate a unique solution behavior at the switching instance. In conclusion: the physics need to tell us whether a disconnection of a coil should be modeled via a switch (as in Figure 1a) or via a complete removal of the coil from the circuit (as in Figure 1b), the mathematics will then tell us whether we will see a Dirac impulse or not in the solution. $\triangleleft$

Finally, the solution theory within the space of piecewise-smooth distribution might also help us to define the "right" impacts even without invoking physical arguments.

Example 5 (Instantanious resetting): Consider the electrical circuit from Figure 1a and we want to model a "double switch", i.e. the opening of the switch for an infinitesimal short amount of time, resulting in a resetting of the current. This implies that the dynamics before and after the double switch are identical and do not yield any jumps or impulses in the state. However, if we introduce a small time delay between the opening and closing of the switch, we obtain jumps and impulses which converge to a well defined jump and impulse if we let the time delay going to zero. In fact, we have the following jumps and impulses where the switching time is at $t = 0$ and the time delay is $\varepsilon > 0$, using the notation from Example 3:

$$\begin{aligned} w(0+) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w(0-), \\ w[0] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -L & 0 \end{bmatrix} w(0-)\delta_0, \\ w(0+\varepsilon+) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} w(0+\varepsilon-), \\ w[0+\varepsilon] &= 0. \end{aligned}$$

Letting ε go to zero and invoking $u = u_L$ if the switch is closed we obtain for the "double switch":

$$\begin{aligned} w(0+) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} w(0-), \\ w[0] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -L & 0 \end{bmatrix} w(0-)\delta_0, \end{aligned}$$

i.e. the current is reset to zero and this jump induces a Dirac-impulse. This solution behavior can now be described by

$$\mathcal{R}_\sigma(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & Ls & -1 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -L & 0 \end{bmatrix} \delta_0,$$

where the switching signal σ is such that the impact happens at $t = 0$. $\triangleleft$

Remark 6 (Distributional behaviors): The class of switched behavior given by (5) is a special case of the more general distributional behavior

$$\mathfrak{B} = \ker \mathfrak{R}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \subseteq (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n,$$

where $\mathfrak{R}(s) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times n}[s]$ is a polynomial matrix whose coefficients are general piecewise-smooth distribution. This generalization encompasses smoothly time varying system as e.g. studied in [6] or distributional DAEs as studied in [12], [13]. However, general results for this class of behaviors are more difficult to obtain and are a topic for future research. $\triangleleft$

V. Autonomy of switched behaviors

In this section we will study autonomy of switched behaviors which is defined by the property

$$(w_1)_{(-\infty,t)} = (w_2)_{(-\infty,t)} \quad \Rightarrow \quad w_1 = w_2$$

which has to hold for all $w_1, w_2 \in \mathfrak{B}_\sigma$ and all $t \in \mathbb{R}$. It is well known [9] that autonomy for classical behaviors in kernel representation, i.e. $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ for some polynomial matrix $R(s) \in \mathbb{R}^{m\times n}[s]$, is equivalent to "regularity" of $R(s)$:

$$R(s) \text{ is regular} \quad \Leftrightarrow \quad R(s) \text{ is square and } \det R(s) \not\equiv 0.$$

It will turn out that autonomy of the individual non-switched behaviors of a switched behavior is necessary and sufficient for autonomy of the switched behavior for all switching signals. We first consider the case without impacts.

Theorem 7: Consider a family of polynomial matrices $R_p(s) \in \mathbb{R}^{m\times n}[s]$, $p \in \mathcal{P}$, and assume that there is an upper bound $d \in \mathbb{N}$ for the polynomial degrees of all $R_p(s)$. Then the corresponding switched behavior $\mathfrak{B}_\sigma$ without impacts (i.e. with $J_p(s) = 0$ for all $p \in \mathcal{P}$) is autonomous for all switching signals σ if, and only if, each classical behavior $\mathfrak{B}_p := \ker R_p(\frac{\mathrm{d}}{\mathrm{d}t})$ is autonomous.

Proof: Necessity is clear, because if one single behavior is not autonomous then choosing the corresponding constant switching signals yields a switched behavior which is identical to the non-autonomous behavior. Hence it remains to show sufficiency. To this end, we first transform the high order description to a first order description via the standard trick to also include derivatives of the original variable into the description. The first order behavioral matrices are then given by

$$\widehat{R}_p = \begin{bmatrix} sI & -I & & \\ & \ddots & \ddots & \\ & & sI & -I \\ R_p^0 & \cdots & R_p^{d-2} & R_p^{d-1} + sR_p^d \end{bmatrix},$$

where $R_p(s) = \sum_{i=0}^d R_p^i s^i$. Then it is easily seen that each behavior $\mathfrak{B}_p$ is isomorphic to the first order behavior $\widehat{\mathfrak{B}}_p := \ker \widehat{R}_p(\frac{\mathrm{d}}{\mathrm{d}t})$ via the isomorphism $w \mapsto \widehat{w} = (w, \dot{w}, \ddot{w}, \ldots, w^{(d-1)})$ and the same is true for the corresponding switched behaviors $\mathfrak{B}_\sigma$ and $\widehat{\mathfrak{B}}_\sigma$ (because the conditions $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{w}_i = \widehat{w}_{i+1}$, $i = 0, 1, \ldots, d-2$, are switch independent). In particular $\mathfrak{B}_\sigma$ is autonomous if, and only if, $\widehat{\mathfrak{B}}_\sigma$ is autonomous. As $\widehat{R}_p(s)$ is of degree one, the corresponding behavior can be written as

$$\widehat{\mathfrak{B}}_p = \left\{ x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{nd} \;\middle|\; E_p\dot{x} = A_p x \right\}$$

with suitable $E_p, A_p \in \mathbb{R}^{nd\times nd}$. In particular, the switched behavior $\widehat{\mathfrak{B}}_\sigma$ is in fact a switched DAE as studied in [12]:

$$\widehat{\mathfrak{B}}_\sigma = \left\{ x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{nd} \;\middle|\; E_\sigma\dot{x} = A_\sigma x \right\}.$$

In [12] it was shown that $\widehat{\mathfrak{B}}_\sigma$ is autonomous for all switching signals if, and only if, the matrix pairs (E_p, A_p) are regular. It therefore remains to show that

$$\det(sE_p - A_p) \equiv 0 \quad \Leftrightarrow \quad \det R_p(s) \equiv 0.$$

Now $\det(sE_p - A_p) \equiv 0$ is equivalent to the property

$$\forall \lambda \in \mathbb{C}\ \exists \widehat{v} = (\widehat{v}_0, \widehat{v}_1, \ldots, \widehat{v}_{d-1}) \in \mathbb{C}^{nd} : \ (\lambda E_p - A_p)\widehat{v} = 0,$$

Due to the definition of E_p, A_p the condition $(\lambda E_p - A_p)\widehat{v} = 0$ is equivalent to

$$\widehat{v}_i = \lambda^i \widehat{v}_0,\ i = 0, \ldots, d-1, \text{ and } \sum_{i=0}^{d} R_p^i \lambda^i \widehat{v}_0 = 0.$$

Hence singularity of (E_p, A_p) is equivalent to

$$\forall \lambda \in \mathbb{C}\ \exists v \in \mathbb{C}^n : R(\lambda)v = 0$$

which is a characterization of $\det R(s) \equiv 0$ and the proof is complete. ■

Remark 8: Autonomy of the switched behavior $\mathfrak{B}_\sigma$ in particular implies that jumps and impulses at the switching instances are already uniquely defined by the "continuous" dynamics of each constituent behavior. This is a remarkable feature of our proposed framework but which is well motivated by physical examples, see e.g. Example 3.

We will now show that autonomy of the switched behavior is independent of the impacts (c.f. [12, Prop. 3.2.2(iv)]).

Theorem 9: Consider a switched behavior $\mathfrak{B}_\sigma = \ker \mathcal{R}_\sigma(\frac{\mathrm{d}}{\mathrm{d}t})$ with $\mathcal{R}_\sigma(s)$ as in (5) and the corresponding impact-free behavior $\mathfrak{B}_\sigma^{\text{if}} = \ker \mathcal{R}_\sigma^{\text{if}}(s)$ with $\mathcal{R}_\sigma^{\text{if}}(s) = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} R_{\sigma(t_i+)}(s)$. Then $\mathfrak{B}_\sigma$ is autonomous if and only if $\mathfrak{B}_\sigma^{\text{if}}$ is autonomous.

Proof: Consider two solutions $w_1, w_2 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ within the switched behavior $\mathfrak{B}_\sigma$. Then due to linearity $w := w_1 - w_2 \in \mathfrak{B}_\sigma$. Assume $w_1 = w_2$ on $(-\infty, t)$ for some $t \in \mathbb{R}$, i.e., $w = 0$ on $(-\infty, t)$. Without restricting generality we can assume that t is a switching time of σ. Now, with $\{\, t_i \in \mathbb{R} \mid i \in \mathbb{Z} \,\}$ the switching times of σ and t_{+1} the next switching time after t,

$$\begin{aligned}
&\mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w)_{(-\infty,t_{+1})} \\
&= \Bigg(\sum_{i\in\mathbb{Z}} \big(\mathbb{1}_{[t_i,t_{i+1})} R_{\sigma(t_i+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) \\
&\quad + J_{\sigma(t_i+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(\delta_{t_i} w) \big) \Bigg)_{(-\infty,t_{+1})} \\
&= \mathbb{1}_{[t,t_{+1})} R_{\sigma(t+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w) + J_{\sigma(t+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(\delta_t)w.
\end{aligned}$$

Invoking (2) and the product rule one obtains (c.f. [13, Rem. 6])

$$\delta_t^{(i)} w = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \underbrace{w^{(j)}(t-)}_{=0} \delta_t^{(i-j)} = 0,$$

hence $J_{\sigma(t+)}(\frac{\mathrm{d}}{\mathrm{d}t})(\delta_t)w = 0$ and

$$\mathcal{R}_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t})(w)_{(-\infty,t_{+1})} = \mathbb{1}_{[t,t_{+1})} R_{\sigma(t+)}(\tfrac{\mathrm{d}}{\mathrm{d}t})(w).$$

Together with an inductive argument this shows that $\mathfrak{B}_\sigma$ is autonomous if, and only if $\mathfrak{B}_\sigma^{\text{if}}$ is autonomous. ■

We can now combine our two main results to obtain a complete characterization of autonomy of general switched behaviors.

Corollary 10: Consider the switched behavior $\mathfrak{B}_\sigma$ given by $\mathcal{R}_\sigma(s)$ as in (5) and denote with $\mathfrak{B}_p$ the classical behaviors induced by $R_p(s)$ and assume that the polynomial degree is upper bounded independently of $p \in \mathcal{P}$. Then $\mathfrak{B}_\sigma$ is autonomous for all switching signals σ if, and only if, $\mathfrak{B}_p$ is autonomous for all p. ◁

VI. Conclusions

We have presented a new framework for switched behaviors, whose major advantage compared to previous framework is the capability to write the behavior (globally) as the kernel of a polynomial matrix. For this the space of piecewise-smooth distributions is utilized. Within this new framework we were able to characterize autonomy of the switched behavior in terms of the constituent non-switched behaviors.

Future work on switched behaviors will concentrate on generalizing classical systems theoretical results like stability, controllability or normal forms. Due to the presence of jumps and impulses in the trajectories it is expected that these generalizations have to involve new arguments and are not always straightforward.

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