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On the stability of switched behavioral systems

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Abstract— **In this paper we study switched systems from a behavioral point of view. We define a switched behavioral system in terms of a bank of behaviors without referring to the notion of state. Moreover we give sufficient conditions for the stability of a switched behavior in terms of the behaviors in the underlying bank.**

I. Introduction

Classical switched systems are usually defined as consisting of a bank of systems described by state-space representations, together with a supervisory system that produces a switching signal that indicates which of the systems in the bank should be active at each time instant. Moreover, when switching occurs the continuity of the state trajectories may or not be required, and, in the latter case, reset maps are specified in order to produce new initial conditions for the post-switching evolution [3].

Here, instead of using state space representations, we propose a notion of switched behavioral system essentially characterized by a bank of behaviors described by higher order differential equations, together with some "gluing conditions" that relate the system variables and their derivatives immediately before and after switching. Moreover, the behaviors are not assumed to share all the same state space, and consequently our definition is more general than the classical one.

One of the most critical issues is the analysis of the stability of switched state space systems, as switching among stable systems may give rise to instability. This does not happen, however, if all the systems in the bank share a common Lyapunov function.

Although other alternative sufficient conditions for the stability of switched systems have been proposed in the literature (see for instance [1]), in this paper, as a first step, we shall concentrate on the Lyapunov approach.

It turns out that the existence of a suitable common (behavioral) Lyapunov function also ensures the stability of the switched behavioral system. In this contribution, we consider the simple particular case of a switched behavioral system based on a bank of two scalar behaviors and study conditions for the existence of such Lyapunov function.

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II. Switched behavioral systems

Before we define switched behavioral systems we introduce some standard behavioral notions.

We denote with $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ the set of smooth functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. We call $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ a linear time-invariant differential behavior (or shorty a *behavior*) if $\mathfrak{B}$ is the set of solutions of a finite system of constant-coefficient differential equations, i.e., if there exists a polynomial matrix $R \in \mathbb{R}^{q\times \mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\} = \ker R(\frac{d}{dt})$. If a behavior is represented by $R(\frac{d}{dt})w = 0$, then we call R a kernel representation of $\mathfrak{B}$. We denote with $\mathfrak{L}^{\mathtt{w}}$ the set of all linear time-invariant differential behaviors with $\mathtt{w}$ variables. Autonomous behaviors will play a special role in this paper. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *autonomous* if

$$[(w_1, w_2 \in \mathfrak{B}) \text{ and } (w_1(t) = w_2(t) \text{ for } t < 0)] \Rightarrow [w_1 = w_2].$$

The autonomy of a behavior can be characterized in as follows.

Theorem 1: ([8]) For $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ the following statements are equivalent

1) $\mathfrak{B}$ is autonomous.
2) The trajectories of $\mathfrak{B}$ have no free components.
3) $\mathfrak{B}$ admits a kernel representation R that is square and such that $\det(R) \neq 0$.

We view a switched behavior as a set of trajectories produced by a switching structure. Such a structure basically consists of a family (or bank) of behaviors, that we shall consider here to be finite, together with a set of switching signals that specify which behavior is to be considered at each time instant, and a set of "gluing conditions" that regulate the concatenation of the trajectories of two behaviors at the switching instants. Contrary to what happens for switched state space systems, the underlying behaviors are not assumed to be described by a state space representation, but rather by higher order differential equations. Moreover, the gluing conditions are not expressed in terms of the system state.

Definition 2: A *switching structure* Σ is a 4-tuple $\Sigma = (\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G})$ where $\mathcal{P} = \{1,\dots,N\} \subset \mathbb{N}$ is the *set of indices*, $\mathcal{F} = (\mathfrak{B}_1,\dots,\mathfrak{B}_N)$, with $\mathfrak{B}_j \in \mathfrak{L}^{\mathtt{w}}$ for $j \in \mathcal{P}$, is the *bank of behaviors*, $\mathcal{S} = \{s : \mathbb{R} \to \mathcal{P} \; : \; s \text{ is piecewise contant and right-continuous}\}$ is the set of admissible *switching signals* and $\mathcal{G} = \{((k,\ell), G^+_{k,\ell}(\xi), G^-_{k,\ell}(\xi)) \mid (G^+_{k,\ell}(\xi), G^-_{k,\ell}(\xi)) \in$

$(\mathbb{R}[\xi]^{g\times \mathtt{w}})^2$ and $(k,\ell) \in \mathcal{P}\times\mathcal{P},\ k\neq \ell\}$ is the set of *gluing conditions*. Moreover, for a given switching signal $s \in \mathcal{S}$, we define the set of *switching instants* with respect to s as $\mathbb{T}_s := \{t \in \mathbb{R} \mid \lim_{\tau\nearrow t} s(\tau) \neq s(t)\} = \{t_1, t_2, \dots\}$ where $t_i < t_{i+1}$.

We next present our definition of switched behavior, where we use the notation $f(t^-) = \lim_{\tau\nearrow t} f(t)$ and $f(t^+) = \lim_{\tau\searrow t} f(t)$.

Definition 3: Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G})$ be a switching structure. For a given switching signal $s \in \mathcal{S}$, we define the *s-switched behavior* $\mathfrak{B}^s$ *with respect to* Σ as the set of trajectories w that satisfy the following two conditions:

1) for all $t_i, t_{i+1} \in \mathbb{T}_s$, there exists $\mathfrak{B}_k$, $k \in \mathcal{P}$ such that
$$w|_{[t_i,t_{i+1})} \in \mathfrak{B}_k|_{[t_i,t_{i+1})},$$
2) w satisfies the gluing conditions at the switching instants $\mathcal{G}$, i.e.,
$$(G^+_{s(t_{i-1}),s(t_i)}(\frac{d}{dt}))w(t_i^+) = (G^-_{s(t_{i-1}),s(t_i)}(\frac{d}{dt}))w(t_i^-)$$
for each $t_i \in \mathbb{T}_s$.

We define the *switched behavior* $\mathfrak{B}^\Sigma$ *of* Σ as
$$\mathfrak{B}^\Sigma := \bigcup_{s\in\mathcal{S}} \mathfrak{B}^s.$$

The next example clarifies the role of the gluing conditions $(G^+_{k,\ell}(\xi), G^-_{k,\ell}(\xi))$ in Definition 3 as concatenation conditions that a trajectory in the switched system must satisfy when switching from $\mathfrak{B}_k$ to $\mathfrak{B}_\ell$ occurs.

Example 4: Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G})$ be a switching structure where $\mathcal{P} = \{1,2\}$, $\mathcal{F} = \Big(\mathfrak{B}_1 = \ker\,(\frac{d^2}{dt^2}+1), \mathfrak{B}_2 = \ker\,(\frac{d}{dt}-2)\Big)$ and
$$\mathcal{G} = \{((2,1), \begin{bmatrix}1\\ \xi\end{bmatrix}, \begin{bmatrix}1\\ \xi\end{bmatrix}), ((1,2),(1,1))\}.$$
Suppose now that we start at time $t=0$ in $\mathfrak{B}_2$ with $w(0) = 1$ and we switch at time $t = \frac{\pi}{2}$ to $\mathfrak{B}_1$. Then, the gluing conditions at $t = \frac{\pi}{2}$ are,
$$\begin{aligned}\lim_{t\searrow\frac{\pi}{2}}(\begin{bmatrix}1\\ \frac{d}{dt}\end{bmatrix}(k_1\cos(t) &+ k_2\sin(t))) = \\ &= \lim_{t\nearrow\frac{\pi}{2}}(\begin{bmatrix}1\\ \frac{d}{dt}\end{bmatrix}e^{2t} \\ &= \begin{bmatrix}e^\pi\\ 2e^\pi\end{bmatrix},\end{aligned}$$
yielding $k_1 = -2e^\pi$ and $k_2 = e^\pi$. If we switch again at $t = \pi$ back to $\mathfrak{B}_2$, then the gluing condition is $w(\pi^+) = w(\pi^-)$, i.e. :
$$\lim_{t\nearrow\pi} ke^{2t} = ke^{2\pi} = \lim_{t\searrow\pi} -2e^\pi\cos(t) + e^\pi\sin(t) = 2e^\pi,$$
which implies that $k = 2e^{-\pi}$. Hence the trajectory $w(t)$ produced by the switching signal
$$s(t) = \begin{cases} 2 & t\in[0,\frac{\pi}{2}) \\ 1 & t\in[\frac{\pi}{2},\pi) \\ 2 & t\geq\pi\end{cases}$$
is given by
$$w(t) = \begin{cases} e^{2t} & t\in[0,\frac{\pi}{2}) \\ -2e^\pi\cos(t) + e^\pi\sin(t) & t\in[\frac{\pi}{2},\pi) \\ 2e^{-\pi}e^{2t} & t\geq\pi\end{cases}.$$

This example illustrates a situation that does not fit into the classical state space framework for switched systems, where the systems are all assumed to have the same state space dimension: here $\mathfrak{B}_1$ is described by a second order differential equation and has therefore minimal state dimension equal to 2, while $\mathfrak{B}_2$ can itself be regarded as a state space system of dimension 1.

Note however that Definition 3 also includes switched state space systems as a special case.

Example 5: The switching structure $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G})$ where $\mathcal{P} = \{1,2\}$, $\mathcal{F} = \big(\mathfrak{B}_1 = \ker\,(\frac{d}{dt}I_\mathtt{n} - A_1), \mathfrak{B}_2 = \ker\,(\frac{d}{dt}I_\mathtt{n} - A_2)\big)$ and $\mathcal{G} = \{((1,2), I_\mathtt{n}, I_\mathtt{n}), ((2,1), I_\mathtt{n}, I_\mathtt{n})\}$, corresponds to a classical switched state space system without state reset, i.e. $x(t^+) = x(t^-)$ at the switching instants. Specifying, for instance, the gluing conditions $\mathcal{G} = \{((1,2), I_\mathtt{n}, S), ((2,1), I_\mathtt{n}, S^{-1})\}$, where S is an invertible matrix of size $\mathtt{n}$, yields the state resets $x(t^+) = Sx(t^-)$ (when switching from system 1 to system 2) and $x(t^+) = S^{-1}x(t^-)$ (when switching from system 2 to system 1).

III. Stability of switched behavioral systems

A set of trajectories $\mathcal{T} \subset \{w : \mathbb{R} \to \mathbb{R}^\mathtt{w}\}$ is said to be *stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathcal{T}$. This implies that none of the components of $w \in \mathcal{T}$ is free, otherwise it could be chosen as not going to zero. Therefore stable behaviors are autonomous and can hence be represented by square, nonsingular, polynomial matrices $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$, cf Theorem 1. From now on we shall consider this type of representations.

It turns out that stability of an autonomous behavior $\mathfrak{B} = \ker R(\frac{d}{dt})$ can be characterized in terms of the determinant of $R(\xi)$.

Theorem 6: ([8]) Let $\mathfrak{B} = \ker R(\frac{d}{dt})$, with $R \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ nonsingular, be an autonomous behavior. Then $\mathfrak{B}$ is stable if and only if $\det R(\lambda) \neq 0$ for all $\lambda \notin \mathbb{C}^- := \{z \in \mathbb{C} \mid Re(z) < 0\}$.

Another way of characterizing stability is in terms of the existence of a Lyapunov function. Lyapunov theory has been mainly developed for systems described by first order differential equations. However, it can be argued that in many situations a model obtained from first principles is not in first order form and it may not be an easy task to transform it in such a special representation. In the behavioral context, the definition of a Lyapunov function is based on the notion of quadratic differential form introduced in [10].

Quadratic differential forms (QDF) are mappings from $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ defined in the following way.

Let

$$\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] := \{\Phi(\zeta,\eta)\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \ : \ \Phi(\zeta,\eta)=\Phi(\eta,\zeta)^\top\}$$

denote the set of symmetric real two-variable (or 2D) $\mathtt{w}\times\mathtt{w}$ polynomial matrices. We say that $\Phi\in\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ has *order* L if it can be written as $\Phi(\zeta,\eta)=\sum_{k,\ell=0}^{L}\Phi_{k,\ell}\zeta^k\eta^\ell$ where $\Phi_{k,L}=\Phi_{L,k}$ is a nonzero matrix for some k. The QDF Q_Φ associated with $\Phi\in\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is defined as

$$\begin{array}{rcl} Q_\Phi : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) & \longrightarrow & \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \\ w & \mapsto & Q_\Phi(w)=\displaystyle\sum_{k,\ell}(\frac{d^k}{dt^k}w)^\top\Phi_{k,\ell}\frac{d^\ell}{dt^\ell}w \ . \end{array}$$

We define the *order* of a quadratic differential form Q_Φ as the order of the associated 2D symmetric polynomial matrix Φ.

Note that $\Phi(\zeta,\eta)$ can be written as

$$\begin{aligned} \Phi(\zeta,\eta) &= \begin{bmatrix} I_{\mathtt{w}} & \zeta I_{\mathtt{w}} & \cdots\zeta^L I_{\mathtt{w}}\end{bmatrix}\tilde{\Phi}\begin{bmatrix} I_{\mathtt{w}} \\ \eta I_{\mathtt{w}} \\ \vdots \\ \eta^L I_{\mathtt{w}}\end{bmatrix} \\ &= S_L^{\mathtt{w}}(\zeta)^\top\tilde{\Phi}\ S_L^{\mathtt{w}}(\eta)\ , \end{aligned} \tag{1}$$

where L is the corresponding order, $\tilde{\Phi}\in\mathbb{R}^{L\mathtt{w}\times L\mathtt{w}}$ is called the *coefficient matrix* of Φ, and $S_L^{\mathtt{w}}(\xi) := \begin{bmatrix} I_{\mathtt{w}} & \xi I_{\mathtt{w}} & \cdots I_{\mathtt{w}}\xi^L\end{bmatrix}^T$.

We say that a QDF Q_Φ is *nonnegative along* $\mathfrak{B}$, denoted $Q_\Phi\overset{\mathfrak{B}}{\geq}0$, if

$$(Q_\Phi(w))(t)\geq 0 \text{ for all } w\in\mathfrak{B} \text{ and } t\in\mathbb{R}\ .$$

When nonnegativity of a QDF Q_Φ holds for every trajectory in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ we write $Q_\Phi\geq 0$ and say that Φ (or Q_Φ) is *nonnegative definite*. Note that, Φ is nonnegative definite if and only if $\tilde{\Phi}\geq 0$. We say that Q_Φ is *positive* along $\mathfrak{B}$, denoted by $Q_\Phi\overset{\mathfrak{B}}{>}0$, if $Q_\Phi\overset{\mathfrak{B}}{\geq}0$ and $Q_\Phi(w)\equiv 0$ with $w\in\mathfrak{B}$ implies that $w\equiv 0$. Moreover, we call a QDF *positive definite* if it is positive along $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Again, Φ is positive definite if and only if $\tilde{\Phi}>0$. We define $Q_\Phi\overset{\mathfrak{B}}{<}0$, $\Phi<0$, etc. accordingly.

A *Lyapunov function* for a behavior $\mathfrak{B}$ is defined as a QDF Q_Φ such that the values of $Q_\Phi(w)$ are nonnegative and decrease with time for the trajectories $w\in\mathfrak{B}$. More concretely:

$$Q_\Phi\overset{\mathfrak{B}}{\geq}0 \quad \text{and} \quad \frac{d}{dt}Q_\Phi\overset{\mathfrak{B}}{<}0\ , \tag{2}$$

where $\frac{d}{dt}Q_\Phi$ denotes the QDF that maps $w\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to $\frac{d}{dt}(Q_\Phi w)$. It is shown in [10] that

$$\frac{d}{dt}Q_{\Phi(\zeta,\eta)}=Q_{(\zeta+\eta)\Phi(\zeta,\eta)} \tag{3}$$

We say that Q_Φ is a *common Lyapunov function* for $\mathcal{F}=(\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_N)$ if it is a Lyapunov function for every $\mathfrak{B}_j,\ j=1,\ldots,N$.

The following result establishes the equivalence between the existence of a Lyapunov function for $\mathfrak{B}$ (as defined in (2)) and the stability of $\mathfrak{B}$.

Theorem 7: ([10, Theorem 4.3]) Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ be a behavior. Then $\mathfrak{B}$ is stable if and only if there exists a Lyapunov function for $\mathfrak{B}$.

We shall say that a switching structure $\Sigma=(\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G})$ is *s*table if the set of trajectories $\mathfrak{B}^\Sigma$ is stable.

Similar to what happens in the case of switched state space systems, although necessary, the stability of each of the behaviors in the bank $\mathcal{F}$ does not guarantee the stability of $\mathfrak{B}^\Sigma$ (and hence of Σ).

Our aim is to obtain sufficient conditions for the stability of a switched structure in terms of Lyapunov functions. We shall focus on scalar behaviors, i.e., we shall consider that the number of variables is $\mathtt{w}=1$. These behaviors are particularly important in the study of stability since, as the result of Theorem 6 shows, the stability of a behavior $\mathfrak{B}=\ker R(\frac{d}{dt})$ with $\mathtt{w}$ variables is equivalent to the stability of the scalar behavior $\mathfrak{B}^{scalar}=\ker\ p(\frac{d}{dt})$, where $p(\xi)=\det R(\xi)$.

Since we now consider behaviors with one variable, the quadratic differential relevant for our purposes are associated with symmetric $2D$ polynomials $V(\zeta,\eta)\in\mathbb{R}_s[\zeta,\eta]$. Such polynomials can be written as

$$V(\zeta,\eta)=S_L^1(\zeta)^\top\ \tilde{V}\ S_L^1(\eta)\ ,$$

for some $L\in\mathbb{N}$, with $\tilde{V}$ the corresponding (symmetric) coefficient matrix. The QDF associated to $V(\zeta,\eta)$ is then

$$\begin{aligned} Q_V(w) &= (S_L^1(\frac{d}{dt})w)^\top\tilde{V}(S_L^1(\frac{d}{dt})w) \\ &= \begin{bmatrix} w & \frac{d}{dt}w & \cdots\frac{d^L}{dt^L}w\end{bmatrix}\tilde{V}\begin{bmatrix} w \\ \frac{d}{dt}w \\ \vdots \\ \frac{d^L}{dt^L}w\end{bmatrix}\ . \end{aligned}$$

Clearly, Q_V is positive definite if and only if $\tilde{V}$ is a positive definite matrix. However, as shown in the following example, in general the positivity of Q_V with respect to a behavior $\mathfrak{B}$ does not imply the positivity of $\tilde{V}$.

Example 8: Let $\mathfrak{B}=\ker p(\frac{d}{dt})$ be a behavior with $p(\xi)=\xi-1$ and

$$\begin{aligned} V(\zeta,\eta) &= -\zeta\eta+2\eta+2\zeta-1 \\ &= \begin{bmatrix}1 & \zeta\end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -1\end{bmatrix}\begin{bmatrix}1 \\ \eta\end{bmatrix}\ . \end{aligned}$$

Then

$$Q_V(w)=-(\frac{d}{dt}w)^2+4(\frac{d}{dt}w)w-w^2,$$

which is not positive, since, for instant $Q_V(w)=-w^2$ for constant trajectories w. However, for the trajectories in $\mathfrak{B}$, we have that $\frac{d}{dt}w = w$ and hence

$$Q_V(w) = 2w^2 \ .$$

Noting that the only trajectory in $\mathfrak{B}$ that takes on the value zero is the zero trajectory, this means that Q_V is positive definite in $\mathfrak{B}$.

The divergence between the positivity of Q_V and its positivity with respect to $\mathfrak{B}$ in the previous example can be explained by the fact that if $w \in \mathfrak{B}$ then the vector

$$\begin{bmatrix} w(t^*) \\ \frac{dw}{dt}(t^*) \end{bmatrix}$$

with $t^* \in \mathbb{R}$, cannot assume arbitrary values in $\mathbb{R}^2$, more concretely, it cannot assume the values $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for which

$$[\alpha \ \ \beta] \ \tilde{V} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \leq 0.$$

The situation illustrated in this example does not occur for a behavior $\mathfrak{B} = \ker\ p(\frac{d}{dt})$ if the order of the QDF Q_V is strictly less than the degree of the polynomial $p(\xi)$. In the sequel we restrict our attention to this case and show that, as for classical state space systems, the stability of a switching structure can be ensured if all the behaviors of the corresponding bank share a suitable Lyapunov function.

Using the same kind of arguments as in the proof of the theorem of Lyapunov (see for instance [4]) it is possible to show the following result.

Theorem 9: Let $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G})$ be a switching structure such that $\mathcal{P} = \{1, \ldots, N\}, N \in \mathbb{N}$, $\mathcal{F} = \{\mathfrak{B}_j, \ j \in \mathcal{P}\}$, $\mathfrak{B}_j = \ker\ p_j(\frac{d}{dt})$ with $\deg p_j(\xi) = n, \ j = 1, \ldots, N$ and

$$\mathcal{G} = \{((k,\ell), \begin{bmatrix} 1 \\ \vdots \\ \xi^{n-1} \end{bmatrix}, \begin{bmatrix} 1 \\ \vdots \\ \xi^{n-1} \end{bmatrix}), k \neq \ell \in \mathcal{P}\}. \qquad (4)$$

Then the switching structure Σ is stable if $\mathcal{F}$ has a common Lyapunov function Q_V of order $n-1$.

Note that the gluing conditions (4) correspond to requiring that the trajectories of the switched behavior are as smooth as possible, by demanding that they have continuous derivatives up to order $n-1$ at the switching instants.

In view of Theorem 9, the natural question to ask is under what conditions a bank of behaviors possesses a common Lyapunov function. A similar question has been widely investigated in the context of switched state-space systems, and several sufficient conditions have been derived in terms of the properties of the system matrices, see for instance [1], [3], [5], [7], [9]. Here we consider the case of a bank consisting of two behaviors. Results for larger banks are under investigation.

Theorem 10: Let $\mathcal{F} = (\mathfrak{B}_1, \mathfrak{B}_2)$ with $\mathfrak{B}_k = \ker\ p_k(\frac{d}{dt})$, where $p_k(\xi) \in \mathbb{R}[\xi]$, $k = 1, 2$, are Hurwitz polynomials of degree n. Assume that $\frac{p_1}{p_2}$ is strictly positive real [1]. Define

$$\Phi(\zeta, \eta) := p_1(\zeta)p_2(\eta) + p_1(\eta)p_2(\zeta).$$

Then, there exists a polynomial $r(\xi) \in \mathbb{R}[\xi]$ such that

1) $r(\xi)r(-\xi) := p_1(\xi)p_2(-\xi) + p_1(-\xi)p_2(\xi)$,
2) $$V(\zeta, \eta) := \frac{\Phi(\zeta, \eta) - r(\zeta)r(\eta)}{\zeta + \eta}$$
belongs to $\mathbb{R}_s[\zeta, \eta]$,
3) Q_V is a common Lyapunov function of order $n-1$ for $\mathcal{F}$.

Proof: The assumption that $\frac{p_1}{p_2}$ is strictly positive real implies that there exists $\epsilon > 0$ such that $\Phi(j\omega, -j\omega) = 2Re(p_1(j\omega)p_2(-j\omega) \geq \epsilon$ for all $\omega \in \mathbb{R}$. Therefore, by standard results on polynomial spectral factorization, we conclude that there exists a polynomial $r(\xi) \in \mathbb{R}[\xi]$ such that 1) holds. Hence,

$$\Phi(\xi, -\xi) - r(\xi)r(-\xi) = 0$$

which implies that the polynomial $\Phi(\zeta, \eta) - r(\zeta)r(\eta)$ is divisible by $(\zeta + \eta)$ and therefore $V(\zeta, \eta)$ is a symmetric polynomial, which proves 2). Note also that 2) implies that Q_V is a storage function for Q_Φ, since it implies that the *dissipation equality* $\frac{d}{dt}Q_V \leq Q_\Phi$ holds (see p. 1720 of [10] for more details).

It follows from [2] that the polynomial $r(\xi)$ may be chosen in such a way that it is an anti-Hurwitz polynomial, i.e., such that all its roots have positive real part. Now note that

$$\begin{aligned}\Phi(\zeta, \eta) = \quad & \frac{1}{2}[(p_1(\zeta) + p_2(\zeta))(p_1(\eta) + p_2(\eta)) \\ & -(p_1(\zeta) - p_2(\zeta))(p_1(\eta) - p_2(\eta))] \ .\end{aligned}$$

Apply Th. 2 of [6] to conclude that $p_1 + p_2$ is Hurwitz. Now use the implication $(3) \Longrightarrow (5)$ of Theorem 6.4 of [10] with $P := p_1 + p_2$ and $N := p_1 - p_2$ in order to conclude that every storage function of Q_Φ, and consequently also Q_V, is nonnegative definite.

Thus, in particular

$$Q_V \overset{\mathfrak{B}_k}{\geq} 0 \quad k = 1, 2.$$

Moreover, taking (3) into account,

$$\begin{aligned}\frac{d}{dt}Q_V(w) \quad &= \quad Q_{\Phi(\zeta,\eta) - r(\zeta)r(\eta)}(w) \\ &= \quad 2(p_1(\frac{d}{dt}w))(p_2(\frac{d}{dt})w) - (r(\frac{d}{dt})w)^2 \\ &\overset{\mathfrak{B}_k}{=} \quad -\left(r(\frac{d}{dt})w\right)^2 ,\end{aligned}$$

[1]We recall that a rational function $g = \frac{p_1}{p_2}$ is strictly positive real if and only if the following three conditions hold: (i) g has no poles s with $\mathrm{Re}(s) \geq 0$, (ii) $\mathrm{Re}(g(j\omega)) > 0$, for all $\omega \geq 0$, and (iii) $g(\infty) > 0$, or $\lim_{\omega \to \infty} \ \omega^2 \mathrm{Re}(g(j\omega)) > 0$.

showing that

$$\frac{d}{dt}Q_V(w) \overset{\mathfrak{B}_k}{\leq} 0, k=1,2.$$

Further, if there exists $w \not\equiv 0$ in $\mathfrak{B}_k$ such that

$$r(\frac{d}{dt})w \equiv 0$$

then $r(\xi)$ and $p_k(\xi)$ have a common root, which contradicts the fact that $p_k(\xi)$ is Hurwitz and $r(\xi)$ is anti-Hurwitz.

Therefore $\frac{d}{dt}Q_V(w) \overset{\mathfrak{B}_k}{<} 0$, showing that Q_V is a Lyapunov function both for $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Moreover, it is not difficult to check that the polynomial $V(\zeta,\eta)$ does contain monomials with powers in ζ or η equal to $n-1$, but not higher, and hence Q_V has order $n-1$. This proves 3). □

Remark 11: As mentioned before, Lyapunov functions are not necessarily nonnegative definite QDFs: they are only required to be nonnegative along the trajectories of the relevant behavior. However, for QDFs of order strictly smaller than the order of the differential equation describing the behavior $\mathfrak{B}$, nonnegativity along $\mathfrak{B}$ is equivalent to nonnegativity. This explains in some sense why the QDF Q_V in the previous theorem is nonnegative definite.

Theorem 10 provides a criterion for the existence of a common Lyapunov function for scalar behaviors. Since this Lyapunov function satisfies the conditions of Theorem 9 the following corollary follows readily.

Corollary 12: Let $\Sigma = (\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G})$ where $\mathcal{P}=\{1,2\}$, $\mathcal{G}$ as in (4) and $\mathcal{F} = \left(\mathfrak{B}_1 = \ker p_1(\frac{d}{dt}), \mathfrak{B}_2 = \ker p_2(\frac{d}{dt})\right)$ where $p_1(\xi),\ p_2(\xi)$ are Hurwitz polynomials of degree n. If $\frac{p_1}{p_2}$ is strictly positive real then Σ ($\mathfrak{B}^{\Sigma}$) is stable.

We illustrate these results in the following example.

Example 13: Let

$$\mathfrak{B}_1 = \ker\ (\frac{d^2}{dt^2} + 2\frac{d}{dt} + 1)$$

and

$$\mathfrak{B}_2 = \ker\ (\frac{d^2}{dt^2} + \frac{3}{2}\frac{d}{dt} + 2)$$

be two behaviors. It is not difficult to verify that both $p_1(\xi) = \xi^2+2\xi+1$ and $p_2(\xi) = \xi^2+\frac{3}{2}\xi+2$ are Hurwitz and that $\frac{p_1}{p_2}$ is strictly positive real. Hence by Theorem 9 we can construct an $r(\xi) \in \mathbb{R}[\xi]$ and a $V(\zeta,\eta) \in \mathbb{R}_s[\zeta,\eta]$ in such a way that Q_V is a Lyapunov function for $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Indeed,

$$p_1(\xi)p_2(-\xi) + p_1(-\xi)p_2(\xi) = 2\xi^4 + 2,$$
$$r(\xi) = \sqrt{2}\xi^2 - 2\cdot 2^{\frac{1}{4}}\xi + 2$$

yielding the common Lyapunov function Q_V where

$$V(\zeta,\eta) = \frac{1}{2}\left(11 + 8\cdot 2^{1/4} + (6-4\sqrt{2})(\eta+\zeta) + (7+4\cdot 2^{3/4})\eta\zeta\right).$$

Applying Theorem 9 we can conclude that the switching structure $\Sigma = (\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G})$ is stable, where $\mathcal{P}=\{1,2\}$, $\mathcal{G}$ as in (4) and $\mathcal{F} = (\mathfrak{B}_1,\mathfrak{B}_2)$.

IV. Concluding Remarks

We have introduced the notion of switched system in the framework of the behavioral approach, and have proved that, similar to what happens in the case of switched state space systems, the existence of a common behavioral Lyapunov function guarantees the stability under switching. Moreover we have given a sufficient condition for the existence of such a Lyapunov function for the case of two behaviors. The generalization to more than two behaviors is under investigation.

Although the presented results are preliminary, it is our conviction that they constitute an important step towards the foundation of a behavioral approach to switched systems.

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