

# On the Dissipativity of Pseudorational Behaviors

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**Abstract**—This paper studies dissipativity for a class of infinite-dimensional systems, called pseudorational, in the behavioral context. A basic equivalence condition for dissipativity is established as a generalization of the finite-dimensional counterpart. For its proof, we derive a new necessary and sufficient condition for entire functions of exponential type (in the Paley-Wiener class) to be symmetrically factorizable. These results play crucial roles in characterizing dissipative behaviors and LQ-optimal behaviors in pseudorational settings.

## I. INTRODUCTION

The notion of dissipativity [14], [15] is one of the most important properties in system theory. It can be viewed as a natural generalization of Lyapunov stability to open systems. Most of robust stability conditions make use of this property.

It is well known that quadratic differential forms (QDF) [18] play an important role in describing dissipativity for linear time-invariant finite-dimensional systems. For example, the theory of analysis and synthesis of dissipative systems are developed in [12], [19] using QDF's. Small gain theorems or the celebrated Popov criterion can also be deduced using such forms [17].

However, for infinite-dimensional systems, the dissipativity described by QDF's is not well explored. In this paper, based on the theory of QDF's developed in [23], we study the dissipativity of a class of infinite-dimensional systems called *pseudorational* [20], [22]. A basic equivalence condition for dissipativity is established as a generalization of the finite-dimensional counterpart. For its proof, we derive a new necessary and sufficient condition for entire functions of exponential type (in the Paley-Wiener class) to be symmetrically factorizable.

Utilizing these results, we then study the problem of characterizing dissipative behaviors with respect to a given quadratic supply rate, which is studied by [8] for finite-dimensional systems. We also give a characterization of LQ-optimal behaviors following [16] in a pseudorational setting.

This paper is organized as follows. After preparing necessary notations in Section II, we investigate the dissipativity of pseudorational behaviors in Section III. Utilizing the result of this section, we study a characterization problem of dissipative behaviors in Section IV. The LQ-control problem is also discussed in Section V.

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## II. NOTATION AND CONVENTION

The real and complex fields are denoted by  $\mathbb{R}$  and  $\mathbb{C}$ , respectively. Let  $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re } s > 0\}$  and  $\mathbb{C}_- := \{s \in \mathbb{C} : \text{Re } s < 0\}$ . For a vector space  $X$ ,  $X^n$  and  $X^{n \times m}$  denote, respectively, the space of  $n$  products of  $X$  and the space of  $n \times m$  matrices with entries in  $X$ . For a complex matrix  $M$ , its transpose is denoted by  $M^T$  and its complex conjugate transpose by  $M^*$ .

$\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$  ( $\mathcal{C}^\infty$  for short) denotes the space of  $\mathbb{R}^q$ -valued  $C^\infty$  functions on  $\mathbb{R}$ . The space of functions having compact support is denoted by  $\mathcal{D}(\mathbb{R}, \mathbb{R}^q)$  (often abbreviated as  $\mathcal{D}^q$ ). By  $\mathcal{E}'(\mathbb{R})$  we denote the space of distributions having compact support in  $\mathbb{R}$ .  $\mathcal{E}'(\mathbb{R})$  is a convolution algebra and every  $p \in \mathcal{E}'(\mathbb{R})$  acts on  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  by the action  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) : w \mapsto p * w$ . The image and kernel of the mapping are denoted by  $\text{im } p$  and  $\text{ker } p$ , respectively. For  $\tau \in \mathbb{R}$ ,  $\delta_\tau$  denotes the Dirac's delta placed on  $\tau$ . The subscript  $\tau$  is omitted when  $\tau = 0$ . Finally  $\mathcal{E}'(\mathbb{R}^2)$  denotes the space of distributions in two variables having compact support in  $\mathbb{R}^2$ .

The Laplace transform of  $p \in \mathcal{E}'(\mathbb{R})$  is defined by

$$\mathcal{L}[p](\zeta) = \hat{p}(\zeta) := \langle p, e^{-\zeta t} \rangle_t$$

where the distribution action is taken with respect to  $t$ . Similarly, for  $p \in \mathcal{E}'(\mathbb{R}^2)$ , its Laplace transform is defined by

$$\hat{p}(\zeta, \eta) := \langle p, e^{-\zeta s - \eta t} \rangle_{s,t}$$

where the action is taken with respect to two variables  $s$  and  $t$ .

By the well-known Paley-Wiener theorem [10], a distribution  $p$  belongs to  $\mathcal{E}'(\mathbb{R})$  if and only if its Laplace transform  $\hat{p}$  is an entire function of exponential type satisfying the *Paley-Wiener estimate*

$$|\hat{p}(\xi)| \leq C(1 + |\xi|)^m e^{a|\text{Re } \xi|} \quad (1)$$

for some  $C \geq 0$ ,  $a \geq 0$ , and a nonnegative integer  $m$ . We denote by  $\mathcal{PW}$  the class of entire functions satisfying the estimate above. In other words,  $\mathcal{PW} = \mathcal{L}[\mathcal{E}'(\mathbb{R})]$ .

For  $\Phi \in \mathcal{E}'(\mathbb{R}^2)^{n \times m}$ , define  $\Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{m \times n}$  and  $\partial\Phi \in \mathcal{E}'(\mathbb{R})^{n \times m}$  by  $\hat{\Phi}^*(\zeta, \eta) := \hat{\Phi}(\eta, \zeta)^T$  and  $(\partial\Phi)^\wedge(\xi) := \hat{\Phi}(-\xi, \xi)$  in the Laplace transform domain.

Let  $F$  be a  $\mathbb{C}^{m \times m}$ -valued function.  $F$  is said to be entire if each entry of  $F$  is entire. If  $F$  is entire,  $F$  is said to be of exponential type if each entry of  $F$  is of exponential type. We say that  $F$  is para-Hermitian if  $F$  equals to its para-Hermitian conjugate  $F^\sim$  defined by  $F^\sim(\xi) := \overline{F(-\xi)}^T$ .

For  $x > 0$  let  $\log^+(x) := \max\{0, \log x\}$ . For a matrix  $A$ ,  $\|A\|$  denotes its maximal singular value. In a vector space  $X$ ,  $\text{span } M$  denotes the vector subspace spanned by a subset  $M$

of  $X$ . When a functional  $f$  takes only nonnegative values, we will write  $f \geq 0$ .

### III. DISSIPATIVITY OF PSEUDORATIONAL BEHAVIORS

The characterization of dissipativity for behaviors of finite-dimensional systems [11, Theorem 4.3] has been extensively utilized in the vast literature. The theorem states that, for the behaviors that admits an image representation, dissipativity with respect to a QDF induced by a polynomial is equivalent to the existence of a storage functions or that of a dissipation function.

In this section, following [11], we study dissipativity for a class of infinite-dimensional systems, called *pseudorational* [21], with respect to the quadratic supply rate described by the QDF introduced in [23]. The QDF is induced by a distribution having compact support and is a natural extension of that induced by a polynomial.

We first introduce the notion of dissipativity, storage functions, and dissipation functions in the pseudorational setting.

*Definition 3.1:* Let  $\mathcal{B} \subset \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$  and  $Q_\Phi$  be the QDF induced by  $\Phi = \Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$ .

- The pair  $(\mathcal{B}, Q_\Phi)$  is said to be *dissipative* if

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0, \quad \forall w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q). \quad (2)$$

- The QDF  $Q_\Psi$  induced by  $\Psi = \Psi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$  is said to be a *storage function* for  $(\mathcal{B}, Q_\Phi)$  if

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w), \quad \forall w \in \mathcal{B}. \quad (3)$$

- The QDF  $Q_\Delta$  induced by  $\Delta = \Delta^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$  is said to be a *dissipation function* for  $(\mathcal{B}, Q_\Phi)$  if

$$Q_\Delta(w) \geq 0, \quad \forall w \in \mathcal{B} \quad (4)$$

and

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt = \int_{-\infty}^{\infty} Q_\Delta(w) dt, \quad \forall w \in \mathcal{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q).$$

The purpose of this section is to show a basic equivalence condition for dissipativity, as a generalization of the finite-dimensional counterpart [11, Theorem 4.3]:

*Theorem 3.2:* Let  $\mathcal{B} = \text{im } M$  be a behavior in image representation with  $M \in \mathcal{E}'(\mathbb{R})^{q \times m}$  and  $\Phi = \Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$ . Suppose that  $M$  has a left inverse in  $\mathcal{E}'(\mathbb{R})^{m \times q}$ , i.e., there exist  $M^\dagger \in \mathcal{E}'(\mathbb{R})^{m \times q}$  such that  $M^\dagger * M = \delta I_m$ . Then the following conditions are equivalent:

- 1)  $(\mathcal{B}, Q_\Phi)$  is dissipative;
- 2) Define  $\hat{\Phi}_0$  by

$$\hat{\Phi}_0(\zeta, \eta) := \hat{M}(\zeta)^\top \hat{\Phi}(\zeta, \eta) \hat{M}(\eta).$$

Then

$$\hat{\Phi}_0(-j\omega, j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}; \quad (5)$$

- 3)  $(\mathcal{B}, Q_\Phi)$  admits a storage function;
- 4)  $(\mathcal{B}, Q_\Phi)$  admits a dissipation function.

The proofs of 4)  $\Rightarrow$  3), 3)  $\Rightarrow$  1), and 1)  $\Rightarrow$  2) can be done in the same way as in the finite-dimensional case [11]. However,

to show the implication 2)  $\Rightarrow$  4) we need a special type of factorization of  $\hat{\Phi}_0(-\xi, \xi)$ , called *symmetric factorization*.

When  $\hat{\Phi}_0(-\xi, \xi)$  is a polynomial, it is well-known [2] that inequality (5) ensures the existence of such a factorization. However, in Theorem 3.2,  $\hat{\Phi}_0(-\xi, \xi)$  is not a polynomial but an *entire function*. In the next subsection we derive a new necessary and sufficient condition for the existence of symmetric factorizations over  $\mathcal{PW}$ .

#### A. Symmetric Factorization over $\mathcal{PW}$

We define the notion of symmetric factorization over  $\mathcal{PW}^{q \times q}$  as follows:

*Definition 3.3:* Let  $\Gamma \in \mathcal{PW}^{q \times q}$  be para-Hermitian.  $F \in \mathcal{PW}^{q \times q}$  is said to induce a *symmetric factorization* of  $\Gamma$  if

$$\Gamma(\xi) = F^\sim(\xi)F(\xi). \quad (6)$$

The aim of this subsection is to prove the following theorem:

*Theorem 3.4:* Let  $\Gamma \in \mathcal{PW}^{q \times q}$  be para-Hermitian.  $\Gamma$  allows a symmetric factorization if and only if

$$\Gamma(j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}. \quad (7)$$

The necessity is trivial in this theorem. For sufficiency, we begin by quoting a basic result from the factorization theory of operator valued entire functions [9]:

*Proposition 3.5 ([9, Theorem 3.6]):* Let  $\Gamma$  be a  $\mathbb{C}^{q \times q}$ -valued entire function of exponential type. Suppose that (7) holds and the integral

$$\int_{-\infty}^{\infty} \frac{\log^+ \|\Gamma(j\omega)\|}{1 + \omega^2} d\omega \quad (8)$$

is finite. Then there exists a  $\mathbb{C}^{q \times q}$ -valued entire function  $F$  of exponential type such that (6) holds and  $\det F$  has no zeros in  $\mathbb{C}_+$ .

This proposition plays a crucial role in proving Theorem 3.4. To make use of this proposition, we additionally need to show that

- 1) the integral (8) always exists for every  $\Gamma \in \mathcal{PW}^{q \times q}$ ;
- 2) the function  $F$  in Theorem 3.4 belongs to  $\mathcal{PW}^{q \times q}$  if  $\Gamma$  belongs to  $\mathcal{PW}^{q \times q}$ .

First, the existence of the integral (8) can be established by the Paley-Wiener estimate (1):

*Proposition 3.6:* The integral (8) is finite if  $\Gamma$  belongs to  $\mathcal{PW}^{q \times q}$ .

*Proof:* Let  $\Gamma$  belong to  $\mathcal{PW}^{q \times q}$ . Then each entry of  $\Gamma$  satisfies the Paley-Wiener estimate (1). From this we can easily check that the function  $\|\Gamma(\xi)\|$  also satisfies the Paley-Wiener estimate; i.e., there exist  $C > 0$ ,  $a > 0$ , and a nonnegative integer  $m$  such that

$$\|\Gamma(\xi)\| \leq C(1 + |\xi|)^m e^{a|\text{Re } \xi|}.$$

Substituting  $j\omega$  into  $\xi$  we have

$$\|\Gamma(j\omega)\| \leq C(1 + |\omega|)^m, \quad \forall \omega \in \mathbb{R}. \quad (9)$$

Integrate over  $\mathbb{R}$  the  $\log^+$  of both sides divided by  $1 + \omega^2$ . Then we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\log^+ \|\Gamma(j\omega)\|}{1 + \omega^2} d\omega \\ & \leq \int_{-\infty}^{\infty} \frac{\log^+ C}{1 + \omega^2} d\omega + m \int_{-\infty}^{\infty} \frac{\log(1 + |\omega|)}{1 + \omega^2} d\omega \\ & \leq \pi \log^+ C + 3m. \end{aligned}$$

Hence the integral (8) is finite.  $\blacksquare$

We then show that, in Proposition 3.5, if  $\Gamma$  belongs to  $\mathcal{PW}^{q \times q}$  then the function  $F$  also belongs to  $\mathcal{PW}^{q \times q}$ :

*Proposition 3.7:* Let  $\Gamma$  and  $F$  be  $\mathbb{C}^{q \times q}$ -valued entire functions of exponential type. Suppose that (6) holds. If  $\Gamma$  belongs to  $\mathcal{PW}^{q \times q}$ , then  $F$  also belongs to  $\mathcal{PW}^{q \times q}$ .

*Proof:* See Appendix.  $\blacksquare$

We are now ready to prove Theorem 3.4:

*Proof of Theorem 3.4:* The necessity is obvious. We prove the sufficiency. Let  $\Gamma \in \mathcal{PW}^{q \times q}$  be para-Hermitian and assume that (7) holds. Since the integral (8) is finite by Proposition 3.6, Proposition 3.5 ensures the existence of a  $\mathbb{C}^{q \times q}$ -valued entire function  $F$  satisfying (6). This function  $F$  actually belongs to  $\mathcal{PW}^{q \times q}$  by Proposition 3.7. This completes the proof of the theorem.  $\blacksquare$

Before closing this subsection, we refer to a more special type of symmetric factorizations, called *symmetric (anti)-Hurwitz factorization*. These factorizations play a key role, for example, in examining the existence of positive storage functions for finite-dimensional systems [18].

*Definition 3.8:* Suppose that  $F \in \mathcal{PW}^{q \times q}$  induces a symmetric factorization (6) for  $\Gamma \in \mathcal{PW}^{q \times q}$ . The factorization is said to be a *symmetric Hurwitz factorization* if  $\det F(\lambda) = 0$  implies  $\operatorname{Re} \lambda < 0$  and a *symmetric anti-Hurwitz factorization* if  $\det F(\lambda) = 0$  implies  $\operatorname{Re} \lambda > 0$ .

The next theorem is an extension of the result given in [2]:

*Theorem 3.9:* Let  $\Gamma \in \mathcal{PW}^{q \times q}$  be para-Hermitian.  $\Gamma$  allows both a symmetric Hurwitz factorization and a symmetric anti-Hurwitz factorization if and only if

$$\Gamma(j\omega) > 0, \quad \forall \omega \in \mathbb{R}.$$

*Proof:* The statement on the symmetric Hurwitz factorization is trivial because, in Theorem 3.2,  $\det F$  already has no zeros in  $\mathbb{C}_+$ . A symmetric anti-Hurwitz factorization can then be obtained from a symmetric Hurwitz factorization of  $\Gamma^\top$ .  $\blacksquare$

### B. Proof of the Main Result

Having established Theorem 3.4, we can proceed to the proof of the main result Theorem 3.2.

*Proof:* We run the cycle 2)  $\Rightarrow$  4)  $\Rightarrow$  3)  $\Rightarrow$  1)  $\Rightarrow$  2).

2)  $\Rightarrow$  4): By Theorem 3.4 there exists  $F_0 \in \mathcal{PW}^{m \times m}$  that induces a symmetric factorization for  $\hat{\Phi}_0(-\xi, \xi)$ . Define  $\hat{\Delta}_0(\zeta, \eta) := \overline{F_0(\bar{\zeta})}^\top F_0(\eta)$  and  $\hat{\Delta}(\zeta, \eta) := \hat{M}^\dagger(\zeta)^\top \hat{\Delta}_0(\zeta, \eta) \hat{M}^\dagger(\eta)$ . Clearly (4) holds. Noting that  $\hat{\Phi}_0(-j\omega, j\omega) = \hat{\Delta}_0(-j\omega, j\omega)$ , by Parseval's identity

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(-j\omega)^* \hat{\Phi}(-j\omega, j\omega) \hat{w}(j\omega) d\omega \quad (10)$$

that holds for all  $\ell \in \mathcal{D}^m$  and  $\Phi = \Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{m \times m}$ , we have  $\int Q_{\Phi_0 - \Delta_0}(\ell) dt = 0$  for all  $\ell \in \mathcal{D}^m$  and hence  $\int Q_{\Phi - \Delta}(w) dt = 0$  for all  $w \in \mathcal{B} \cap \mathcal{D}^q$ .

4)  $\Rightarrow$  3): Suppose that  $(\mathcal{B}, Q_\Phi)$  admits the dissipation function  $Q_\Delta$  with  $\Delta = \Delta^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$ . Defining  $\hat{\Delta}_0(\zeta, \eta) := \hat{M}(\zeta)^\top \hat{\Delta}(\zeta, \eta) \hat{M}(\eta)$  we have

$$\int_{-\infty}^{\infty} Q_{\Phi_0 - \Delta_0}(\ell) dt = 0, \quad \forall \ell \in \mathcal{D}^m.$$

By [23, Theorem 6.2] there exists  $\Psi_0 = \Psi_0^* \in \mathcal{E}'(\mathbb{R}^2)^{m \times m}$  such that

$$\frac{d}{dt} Q_{\Psi_0}(\ell) = Q_{\Phi_0 - \Delta_0}(\ell) \leq Q_{\Phi_0}(\ell)$$

for all  $\ell \in (\mathcal{C}^\infty)^m$ . Let  $\hat{\Psi}(\zeta, \eta) := \hat{M}^\dagger(\zeta)^\top \hat{\Psi}_0(\zeta, \eta) \hat{M}^\dagger(\eta)$ . Then, by the image representation  $\mathcal{B} = \operatorname{im} M$ ,  $Q_\Psi$  gives a storage function for  $(\mathcal{B}, Q_\Phi)$ .

3)  $\Rightarrow$  1): Let  $\Psi = \Psi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$  induce a storage function for  $(\mathcal{B}, Q_\Phi)$ . Then the integration of (3) for  $w \in \mathcal{B} \cap \mathcal{D}^q$  readily yields (2) and hence  $(\mathcal{B}, Q_\Phi)$  is dissipative.

1)  $\Rightarrow$  2): We can show its contraposition by using Parseval's identity (10). For the detail, see the appendix.  $\blacksquare$

### C. Example

1) *Acoustic waves in a duct:* Let us study the dissipativity of the acoustic waves in the duct (see, for example, [3]) modeled by the following wave equation

$$\frac{1}{c^2} \frac{\partial p}{\partial t} = -\rho_0 \frac{\partial v}{\partial x}, \quad \rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}$$

where  $c > 0$  is the speed of sound,  $\rho_0 > 0$  is the air density,  $p$  is the pressure in the duct, and  $v$  is the particle velocity. Let  $L$  be the length of the duct. Under the constant impedance condition  $p(L, t) = Zv(L, t)$  at the open end, the transfer function from  $v_0 = v(0, \cdot)$  to  $p_0 = p(0, \cdot)$  is given by

$$G(s) = \rho_0 c \frac{1 + \alpha e^{-2L\frac{s}{c}}}{1 - \alpha e^{-2L\frac{s}{c}}} \quad (11)$$

where

$$\alpha = \frac{Z - \rho_0 c}{Z + \rho_0 c}.$$

We show that the behavior of  $\begin{bmatrix} v_0 & p_0 \end{bmatrix}^\top$  admits an image representation  $\mathcal{B} = \operatorname{im} M$  with  $M$  having a left inverse in  $\mathcal{E}'$ . By (11) we have

$$\mathcal{B} = \ker \begin{bmatrix} \rho_0 c(\delta + \alpha \delta_{2L/c}) & -(\delta - \alpha \delta_{2L/c}) \end{bmatrix}.$$

Since both distributions  $\delta + \alpha \delta_{2L/c}$  and  $\delta - \alpha \delta_{2L/c}$  yield surjections on  $\mathcal{C}^\infty$  via convolution [5, Theorem 2.5], we can actually show that  $\mathcal{B}$  admits an image representation  $\mathcal{B} = \operatorname{im} M$  with

$$M = \begin{bmatrix} \delta - \alpha \delta_{2L/c} \\ \rho_0 c(\delta + \alpha \delta_{2L/c}) \end{bmatrix}$$

that has a left inverse  $\begin{bmatrix} \delta & \delta/\rho_0 c \end{bmatrix} / 2 \in \mathcal{E}'(\mathbb{R})^{1 \times 2}$ .

Regarding the product  $v_0 p_0$  as the energy supply rate, we check the dissipativity of the pair  $(\mathcal{B}, Q_\Phi)$ . Defining

$$\Phi := \frac{1}{2} \begin{bmatrix} 0 & \delta \otimes \delta \\ \delta \otimes \delta & 0 \end{bmatrix}$$

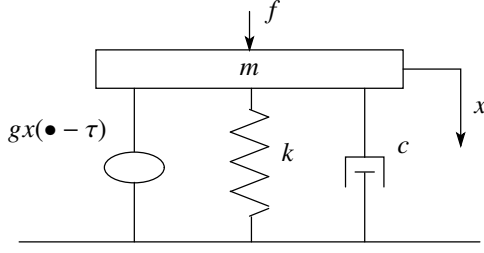


Fig. 1. Delayed resonator

we have  $Q_\Phi(w) = v_0 p_0$ . Then an easy calculation gives  $\hat{\Phi}_0(-j\omega, j\omega) = \rho_0 c(1 - \alpha^2)$ . Hence, from Theorem 3.2, the system is dissipative if and only if  $-1 \leq \alpha \leq 1$  or, equivalently,

$$Z \geq 0.$$

2) *Delayed resonator*: Let us consider the mechanical system depicted in Fig. 1. In this figure,  $m > 0$  denotes the mass,  $k \geq 0$  the spring constant, and  $c > 0$  the damping coefficient.  $f$  is the force applied to the mass and  $x$  is the relative position of the mass from the equilibrium.  $gx(t - \tau)$  represents a delayed feedback with  $g \geq 0$ . Such a feedback is used in, for example, delayed resonators [7].

Since the dynamics of the system can be written by the equation  $m\ddot{x} = f - kx - c\dot{x} - gx(\cdot - \tau)$ , the set of all the trajectories taken by  $w := \begin{bmatrix} x & f \end{bmatrix}^\top$  admits a kernel representation

$$\mathcal{B} = \ker \begin{bmatrix} m\delta'' + c\delta' + k\delta + g\delta_\tau & -\delta \end{bmatrix},$$

which clearly admits an image representation  $\mathcal{B} = \text{im } M$  with

$$M = \begin{bmatrix} \delta \\ m\delta'' + c\delta' + k\delta + g\delta_\tau \end{bmatrix}$$

having a left inverse  $\begin{bmatrix} \delta & 0 \end{bmatrix} \in \mathcal{E}'(\mathbb{R})^{1 \times 2}$ . Now let

$$\Phi := \frac{1}{2} \begin{bmatrix} 0 & \delta' \otimes \delta \\ \delta \otimes \delta' & 0 \end{bmatrix}.$$

Then  $Q_\Phi(w) = f\dot{x}$  represents the mechanical energy supplied to the mass.

We check the dissipativity of the pair  $(\mathcal{B}, Q_\Phi)$ . A straightforward computation gives

$$\hat{\Phi}_0(-j\omega, j\omega) = \omega(c\omega - g \sin(\tau\omega)).$$

From this equation we can see that  $(\mathcal{B}, Q_\Phi)$  is dissipative if and only if  $(c\omega - g \sin(\tau\omega)) \geq 0$  for all  $\omega \geq 0$ . This condition can be shown to be equivalent to

$$g\tau \leq c.$$

#### IV. CHARACTERIZATION OF DISSIPATIVE BEHAVIORS

Theorem 3.2 answers the question when a given behavior is dissipative with respect to a given quadratic supply rate. Then there naturally arises the following question: *given a quadratic supply rate, can one characterize all the behavior that is dissipative with respect to the given quadratic supply rate?* Theorem 3.2 enables us to answer this question.

For any nonnegative integers  $m$  and  $n$ , define the matrix  $J_{mn}$  by  $J_{mn} := \text{diag}(I_m, -I_n)$ . We omit subscripts  $m$  and  $n$  when they are unimportant. Given  $\Gamma \in \mathcal{P}\mathcal{W}^{q \times q}$ , a nonsingular matrix  $K \in \mathcal{P}\mathcal{W}^{q \times q}$  is said to induce a *J-spectral factorization* of  $\Gamma$  if  $\Gamma(\xi) = K(-\xi)^\top J_{mn} K(\xi)$ .

Extending [8, Theorem 3.2] for finite dimensional behaviors, we can give a characterization of dissipative pseudorational behaviors.

*Theorem 4.1*: Let  $M \in \mathcal{E}'(\mathbb{R})^{q \times m}$  and  $\Phi = \Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$ . Suppose that  $\hat{K} \in \mathcal{P}\mathcal{W}^{q \times q}$  induces the *J-spectral factorization*  $\hat{\Phi}(-\xi, \xi) = \hat{K}(-\xi)^\top J \hat{K}(\xi)$ . Then the following statements are true:

- 1)  $(\text{im } M, Q_\Phi)$  is dissipative if and only if  $(\text{im}(K * M), Q_J)$  is dissipative.
- 2) Let  $L := \text{cofac } K$ . If the mapping  $\mathcal{C}^\infty \rightarrow \mathcal{C}^\infty : f \mapsto (\det K) * f$  is surjective, then  $(\text{im } M, Q_J)$  is dissipative if and only if  $(\text{im}(L * M), Q_\Phi)$  is dissipative.

*Proof*: 1): Let  $M_0 := K * M$ . Then we can easily obtain

$$M^\top(-j\omega)\Phi(-j\omega, j\omega)M(j\omega) = M_0(-j\omega)^\top J M_0(j\omega).$$

From this equation and since 1)  $\Leftrightarrow$  2) of Theorem 3.2 holds without the invertibility assumption, the statement follows.

- 2): Since  $\det K$  induces a surjection on  $\mathcal{C}^\infty$ ,

$$\begin{aligned} \text{im } M &= M * ((\det K) * (\mathcal{C}^\infty)^d) \\ &= (\det K) * M * (\mathcal{C}^\infty)^d \\ &= K * L * M * (\mathcal{C}^\infty)^d = K * \text{im}(L * M). \end{aligned}$$

This implies that  $(\text{im } M, Q_J)$  is dissipative if and only if  $(K * \text{im}(L * M), Q_J)$  is dissipative. This is equivalent to saying, from the first statement, that  $(\text{im}(L * M), Q_\Phi)$  is dissipative.  $\blacksquare$

*Remark 4.2*: The surjectivity of the convolution mapping induced by  $\text{cofac } K$  can be checked by, for example, [4, Theorem 2.5], which states that a distribution of type  $\sum_{l=1}^N \alpha_l \delta_{\tau_l}^{(k_l)}$  with  $\alpha_l, \tau_l \in \mathbb{R}$  and nonnegative integers  $k_l$  always induces a surjection.

#### V. LQ-CONTROL

Following [16], we study pseudorational LQ-optimal behaviors utilizing Theorem 3.9. Let  $\mathcal{B} \in (\mathcal{C}^\infty)^q$  be a behavior and  $\Phi = \Phi^* \in \mathcal{E}'(\mathbb{R}^2)^{q \times q}$ . Define for each  $w \in \mathcal{B}$  and  $\Delta \in \mathcal{B} \cap \mathcal{D}^q$  the *cost-degradation* [16],  $J_{\Phi, w}(\Delta)$ , as

$$J_{\Phi, w}(\Delta) := \int_{-\infty}^{\infty} Q_\Phi(w + \Delta) - Q_\Phi(w) dt.$$

Now define the *optimal behavior* as  $\mathcal{B}_{\text{opt}} := \{w \in \mathcal{B} : J_{\Phi, w} \geq 0, \lim_{t \rightarrow \infty} w(t) = 0\}$ . A characterization of the optimal behavior for finite-dimensional systems is given in [16].

As in Theorem 3.2, we consider only the behaviors in image representation  $\text{im } M$  with a left-invertible  $M \in \mathcal{E}'(\mathbb{R})^{q \times m}$ . In the similar way as in [16], the invertibility of  $M$  actually enables us to reduce the problem to the special case of  $M = I$ . Hence hereafter we assume  $M = I$ .

First we state an analogue of [16, Proposition 1]. The proof can be done in the same way and hence is omitted.

*Proposition 5.1:* There exists  $w \in \mathcal{B}$  such that  $J_{\Phi,w} \geq 0$  if and only if

$$\hat{\Phi}(-j\omega, j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

Furthermore, under this condition we have

$$\{w \in \mathcal{B} : J_{\Phi,w} \geq 0\} = \ker \partial\Phi. \quad (12)$$

Using this proposition we will give an estimate for the optimal behavior. Before stating the result, we need to examine the structure of the space  $\ker \partial\Phi$ . For simplicity, we assume that all the multiplicities of the zeros of  $\det \partial\hat{\Phi}$  are equal to 1 and all the derivatives of  $\partial\hat{\Phi}$  are nonsingular at each zero of  $\det \partial\hat{\Phi}$ . We call functions of type  $p(t)e^{\lambda t}$  with  $p$  being polynomial in  $t$  as *polynomial-exponential* functions.

*Proposition 5.2:* Let  $T \in \mathcal{E}'(\mathbb{R})^{q \times q}$  and suppose that the multiplicities of the zeros of  $\det \hat{T}$  are equal to 1 and all the derivatives of  $\hat{T}$  are regular at each zero of  $\det \hat{T}$ . Then

$$\ker T = \overline{\text{span}\{e^{\lambda t} v : \hat{T}(\lambda)v = 0\}}$$

where the closure is taken with respect to the topology of  $(\mathcal{C}^\infty)^q$ , i.e., that of uniform convergence in all derivatives on every compact set.

*Proof:* Since the linear mapping  $f \mapsto T * f$  on  $(\mathcal{C}^\infty)^q$  for  $f \in (\mathcal{C}^\infty)^q$  is continuous, its kernel  $\ker T$  is a closed linear subspace of  $(\mathcal{C}^\infty)^q$ . Moreover  $\ker T$  is clearly shift-invariant. Therefore  $\ker T$  is spanned by the polynomial-exponential functions it contains [6].

Thus it suffices to show that any exponential-polynomial function belonging to  $\ker T$  can be written as  $e^{\lambda t} v$  with  $\lambda \in \mathbb{C}$  and  $v \in \mathbb{C}^q$  satisfying  $\hat{T}(\lambda)v = 0$ . Suppose that a polynomial-exponential function  $f(t) := p(t)e^{\lambda t}$  belongs to  $\ker T$ . Then, in the same way as in [23, Lemma 8.1], we have

$$0 = (T * f)(t) = \left( \sum_{k=0}^N \frac{\hat{T}^{(k)}(\lambda)}{k!} p^{(k)}(t) \right) e^{\lambda t} \quad (13)$$

where  $N$  is the degree of  $p$  as a polynomial.

Then we can show  $N = 0$ . Note that the multiplicity of the eigenvalue 0 of  $\hat{T}(\lambda)$  equals 1 and  $\hat{T}'(\lambda)$  is nonsingular. Let us write  $p(t) = \sum_{k=0}^N p_k t^k$  with  $p_k \in \mathbb{C}^q$ . It suffices to show that  $N \geq 1$  implies  $p_N = 0$ . Suppose  $N \geq 1$ . Since  $0 = \hat{T}(\lambda)p_N = \hat{T}(\lambda)p_{N-1} + N\hat{T}'(\lambda)p_N$  from (13), we have  $p_{N-1} \in \ker A^2$  where  $A := \hat{T}'(\lambda)^{-1}\hat{T}(\lambda)$ . Since the multiplicity of the eigenvalue 0 of  $A$  is equal to 1, we have  $p_{N-1} \in \ker A$ , which can be shown to be equivalent to  $p_N = 0$ .

Let  $p(t) =: v \in \mathbb{C}^q$ . Substituting this into (13) we have  $0 = \hat{T}(\lambda)v e^{\lambda t}$ . Hence  $\hat{T}(\lambda)v = 0$ . ■

This proposition leads us to the following result:

*Theorem 5.3:* Suppose that  $\hat{H} \in \mathcal{P}\mathcal{W}^{q \times q}$  induces the symmetric Hurwitz factorization as  $\hat{\Phi}(-\xi, \xi) = \hat{H}^*(\xi)\hat{H}(\xi)$ . Then

$$\mathcal{B}_{\text{opt}} \subset \ker H.$$

*Proof:* From Proposition 5.2,

$$\ker \partial\Phi = \overline{\text{span}\{e^{\lambda t} v : \partial\hat{\Phi}(\lambda)v = 0\}}. \quad (14)$$

Since  $\hat{H}$  induces a symmetric Hurwitz factorization for  $\partial\hat{\Phi}$ , we can show

$$\ker H = \overline{\text{span}\{e^{\lambda t} v : \partial\hat{\Phi}(\lambda)v = 0, \lambda \in \mathbb{C}_-\}}. \quad (15)$$

Let us write  $(\mathcal{C}_s^\infty)^q := \{w \in (\mathcal{C}^\infty)^q : \lim_{t \rightarrow \infty} w(t) = 0\}$ . Then (12) and (14) yields  $\mathcal{B}_{\text{opt}} = (\mathcal{C}_s^\infty)^q \cap \overline{\text{span}\{e^{\lambda t} v : \partial\hat{\Phi}(\lambda)v = 0\}}$ . Since in general  $A \cap \bar{B} \subset \overline{A \cap B}$  for open subsets  $A$  and  $B$  in a topological space [13],

$$\begin{aligned} \mathcal{B}_{\text{opt}} &\subset \overline{(\mathcal{C}_s^\infty)^q \cap \text{span}\{e^{\lambda t} v : \partial\hat{\Phi}(\lambda)v = 0\}} \\ &= \overline{\text{span}\{e^{\lambda t} v : \partial\hat{\Phi}(\lambda)v = 0, \lambda \in \mathbb{C}_-\}} = \ker H. \end{aligned}$$

where the last equation follows from (15). ■

## VI. CONCLUSION

We have studied dissipativity for a class of infinite-dimensional systems, called pseudorational, in the behavioral context. We have established a basic equivalence condition for dissipativity as a generalization of the finite-dimensional counterpart. For its proof, we derived a new necessary and sufficient condition for entire functions in the Paley-Wiener class to be symmetrically factorizable. Using these results, we then studied the characterizations of dissipative behaviors and LQ-optimal behaviors in pseudorational settings.

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## APPENDIX

### A. Proof of Proposition 3.7

We start with the following lemma which states that the growth rate of entire functions of exponential type can be governed by those on real and imaginary axes:

*Lemma A.1 ([1, Theorem 6.2.4]):* Let  $f$  be a complex function defined at least on the closed right half plane. Suppose that following conditions hold:

- 1)  $f$  is holomorphic on  $\mathbb{C}_+$ ;
- 2)  $f$  is of exponential type on  $\mathbb{C}_+$ ; i.e., there exist  $K \geq 0$  and  $\tau \geq 0$  such that  $|f(\xi)| \leq Ke^{\tau|\xi|}$  for all  $\xi \in \mathbb{C}_+$ .
- 3)  $|f(j\omega)|$  is bounded as a function of  $\omega \in \mathbb{R}$ .

Then there exists  $M > 0$  such that

$$|f(\xi)| \leq Me^{\tau \operatorname{Re} \xi}, \quad \forall \xi \in \overline{\mathbb{C}_+}.$$

With this lemma, we can prove the next lemma that enables us to judge whether or not a given entire function of exponential type belongs to  $\mathcal{PW}$  from the growth rate of the function on the imaginary axis:

*Lemma A.2:* Let  $f$  be an entire function of exponential type. Suppose that there exist  $C > 0$  and a nonnegative integer  $m$  such that

$$|f(j\omega)| \leq C(1 + |\omega|)^m, \quad \forall \omega \in \mathbb{R}. \quad (16)$$

Then  $f$  belongs to  $\mathcal{PW}$ .

*Proof:* Let  $f$  be an entire function of exponential type. Suppose that there exist  $C > 0$  and a nonnegative integer  $m$  satisfying (16). Define a meromorphic function  $f_0$  by

$$f_0(\xi) := \frac{f(\xi)}{(\xi + 1)^m}.$$

We check that  $f_0$  satisfies the assumptions of Lemma A.1. Since  $f$  is an entire function of exponential type, clearly  $f_0$  is defined on the closed right half plane, and holomorphic and of exponential type on  $\mathbb{C}_+$ . From (16) we can check that  $f_0$  is bounded on the imaginary axis.

Then by Lemma A.1 there exist  $M > 0$  and  $\tau > 0$  such that  $|f_0(\xi)| < Me^{\tau \operatorname{Re} \xi}$ ,  $\forall \xi \in \overline{\mathbb{C}_+}$ . Therefore

$$|f(\xi)| = |f_0(\xi)|(1 + \xi)^m < M(1 + |\xi|)^m e^{\tau \operatorname{Re} \xi}, \quad \forall \xi \in \overline{\mathbb{C}_+}.$$

This is nothing but the Paley-Wiener estimate (1) on the closed right half plane. In the similar way, we can show the Paley-Wiener estimate on the closed left half plane using the left-half plane version of Lemma A.1. Combining these estimates, we obtain a Paley-Wiener estimate of  $f$  on the entire complex plane. Hence  $f$  belongs to  $\mathcal{PW}$ . ■

Now we can prove Proposition 3.7:

*Proof of Proposition 3.7:* Let  $f$  be any entry of  $F$ . We show that  $f$  belongs to  $\mathcal{PW}$ . Since  $F$  is of exponential type,  $f$  is also of exponential type. Hence, by Lemma A.2, it is sufficient to show that there exist  $C > 0$  and a nonnegative integer  $m$  satisfying (16).

From the definition of the norm for matrices, there exists a constant  $M > 0$  such that

$$|f(j\omega)| \leq M\|F(j\omega)\|, \quad \forall \omega \in \mathbb{R}. \quad (17)$$

Since (6) holds from the assumption, we have

$$\|F(j\omega)\|^2 = \|\Gamma(j\omega)\|, \quad \forall \omega \in \mathbb{R}. \quad (18)$$

From inequalities (17), (18), and (9), we can obtain the estimate of type (16) as follows:

$$|f(j\omega)| \leq MC^{1/2}(1 + |\omega|)^{m/2}, \quad \forall \omega \in \mathbb{R}.$$

This completes the proof. ■

### B. Proof of 1) $\Rightarrow$ 2) in Theorem 3.2

We prove the implication 1)  $\Rightarrow$  2) in Theorem 3.2 by showing its contraposition. Suppose that there exists  $\omega_0 \in \mathbb{R}$  such that  $\hat{\Phi}_0(-j\omega_0, j\omega_0) < 0$ . Then there exists  $v \in \mathbb{C}^q$  such that

$$v^* \hat{\Phi}_0(-j\omega_0, j\omega_0)v < 0. \quad (19)$$

Take any  $\rho \in \mathcal{D}(\mathbb{R}, \mathbb{R})$  with

$$\hat{\rho}(0) \neq 0. \quad (20)$$

For a positive integer  $N$  define

$$w_N := \rho * \frac{(e^{j\omega_0 t})|_{[-N, N]}}{\sqrt{2N}} v.$$

Using Parseval's identity (10) we can obtain

$$\int_{-\infty}^{\infty} Q_{\Phi_0}(w_N) dt = \int_{-\infty}^{\infty} f(\omega) \frac{N}{\pi} \operatorname{sinc}^2(N\omega) d\omega, \quad (21)$$

where  $\operatorname{sinc} \omega := \omega^{-1} \sin(\omega)$  and

$$f(\omega) := v^* \hat{\rho}(j\omega) \partial \hat{\Phi}_0(j\omega + j\omega_0) \hat{\rho}(j\omega)v$$

We show that  $f$  belongs to the space  $\mathcal{S}$  of testing functions of rapid descent. First  $\hat{\rho}$  belongs to  $\mathcal{S}$  because  $\mathcal{S}$  is invariant under the Fourier transform [13]. Second, the growth rate of  $\partial \hat{\Phi}_0(j \cdot)$  is at most that of polynomials because  $\partial \hat{\Phi}_0$  satisfies the Paley-Wiener estimate (1). Therefore  $f$  belongs to  $\mathcal{S}$ .

Because  $(N/\pi) \operatorname{sinc}^2(N\omega)$  converges to  $\delta$  as  $N$  goes to  $\infty$  with respect to the topology of  $\mathcal{S}'$  [13], the right hand side of (21) converges to  $f(0) = |\hat{\rho}(0)|^2 v^* \partial \hat{\Phi}_0(j\omega_0)v$ , which is negative from (19) and (20). Therefore there exists  $w_N$  such that  $\int_{-\infty}^{\infty} Q_{\Phi_0}(w_N) dt < 0$  and hence  $(\mathcal{B}, Q_{\Phi_0})$  is not dissipative. ■