Terminals and Ports

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***Abstract*—The behavioral approach to dynamical systems is applied to electrical circuits. This offers an attractive way to introduce circuits pedagogically. An electrical circuit is a device that interacts with its environment through wires, called terminals. Associated with each terminal, there are two variables, a potential and a current. Interconnection is viewed as terminals that share their potential and their current after interconnection. A *port* is a set of terminals that satisfies port-KCL. If terminals $\{1,2,\ldots,p\}$ form a port, and V_k denotes the potential and I_k the current at the k-th terminal, then we define the power that flows into the circuit at time t along these p terminals as $V_1(t)I_1(t)+V_2(t)I_2(t)+\cdots+V_p(t)I_p(t)$, and the energy that flows into the circuit along these p terminals during the time-interval $[t_1,t_2]$ as $\int_{t_1}^{t_2}\big(V_1(t)I_1(t)+V_2(t)I_2(t)+\cdots+V_p(t)I_p(t)\big)\,dt$. These expressions for power and energy are not valid unless the set of terminals forms a port. We conclude that terminals are for interconnection, and ports are for energy transfer. We formulate a theorem stating that a connected RLC circuit forms a 1-port.**

I. Introduction

The aim of this article is to explain the distinction that should be made in physical systems between interconnection of systems on the one hand, and energy transfer between systems on the other hand. Interconnection happens via terminals, while energy transfer happens via ports. We consider systems that interact through terminals, as wires for electrical circuits.

We use the behavioral approach [8] as a pedagogically attractive way to discuss mathematical models and dynamical systems, and, in particular, electrical circuits.

II. Circuits

We view an electrical circuit as a device, a black-box, with wires, called terminals, through which the circuit can interact with its environment (see Figure 1). The interaction takes place

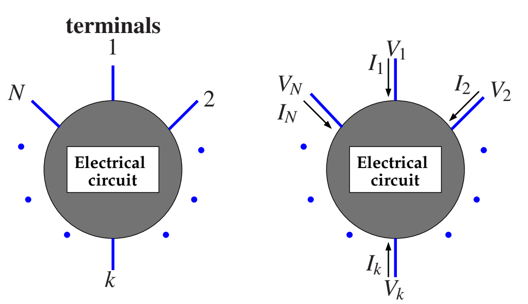

Fig. 1. An electrical circuit

through two real variables, *a potential and a current*, at each terminal. Throughout, the current is counted positive when it flows into the circuit. For the basic concepts of circuit theory, see [2] and [3]. The setting developed in [5] and [6] has the same flavor as our approach.

An N-terminal electrical circuit is a dynamical system $\Sigma = (\mathbb{R},\mathbb{R}^{2N},\mathscr{B})$, with time axis $\mathbb{R}$, signal space $\mathbb{R}^{2N}$, and behavior $\mathscr{B}$ a subset $\mathscr{B}\subseteq(\mathbb{R}^{2N})^{\mathbb{R}}$; $(V,I)\in\mathscr{B}$ means that the time-function $(V,I)=(V_1,V_2,\ldots,V_N,I_1,I_2,\ldots,I_N):\mathbb{R}\to\mathbb{R}^N\times\mathbb{R}^N$ is compatible with the architecture and the element values of the circuit.

Circuit properties are defined in terms of the behavior.

- A circuit obeys *Kirchhoff's voltage law* (KVL) if $(V_1,\ldots,V_N,I_1,\ldots,I_N)\in\mathscr{B}$ and $\alpha:\mathbb{R}\to\mathbb{R}$ imply $(V_1+\alpha,\ldots,V_N+\alpha,I_1,\ldots,I_N)\in\mathscr{B}$.
- A circuit obeys *Kirchhoff's current law* (KCL) if $(V_1,\ldots,V_N,I_1\ldots,I_N)\in\mathscr{B}$ implies $I_1+\cdots+I_N=0$.
- A circuit is *linear* if $\mathscr{B}$ is a linear subspace of $(\mathbb{R}^{2N})^{\mathbb{R}}$.

A circuit obeys KVL if the behavioral equations contain only the differences V_i-V_j for $i,j\in\{1,2,\ldots,N\}$. KVL means that the potentials are defined up to an arbitrary additive constant (that may change in time). KCL means that the circuit stores no net charge. Linearity means that the superposition principle holds.

The behavior of the classical linear circuit elements are defined by equations. For the 2-terminal elements, we have

$$\begin{array}{lll} \textit{resistor:} & V_1-V_2=RI_1, & I_1+I_2=0,\\ \textit{capacitor:} & C\dfrac{d}{dt}(V_1-V_2)=I_1, & I_1+I_2=0,\\ \textit{inductor:} & V_1-V_2=L\dfrac{d}{dt}I_1, & I_1+I_2=0,\end{array}$$

while for the 4-terminal elements, we have

transformer:

$$V_1-V_2=n(V_3-V_4),\ nI_1+I_3=0,\ I_1+I_2=0,\ I_3+I_4=0,$$

gyrator:

$$V_1-V_2=gI_3,\ V_3-V_4=-gI_1,\ I_1+I_2=0,\ I_3+I_4=0.$$

The *transistor* is a 3-terminal element. Denote the terminals by $\{e,c,b\}$. In the case of a *pnp* transistor, the behavioral equations are of the form $I_e=f_e(V_e-V_b,V_c-V_b), I_c=f_c(V_e-V_b,V_c-V_b)$, $I_e+I_c+I_b=0$. An n-terminal *connector* is an element with equations

$$V_1=V_2=\cdots=V_n,\qquad I_1+I_2+\cdots+I_n=0.$$

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III. Interconnection

We view interconnection as connecting terminals, like soldering wires together. Assume that we have a circuit with N terminals. Connecting terminals $N-1$ and N, as shown in Figure 2, leads to imposing the relations

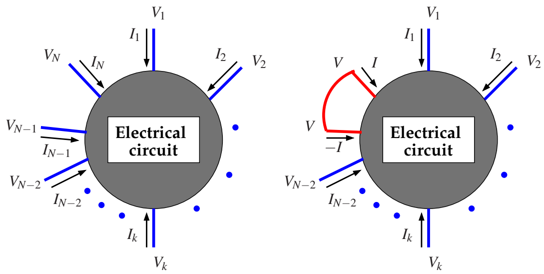

Fig. 2. Terminal connection

$$\boxed{V_{N-1}=V_N, \quad I_{N-1}+I_N=0.}$$

This implies that after interconnection the terminals share the variables V_{N-1},V_N, and I_{N-1},I_N (up to a sign). Interconnection is therefore *variable sharing*. The interconnected circuit has $N-2$ terminals. Its behavior is

$$\mathscr{B}'=\{(V_1,I_1,V_2,I_2,\ldots,V_{N-2},I_{N-2}):\mathbb{R}\to\mathbb{R}^{2(N-2)} \mid \exists V,I \text{ such that } (V_1,I_1,V_2,I_2,\ldots,V_{N-2},I_{N-2},V,I,V,-I)\in\mathscr{B}\}.$$

Once we have defined the connection of two terminals of the same circuit, we obtain what happens when we connect two terminals of two different circuits, or many terminals of many circuits (see Figure 3).

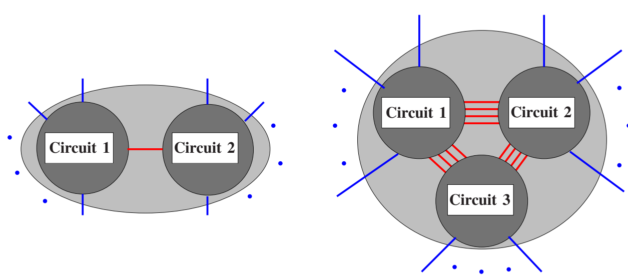

Fig. 3. Interconnection of circuits.

Interconnection preserves many circuit properties. In particular, if $\mathscr{B}$ obeys KVL, or KCL, or is linear, then so does $\mathscr{B}'$.

IV. Ports

In this section, we introduce a notion that is essential to the energy exchange of a circuit with its environment and between circuits. Consider an N-terminal circuit, and single out p terminals, which we take to be the first p terminals.

The set of terminals $\{1,2,\ldots,p\}$ forms a *port* :$\Leftrightarrow$

$$(V_1,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N)\in\mathscr{B} \Rightarrow \boxed{I_1+I_2+\cdots+I_p=0.}$$

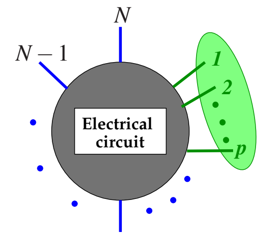

Fig. 4. Port

We call this relation *port-KCL*.

KCL implies that all the terminals combined form a port, and if terminals $\{1,2,\ldots,p\}$ form a port, then so do terminals $\{p+1,p+2,\ldots,N\}$. If terminals $\{1,2,\ldots,p\}$ form a port, then we call this set of terminals a p-terminal port. If the circuit terminals are partitioned into the ports $\{1,\ldots,p_1\},\{p_1+1,\ldots,p_1+p_2\},\ldots,\{p_1+\cdots+p_{k-1}+1,\ldots,p_1+\cdots+p_{k-1}+p_k=N\}$, then we call the circuit a k-port consisting of p_1-, $\ldots$, p_k-terminal ports.

If the set of terminals $\{1,2,\ldots,p\}$ form a port, then we define the *power* that flows into the circuit at time t along these p terminals to be equal to

$$\boxed{\text{power} \;=\; V_1(t)I_1(t)+V_2(t)I_2(t)+\cdots+V_p(t)I_p(t),}$$

and the *energy* that flows into the circuit along these p terminals during the time-interval $[t_1,t_2]$ to be equal to

$$\boxed{\text{energy} \;=\; \int_{t_1}^{t_2}\left(V_1(t)I_1(t)+V_2(t)I_2(t)+\cdots+V_p(t)I_p(t)\right)dt.}$$

These formulas for power and energy are not valid *unless these terminals form a port!* In particular, it is not possible to speak about the energy that flows into the circuit along a single wire — a conclusion that is physically evident. Power and energy flow are not 'local' physical entities, but they involve 'action at a distance', they require more than one terminal.

Resistors, capacitors, and inductors are 2-terminal 1-ports. Transformers and gyrators are 2-terminal 2-ports. Terminals $\{1,2\}$ and $\{3,4\}$ of a transformer and a gyrator form 1-ports, and the energy that flows into the port $\{1,2\}$ is equal to the energy that flows out of the port $\{3,4\}$. A transistor is a 3-terminal 1-port, and a connector that connects n terminals is an n-terminal 1-port. A 2-terminal circuit that consists of the interconnection of circuits that all satisfy KCL forms a 1-port, since KCL is preserved under interconnection. In particular, a 2-terminal circuit that is composed of resistors, capacitors, inductors, transformers, gyrators, connectors, etc. forms a port. However, a pair of terminals of a circuit with more than two terminals rarely form a port.

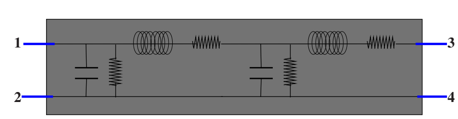

Fig. 5. A transmission line

For the circuit shown in Figure 5, the terminals $\{1,2,3,4\}$ form a port, but there is no reason why the terminal pairs $\{1,2\}$ and $\{3,4\}$ should form ports. In particular, it is not possible to discuss the relation between the energy that flows from the terminals $\{1,2\}$ to the terminals $\{3,4\}$.

In order to make the terminal pairs $\{1,2\}$ and $\{3,4\}$ of the transmission line in Figure 5 into ports, one can add unit transformers, as shown in Figure 6.

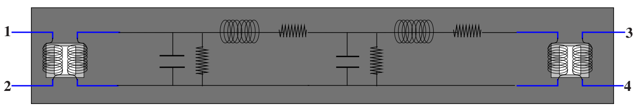

Fig. 6. A transmission line with unit transformers

V. Internal ports

In order to study the energy flow inside a circuit, we introduce in this section circuits with both external and internal terminals. In many circumstances, a model of an electrical circuit is needed that describes not only the behavior of the potential and current on the external terminals, but also the behavior of certain variables inside the circuit. These internal variables could in principle be any combination or function of internal variables, but for simplicity we assume that we are concerned with the potential and current on certain of the internal terminals, as shown in Figure 7.

Assume that there are N external terminals, $\{1,2,\ldots,N\}$, and N' internal terminals, $\{1',2',\ldots,N'\}$. The dynamic laws of the circuit define a dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^{2(N+N')}, \mathscr{B})$, where $(V,I) \in \mathscr{B}$ means that the time-function $(V,I) = (V_1,\ldots,V_N,V_{1'},\ldots,V_{N'},I_1,\ldots,I_N,I_{1'},\ldots,I_{N'}) : \mathbb{R} \to \mathbb{R}^{2(N+N')}$ is compatible with the architecture of the circuit and the values of the elements in the circuit.

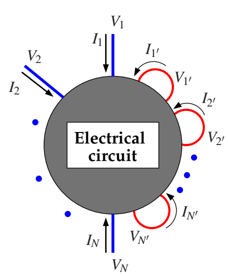

Fig. 7. A circuit with external and internal terminals.

Terminals $\{1',2',\ldots,N'\}$ forms an *internal port* $:\Leftrightarrow$

$$(V_1,\ldots,V_N,V_{1'},\ldots,V_{N'},I_1,\ldots,I_N,I_{1'},\ldots,I_{N'}) \in \mathscr{B}$$
$$\Rightarrow \boxed{I_{1'} + I_{2'} + \cdots + I_{N'} = 0.}$$

A circuit has in general *external ports*, consisting of only external terminals, *internal ports*, consisting of only internal terminals, and *mixed ports*, consisting of both external and internal terminals. The internal ports allow to consider the power and energy flow between internal parts of a circuit. For example, it is possible to consider the energy transferred into the terminals $\{1,2\}$ and $\{3,4\}$ of the circuit shown in Figure 8, since these pairs of terminals form internal ports.

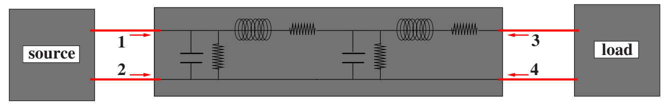

Fig. 8. A terminated transmission line

In addition to power and energy flow along ports, one can consider *port-like trajectories*, that is elements of $\mathscr{B}$ which have $I_1(t) + I_2(t) + \cdots + I_p(t) = 0$ for $t \in [t_1,t_2]$. In this case, rather than port-KCL being implied by $\mathscr{B}$ as a consequence of properties of the circuit, it is imposed on $\mathscr{B}$, for example by the intended termination that one has in mind. In such situations, it is legitimate to discuss power and energy flow along a set of terminals. This situation is frequently encountered in the classical literature, for example in the theory of N-ports.

VI. Terminals are for interconnection, ports for energy transfer

As explained before, interconnection means that certain terminals share the same potential and current (up to a sign). This is distinctly different from stating that power or energy flows from one side of an interconnection to the other side. Power and energy involve ports, and this requires consideration of more than one terminal at the time. For example, the two

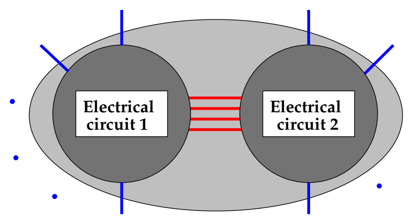

Fig. 9. Interconnected circuits

circuits in Figure 9 share four terminals, but it is not possible to speak of the energy that flows from circuit 1 to circuit 2, unless the connected terminals form internal ports, and it is not possible to speak about the energy that flows from the environment into circuit 1, or from the environment into circuit 2, unless the external terminals of circuit 1 and of circuit 2 form ports (or the behavior is restricted to port-like behavior). Of course KVL and KCL imply that the complete set of external terminals of the interconnected system forms a port.

Setting up behavioral equations of a circuit involves interconnection and variable sharing. Exchange of power and energy involves ports. Interconnections need not involve ports or power and energy transfer. These observations put into perspective power-based modeling methodologies of interconnected systems, as bond graphs [7], [4]. In [8] we propose a modeling methodology for interconnected systems based on *tearing, zooming, and linking*, which involves interconnection by sharing variables, but in which power considerations do not take a central place.

VII. CIRCUITS WITH 2-TERMINAL PORTS

A $2N$-terminal circuit that consists of N pairs of terminals $\{1,2\},\{3,4\},\ldots,\{2N-1,2N\}$ that all form 1-ports can be described in terms of $2N$ port variables, N voltages and N currents, instead of $4N$ terminal variables, $2N$ potentials and $2N$ currents. The $2N$-terminal circuit $\Sigma = \left(\mathbb{R},\mathbb{R}^{4N},\mathscr{B}\right)$ leads to the N-port circuit $\Sigma_{\text{port}} = \left(\mathbb{R},\mathbb{R}^{2N},\mathscr{B}_{\text{port}}\right)$ with behavior

$$\begin{aligned}\mathscr{B}_{\text{port}} := \{&(V_1,V_2,\ldots,V_N,I_1,I_2,\ldots,I_N) : \mathbb{R} \to \mathbb{R}^{2N} \mid \\ &(V_1,0,V_2,0,\ldots,V_N,0,I_1,-I_1,I_2,-I_2,\ldots,I_N,-I_N) \in \mathscr{B}\}.\end{aligned}$$

The port description is more parsimonious than the terminal description, since it involves only half as many variables.

Classical circuit theory has focusses on elements that have 2-terminal ports (resistors, capacitors, inductors, transformers, gyrators), interconnected by connectors. The architecture of such circuits can be nicely described in terms of digraphs with leaves, with 2-terminal ports in the edges, connectors in the vertices, and leaves for the external terminals.

Unfortunately, the set-up used in classical circuit theory has serious drawbacks. To begin with, important circuit elements, as transistors, do not consist of 2-terminal ports. Also, the interconnection of 2-terminal ports does not lead to circuits with 2-terminal ports. Such circuits need not even have an even number of terminals. This is illustrated by circuits, as Y's or Δ's, that have 3 external terminals. The terminal approach for the description of circuits is much more suited for hierarchical modeling [8] than the port description, even though, as we have seen, the latter leads to models that are more parsimonious in terms of the number of variables needed.

Often, the port structure studied does not result from the architecture and the element values of the circuit, but from the intended use of terminal pairs as 2-terminal 1-ports. In other words, we are not dealing with proper ports, but with port-like behavior. Unfortunately, the classical literature is often quite confusing about this point.

VIII. TWO THEOREMS

We end this article with some concrete results regarding ports. Informally, the first theorem states that a connected RLC circuit forms a 1-port. In order to have a multiport, we need to use multiport building blocks, as transformers or gyrators. In order to make this into a precise result, we need to introduce a bit of graph theory.

A *graph with leaves* is defined as $\mathscr{G} = (\mathbb{V},\mathbb{E},\mathbb{L},f_{\mathbb{E}},f_{\mathbb{L}})$, with $\mathbb{V}$ a finite set of vertices, $\mathbb{E}$ a finite set of edges, $\mathbb{L}$ a finite set of leaves, $f_{\mathbb{E}}$ the edge incidence map, and $f_{\mathbb{L}}$ the leaf incidence map. $f_{\mathbb{E}}$ maps each element $e \in \mathbb{E}$ into an unordered pair $[v_1,v_2]$, with $v_1,v_2 \in \mathbb{V}$, and $f_{\mathbb{L}}$ is a map from $\mathbb{L}$ to $\mathbb{V}$. A path from leaf $\ell_1 \in \mathbb{L}$ to leaf $\ell_2 \in \mathbb{L}$ is a sequence $(\ell_1,v_1,e_1,v_2,e_2,\ldots,v_n,e_n,v_{n+1},\ell_2)$ with $v_m \in \mathbb{V}$ for $m = 1,2,\ldots,n+1$, $e_m \in \mathbb{E}$ for $m = 1,2,\ldots,n$, ℓ_1 incident to v_1, v_m, v_{m+1} incident to e_m for $m = 1,2,\ldots,n+1$, and ℓ_2 incident to v_{n+1}. A graph with leaves is said to have *connected leaves* if there is a path between every pair of leaves.

For RLC circuits, the architecture is defined by a graph with leaves $\mathscr{G} = (\mathbb{V},\mathbb{E},\mathbb{L},f_{\mathbb{E}},f_{\mathbb{L}})$. The leaves correspond to external terminals, and the vertices to connectors. Associate with each edge a positive linear resistor, or a positive linear capacitor, or a positive linear inductor.

Theorem: Assume that the graph with leaves that defines the architecture of a linear passive N-terminal RLC circuit has connected leaves. Then this circuit has no ports other than the one consisting of the complete set of terminals $\{1,2,\ldots,N\}$.

In [9], this theorem is proven for resistive circuits. The Bott-Duffin synthesis result immediately leads to a generalization of this theorem to circuits with arbitrary linear passive impedances in the edges.

A second result involves the relation between port-KCL and port-KVL. Consider an N-terminal circuit, and single out the first p terminals. The set of terminals $\{1,2,\ldots,p\}$ satisfies *port-KVL*

$$\begin{aligned}&:\Leftrightarrow [\![(V_1,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N) \in \mathscr{B} \\ &\text{and } \alpha : \mathbb{R} \to \mathbb{R}]\!] \quad \Rightarrow \quad [\![(V_1+\alpha,\ldots,V_p+\alpha,V_{p+1},\ldots,V_N, \\ &\qquad\qquad I_1,\ldots,I_p,I_{p+1},\ldots,I_N) \in \mathscr{B}.\end{aligned}$$

This condition is equivalent to asking that the behavioral equations contain the variables V_i for $i \in \{1,2,\ldots,p\}$ only through the differences $V_i - V_j$ for $i,j \in \{1,2,\ldots,p\}$.

Theorem: For a linear passive circuit port-KCL $\Leftrightarrow$ port-KVL.

The notion of a port may be generalized to other types (mechanical, thermal, hydraulic) of systems. The generalization to mechanical systems is especially interesting due to the fact that a mass does not form a port.

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