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Path Integrals and Bézoutians for Pseudorational Transfer Functions

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Abstract— **There is an effective way of constructing a Lyapunov function without recourse to a state space construction. This is based upon an integral of special type called a path integral, and this approach is particularly suited for behavior theory. The theory successfully exhibits a deep connection between Lyapunov theory and Bézoutians, but it remained mostly in the finite-dimensional context. This paper extends the theory to a class of systems described by a wider class of transfer functions called *pseudorational*, which contains an interesting class of distributed parameter systems, e.g., delay systems. The paper extends the notion of path integrals using an convolution algebra of distributions, and then relates this theory to an infinite-dimensional version of Bézoutians, which in turn gives rise to a new interesting class of Lyapunov functions.**

I. Introduction

It is well known and appreciated that Lyapunov theory plays a key role in stability theory of dynamical systems. Lyapunov functions defined on the state space are central tools in linear and nonlinear system theory.

It is perhaps less appreciated that there is an effective way of constructing a Lyapunov function and discussing stability without recourse to a state space formalism. This approach is based upon an integral of special type, called *path integral*. Given a dynamical system and trajectories associated with it, an integral is said to be a path integral if its value is independent of the trajectory except that it depends only on its values (including derivatives) at the end points.

This leads to an elegant theory of constructing Lyapunov functions for linear systems; it was developed in late 60s by R.W. Brockett [1]. This approach had been somewhat forgotten for quite some time since then, but recently new light is shed on this approach in the behavioral context [4], [5]. The approach is particularly suitable for behavioral theory, and it provides a basis-free approach for the general theory of stability and Lyapunov functions.

So far the theory has only been developed for finite-dimensional systems for various technical reasons. Recently, the authors developed a new framework for studying behaviors for infinite-dimensional systems [12] in the context of pseudorational transfer functions. This class of systems is described as the kernel of a convolution operator as $\{w : p*w = 0\}$ with p a distribution with compact support. Delay systems, retarded or neutral, or systems with bounded impulse response can be well handled by this class (see, e.g., [9]), and provides a suitable framework for generalizing path integrals and related Lyapunov theory; see also [13].

II. Notation and Nomenclature

$\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ ($\mathscr{C}^\infty$ for short) is the space of C^∞ functions on $(-\infty,\infty)$. Similarly for $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ with higher dimensional codomains. $\mathscr{D}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ denote the space of $\mathbb{R}^{\mathtt{q}}$-valued C^∞ functions having compact support in $(-\infty,\infty)$. $\mathscr{D}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is its dual, the space of distributions. $\mathscr{D}'_+(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is the subspace of $\mathscr{D}'$ with support bounded on the left. $\mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ denotes the space of distributions with compact support in $(-\infty,\infty)$. $\mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is a convolution algebra and acts on $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ by the action: $p* : \mathscr{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathscr{C}^\infty(\mathbb{R},\mathbb{R}) : w \mapsto p*w$. $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ is a module over $\mathscr{E}'$ via this action. Similarly, $\mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$ denotes the space of distributions in two variables having compact support in $\mathbb{R}^2$. For simplicity of notation, we may drop the range space $\mathbb{R}^{\mathtt{q}}$ and write $\mathscr{E}'(\mathbb{R})$, etc., when no confusion is likely,

A distribution α is said to be of *order at most m* if it can be extended as a continuous linear functional on the space of m-times continuously differentiable functions. Such a distribution is said to be of *finite order*. The largest number m, if one exists, is called the *order* of α ([2], [3]). The delta distribution δ_a ($a \in \mathbb{R}$) is of order zero, while its derivative δ'_a is of order one, etc. A distribution with compact support is known to be always of finite order ([2], [3]).

The Laplace transform of $p \in \mathscr{E}'(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ is defined by

$$\mathscr{L}[p](\zeta) = \hat{p}(\zeta) := \langle p, e^{-\zeta t}\rangle_t \tag{1}$$

where the action is taken with respect to t. Likewise, for $p \in \mathscr{E}'(\mathbb{R}^2,\mathbb{R}^{\mathtt{q}})$, its Laplace transform is defined by

$$\hat{p}(\zeta,\eta) := \langle p, e^{-(\zeta s+\eta t)}\rangle_{s,t} \tag{2}$$

where the distribution action is taken with respect to two variables s and t. For example, $\mathscr{L}[\delta''_s \otimes \delta'_t] = \zeta^2 \cdot \eta$.

By the well-known Paley-Wiener theorem [2], [3], $\hat{p}(\zeta)$ is an entire function of exponential type satisfying the *Paley-Wiener estimate*

$$|\hat{p}(\zeta)| \le C(1+|\zeta|)^r e^{a|\operatorname{Re}\zeta|} \tag{3}$$

for some $C, a \geq 0$ and a nonnegative integer r.

Likewise, for $p \in \mathscr{E}'(\mathbb{R}^2, \mathbb{R}^{\mathtt{q}})$, there exist $C, a \geq 0$ and a nonnegative integer r such that its Laplace transform

$$|\hat{p}(\zeta,\eta)| \leq C(1+|\zeta|+|\eta|)^r e^{a(|\mathrm{Re}\,\zeta|+|\mathrm{Re}\,\eta|)}. \quad (4)$$

This is also a sufficient condition for a function $\hat{p}(\cdot,\cdot)$ to be the Laplace transform of a distribution in $\mathscr{E}'(\mathbb{R}^2, \mathbb{R}^{\mathtt{q}})$. We denote by $\mathscr{PW}$ the class of functions satisfying the estimate above for some C, a, m. In other words, $\mathscr{PW} = \mathscr{L}[\mathscr{E}']$.

Other spaces, such as L^2, L^2_{loc} are all standard. For a vector space X, $X^{\mathtt{n}}$ and $X^{\mathtt{n}\times\mathtt{m}}$ denote, respectively, the spaces of $\mathtt{n}$ products of X and the space of $\mathtt{n}\times\mathtt{m}$ matrices with entries in X. When a specific dimension is immaterial, we will simply write $X^{\bullet}$ $X^{\bullet\times\bullet}$.

III. Quadratic Differential Forms

In the classical context, path integrals and quadratic differential forms are studied over the ring of polynomials in two variables $\mathbb{R}[\zeta,\eta]$ [4], [5]. Consider the symmetric two-variable polynomial matrix $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$, where $\Phi^*[\zeta,\eta] := \Phi^T[\eta,\zeta]$, with coefficient matrices as $\Phi(\zeta,\eta) = \sum_{k,\ell}\Phi_{k,\ell}\zeta^k\eta^\ell$. The quadratic differential form (QDF for short) $Q_\Phi : (\mathscr{C}^\infty)^q \to (\mathscr{C}^\infty)^q$ is defined by

$$Q_\Phi(w) := \sum_{k,\ell}\left(\frac{d^k}{dt^k}\overline{w}\right)^T \Phi_{k,\ell}\left(\frac{d^\ell}{dt^\ell}w\right).$$

For example, $\Phi = (\zeta+\eta)/2$ yields the QDF $Q_\Phi = w(dw/dt)$.

Observing this example, we notice that we can view Φ as the Laplace transform of two-variable distributions $(\delta'_s \otimes \delta_t + \delta_s \otimes \delta'_t)/2$ where δ'_s denotes the derivative of the delta distribution in the variable s, and likewise for δ'_t, δ_s, δ_t, etc.; $\alpha_s \otimes \beta_t$ denotes the tensor product of two distributions α and β. (In fact, $\mathscr{L}[\delta'_s] = \zeta$, and $\mathscr{L}[\delta'_t] = \eta$.)

Generalizing this, we can easily extend the definition above to tensor products of distributions in variables s and t, and then to distributions $\Phi \in \mathscr{E}'(\mathbb{R}^2)$. Indeed, if $\Phi = \alpha_s \otimes \beta_t$, $\alpha, \beta \in \mathscr{E}'(\mathbb{R})$

$$Q_\Phi(w) = (\overline{w} * \alpha)\cdot(\beta * w),$$

and extend linearly for the elements of form $\sum_{k,\ell}\alpha^i_s \otimes \beta^j_t$. Since $\mathscr{E}'(\mathbb{R}) \otimes \mathscr{E}'(\mathbb{R})$ is dense in $\mathscr{E}'(\mathbb{R}^2)$ (cf., [3]), we can extend this definition to the whole of $\mathscr{E}'(\mathbb{R}^2)$. Finally, for the matrix case, we apply the definition above to each entries.

In short, given $\Phi \in \mathscr{E}'(\mathbb{R}^2, \mathbb{R}^{\mathtt{q}})$,

$$\Phi(v,w) = \overline{v}_s * \Phi * w_t \quad (5)$$

where the convolution from the left is taken with respect to the variable s while that on the right is taken with respect to t. For example, $\overline{v} * (\sum \alpha_k \otimes \beta_\ell) * w = \sum_{k,\ell}(\overline{v} * \alpha_k)_s(\beta_\ell * w)_t$. This gives a bilinear mapping from $(\mathscr{C}^\infty)$ to $(\mathscr{C}^\infty)$. Then the *quadratic differential form* Q_Φ associated with Φ is defined by

$$Q_\Phi(w) := \Phi(w,w) = \overline{v}_s * \Phi * w_t|_{s=t}. \quad (6)$$

Given $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q}\times\mathtt{q}}$ such that $\Phi^* = \Phi$, we define the *quadratic differential form* $Q_\Phi : (\mathscr{C}^\infty)^{\mathtt{q}} \to (\mathscr{C}^\infty)^{\mathtt{q}}$ associated with Φ by

$$Q_\Phi(w) := \Phi(w,w) = (\overline{w}_s * \Phi * w_t)|_{s=t} \quad (7)$$

as a function of a single variable $t \in \mathbb{R}$.

Example 3.1: Define $\Phi := (1/2)[\delta'_s \otimes \delta_t + \delta_s \otimes \delta'_t]$. Then $\Phi(v,w) = (1/2)[(dv/ds)(s)\cdot w(t) + v(s)\cdot(dw/dt)(t)]$ and $Q_\Phi(w) = (1/2)[(dw/dt)(t)\cdot w(t) + w(t)\cdot(dw/dt)(t)]$.

Example 3.2: For $\Phi := \delta''_{-1} \otimes \delta'_{-1}$,

$$Q_\Phi(w) = \Phi(w,w) = \frac{d^2w}{dt^2}(t+1)\cdot\frac{dw}{dt}(t+1).$$

A. Basic Operations on $\mathscr{E}'(\mathbb{R}^2, \mathbb{R}^{\mathtt{q}})$ or $\mathscr{PW}$

Let $P \in (\mathscr{E}')(\mathbb{R}^2)^{\mathtt{n}_1\times\mathtt{n}_2}$. Define $P^\sim \in (\mathscr{E}')^{n_2\times n_1}$ by

$$P^\sim := (\overline{\check{P}})^T \quad (8)$$

where $\check{\alpha}$ is defined by

$$\langle\check{\alpha},\phi\rangle := \langle\alpha,\phi(-\cdot)\rangle, \alpha \in \mathscr{E}', \phi \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}).$$

Hence for $\hat{P} \in (\mathscr{PW})^{n_1\times n_2}$, $\hat{P}^\sim(\zeta) = (\check{P}^T)\hat{} = (P\hat{)}(\zeta) = \hat{P}^T(-\zeta)$.

For $\hat{P} \in \mathscr{PW}^{\bullet\times\bullet}[\zeta,\eta]$, $\hat{P}^*(\zeta,\eta) := \hat{P}^T(\eta,\zeta)$. Also,

$$\hat{P}^{\bullet}(\zeta,\eta) := (\zeta+\eta)\hat{P}(\zeta,\eta).$$

In the (s,t)-domain, this corresponds to

$$\overset{\bullet}{P} = (\delta'_s * P) + (\delta'_t * P) = \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)P. \quad (9)$$

$$\partial\hat{P}(\xi) := \hat{P}(-\xi,\xi).$$

For an element P of type $P = \alpha_s \otimes \beta_t$, this means

$$\partial P = \check{\alpha}_t \otimes \beta_t.$$

The formula for the general case is obtained by extending this linearly.

We note the following lemma for the expression $\hat{\Phi}(\zeta,\eta)/(\zeta+\eta)$ to belong to the class $\mathscr{PW}$:

Lemma 3.3: Let $f \in (\mathscr{PW})^{\bullet\times\bullet}$. $f(\zeta,\eta)/(\zeta+\eta)$ belongs to the class $\mathscr{PW}$ if and only if $\partial f = 0$, i.e., $f(-\xi,\xi) = 0$.

Proof If $f(\zeta,\eta) = (\zeta+\eta)g(\zeta,\eta)$ for some entire function g, then clearly $f(-\xi,\xi) = 0$ for every ξ. Conversely, suppose $f(-\xi,\xi) = 0$ for every ξ. For each $\eta \in \mathbb{C}$, define

$$f_\eta(\zeta) := f(\zeta,\eta).$$

Then by $f(-\xi,\xi) = 0$, $f_\eta(\zeta)$ has a factor $(\zeta+\eta)$, and can be written as $f_\eta(\zeta) = (\zeta+\eta)g_\eta(\zeta)$. The analyticity of g in ζ and in η follows from that of f. Write $g_\eta(\zeta)$ as $g(\zeta,\eta)$. We must show the Paley-Wiener estimate (4) for g. Since f satisfies (4), we have, for $|\zeta+\eta| \geq 1$, that

$$|g(\zeta,\eta)| = \left|\frac{f(\zeta,\eta)}{\zeta+\eta}\right| \leq C(1+|\zeta|+|\eta|)^r e^{a(|\mathrm{Re}\,\zeta|+|\mathrm{Re}\,\eta|)}, \quad (10)$$

because $|\zeta+\eta| \geq 1$. If we show the same type of estimate for the region $|\zeta+\eta| \leq 1$, the proof would be complete. Now fix any $\zeta \in \mathbb{C}$, and consider the region $D_\zeta := \{\eta : |\eta+\zeta| \leq 1\}$, whose boundary is $\gamma_\zeta = \{\eta : |\eta+\zeta| = 1\}$. According to (10), $|g(\zeta,\eta)| \leq C(1+|\zeta|+|\eta|)^r e^{a(|\mathrm{Re}\,\zeta|+|\mathrm{Re}\,\eta|)}$ on γ_ζ. Then by the maximum modulus principle, $g(\zeta,\eta)$ satisfies the same estimate in the region D_ζ. Since ζ is arbitrary, it satisfies the estimate (10) irrespective of $|\zeta+\eta| \leq 1$ or not. This shows the Paley-Wiener estimate for g, and the claim is proved. □

The following lemma is a direct consequence of the definition of $\overset{\bullet}{\Psi}$:

Lemma 3.4: For $\Psi \in \mathscr{E}'(\mathbb{R}^2, \mathbb{R}^{\mathtt{q}})^{\bullet\times\bullet}$,

$$\frac{d}{dt} Q_\Psi = Q_{\overset{\bullet}{\Psi}}.$$

Proof Consider $\alpha_s \otimes \beta_t$, and consider the action $w \mapsto (w*\alpha)\cdot(\beta*w)$. According to (9), differentiation of this yields $(w*(d\alpha/ds))\cdot(\beta*w) + (w*\alpha)\cdot((d\beta/dt)*w)|_{s=t} = (w*\delta_s'*\alpha\cdot(\beta*w) + (w*\alpha)\cdot((\delta_t'*\beta)*w)|_{s=t} = Q_{\overset{\bullet}{\Psi}}(w)$. Extend linearly and then also extend continuously to complete the proof. □

IV. Path Integrals

The integral

$$\int_{t_1}^{t_2} Q_\Phi(w)dt \tag{11}$$

(or briefly $\int Q_\Phi$) is said to be *independent of path*, or simply a *path integral* if it depends only on the values taken on by w and its derivatives at end points t_1 and t_2 (*but not on the intermediate trajectories between them*).

The following theorem gives equivalent conditions for Φ to give rise to a path integral.

Theorem 4.1: Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q}\times\mathtt{q}}$, and Q_Φ the quadratic differential form associated with Φ. The following conditions are equivalent:

(i) $\int Q_\Phi$ is a path integral;
(ii) $\partial\Phi = 0$;
(iii) $\int_{-\infty}^{\infty} Q_\Phi(w)dt = 0$ for all $w \in \mathscr{D}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$;
(iv) the expression $\hat{\Phi}(\zeta,\eta)/(\zeta+\eta)$ belongs to the class $\mathscr{PW}$.
(v) there exists a two-variable matrix $\Psi \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q}\times\mathtt{q}}$ that defines a Hermitian bilinear form on $(\mathscr{C}^\infty)^{\mathtt{q}} \otimes (\mathscr{C}^\infty)^{\mathtt{q}}$ such that

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) \tag{12}$$

for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$.

Proof

(i) $\Rightarrow$ **(iii)** is trivial by taking t_1 and t_2 outside the support of w.

(ii) $\Leftrightarrow$ **(iii)** is obvious from Parseval's identity

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{w}}^T(-i\omega)\hat{\Phi}(-i\omega, i\omega)\hat{w}(i\omega)d\omega.$$

(The implication (iii) $\Rightarrow$ (ii) requires a technical argument that nonvanishing $\hat{\Phi}$ for some ω_0 yields a nonzero integral on the right for some w, but this follows from a standard real analysis argument.)

(ii) $\Leftrightarrow$ **(iv)** This is obvious from Lemma 3.3.

(iv) $\Leftrightarrow$ **(v)** follows trivially from Lemma 3.4.

(v) $\Rightarrow$ **(i)** is trivial. □

V. Pseudorational Behaviors

We review a few rudiments of pseudorational behaviors as given in [12].

Definition 5.1: Let R be an $\mathtt{p}\times\mathtt{w}$ matrix ($\mathtt{w} \geq \mathtt{p}$) with entries in $\mathscr{E}'$. It is said to be *pseudorational* if there exists a $\mathtt{p}\times\mathtt{p}$ submatrix P such that

1) $P^{-1} \in \mathscr{D}_+'(\mathbb{R})$ exists with respect to convolution;
2) $\mathrm{ord}(\det P^{-1}) = -\mathrm{ord}(\det P)$, where $\mathrm{ord}\,\psi$ denotes the order of a distribution ψ [2], [3] (for a definition, see the Appendix).

Definition 5.2: Let R be pseudorational as defined above. The *behavior* $\mathscr{B}$ defined by R is given by

$$\mathscr{B} := \{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) : R*w = 0\} \tag{13}$$

The convolution $R*w$ is taken in the sense of distributions. Since R has compact support, this convolution is always well defined [2].

Remark 5.3: We here took $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ as the signal space in place of $L^2_{loc}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ in [12], but the basic structure remains intact.

A state space formalism is possible for this class and it yields various nice properties as follows:

Suppose, without loss of generality, that R is partitioned as $R = \begin{bmatrix} P & Q \end{bmatrix}$ such that P satisfies the invertibility condition of Definition 5.1, i.e., we consider the kernel representation

$$P*y + Q*u = 0 \tag{14}$$

where $w := \begin{bmatrix} y & u \end{bmatrix}^T$ is partitioned conformably with the sizes of P and Q.

A nice consequence of pseudorationality is that this space X is always a closed subspace of the following more tractable space X^P:

$$X^P := \{x \in (L^2_{[0,\infty)})^{\mathtt{p}} \,|\, P*x|_{[0,\infty)} = 0\}, \tag{15}$$

and it is possible to give a realization using X^P as a state space. The state transition is generated by the left shift semigroup:

$$(\sigma_\tau x)(t) := x(t+\tau)$$

and its infinitesimal generator A determines the spectrum of the system ([6]). We have the following facts concerning the spectrum, stability, and coprimeness of the representation $\begin{bmatrix} P & Q \end{bmatrix}$ ([6], [7], [8], [9]):

Facts 5.4: 1) The spectrum $\sigma(A)$ is given by

$$\sigma(A) = \{\lambda \,|\, \det\hat{P}(\lambda) = 0\}. \tag{16}$$

Furthermore, every $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. The corresponding eigenfunction for

$\lambda \in \sigma(A)$ is given by $e^{\lambda t} v$ where $\hat{P}(\lambda)v = 0$. Similarly for generalized eigenfunctions such as $te^{\lambda t} v'$.

2) The semigroup σ_t is exponentially stable, i.e., satisfies for some $C, \beta > 0$

$$\|\sigma_t\| \leq Ce^{-\beta t}, \quad t \geq 0,$$

if and only if there exists $\rho > 0$ such that

$$\sup\{\operatorname{Re}\lambda : \det \hat{P}(\lambda) = 0\} \leq -\rho.$$

VI. Path Integrals along a Behavior

Generalizing the results of Section IV on path integrals in the unconstrained case, we now study path integrals *along a behavior* $\mathscr{B}$.

Definition 6.1: Let $\mathscr{B}$ be the behavior (13) with pseudorational R. The integral $\int Q_\Phi$ is said to be *independent of path* or a *path integral along* $\mathscr{B}$ if the path independence condition holds for all $w_1, w_2 \in \mathscr{B}$.

Let $\mathscr{B}$ be as above. We assume that $\mathscr{B}$ also admits an *image representation*, i.e., $\mathscr{B} = M * \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$. This implies that $\mathscr{B}$ is controllable. In fact, for a polynomial R, controllability of $\mathscr{B}$ is also sufficient for the existence of an image representation, but in the present situation, it is not fully known. A partial necessary and sufficient result for the scalar case is given in [12].

We then have the following theorem.

Theorem 6.2: Let $\mathscr{B}$ be a behavior defined by a pseudorational R, and suppose that $\mathscr{B}$ admits an image representation $\mathscr{B} = \operatorname{im} M*$. Let Φ be as above. Then the following conditions are equivalent:

(i) $\int Q_\Phi$ is a path integral along $\mathscr{B}$;

(ii) there exists $\Psi = \Psi^* \in \mathscr{PW}^{q\times q}[\zeta, \eta]$ such that

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) \tag{17}$$

for all $w \in \mathscr{B}$;

(iii) $\int Q_{\Phi'}$ is a path integral where Φ' is defined by $\Phi'(\zeta, \eta) := M^T(\zeta)\Phi(\zeta, \eta)M(\eta)$;

(iv) $\partial \Phi' = 0$;

(v) there exists $\Psi' = (\Psi')^\bullet = \mathscr{PW}^{q\times q}[\zeta, \eta]$ such that

$$\frac{d}{dt} Q_{\Psi'}(\ell) = Q_{\Phi'}(\ell)$$

for all $\ell \in \mathscr{C}^\infty$, i.e., $\Psi'^\bullet = \Phi'$.

Proof The equivalence of (iii), (iv) and (v) is a direct consequence of the image representation $\mathscr{B} = M * \mathscr{C}^\infty$ and Theorem 4.1. The crux here is that the image representation reduces these statements on $w \in \mathscr{B}$ to the unconstrained ℓ via $w = M * \ell$. The equivalence of (ii) and (v) is also an easy consequence of the image representation: for every $w \in \mathscr{B}$ there exists $\ell \in \mathscr{C}^\infty$ such that $w = M * \ell$.

Now the implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) are obvious. $\square$

We also have the following proposition:

Proposition 6.3: Let $\mathscr{B}$ be as above, admitting an image representation $\mathscr{B} = \operatorname{im} M*$. Suppose that the extended Lyapunov equation

$$X^* * R + R^* * X = \partial \Phi \tag{18}$$

has a solution $X \in \mathscr{E}'(\mathbb{R}^2)^{\mathtt{q}\times\mathtt{q}}$. Then $\int Q_\Phi$ is a path integral.

Outline of Proof Take $w_1, w_2 \in \mathscr{B}$ and $t_1, t_2 \in \mathbb{R}$, and consider $\int_{t_1}^{t_2} Q_\Phi(w_1)$ and $\int_{t_1}^{t_2} Q_\Phi(w_2)$. Since $\mathscr{B}$ admits the image representation $\mathscr{B} = \operatorname{im} M*$, there exist $\ell_1, \ell_2 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ such that $w_i \in M * \ell_i$, $i = 1, 2$.

Suppose that w_1 and w_2 have the same values and derivatives at end points t_1 and t_2. Clearly $w = M * \ell$, for $\ell = \ell_1 - \ell_2$. Then $\ell \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$. This ℓ does not necessarily have a compact support, but suppose for the moment that $\operatorname{supp} \ell \subset [-t_1, t_2]$. Since $R * M = 0$, we have

$$M^* * (\partial \Phi) * M = M^* * (X^* * R + R^* * X) * M = 0. \tag{19}$$

Then the assertion readily reduces to Theorem 4.1. Indeed, we have from Parseval's identity and (19)

$$\begin{aligned} \int_{t_1}^{t_2} Q_\Phi(w) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{w}}^T(-i\omega) \hat{\Phi}(-i\omega, i\omega) \hat{w}(i\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{\ell}}^T(-i\omega) \hat{M}^* (\hat{X}^* \hat{R} + \hat{R} \hat{X}) \hat{M} \hat{\ell}(i\omega) d\omega \\ &= 0 \end{aligned}$$

Hence the integrals $\int_{t_1}^{t_2} Q_\Phi(w_1)$ and $\int_{t_1}^{t_2} Q_\Phi(w_2)$ are equal.

When ℓ does not have compact support, take any $\varepsilon > 0$. It is possible to multiply a $\mathscr{C}^\infty$ function $\chi_\varepsilon(t)$ taking values in $[0, 1]$ as

$$\chi_\varepsilon(t) := \begin{cases} 1, & t \in [t_1, t_2] \\ 0, & t < t_1 - \varepsilon \text{ or } t > t_2 + \varepsilon. \end{cases}$$

Then $\chi_\varepsilon \ell \to \ell$ as $\varepsilon \to 0$. Also,

$$\int_{-\infty}^{\infty} Q_\Phi(M * \chi_\varepsilon \ell) \to \int_{t_1}^{t_2} Q_\Phi(w) dt$$

as $\varepsilon \to 0$. Hence the claim holds for the general case. $\square$

VII. Stability

Let $R \in (\mathscr{E}'(\mathbb{R}, \mathbb{R}^{\mathtt{q}}))^{\mathtt{p}\times\mathtt{q}}$ be pseudorational, and let $\mathscr{B}$ be the autonomous behavior defined by R, i.e.,

$$\mathscr{B} = \{w : R * w = 0\}. \tag{20}$$

We discuss stability conditions in terms of R.

Lemma 7.1: The behavior $\mathscr{B}$ is exponentially stable if and only if

$$\sup\{\operatorname{Re}\lambda : \det \hat{R}(\lambda) = 0\} < 0. \tag{21}$$

Outline of Proof Without loss of generality, we can shift R to left so that $\operatorname{supp} R \subset (-\infty, 0]$. Consider $\overline{R} := \begin{bmatrix} R & I \end{bmatrix}$, and define

$$\overline{\mathscr{B}} := \{\begin{bmatrix} y & u \end{bmatrix}^T : \overline{R} * \begin{bmatrix} y & u \end{bmatrix}^T = 0\}.$$

Then $\mathscr{B} \subset \pi_1(\overline{\mathscr{B}})$, where π_1 denotes the projection to the first component. Hence $\mathscr{B}$ is asymptotically stable if every element of $\pi_1(\overline{\mathscr{B}})$ decays to zero asymptotically. Now note that $\overline{\mathscr{B}}$ is trivially controllable, every trajectory $w \in \overline{\mathscr{B}}$ can be concatenated with zero trajectory as

$$w'(t) = \begin{cases} w(t), & t \geq 0 \\ 0, & t \leq -T \end{cases}$$

for some $T > 0$. Then $\pi_1(w')$ clearly belongs to X^R because $R * w = 0$. According to Facts 5.4, $w(t)$ goes to zero as $t \to \infty$, and this decay is exponential. This proves the claim. □

VIII. LYAPUNOV STABILITY

A characteristic feature in stability for the class of pseudorational transfer functions is that asymptotic stability is determined by the location of poles, i.e., zeros of $\det \hat{R}(\zeta)$. Indeed, as we have seen in Lemma 7.1, the behavior

$$\mathscr{B} = \{w : R * w = 0\},$$

is exponentially stable if and only if $\sup\{\mathrm{Re}\,\lambda : \det \hat{R}(\lambda) = 0\} < 0$, and this is determined how each characteristic solution $e^{\lambda t}a$, $a \in \mathbb{C}^{\mathrm{q}}$ ($\det \hat{R}(\lambda) = 0$), behaves. This plays a crucial role in discussing stability in the Lyapunov theory. We start with the following lemma which tells us how $p \in \mathscr{E}'(\mathbb{R}, \mathbb{R}^{\mathrm{q}})$ acts on $e^{\lambda t}$ via convolution:

Lemma 8.1: For $p \in \mathscr{E}'(\mathbb{R}, \mathbb{R}^{\mathrm{q}})$, $p * e^{\lambda t} = \hat{p}(\lambda) e^{\lambda t}$.

Proof This is obvious for elements of type $\sum \alpha_i \delta_{t_i}$. Since such elements form a dense subspace of $\mathscr{E}'$ ([2]), the result readily follows. □

We now give some preliminary notions on positivity (resp. negativity).

Definition 8.2: The QDF Q_Φ induced by Φ is said to be *nonnegative* (denoted $Q_\Phi \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{q}})$, and *positive* (denoted $Q_\Phi(w) > 0$) if it is nonnegative and $Q_\Phi(w) = 0$ implies $w = 0$.

Let $\mathscr{B} = \{w : R * w = 0\}$ be a pseudorational behavior. The QDF Q_Φ induced by Φ is said to be *$\mathscr{B}$-nonnegative* (denoted $Q_\Phi \overset{\mathscr{B}}{\geq} 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathscr{B}$, and *$\mathscr{B}$-positive* (denoted $Q_\Phi(w) \overset{\mathscr{B}}{>} 0$) if it is $\mathscr{B}$-nonnegative and if $Q_\Phi(w)$ and $w \in \mathscr{B}$ imply $w = 0$. $\mathscr{B}$-nonpositivity and $\mathscr{B}$-negativity are defined if the respective conditions hold for $-Q_\Phi$.

We say that Q_Φ *weakly strictly positive along $\mathscr{B}$* if

- Q_Φ is $\mathscr{B}$-positive; and
- for every $\gamma > 0$ there exists c_γ such that $\overline{a}^T \hat{\Phi}(\overline{\lambda}, \lambda) a \geq c_\gamma \|a\|^2$ for all λ with $\hat{p}(\lambda) = 0$, $\mathrm{Re}\,\lambda \geq -\gamma$ and $a \in \mathbb{C}^q$.

Similarly for *weakly strict negativity along $\mathscr{B}$*.

For a polynomial $\hat{\Phi}$, $\mathscr{B}$-positivity clearly implies the second condition. However, for pseudorational behaviors, this may not be true. Note that we require the above estimate only for the eigenvalues λ, whence the term "weakly".

Theorem 8.3: Let $\mathscr{B}$ be as above. $\mathscr{B}$ is asymptotically stable if there exists $\Psi = \Psi^* \in \mathscr{E}'(\mathbb{R}^2)^{\mathrm{q} \times \mathrm{q}}$ whose elements are measures (i.e., distributions of order 0) such that Q_Ψ is weakly strictly positive along $\mathscr{B}$ and $\overset{\bullet}{\Psi}$ weakly strictly negative along $\mathscr{B}$.

Proof Let $\exp_\lambda : \mathbb{R} \to \mathbb{C} : t \mapsto e^{\lambda t}$ be the exponential function with exponent parameter λ. Lemma 7.1 implies that we can deduce stability of $\mathscr{B}$ if there exists $c > 0$ such that $a \exp_\lambda(\cdot) \in \mathscr{B}$, $a \neq 0$ implies $\mathrm{Re}\,\lambda \leq -c < 0$. Now take any $\gamma > 0$ and consider $a \exp_\lambda(\cdot) \in \mathscr{B}$ with $\mathrm{Re}\,\lambda \geq -\gamma$. Then

$$Q_\Psi(a \exp_\lambda) = \overline{a}^T \hat{\Psi}(\overline{\lambda}, \lambda) a (\exp_{2\mathrm{Re}\,\lambda}(\cdot)),$$

and

$$Q_{\overset{\bullet}{\Psi}}(a \exp_\lambda) = (2\mathrm{Re}\,\lambda) \overline{a}^T \hat{\Psi}(\overline{\lambda}, \lambda) a (\exp_{2\mathrm{Re}\,\lambda}(\cdot)).$$

Hence the weak strict positivity of $Q_\Psi(w)$ implies $a^T \hat{\Psi}(\overline{\lambda}, \lambda) a \geq c_\gamma \|a\|^2 \geq 0$. Also since the elements of $\hat{\Psi}$ are measures, $a^T \hat{\Psi}(\overline{\lambda}, \lambda) a \leq \beta \|a\|^2$. On the other hand, weak strict negativity of $Q_{\overset{\bullet}{\Psi}}$ implies

$$Q_{\overset{\bullet}{\Psi}}(a \exp_\lambda(\cdot)) \leq -\rho \|a\|^2.$$

Combining these, we obtain

$$(2\mathrm{Re}\,\lambda) \cdot c \|a\|^2 \leq -\rho \|a\|^2$$

and hence $\mathrm{Re}\,\lambda \leq -\rho/(2c) < 0$ for such λ. Since other λ's satisfying $\hat{p}(\lambda) = 0$ satisfy $\mathrm{Re}\,\lambda < -\gamma$, this yields exponential stability of $\mathscr{B}$. □

Remark 8.4: In the theorem above, the condition that the elements of Ψ be measures is necessary to guarantee the boundedness of $\Psi(\overline{\lambda}, \lambda)$. However, for the single variable case, one can reduce the general case to this case. See the next section.

Proposition 8.5: Under the hypotheses of Theorem 8.3,

$$Q_\Psi(w)(0) = -\int_0^\infty Q_{\overset{\bullet}{\Psi}}(w) dt \tag{22}$$

Proof Note that

$$Q_\Psi(w)(t) - Q_\Psi(w)(0) = \int_0^t Q_{\overset{\bullet}{\Psi}}(w) dt.$$

By Theorem 8.3, $Q_\Psi(w)(t) \to 0$ as $t \to \infty$, the result follows. □

IX. THE BÉZOUTIAN

We have seen that exponential stability can be deduced from the existence of a suitable positive definite quadratic form Ψ that works as a Lyapunov function. The question then hinges upon how one can find such a Ψ. The objective of this section is to show that for the single-variable case, the Bézoutian gives a universal construction for obtaining a Lyapunov function.

In this section we confine ourselves to the case $q = 1$, that is, given $p \in \mathscr{E}'$, we consider the behavior

$$\mathscr{B} = \{w : p * w = 0\}.$$

Define the Bézoutian $b(\zeta, \eta)$ by

$$b(\zeta, \eta) := \frac{p(\zeta)p(\eta) - p(-\zeta)p(-\eta)}{\zeta + \eta}. \tag{23}$$

Note that this expression belongs to the class $\mathscr{PW}[\zeta, \eta]$, and hence its inverse Laplace transform is a distribution having compact support. Let us further assume that p is a measure, i.e., distribution of order 0. If not, $\hat{p}(s)$ possess (stable) zeros, and we can reduce $\hat{p}(s)$ to a measure by extracting such zeros. For details, see [10].

We now have the following theorem:

Theorem 9.1: Suppose that $p \in \mathscr{E}'$ is a measure. The following conditions are equivalent:

(i) $\mathscr{B} = \{w : p * w = 0\}$ is exponentially stable;
(ii) there exists $\rho > 0$ such that $\sup\{\lambda : \hat{p}(\lambda) = 0\} \le -\rho$;
(iii) $Q_b \ge 0$ and the pair $(p, p^{\check{}})$ is coprime in the following sense: there exists $\phi, \psi \in \mathscr{E}'$ such that

$$p * \phi + p^{\check{}} * \psi = \delta \tag{24}$$

(iv) Q_b is weakly strictly positive definite, and $Q_{\underset{b}{\bullet}}$ is weakly strictly negative definite.

Proof The equivalence of (i) and (ii) are already shown.

Note first that for $w \in \mathscr{B}$, we have

$$\frac{d}{dt} Q_b(w) = |p * w|^2 - |p^{\check{}} * w|^2 = -|p^{\check{}} * w|^2 \tag{25}$$

because $p * w = 0$.

(i) ⇒ (iii) Since $\mathscr{B}$ is asymptotically stable, we have from (25)

$$Q_b(w)(0) = \int_0^\infty |p^{\check{}} * w)|^2 dt \ge 0.$$

Now exponential stability implies that $\sup\{\lambda : \hat{p}(\lambda) = 0\} \le -\rho$ for some $\rho > 0$ and also ([10])

$$\left| \frac{1}{\hat{p}(\zeta)} \right| \le C, \quad \operatorname{Re}\zeta \ge 0. \tag{26}$$

This implies that for λ_n, $n = 1, 2, \ldots$ with $\hat{p}(\lambda_n) = 0$, $|\hat{p^{\check{}}}(\lambda_n)| = |\hat{p}(-\lambda_n)| \ge (1/C)$. Then by the coprimeness condition [12, Theorem 4.1], $(p, p^{\check{}})$ satisfies the Bézout identity (24).

(iii) ⇒ (i) and (iv) By (25), we have for $w \in \mathscr{B}$,

$$\frac{d}{dt} Q_b(w) \le 0.$$

We show that $(d/dt)Q_b(w) < 0$. Suppose that $(d/dt)Q_b(w) = 0$ for some w, i.e., $p^{\check{}} * w = 0$ according to (25). Then $w \in \mathscr{B} \cap \mathscr{B}_{p^{\check{}}}$, where

$$\mathscr{B}_{p^{\check{}}} := \{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}) : p^{\check{}} * w = 0\}.$$

Since $(p, p^{\check{}})$ satisfies (24), $\mathscr{B} \cap \mathscr{B}_{p^{\check{}}} = 0$ because for $w \in \mathscr{B} \cap \mathscr{B}_{p^{\check{}}}$

$$w = (\phi * p + \psi * p^{\check{}}) * w = 0.$$

Hence $(d/dt)Q_b(w) < 0$. Again by [12, Theorem 4.1] and (24) there exists $c > 0$ such that

$$|\hat{p^{\check{}}}(\lambda_n)| \ge c > 0$$

for all λ_n with $\hat{p}(\lambda_n) = 0$. Then

$$-|\hat{p^{\check{}}}(\lambda_n)|^2 = -|\hat{p}(-\lambda_n)|^2 \le -c^2. \tag{27}$$

Hence $Q_{\underset{b}{\bullet}}$ is weakly strictly negative definite. Furthermore,

$$Q_b(\exp_{\lambda_n}(\cdot)) = \frac{-\hat{p}(-\lambda_n)\hat{p}(-\overline{\lambda}_n)}{2\operatorname{Re}\lambda_n} \exp_{2\operatorname{Re}\lambda_n}(\cdot)$$

Now take any $\gamma > 0$, and suppose $\operatorname{Re}\lambda_n \ge -\gamma$. Then by (27)

$$\frac{-\hat{p}(-\lambda_n)\hat{p}(-\overline{\lambda}_n)}{2\operatorname{Re}\lambda_n} \ge \frac{|\hat{p}(-\lambda_n)|^2}{2\gamma} \ge \frac{c^2}{2\gamma} > 0.$$

Hence Q_b is weakly strictly positive definite. Hence by Theorem 8.3, $\mathscr{B}$ is asymptotically stable. This proof also shows that (iii) implies (iv).

(iv) ⇒ (i) This is already proved in Theorem 8.3. □

When p belongs to the class $\mathscr{R}$ as defined in [11], we can relax condition (iv) as follows:

Corollary 9.2: Let p be pseudorational, and suppose that p belong to the class $\mathscr{R}$ as defined in [11]. Then $\mathscr{B}$ is exponentially stable if Q_b is $\mathscr{B}$-positive.

This is obvious since there are only finitely many zeros of $\hat{p}(\zeta)$ in $\{\zeta : -\rho < \operatorname{Re}\zeta < 0\}$ for arbitrary ρ. A simplified proof of Theorem 8.3 without requiring uniformity works, just as in the finite-dimensional case. Note that we do not have to require weak strict positivity.

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