Remarks on Modeling Interconnected Systems

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1 Introduction

One of the opportunities created by the ever-continuing increase in computing power is the possibility to model physical systems accurately and in detail. Mathematical models are the basis for simulation and of model-based analysis and design. Complex interconnected systems are usually modeled by tearing, zooming, and linking. *Tearing* refers to viewing a system as an interconnection of sub-systems, *zooming* to the process of modeling the individual sub-systems, and *linking* to setting up the relations that describe the interaction between the sub-systems.

In order for such modeling procedures to be useable and valuable, appropriate mathematical concepts are needed. Appropriate concepts for describing a system and its sub-systems, appropriate concepts for describing the architecture how the sub-systems are laid out, and appropriate concepts for describing the interaction between sub-systems. The viewpoint put forward in classical system theory is to regard a system in terms of inputs and outputs, to describe the architecture in terms of signal flow graphs, and to think of the interaction of sub-systems as assigning the output of one system to be the input of another one.

The aim of this note (see [1] for a more elaborate discussion) is to put forward the thesis that for the modeling of physical systems the input/output point of view misses the essence of what is involved. Input/output thinking is in general inappropriate for describing an individual physical system, signal flow graphs do not capture the architecture of an interconnected system, and output-to-input assignment is a limited way of viewing the interaction between physical systems. The classical input/output view fails in the simplest examples, as electrical circuits, mechanical devices, and civil engineering structures. It is therefore unthinkable that this view could suddenly become appropriate for complex systems, as those met in biology, ecology, and economics. The input/output view is perhaps suitable for signal processing and for certain aspects of feedback control, but for the modeling of physical systems this view falls short and is even misleading.

2 The behavior of a system

Consider a system that interacts with its environment through terminals, and assume that the aim is to describe the dynamics of the variables involved in this interaction. For the sake of concreteness, it is convenient to think for instance of an electrical circuit with wires (‘terminals’) sticking out of it, and the aim is to describe the voltages and currents on these terminals, or of a mechanical system with pins (‘terminals’) attached to it, and the aim is to describe the positions of these terminals and the forces acting on them.

Associated with each terminal there are a number of variables, as a potential and a current for an electrical wire, and, for a mechanical pin, a position and a force in the 1D case, or a position, a force, an attitude, and a torque in the 2D or 3D case. For the great majority of applications more than one variable is involved for a single terminal. Assume that there are N terminals and that the variables associated with the k-th terminal belong to the space $\mathbb{W}_k$. The internal structure of the system and the parameter values of the elements lead to constraints on the possible time-functions $w = (w_1, w_2, \dots, w_N) : \mathbb{R} \to \mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \cdots \times \mathbb{W}_N$

that are possible. The set of possible time-trajectories is called the *behavior* and is denoted by $\mathscr{B}$. Hence $w \in \mathscr{B}$ means that the dynamical laws of the system allow the trajectory w, while $w \notin \mathscr{B}$ means that the dynamical laws forbid the trajectory w. The behavior is the end point of a modeling process.

It could very well be that, after scrutinizing $\mathscr{B}$, $\mathbb{W}$ can be partitioned into the product of two sets, $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$, with the variables in $\mathbb{U}$ acting as inputs and the variables in $\mathbb{Y}$ acting as outputs, with inputs and outputs defined in the usual way, and with $\mathscr{B}$ specified through a transfer function, or an inout/output map, or another input/output structure. It could even be that in addition to the input and output spaces $\mathbb{U}$ and $\mathbb{Y}$, there is a state space $\mathbb{X}$ such that $\mathscr{B}$ can be described through a system of differential equations $\frac{d}{dt}x = f(x,u),\ y = h(x,u)$ or a DAE. However, such structured models are not the end point of first principles modeling, and they should not be the starting point of a theory of dynamical systems.

Physical laws impose constraints on variables. Laws do not state that one set of variables causes another set of variables, but that variables can happen simultaneously only if certain relations are satisfied. That is what the gas law states, that is Newton's second law says, that is Maxwell's equations express. Arrows, signal directions, and pathways can help in explaining the functioning of some phenomena in a complex systems, but for most interactions they are inappropriate and a consequence of *post hoc ergo propter hoc* reasoning. Viewing a dynamical system as a behavior treats all variables on the same level. Input/output thinking requires splitting variables that act on one and the same terminal, and to depict these variables on different terminals. This has understandably led to much confusion pedagogically.

How does one decide whether a circuit is voltage or current driven? Even a simple resistor can be viewed as either current or voltage driven, so what should be taken to be the input and the output? If we consider a simple mechanical system with two pins, then the forces acting on these springs could be constrained to be equal, as is the case for a system consisting of massless springs and dampers, or they could be independent, as is the case when there is a mass in the system. In the former cases, we cannot view both forces as inputs, while in the latter case, we can view both forces as inputs. And this input/output partitioning question becomes more and more intractable as a system becomes more and more complex. The input/output point of view is simply not appropriate as a starting point for the description of physical models. A physical system is not a signal processor.

Caveat. Even the choice of the terminal variables is not a trivial matter. For electrical terminals it is natural to choose the potential of the terminal and the current flowing into the circuit as the terminal variables, even though only potential differences are physically measurable. For 1D mechanical systems it is natural to choose the position of the terminal and the force acting on the system as the terminal variables, even though only displacements (position differences) are physically measurable. The reason of this choice of variables will come up again when discussing interconnections.

3 The interconnection architecture

The layout of an interconnected system can be formalized as a *graph with leaves* (a graph with leaves is like an ordinary graph but in which some of the edges, the leaves, are incident to only one vertex, rather than two as is the case for ordinary edges). The vertices of the graph correspond to sub-systems, the edges correspond to connected terminals, and the leaves correspond to external terminals that allow the interconnected system to interact with its environment.

Note that this architecture has a hierarchical structure, since the sub-system in a particular vertex can in turn be viewed as an interconnection architecture of sub-sub-systems. The edges and the leaves that are incident to the vertex corresponding to the sub-system in the original graph become the leaves of the sub-system architecture in the new graph.

Caveat. In this interconnection architecture the sub-systems are in the vertices, and the connections are in the edges. This is in contrast with the graph structure used in classical circuit theory, where the sub-systems are in the edges and the vertices take care of the connections. The graph structure used in classical circuit theory is convenient for modeling circuits consisting of elements that are 2-terminal 1-ports, as R's, L's, and C's, and 2-terminal multiports, as transformers and gyrators. However, the graph structure of classical circuit theory has serious disadvantages. For example, it does not deal with 3-terminal elements as transistors, Y's, and Δ's. Furthermore, it misses hierarchy, since an interconnection of 2-terminal 1-ports can have any number of external terminals, and does not need to be a 2-terminal multiport. It does not even need to have an even number of external terminals.

4 Interconnection as variable sharing

Assume that two terminals of two sub-systems are interconnected. What relations does this impose on the variables on these two terminals? Usually these relations are very simple and can be viewed as *variable sharing*. If both terminals are electrical terminals, interconnection imposes the constraints $V_1 = V_2, I_1 + I_2 = 0$, with V_1 the potential of the first and V_2 of the second terminal, and with I_1 the current flowing in the first and I_2 in the second terminal (both counted > 0 when the current flows into the respective circuits). Note that interconnection identifies potentials, one of the reasons why it is convenient to work with potentials (rather than potential differences) as terminal variables. If both terminals are 1D mechanical terminals, interconnection imposes the constraints $q_1 = q_2, F_1 + F_2 = 0$, with the q's positions and the F's forces. Note that interconnection identifies positions (rather than displacements), one of the reasons why it is convenient to work with positions as terminal variables. If both terminals are thermal terminals, interconnection imposes the constraints $T_1 = T_2, Q_1 + Q_2 = 0$, with the T's temperatures and the Q's heat flows. If both terminals are hydraulic, interconnection imposes the constraints $p_1 = p_2, f_1 + f_2 = 0$, with the p's pressures and the f's mass-flows.

The interconnection of two terminals results in variable sharing (up to the choice of signs). This is distinctly different from the output-to-input assignment implied by signal flow graphs. Connection does not mean signal transmission from one system to another, but it means that variables are equated, that additional constraints are imposed. A physical system is not a signal processor.

Caveat. The input/output approach leads to elementary difficulties of the following sort. Suppose that two mechanical systems are interconnected by welding or screwing or gluing two pins (terminals) together. Assume that intuition, or analysis of the behavior has led to the conclusion that the forces on these terminals act as inputs, and that the positions are outputs. Interconnection of the two terminals leads to $q_1 = q_2$ and $F_1 + F_2 = 0$, that is, it leads to equating two inputs and to equating two outputs. But this is forbidden by input/output thinking. Signal flow graphs cannot cope in a symmetric way with the interconnection of two identical masses, an operation carried out by a welder and a bricklayer dozens of times a day. Why should such elementary deficiencies not carry over to biological systems? Why should chemistry be different?

5 The behavior of the interconnected system

The models for the sub-systems specify the behavior of the variables on the terminals corresponding to the edges and the leaves that are incident to the vertex corresponding to the sub-system. These behaviors involve the variables on the edges and the leaves incident to that vertex. Each edge in turn corresponds to two terminals belonging to the subsystems that correspond to the vertices to which that edge is incident. Expressing the variable-sharing constraints on the variables on these terminals leads to the interconnection laws. The combination of the behaviors of the subsystems and the interconnection laws defines the over-all behavior involving the variables on the terminals corresponding to all the edges and leaves. Viewing the variables on the edges as auxiliary, *latent*, variables and the variables on the leaves as *manifest*, as the variables the model aims at, leads to the behavior of the variables on the external terminals of the interconnected system. This procedure usually involves many equations with many auxiliary variables, but for certain classes of equations, there exist effective algorithms for elimination of these auxiliary variables (see [1] for details).

6 Let's get the physics right

Modeling is the most neglected aspect of theoretical engineering in general and, more specifically, in Systems and Control. As argued above, the input/output point of view is unsuitable for modeling physical systems and signal flow graphs are unsuitable for modeling their interconnection. A physical system is not a signal processor. The many books and courses entitled *Signals and Systems* do not deal with physical systems and their interconnection in a realistic way, and could better be renamed *Signals and Signal Processors*. To do justice to models involving physical systems, one should use a different set of ideas. In the previous sections the basic ingredients of a more cogent approach have been sketched. Backing off from input/output thinking is required if Systems and Control takes modeling of complex systems, as those met in biology, and the applications of the ideas of the field in physical domains earnestly.

The neglect of physical modeling is also evident in areas which use probabilistic models. To begin with, the interpretation of probability is seldom explained. This would pose no problem if the interpretation of a particular concept is evident, but in the case of probability with its highly divergent interpretations, this neglect is objectionable. Often, it is vaguely implied that a frequentist interpretation is used. But then, why can measurement inaccuracy be modeled as an additive stochastic process? Why should an unmeasured nuisance signal in system identification be a realization of a stochastic process? Why should a communication channel change a 0 to a 1 and a 1 to a 0 with a fixed relative frequency? Where would this regularity of error generation come from? Undoubtedly, in many circumstances, these probabilistic methods can be rationalized, but this should be done. By and large, research and teaching happens without bothering to explain the physics that leads to probabilistic models.

How can such a situation have occurred? Why is the physics of models not more prominently present in areas as Systems and Control? Why are probability, inputs, outputs, and signal flow graphs used without scrutinizing the physical situations to which they claim to pertain? The explanation, it seems, lies in the sociology of science. Normal science uses an established paradigm in which to operate. Courses use existing textbooks with exercises that are clearly posed and that emphasize mathematical issues. When a problem is cast in an input/output setting with disturbances modeled as stochastic processes, the context becomes an established, often sophisticated, mathematical framework, with results that may be difficult to obtain and to prove, but that are unambiguously verifiable mathematically. The results are judged by their mathematical depth and difficulty. In other words, the explanation lies in the *Lure of Mathematics*. There is no other explanation.

References

[1] J.C. Willems, The behavioral approach to open and interconnected systems, *Control Systems Magazine*, volume 27, pages 46–99, 2007.