Behavioral Controllability and Coprimeness for A Class of Infinite-Dimensional Systems

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***Abstract*— Behavioral system theory has become a successful framework in providing a viewpoint that does not depend on a priori notions of inputs/outputs. In particular, this theory provides such notions as controllability, without an explicit reference to state space formalism. One also obtains several interesting consequences of controllability, for example, direct sum decomposition of the signal space with a controllable behavior $\mathscr{B}$ as a direct summand. While there are some attempts to extend this theory to infinite-dimensional systems, for example, delay systems, the overall picture seems to remains incomplete. This article extends this theory, particularly the notion of controllability, to a well-behaved class of infinite-dimensional systems called pseudorational. A crucial notion in connection with this is the Bézout identity, and we relate a recent result to the context of behavioral controllability. We establish the relationships with such notions as image representation, direct sum decompositions.**

I. Introduction

Behavioral system theory has become a successful framework in providing a viewpoint that does not depend on the a priori notions of inputs/outputs. An introductory and tutorial account is given in [7], [3]. In particular, this theory successfully provides such notions as controllability, without an explicit reference to state space formalism. One also obtains several interesting and illuminating consequences of controllability, for example, direct sum decomposition of the signal space with a controllable behavior $\mathscr{B}$ as a direct summand.

There are some attempts to extend this theory to infinite-dimensional systems, for example, delay systems, and some rank conditions for behavioral controllability have been obtained; see, e.g., [4], [2]. While these results give a nice generalization of their finite-dimensional counterparts, the overall picture still needs to be further studied in a more general and perhaps abstract setting. For example, one wants to see how the notion of zeros and poles can affect controllability in an abstract setting. This is to some extent accomplished in [4], [2], but we here intend to give a theory in a more general, and unified setting, and provide a framework in a well-behaved class of infinite-dimensional systems called pseudorational.

In [8], [9], the first author introduced the notion of pseudorational impulse responses. Roughly speaking, an impulse response is said to be pseudorational if it is expressible as a ratio of distributions with compact support, e.g., $G = p^{-1} * q$(While we used $q^{-1} * p$ in [8], [9] and in other papers, it is customary to use p for a denominator, so we have switched the notation to $p^{-1} * q$.) This leads to an input/output relation

$$p * y = q * u, \tag{1}$$

and various system properties have been studied associated to it: for example,

1) realization procedure
2) complete characterization of spectra in terms of the denominator of the transfer function
3) stability characterization in terms of the spectrum location
4) relations between controllability and coprimeness conditions.

These are summarized in a survey paper [11].

The representation (1) is also suitable for behavioral study. The difference here is that behavioral theory is not restricted by the causality constraints, and hence somewhat a crucial condition on supports of p and q in [8], [9] can then be removed. This leads to a different condition for unimodularity of distributions, and hence coprimeness conditions.

The paper is organized as follows: Section 2 introduces pseudorationality, and then generalizes this notion to the behavioral context. We briefly describe a state space formalism and realization procedures in Section 3. Spectral properties and eigenfunction completeness are also reviewed, and they are crucial in characterizing coprimeness properties. Section 4 introduces the notions of behavioral controllability in the present context, and gives various criteria for controllability. Of particular importance is the Bézout identity. Section 5 gives a proof for a condition for the Bézout identity, with generalization to the multivariable case.

II. PSEUDORATIONALITY

We first review the classical notion of pseudorationality as introduced in [8]. Let $\mathscr{E}'(\mathbb{R}_-)$ denote the space of distributions having compact support contained in the negative half line $(-\infty,0]$. Distributions such as Dirac's delta δ_a placed at $a \le 0$, its derivative δ_a' are examples of elements in $\mathscr{E}'(\mathbb{R}_-)$. For basic notation and nomenclature, see the Appendix.

An impulse response function $\mathtt{p}\times\mathtt{m}$ matrix G ($\operatorname{supp} G \subset [0,\infty)$) is said to be *pseudorational* ([8]) if there exist matrices P and Q having entries in $\mathscr{E}'(\mathbb{R}_-)^{\mathtt{p}\times\mathtt{p}}$ and $\mathscr{E}'(\mathbb{R}_-)^{\mathtt{p}\times\mathtt{m}}$, respectively, such that

1) $G = P^{-1} * Q$ where the inverse is taken with respect to convolution;
2) $\operatorname{ord}\det P^{-1} = -\operatorname{ord}\det P$, where $\operatorname{ord}\psi$ denotes the order of a distribution ψ [5], [6] (for a definition, see the Appendix).

As an example, consider the delay-differential equation:

$$\begin{aligned} \dot{x}(t) &= x(t-1) + u(t) \\ y(t) &= x(t). \end{aligned}$$

This can be expressed as $x = (\delta' - \delta_1)^{-1} * u$. Shifting the time axis by 1, we obtain $x = (\delta_{-1}' - \delta)^{-1} * \delta_{-1} * u$, and this is pseudorational.

We will extend this notion as to be appropriate to the study of behaviors. To this end, we introduce the following.

Definition 2.1: Let R be an $\mathtt{p}\times\mathtt{w}$ matrix with entries in $\mathscr{E}'(\mathbb{R})$. It is said to be *pseudorational* if there exists a $\mathtt{p}\times\mathtt{p}$ submatrix P such that

1) $P^{-1} \in \mathscr{D}_+'(\mathbb{R})$ exists with respect to convolution
2) $\operatorname{ord}\det P^{-1} = -\operatorname{ord}\det P$.

Note that we have removed the constraint that the support of R be contained in $(-\infty,0]$. However, note that we can make it belong to $\mathscr{E}'(\mathbb{R}_-)^{\mathtt{p}\times\mathtt{w}}$ by suitably shifting its element to the left.

To introduce behaviors in this context, let $L^2_{loc}(-\infty,\infty)$ be the space of locally square integrable functions. We give the following definition:

Definition 2.2: Let R be pseudorational as defined above. The *behavior* $\mathscr{B}$ defined by R is given by

$$\mathscr{B} := \{w \in (L^2_{loc}(-\infty,\infty))^{\mathtt{w}} | R * w = 0\} \tag{2}$$

The convolution $R * w$ is taken in the sense of distributions. Since R has compact support, this convolution is always well defined [5].

Example 2.3: Let R be defined as

$$R := [\delta' - \delta_1, -\delta']$$

This yields a behavioral equation

$$\frac{d}{dt} w_1(t) - w_1(t-1) - w_2'(t) = 0. \tag{3}$$

Clearly, this can be also written as

$$\frac{d}{dt} w_1(t+1) - w_1(t) - w_2'(t+1) = 0,$$

because the behavior defined by (3) is shift-invariant. In the latter expression, R is given by

$$R := [\delta_{-1}' - \delta, -\delta_{-1}'].$$

The behavior $\mathscr{B}$ is *time-invariant* in the sense that $\sigma_t \mathscr{B} \subset \mathscr{B}$ for every $t \in \mathbb{R}$, where σ_t is the left shift semigroup in $L^2_{loc}(-\infty,\infty)$ defined by

$$(\sigma_t w)(s) := w(s+t). \tag{4}$$

This clearly follows from the definition (2) since $R * (\sigma_t w) = R * \delta_{-t} * w = \delta_{-t} * R * w = 0$.

We introduce behaviors in a wider space of signals, namely in the space of distributions. Let $\mathscr{D}'$ be the space of distributions on $\mathbb{R}$, and let R be pseudorational. The *distributional behavior* $\mathscr{B}_{\mathscr{D}'}$ defined by R is given by

$$\mathscr{B}_{\mathscr{D}'} := \{w \in (\mathscr{D}')^{\mathtt{w}} | R * w = 0\}. \tag{5}$$

III. STATE SPACE REPRESENTATIONS

Let $R \in \mathscr{E}'(\mathbb{R})^{\mathtt{p}\times\mathtt{w}}$ be pseudorational. Suppose, without loss of generality, that R is partitioned as $R = \begin{bmatrix} P & Q \end{bmatrix}$ such that P satisfies the invertibility condition of Definition 2.1, i.e., we consider the kernel representation

$$P * y + Q * u = 0 \tag{6}$$

where $w := \begin{bmatrix} y & u \end{bmatrix}^T$ is partitioned conformably with the sizes of P and Q.

When $G := P^{-1} * Q$ belongs to $L^2_{loc}(-\infty,\infty)^{\mathtt{p}\times\mathtt{m}}$, and $\operatorname{supp} G$ is contained in $[0,\infty)$, it is possible to give a state space model to (6).

To this end, it is possible to invoke realization theory developed in [8]; see also [11] for a comprehensive survey materials. We here content ourselves with a simplest model.

Let $\Gamma := L^2_{loc}[0,\infty)$ be the space of all *locally* Lebesgue square integrable functions with obvious family of semi-norms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}.$$

This is the *projective limit* of spaces $\{L^2[0,n]\}_{n>0}$. This space is equipped with a shift operator

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \ge 0, s \ge 0. \tag{7}$$

Define X^P by

$$X^P := \{x \in \Gamma^{\mathtt{p}} \,|\, \pi(P * x) = 0\}, \tag{8}$$

where π is the truncation to $(0,\infty)$. It is easy to check X^P is a σ_t-invariant closed subspace of $\Gamma^{\mathtt{p}}$. Take this X^q as the state space, and let $T(t) := \sigma_t$ be the state transition semigroup. Since X^P is easily seen to be σ_t-invariant, $T(t)$ defines a C_0-semigroup. Denote by A the infinitesimal generator of T. Let $B := G(\cdot)$. Since $G = P^{-1} * Q$, Gu belongs to X^P for every $u \in \mathbb{R}^{\mathtt{m}}$. Then the state space model

$$\frac{d}{dt} x_t = A x_t + B u(t) \tag{9}$$

$$y(t) = x_t(0) \tag{10}$$

for $x_t(\cdot) \in X^P$ realizes the convolution input/output relation (6) [8]. The evaluation mapping

$$X^P \ni x \mapsto x(0)$$

is a densely defined closed operator in X^P, and the domain of A is

$$D(A) = \{x \in X^P \,|\, dx/dt \in X^P\}.$$

Given $x_0 \in X^P$, and an input u, the solution of the state space model (9) is given by

$$x_t = T(t)x_0 + \int_0^t G(t-\tau)u(\tau)d\tau = T(t)x_0 + \pi(P^{-1} * Q * u). \tag{11}$$

where $T(t)$ is the shift semigroup generated by A. $T(t)$ is actually the left shift semigroup σ_t restricted to X^P.

A remarkable feature is that the spectrum of A is completely characterized in terms of the zeros of the Laplace transform of P.

Theorem 3.1: The spectrum $\sigma(A)$ is given by

$$\sigma(A) = \{\lambda \,|\, \det \hat{P}(\lambda) = 0\} \tag{12}$$

Furthermore, every $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. The corresponding eigenfunction for $\lambda \in \sigma(A)$ is given by $e^{\lambda t}v$ where $\hat{P}(\lambda)v = 0$. Similarly for generalized eigenfunctions such as $te^{\lambda t}v'$. See [9] for details. The resolvent set $\rho(A)$ is its complement. For each $\lambda \in \rho(A)$, the resolvent operator $(\lambda I - A)^{-1}$ is compact.

Since $\hat{P}$ (and hence $\det \hat{P}$) is an entire function of exponential type by the Paley-Wiener theorem 8.1, the spectrum is discrete, and with finite multiplicities.

IV. Controllability and Coprimeness

We now introduce the notion of controllability [3] in the present context.

Definition 4.1: Let R be pseudorational, and $\mathscr{B}$ the behavior associated to it. $\mathscr{B}$ is said to be *controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq T$ (see Fig. IV).

In other words, every pair of trajectories can be concatenated into one trajectory that agrees with them in the past and future.

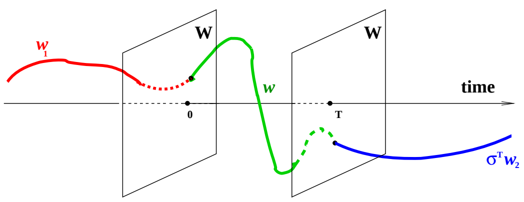

Fig. 1. Concatenation of trajectories

We also introduce an extended notion of controllability as follows:

Definition 4.2: Let R be pseudorational, and $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (5). $\mathscr{B}_{\mathscr{D}'}$ is said to be *distributionally controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w|_{(-\infty,0)} = w_1$ on $(-\infty, 0)$, and $w|_{(T,\infty)} = \sigma_{-T}w_2$ on (T, ∞).

We now introduce various notions of coprimeness.

Definition 4.3: The pair (P,Q), $P,Q \in \mathscr{E}'(\mathbb{R})$ is said to be *spectrally coprime* if $\hat{P}(s)$ and $\hat{Q}(s)$ have no common zeros. It is *approximately coprime* if there exist sequences $\Phi_n, \Psi_n \in \mathscr{E}'(\mathbb{R})$ such that $P * \Phi_n + Q * \Psi_n \to \delta I$ in $\mathscr{E}'(\mathbb{R})$. The pair (P,Q) is said to satisfy the *Bézout identity* (or simply *Bézout*), if there exists $\Phi, \Psi \in \mathscr{E}'(\mathbb{R})$ such that

$$P * \Phi + Q * \Psi = \delta I, \tag{13}$$

Or equivalently,

$$\hat{P}(s)\hat{\Phi}(s) + \hat{Q}(s)\hat{\Psi}(s) = I \tag{14}$$

for some entire functions $\hat{\Phi}, \hat{\Psi}$ satisfying the Paley-Wiener estimate (39).

It is well known [3] that controllability admits various nice characterizations in terms of coprimeness, image representation, full rank conditions, etc. We here attempt to give a generalization of such results to the present context. To this end, we confine ourselves to the simplest scalar case, i.e., $p = m = 1$. We will also assume that q also satisfies the condition that the zeros of $\hat{q}(s)$ is contained in a half plane $\{s \,|\, \mathrm{Re}\, s < c\}$ for some $c \in \mathbb{R}$.

Theorem 4.4: Let R be pseudorational, and suppose without loss of generality that R is of form $R := \begin{bmatrix} p & q \end{bmatrix}$ where p satisfies the invertibility condition in Definition 2.1. Let $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (5). Then the following statements are equivalent:

1) $\mathscr{B}_{\mathscr{D}'}$ is controllable.
2) There exist $\psi, \phi \in \mathscr{E}'(\mathbb{R})$ such that $p * \phi + q * \psi = \delta$.
3) $\mathscr{B}_{\mathscr{D}'}$ admits an image representation, i.e., there exists M over $\mathscr{E}'(\mathbb{R})$ such that for every $w \in \mathscr{B}_{\mathscr{D}'}$, there exists $\ell \in C^\infty(\mathbb{R})$ such that $w = M * \ell$.
4) $\mathscr{B}_{\mathscr{D}'}$ is a direct summand of $\mathscr{D}'$, i.e., there exists an distributional behavior $\mathscr{B}'$ such that $\mathscr{D}' = \mathscr{B}_{\mathscr{D}'} \oplus \mathscr{B}'$.
5) Let $\Lambda := \{\lambda \in \mathbb{C} | \hat{p}(\lambda) = 0\}$. Suppose that the algebraic multiplicity of each zero $\lambda \in \Lambda$ is globally bounded. There exist $k \geq 0$ and $c > 0$ such that

$$|\lambda^k \hat{q}(\lambda)| \geq c, \quad \forall \lambda \in \Lambda. \tag{15}$$

Proof of 2) $\Rightarrow$ 3), 4), 5), and 3) $\Rightarrow$ 1) Suppose 2) holds. Substituting $\lambda \in \Lambda$, we obtain $\hat{q}(\lambda)\hat{\phi}(\lambda) = 1$. Since ϕ has compact support, $\hat{\Phi}$ is at most of polynomial order [5]. Taking λ^k to be such an order, 5) follows.

Consider the mapping

$$\pi_{\mathscr{B}_{\mathscr{D}'}} : \mathscr{D}' \ni \ell \mapsto \begin{bmatrix} q \\ -p \end{bmatrix} * \ell \in \mathscr{D}'. \tag{16}$$

We claim that this gives an image representation. Since

$$\begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} q \\ -p \end{bmatrix} * \ell = 0,$$

the image of (16) clearly belongs to $\mathscr{B}_{\mathscr{D}'}$. We need only to prove that this mapping is surjective. Take any $\begin{bmatrix} y \\ u \end{bmatrix}$ in

$\mathscr{B}_{\mathscr{D}'}$, and set

$$\ell := \begin{bmatrix} \psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix}. \tag{17}$$

It follows that

$$\begin{aligned} \begin{bmatrix} q \\ -p \end{bmatrix} * \begin{bmatrix} \psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} q*\psi & q*\phi \\ -p*\psi & -p*\phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} \\ &= \begin{bmatrix} \delta - p*\phi & q*\phi \\ -p*\psi & q*\psi-\delta \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} \\ &= \begin{bmatrix} y-\phi*(q*u-p*y) \\ u-\psi*(q*u-p*y) \end{bmatrix} \\ &= \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned}$$

Hence $\pi_{\mathscr{B}_{\mathscr{D}'}}$ is surjective and 3) follows.

To prove 4), first note that

$$\begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix}$$

is a unimodular matrix in $\mathscr{E}'(\mathbb{R})$. In fact, its determinant is $p*\phi+q*\psi=\delta$. Define $\tilde{\mathscr{B}}_{\mathscr{D}'}$ by

$$\tilde{\mathscr{B}}_{\mathscr{D}'} := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid \begin{bmatrix} -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} = 0 \right\}.$$

We first claim $\mathscr{B}_{\mathscr{D}'} \cap \tilde{\mathscr{B}}_{\mathscr{D}'} = \{0\}$. Indeed, If $\begin{bmatrix} y & u \end{bmatrix}^T$ belongs to both $\mathscr{B}_{\mathscr{D}'}$ and $\tilde{\mathscr{B}}_{\mathscr{D}'}$,

$$\begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} = 0$$

which readily yields $\begin{bmatrix} y & u \end{bmatrix}^T = 0$ because of the unimodularity of the matrix on the right.

Now take any $\begin{bmatrix} y & u \end{bmatrix}^T$ in $(\mathscr{D}')^{\mathtt{w}}$. Define

$$\begin{bmatrix} v \\ x \end{bmatrix} := \begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix}. \tag{18}$$

Then

$$\begin{aligned} \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix}^{-1} * \begin{bmatrix} v \\ x \end{bmatrix} \\ &= \begin{bmatrix} \phi & -q \\ \psi & p \end{bmatrix} * \begin{bmatrix} v \\ x \end{bmatrix} \\ &= \begin{bmatrix} -q \\ p \end{bmatrix} * x + \begin{bmatrix} \phi \\ \psi \end{bmatrix} v. \end{aligned}$$

The first term belongs to $\mathscr{B}_{\mathscr{D}'}$ while the second term to $\tilde{\mathscr{B}}_{\mathscr{D}'}$. Hence the correspondence (18) is surjective to $\mathscr{D}'^{w}$, and $\mathscr{D}'^{w} = \mathscr{B}_{\mathscr{D}'} \oplus \tilde{\mathscr{B}}_{\mathscr{D}'}$. Furthermore, since this correspondence is clearly continuous with respect to the topology of $\mathscr{D}'$, this direct sum decomposition is topological.

3) $\Rightarrow$ 1) Now if 4) holds, then the behavioral representation is

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -M \end{bmatrix} \begin{bmatrix} y \\ u \\ \ell \end{bmatrix} = 0.$$

Since I is unimodular, the behavior is trivially controllable ([3]). □

To prove the implication 1) $\Rightarrow$ 2), we first prove the following:

Proposition 4.5: Let R be pseudorational, and suppose that $\mathscr{B}$ is controllable. Suppose further that P^{-1} belongs to $L^2_{loc}(-\infty,\infty)$. Then there exist matrices Ψ, Φ with elements in $L^2[a,b]$ for some $a,b>0$ such that $P*\Psi+Q*\Phi=\delta I$.

Proof Since we can shift Q^{-1} arbitrarily, we may assume without loss of generality that Q^{-1} belongs to $L^2_{loc}[0,\infty)$ and $P,Q \in \mathscr{E}'(\mathbb{R}_-)$. Partition w conformably with P and Q as $w = \begin{bmatrix} y \\ u \end{bmatrix}$. Then $\mathscr{B}$ is described by

$$P*y+Q*u=0. \tag{19}$$

We can invoke realization theory for $P^{-1}*Q$ as described in Section III. Then by (11) every solution of (19) can be written as

$$x(t) = x_{\text{free}}(t) + \pi(P^{-1}*Q*u) \tag{20}$$

where π is the truncation to $(0,\infty)$, and $x_{\text{free}}(t)$ is the solution to

$$P*x=0.$$

Hence every $x_{\text{free}}(t)$ should take the form $P^{-1}*x_0$ for some x_0. Since $\mathscr{B}$ is controllable, there exist $T>0$ and $(y,u)\in\mathscr{B}$ such that

$$(y,u) = \begin{cases} (0,0) & t<-T \\ (P^{-1}e_{ij},0) & t>0 \end{cases}$$

This readily implies that there exists $\Psi \in (L^2[-T,0])^{\bullet}$ such that $\pi P^{-1}*Q*\Psi = P^{-1}$. In other words,

$$P^{-1}*Q*\Psi = P^{-1} - \Phi$$

for some $\Phi \in (L^2[-T,0])^{\bullet}$. Multiplying P from the left yields

$$P*\Phi+Q*\Psi=\delta I.$$

□

Proof of 1) $\Rightarrow$ 2) To show this implication, one needs only to extend the above argument to the case $P^{-1}\in\mathscr{D}'_+(\mathbb{R})$. By taking the "state space"

$$\overline{X}^P := \{x \in \mathscr{D}' \mid \operatorname{supp} x \subset [0,\infty), Q*x \in \mathscr{E}'(\mathbb{R}_-)\}. \tag{21}$$

One readily see that this is a completion of X^P in $\mathscr{D}'$. The state transition formula (11) works equally well, and then we obtain matrices Φ and Ψ not in L^2 but in $\mathscr{E}'(\mathbb{R}_-)$. This readily yields the desired conclusion. □

Remark 4.6: Note that we do not need $\mathscr{B}_{\mathscr{D}'}$ to be a scalar behavior in the proof above. We can also modify some results with C^∞ or $L^2_{loc}(-\infty,\infty)$ behaviors, but we omit the details.

To complete the equivalence in Theorem 4.4, we need to prove **5) $\Rightarrow$ 2)**. This will be given in the next section.

V. Bézout Identity

As we have seen in the previous section, the Bézout identity plays a crucial role in characterizing controllability.

This is first obtained in [10] for the case of $\mathscr{E}'(\mathbb{R}_-)$. We here extend this result to $\mathscr{E}'(\mathbb{R})$ with an indication of a generalization to the multivariable case.

For the case of measures, characterizing the Bézout identity

$$p*\phi + q*\psi = \delta \tag{22}$$

is essentially the question of characterizing maximal ideals in the quotient space the space of measures modulo (p), and this is a question related to the Gel'fand representation theory [10].

Let us first describe the relationship of the Bézout condition in $\mathscr{E}'(\mathbb{R})$ to that in $\mathscr{E}'(\mathbb{R}_-)$.

Lemma 5.1: Let (p,q) be as in Theorem 4.4. Then (p,q) is a Bézout pair if and only if there exists $L>0$, and $\alpha,\beta \in \mathscr{E}'(\mathbb{R}_-)$ such that

$$p*\alpha + q*\beta = \delta_{-L}. \tag{23}$$

Proof Suppose (23) holds. Then by taking the convolutions with δ_L on both sides, we obtain

$$p*\alpha*\delta_L + q*\beta*\delta_L = \delta.$$

Conversely, if

$$p*\phi + q*\psi = \delta$$

for $\phi,\psi \in \mathscr{E}'(\mathbb{R})$, then it is clear that by suitably convolving δ_{-L} with both sides we obtain

$$p*\delta_{-L_1}*\phi*\delta_{-L_2} + q*\delta_{-L_1}*\psi*\delta_{-L_2} = \delta_{-L}$$

where $L = L_1 + L_2$ and $p*\delta_{-L_1}$, $q*\delta_{-L_1}$, $\phi*\delta_{-L_2}$, $\psi*\delta_{-L_2}$ all belong to $\mathscr{E}'(\mathbb{R}_-)$. This completes the proof. □

This lemma states that the Bézout condition for elements in $\mathscr{E}'(\mathbb{R})$ can be tested whether (23) holds by suitably shifting p and q to make them belong to $\mathscr{E}'(\mathbb{R}_-)$. This is because δ_a is a unit in $\mathscr{E}'(\mathbb{R})$ (although it is never so in $\mathscr{E}'(\mathbb{R}_-)$ unless $a=0$).

Hence it is enough to check condition (23) for elements p and q already belonging to $\mathscr{E}'(\mathbb{R}_-)$. Now note that (23) holds if and only if there exists $a<0$ such that $\max\{r(p), r(q)\} = a$. But this can be avoided by suitably shifting p and q to the right to make $\max\{r(p), r(q)\} = 0$. So let us hereafter assume that one of p and q, say, p satisfies $r(p)=0$.

We now want to characterize the identity (22).

The following theorem is obtained in [10]:

Theorem 5.2: Let $p^{-1}*q$ be pseudorational such that $r(p)=0$. Suppose that there exists a nonnegative integer m such that

$$|\lambda_n^m \hat{q}(\lambda_n)| \geq c, n = 1,2,\ldots \tag{24}$$

Then the pair (p,q) is Bézout.

The rest of this section is devoted to the proof of this theorem.

Note first that (22) means $[q] \cong [\delta]$ modulo p, namely $[q]$ is invertible over the quotient space $\mathscr{E}'(\mathbb{R}_-)/(p)$. This is characterized in [10]. We here briefly review the main outline of the proof and indicate the basic idea, with indications for the generalization to the multivariable case.

We first observe that $\mathscr{E}'(\mathbb{R}_-)$ and $\mathscr{E}[0,\infty)$ are dual to each other with respect to the following duality:

$$\langle \alpha, f\rangle := (\alpha * f)(0), \quad \alpha \in \mathscr{E}'(\mathbb{R}_-), f \in \mathscr{E}[0,\infty). \tag{25}$$

It is easy to see that (25) defines a separately continuous bilinear form on $\mathscr{E}'(\mathbb{R}_-) \times \mathscr{E}[0,\infty)$, and they are indeed dual to each other.

The outline of the proof is as follows:

1) To characterize the invertibility of $[q]$ in $\mathscr{E}'(\mathbb{R}_-)/(p)$, we view $\mathscr{E}'(\mathbb{R}_-)/(p)$ as the dual of a closed subspace (denoted $\mathscr{E}^{(p)}$) of $\mathscr{E}[0,\infty)$.
2) $\mathscr{E}^{(p)}$ admits a very simple representation. Due to the condition $r(p)=0$, $\mathscr{E}^{(p)}$ is eigenfunction complete [9], and every element admits an infinite series expansion: $x = \sum_n \alpha_n e^{\lambda_n t}$.
3) With respect to the duality (25), the action of q on $e^{\lambda_n t}$ is given by
$$\langle q, e^{\lambda_n t}\rangle = (q * e\lambda_n t)(0) = \hat{q}(\lambda_n). \tag{26}$$
4) Using (26), we see that the candidate for $\psi := [q]^{-1}$ should satisfy $\hat{\psi}(\lambda_n) = 1/\hat{q}(\lambda_n)$.
5) Whether this formula leads to a well defined element in $\mathscr{E}'(\mathbb{R}_-)/(p)$ is the crucial step.

Let us start with the following lemma:

Lemma 5.3: The dual space of $\mathscr{E}'(\mathbb{R}_-)/(p)$ is given by

$$\begin{aligned}(\mathscr{E}'(\mathbb{R}_-)/(p))' &= \{x \in \mathscr{E}[0,\infty) | p*x \in \mathscr{E}'(\mathbb{R}_-)\} \\ &=: \mathscr{E}^{(p)}.\end{aligned} \tag{27}$$

Proof Since $(\mathscr{E}'(\mathbb{R}_-))' = \mathscr{E}[0,\infty)$, we have

$$\begin{aligned}(\mathscr{E}'(\mathbb{R}_-)/(p))' &= \{x \in \mathscr{E}[0,\infty) \,|\, \langle \alpha, x\rangle = 0 \forall \alpha \in (q)\} \\ &= \{x \in \mathscr{E}[0,\infty) \,|\, \delta_{-t}*q*x = 0\} \\ &= \{x \in \mathscr{E}[0,\infty) \,|\, p*x \in \mathscr{E}'(\mathbb{R}_-)\}.\end{aligned}$$

□

From here on suppose for simplicity that the zeros λ_n of $\hat{q}(s)$ are all simple zeros, and that m in (24) is 0 (although these are not at all necessary).

Lemma 5.4: Under the hypothesis of $r(p)=0$,

$$\mathrm{span}\{e^{\lambda_n t}\}_{n=1}^{\infty} \tag{28}$$

is dense in $\mathscr{E}^{(p)}$. Furthermore, every $x \in \mathscr{E}^{(p)}$ admits an expansion of type

$$x = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t} \in \mathscr{E}^{(p)} \tag{29}$$

that converges with respect to the topology of $\mathscr{E}[0,\infty)$.

Proof That the subset (28) is dense is similar to that given in [9]. (The proof given there is for $L^2_{loc}[0,\infty)$ instead of $\mathscr{E}[0,\infty)$ but the proof is similar).

We want to show (29).

Take any $x \in \mathscr{E}^{(p)}$. Then there exists a sequence x_i such that

$$x_i(t) = \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t}$$

and $x_i \to x \in \mathscr{E}^{(p)}$ as $i \to \infty$. This means that every derivative of finite order $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m e^{\lambda_n t}$ converges to $(d/dt)^m x$. In particular, $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m$ is convergent for every $m \geq 0$. By the same argument as given for (32) below, $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m e^{\lambda_n t}$ is uniformly and absolutely convergent on every bounded interval $[0,T]$.

We first claim that for each fixed n, the sequence $\{\alpha_n^{(i)}\}$ is convergent as $i \to \infty$. By the Hahn-Banach theorem, take a continuous linear functional $f_n \in (\mathscr{E}^{(p)})'$ such that

$$\langle f_n, e^{\lambda_j t} \rangle = \delta_{jn}$$

where δ_{jn} denotes Kronecker's delta. Then $\langle f_n, \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t} \rangle = \alpha_n^{(i)}$. By continuity, the left-hand side converges to $\langle f_n, x \rangle$, so that $\alpha_n^{(i)}$ is convergent, as $i \to \infty$.

Now define $\alpha_n := \lim_{i \to \infty} \alpha_n^{(i)}$. Then

$$x(t) = \lim_{i \to \infty} x_i(t) = \lim_{i \to \infty} \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t}.$$

Since the last term converges locally uniformly and absolutely, we can exchange the order of lim and $\sum$, and see that the last term is equal to $\sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}$. The same can be said of every finite-order derivative, and this shows that the series

$$\sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}$$

actually converges in $\mathscr{E}^{(p)}$. This completes the proof. □

Note that the proof above works equally well for the multivariable case. All we need to do is to replace α_n by a corresponding eigenvector.

In view of the Lemma above, we are led to the definition

$$\langle q, \psi \rangle = \sum_{n=1}^{\infty} \alpha_n / q(\lambda_n). \tag{30}$$

We need to show that this gives a continuous linear form on $\mathscr{E}^{(p)}$.

This is guaranteed by the following lemma:

Lemma 5.5: Let

$$x = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t} \in \mathscr{E}^{(p)}. \tag{31}$$

Then for every r,

$$\sum_{n=1}^{\infty} \alpha_n n^r < \infty. \tag{32}$$

In particular,

$$\sum_{n=1}^{\infty} |\alpha_n| < \infty. \tag{33}$$

Sketch of Proof The idea of the proof is that if (31) is convergent (which is guaranteed by Lemma 5.4), then it means a very strong convergence since it should converge with respect to the topology of $\mathscr{E}[0,\infty)$. In particular, the derivative of an arbitrary order should converge. Since λ_n are the zeros of an entire function $\hat{p}(s)$ of exponential type, it grows with order as fast as n [1, Chapter 8]. This essentially yields (32). A complete proof may be found in [10]. □

VI. Systems with Commensurable Delays

It is proven in [2], [4] that systems with commensurable delays are controllable if and only if the matrix R has constant rank for all $\lambda \in \mathbb{C}$. This is somewhat mysterious in the light of Theorem 4.4, since condition 5) requires that there be no "asymptotic cancellation at ∞," while the result by [2], [4] requires only "no cancellation in $\mathbb{C}$."

Roughly speaking, this is due to the following structure. Consider $q(s,z)$ as a polynomial of two variables. Then $q(s,z)$ as $s \to \infty$ can go to zero only at most with polynomial order in s,z. Hence if there is an asymptotic cancellation as $s \to \infty$, this can be removed by multiplying a suitable factor s^m, because such a cancellation must be of polynomial order. Hence condition (15) works.

Example 6.1: Consider the pair $(z, sz-1)$, $z = e^s$. This pair has an asymptotic cancellation for $z = 1/s$, as $s \to \infty$. But this cancellation can be removed by multiplying s to the first component z. This is why the pair $(e^s, se^s - 1)$ is Bézout over $\mathscr{E}'(\mathbb{R}_-)$ while it is not over the space of measures where such a multiplication by s is not allowed.

VII. Concluding Remarks

We have shown some basic facts about pseudorational behaviors. While we are mostly confined to scalar systems, the proofs given here depart quite much from the classical ones in that they do not make use of canonical forms (e.g., Smith-MacMillan form) or rank-test conditions which are confined to more restricted contexts. It is hoped that some controllability criteria can be generalized to the multivariable case, as indicated in the last section.

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Appendix: Notation and Nomenclature

Let $\mathscr{E}'(\mathbb{R}_-)$ denote the space of distributions having compact support contained in the negative half line $(-\infty,0]$. Distributions such as Dirac's delta δ_a placed at $a \leq 0$, its derivative δ'_a are examples of elements in $\mathscr{E}'(\mathbb{R}_-)$. In contrast, $\mathscr{E}'(\mathbb{R})$ denotes the space of distributions with compact support, not necessarily contained in $(-\infty,0]$. A distribution α is said to be of *order* at most m if it can be extended as a continuous linear functional on the space of m-times continuously differentiable functions. Such a distribution is said to be of *finite order*. The largest number m, if one exists, is called the *order* of α ([5], [6]). The delta distribution δ_a, $a \in \mathbb{R}$ is of order zero, and its derivative δ'_a is of order one, etc. A distribution with compact support is known to be always of finite order ([5], [6]).

For a distribution $\alpha \in \mathscr{E}'(\mathbb{R})$, define real numbers $\ell(\alpha)$ and $r(\alpha)$ by

$$\ell(\alpha) := \inf\{t | t \in \operatorname{supp} \alpha\}, \tag{34}$$

$$r(\alpha) := \sup\{t | t \in \operatorname{supp} \alpha\}. \tag{35}$$

We need various properties of the Laplace transform of elements in $\mathscr{E}'(\mathbb{R})$. Above all, the following Paley-Wiener theorem is most important:

Theorem 8.1 ([5]): A complex analytic function $f(s)$ is the Laplace transform of a distribution $\phi \in \mathscr{E}'(\mathbb{R})$ if and only if $f(s)$ is an entire function that satisfies the following growth estimate for some $C > 0, a > 0$ and integer $m \geq 0$:

$$|f(s)| \leq C(1+|s|)^m e^{a|\mathrm{Re}\, s|}. \tag{36}$$

In particular, $f(s) = \hat{\phi}(s)$ for some $\phi \in \mathscr{E}'(\mathbb{R}_-)$ if and only if it satisfies the estimate

$$\begin{aligned} |\hat{f}(s)| &\leq C(1+|s|)^m e^{a \mathrm{Re}\, s}, \mathrm{Re}\, s \geq 0, \\ &\leq C(1+|s|)^m, \mathrm{Re}\, s \leq 0 \end{aligned} \tag{37}$$

for some $C > 0, a > 0$ and integer $m \geq 0$. In this case, the support of ϕ is contained in $[-a,0]$

The zeros of $\hat{f}(s)$ are discrete, and each zero has a finite multiplicity. This in particular implies the following Hadamard factorization for $\hat{f}(s)$ [1]:

$$\hat{f}(s) = s^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{38}$$

Since there are no finite accumulation point for $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$.

Theorem 8.2 ([5]): A necessary and sufficient condition for a complex function $\chi(s)$ to be the Laplace transform of a distribution $f \in \mathscr{E}'(\mathbb{R}_-)$ is that

1) $\chi(s)$ is an entire function; and
2) $\chi(s)$ satisfies the growth estimate

$$\begin{aligned} |\chi(s)| &\leq C(1+|s|)^m e^{a \mathrm{Re}\, s}, \mathrm{Re}\, s \geq 0, \\ &\leq C(1+|s|)^m, \mathrm{Re}\, s \leq 0. \end{aligned} \tag{39}$$

for some $C > 0, a > 0$ and integer $m \geq 0$.
We will refer to (39) as the Paley-Wiener estimate.

Note that the zeros of $\chi(s)$ are discrete, and each zero has a finite multiplicity, because $\chi(s)$ is entire.

Since $\chi(s)$ is an entire function of exponential type, the following Hadamard factorization holds ([1]):

$$\chi(s) = s^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{40}$$

Since there are no finite accumulation point for $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$.

Hence for a pseudorational impulse response G, its Laplace transform, i.e., *transfer function*, $\hat{G}(s)$ is $\hat{p}(s)/\hat{q}(s)$, and hence it is the ratio of entire functions satisfying the estimate (39) above.

Let $\Omega := \lim_{\rightarrow} L^2[-n,0]$ denote the *inductive limit* of the spaces $\{L^2[-n,0]\}_{n>0}$; it is the union $\cup_{n=1}^{\infty} L^2[-n,0]$, endowed with the finest topology that makes all injections $j_n : L^2[-n,0] \to \Omega$ continuous; see, e.g., [6]. Dually, $\Gamma := L^2_{loc}[0,\infty)$ is the space of all *locally* Lebesgue square integrable functions with obvious family of seminorms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}.$$

This is the *projective limit* of spaces $\{L^2[0,n]\}_{n>0}$. Ω is the space of past inputs, and Γ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the following natural *left shift* semigroups:

$$(\sigma_t \omega)(s) := \begin{cases} \omega(s+t), & s \leq -t, \\ 0, & -t < s \leq 0, \end{cases} \quad \omega \in \Omega, t \geq 0, s \leq 0. \tag{41}$$

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \geq 0, s \geq 0. \tag{42}$$

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